On the plane representation of the homaloidal surfaces which have a twisted cubic as multiple curve.

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1. *Introductory.*

The most important application of the rational point transformations between two spaces lies in the construction of algebraical surfaces possessing singularities of various kinds and the investigation of their properties.

When a plane is transformed by such a transformation into a rational algebraical surface (homaloid) the geometry of the surface is immediately derivable from the geometry of the plane.

This derivation is facilitated by the introduction of an additional step, viz., the discovery of the plane representation of lowest order. The rational space transformation of course supplies us with a plane representation of the surface, but in general it is not that of lowest order.

A plane section of the surface will have as corresponding curve on the plane which corresponds to the surface a curve having an order equal to that of the surface and possessing multiple points at the intersections of the F-curve system with the plane. We have to transform this curve by Cremona transformations till its order is the lowest possible.

The representation of certain quartic and quintic homaloids has been discussed analytically by Clebsch.* Amongst the surfaces treated by him are (a) quartics with a single nodal line, (b) ruled quartics with a double twisted cubic, (c) quintics with a double twisted cubic, (d) quintics with two non-intersecting nodal lines. In the following paper the plane representation of a widely extended class of homaloidal surfaces having a twisted cubic,

* *Math. Ann.*, Bd. I. and II.

either proper or degenerate, as curve of any order of multiplicity will be investigated. These surfaces may have also chords of the cubic as multiple lines. For the derivation of the surfaces by the method of rational point transformations the reader is referred to a paper on "The Three-Dimensional Transformations founded on the Twisted Cubic and its Chord System," (*Proc.* E. M. S., Vol. XXXIV., p. 133). It will be found that the surfaces of Clebsch are the simplest cases of the class referred to. The notation here employed is that of the paper quoted above.

I.

Surfaces with a proper twisted cubic as multiple curve.
First Series.

2. The series is doubly infinite, and the representative surface ${}_pF_n$ is of order $(2p+2)\,n-3$ and has a twisted cubic Σ_3 as curve of multiplicity $(p+1)\,n-2$. It has also α_i chords of Σ_3 as lines of multiplicity $i\,n$ where α_i and i satisfy the equations

$$\Sigma\alpha_i\,i^2 = p^2-1 \quad \text{and} \quad \Sigma\alpha_i\,i = 3\,(p-1),$$

and in addition $(2p+2)(n^2-n)+3$ chords of Σ_3 as simple lines. p and n are any positive integers.

To obtain the plane representation of lowest order we introduce the mechanism of the space transformation, the two spaces being in this case coincident. The F-systems, however, need not be supposed to be coincident.

Take an arbitrary plane π and its corresponding surface ${}_pF_n$. A plane section of ${}_pF_n$ transforms into a curve on π of order $(2p+2)\,n-3$ with three nodes of order $(p+1)\,n-2$ at the points A, B, C, where Σ_3' meets π, α_i nodes of order $i\,n$ and $3+(2p+2)(n^2-n)$ simple points. Taking A, B, C as F-points of a quadratic transformation, this curve becomes a curve of order $(p+1)\,n$ having α_i nodes of order $i\,n$ and $3+(2p+2)(n^2-n)$ simple points. A further transformation of this curve by a Cremona transformation of order p having as F-system the α_i points yields a curve of order $(p+1)\,n$ of precisely similar nature. Hence the order $(p+1)\,n$ is the lowest possible.

3. *Plane representation of* Σ_3.

Corresponding to Σ_3 we get a surface of order $(4p+4)\,n-(2p+6)$ which meets π in a curve of this order having α_i nodes of order

$i(2n-1)$, nodes at A, B, C of order $2(p+1)n-(p+4)$, and $3+(2p+2)(n^2-n)$ double points.

This curve becomes, on applying the quadratic transformation, a curve of order $2(p+1)n-p$ having α_i nodes of order $i(2n-1)$ and $3+(2p+2)(n^2-n)$ double points. A further transformation of this curve by the Cremona transformation of order p gives a curve of order $2(p+1)n-1$ having α_i nodes of order $2in$ and $3+(2p+2)(n^2-n)$ double points. No further diminution of this order is possible, and hence this curve is the required representation of Σ_3.

Again, corresponding to a point on Σ_3 we get a curve of order $(p+1)n-2$ lying on a rational ruled surface of order $2p$ having α_i chords of Σ_3 as lines of multiplicity i. This surface meets π in a curve of order $2p$ having nodes of order p at A, B, C and α_i nodes of order i. Applying to this the above transformations, we obtain a straight line. Consequently, the points on π which correspond to any point of Σ_3 form a group of $(p+1)n-2$ collinear points. The lines which contain these groups envelope a conic. For if a point describe Σ_3, the surfaces of order $2p$ envelope a surface of order $4p$ having Σ_3 as curve of multiplicity $2p$ and the α_i lines as lines of multiplicity $2i$.

This envelope is the correspondent of the quartic developable generated by the tangents to Σ_3. The curves of order $2p$ on π therefore envelope a curve of order $4p$ having nodes of order $2p$ at A, B, C and α_i nodes of order $2i$. Transforming this curve as before, we obtain a conic. This conic will be denoted by C_2.

4. *Plane representation of the multiple chords.*

The curve corresponding to a point on a multiple chord of order in is of order in. It lies on a ruled surface of order $2i$ having Σ_3 as curve of multiplicity i and α_i chords of multiplicity β_{ij} where $\sum_{j=1} i\beta_{ij} = pi$. Transforming the section of this surface by π, we obtain a point, viz., one of the multiple points of order in.

Corresponding to the points of intersection of a multiple chord of order in with Σ_3 we get a degenerate curve of order $(p+1)n-2$ composed of a curve of order $(p+1-i)n-2$ and a curve of order in. The collinear groups on π corresponding to these points consist of in points coalescing at one of the α_i points and a group of $(p+1-i)n-2$ points whose line of collinearity passes through the i-point.

The F-point system on π consequently consists of α_i points of order in (which will be referred to as the i-points) and $3+(2p+2)(n^2-n)$ simple points.

5. *Systems of curves on* ${}_pF_n$.

An arbitrary line on π transforms into a rational curve of order $(p+1)n$ on ${}_pF_n$ meeting Σ_3 in $2(p+1)n-1$ points. This curve is the intersection of a conicoid which contains Σ_3 with ${}_pF_n$. The system is doubly infinite.

A rational curve of order p on π having α_i nodes of order i also transforms into a curve of order $(p+1)n$, and the system is also doubly infinite. Any curve of the first system intersects a curve of the second in p points. The curves of the first do not in general meet any of the multiple chords, whereas the curves of the second meet a chord of order in in i points. Any two curves belonging to the same system intersect in one point.

Generally, a curve of order l on π which has an i-point as node of order γ_i and nodes of orders a, b, c, d, etc., at the simple F-points transforms into a curve of order $l(p+1)n-\Sigma\alpha_i\gamma_i in-\Sigma a$. In particular, a line through an i-point transforms into a curve of order $(p+1-i)n$, which meets the corresponding multiple chord in one point.

If a line touch the conic C_2, the corresponding curve has a node of order $(p+1)n-2$ on Σ_3. It also touches a fixed curve of order $2(p+1)n$, which corresponds to C_2. The latter curve is the intersection of the surface with the quartic developable of the tangents to Σ_3. This system of nodal curves is singly infinite.

A tangent to C_2 through a i-point transforms into a curve of order $(p+1-i)n$, having a node of order $(p+1-i)n-2$ at the intersection of the corresponding i-chord with Σ_3. The number of these curves is $2\Sigma\alpha_i$.

Let the curve on π which represents Σ_3 be denoted by $C\Sigma_3$. A tangent to C_2, which also touches $C\Sigma_3$ at a point belonging to a collinear group of $(p+1)n-2$ points transforms into a curve having a node on Σ_3, at which two of the branches touch each other. Two of the sheets of the surface at this point have the same tangent plane, and such a point is a pinch point on Σ_3.

6. *Pinch points on* Σ_3.

The number of pinch points on Σ_3 is not easy to determine directly.*

For a surface of order $2m-1$ having Σ_3 as curve of multiplicity $m-1$, but without multiple chords, I have found by an extension of the analytical method of Clebsch that the number of pinch points on Σ_3 is $2m^2-8$. The presence of multiple chords reduces this number.

The diminution in the number of common tangents to C_2 and $C\Sigma_3$ due to the presence of an i-point of order in is $2in(in-1)$. The total diminution is therefore $2\Sigma\alpha_i in(in-1)$ or

$$2n^2(p^2-1)-6n(p-1).$$

The number of pinch points on Σ_3 is consequently

$$2[(p+1)n-1]^2-8-2n^2(p^2-1)+6n(p-1)$$

or
$$(4p+4)n^2+2n(p-5)-6.$$

7. *Triple tangent planes.*

A plane section of ${}_pF_n$ through a simple chord of Σ_3 is represented on π by a curve of order $(p+1)n$ having a node at one of the simple F-points. The planes of these sections are bitangent planes to the surface, and we have consequently $3+(2p+2)(n^2-n)$ pencils of such planes. In each pencil a finite number of planes will be triple tangent planes, and when this is the case the curve of order $(p+1)n$ will have an additional node. The number of curves of order m belonging to a pencil and having λ_i nodes of order i which possess an additional node is $3(m-1)^2-\Sigma(i-1)(3i+1)\lambda_i$.† Substituting for m, λ and i, we get for the number of triple tangent planes through a simple chord

$$(6p+6)n^2-12n-4+\Sigma\alpha_i.$$

The total number of such planes is therefore

$$[(2p+2)(n^2-n)+3][(6p+6)n^2-12n-4+\Sigma\alpha_i].$$

This number does not in general exhaust the triple tangent planes which the surface can possess. For particular surfaces

* A direct determination is given in § 22.

† Noether (*Math. Ann.*, Bd. III., p. 223), and Cayley (*Crelle*, Bd. 63). For a simple analytical proof see *Proc. L. M. S.*, III., p. 14.

of the series a plane section of order $(2p+2)n-3$ may be degenerate and composed of two curves which intersect in 3 points which do not lie on Σ_3 or on a multiple chord. Such planes are also triple tangent planes. Or again, the surface may have additional lines, though these will not be chords of Σ_3, and through them it will be possible in general to describe a definite number of triple tangent planes.

8. *Nature of the F-system of points on π.*

Since the plane sections of ${}_pF_n$ are triply infinite in number, it follows that the system of F-points on π, consisting of α_i points of order in and $3+(2p+2)(n^2-n)$ simple points, must be such that the system of curves of order $(p+1)n$ through them is also triply infinite.

For values of $n > 2$ this condition will not be satisfied by an arbitrary choice of the F-points.

The F-point system forms in this case what Cayley called a special system. Then if we put $p=1$ and $n=3$, we get a nonic surface whose plane representation is of order 6, and there are 27 simple F-points. Now, a sextic being determined completely by 27 arbitrary points, it follows that the 27 F-points are of a special kind. Instead of imposing 27 conditions on the sextic, they must in effect impose only 24. To find an explanation for this we refer again to the space transformation. For the particular case of the nonic surface we have a triply infinite number of such surfaces having Σ_3 as curve of multiplicity 4, and containing an F-curve of order 24. Hence on the plane π we have a triply infinite system of curves of order 9 with nodes of 4 at A, B, C and 24 simple points. Transforming by a quadratic transformation with A, B, C as F-points, we obtain a system triply infinite of sextics through 27 points. We thus infer that it is possible through the 27 F-points to pass a triply infinite system of sextics.

9. *Direct derivation of the plane representation of ${}_pF_n$.*

The results contained in the previous articles may be obtained as follows.

Any point on ${}_pF_n$ is connected with a point Q of π by means of the unique chord of Σ_3 through P. The quadratic transformation with A, B, C as F-points yields as the correspondent of Q at a point

Q'. P and Q' are taken as corresponding points on ${}_pF_n$ and π respectively. The correspondence is evidently (1, 1).

A plane section of ${}_pF_n$ becomes on π a curve of order $(p+1)n$, having α_i nodes of order in and $3+(2p+2)(n^2-n)$ simple points. If H be any point on Σ_3, not on a multiple chord, we obtain, by taking the chords of Σ_3 through the points of the different sheets of the surface consecutive to H, a group of $(p+1)n-2$ points lying on a conic through A, B, C. These conics envelope a tricuspidal quartic with cusps at A, B, C. This quartic transforms into a conic touching the sides of the triangle A, B, C. The analytical method employed by Clebsch (*Math. Ann.*, I.) in the case of the quintic is substantially identical with this.

A direct method such as this is, of course, only of limited application.

Second Series.

10. Another remarkable series of surfaces of the same type as the above is obtained by transforming a cubic surface which contains Σ_3 in space (1). Choosing 3 of its 6 lines which are chords of Σ_3 as F-lines and Σ_3 as F-cubic, and transforming it by a cubic transformation into a space (1′), we get a plane. Transforming the cubic into space (2), we get a surface of order $(2p+2)n+2p-3$ having Σ_3 as curve of multiplicity $(p+1)n+p-2$, α_i chords of order $i(n+1)$ and $6+(2p+2)(n^2-1)$ simple chords. In what follows the results only will be given, the details of their derivation being omitted.

The plane representation of lowest order is of order $(p+1)n+p$. The F-point system on π consists of α_i points of multiplicity $i(n+1)$ and $6+(2p+2)(n^2-1)$ simple points. Σ_3 is represented by a curve of order $2(p+1)n+2p-1$ having the i-points as points of order $2i(n+1)$ and the simple points as double points.

The number of proper pinch points on Σ_3 is

$$(4p+4)n^2+n(6p-6)+2p-10.$$

The number of triple tangent planes through simple chords is

$$[6+(2p+2)(n^2-1)]\,[n^2(6p+6)-18n-6pn+\Sigma\alpha_i-5].$$

The surfaces of the series are in order intermediate to those of the first.

11. *General observations on the two series.*

The most general type of homaloid having Σ_3 as multiple curve is obtained by putting $p=1$. The two series can in this case be combined into a single one. The order of the representative surface is $2m-1$, Σ_3 being of multiplicity $m-1$, where m is any positive integer. There are no multiple chords, and the number of simple chords is m^2+2. The order of the plane representation is $m+1$. There are $2m^2-8$ pinch points on Σ_3, and there are $(m^2+2)(3m^2-7)$ triple tangent planes through simple chords of Σ_3.

Putting $m=2$ we get the well-known representation of the non-singular cubic surface.

Putting $m=3$ we obtain the results of Clebsch for the quintic. On the quintic there are 11 lines and 55 conics, and the planes of these conics are also triple tangent planes. The representative of Σ_3 in the case of the cubic is a unicursal curve of order 5 with 6 double points. In the case of the quintic it is a hyperelliptic curve of deficiency 4 and order 7 with 11 double points. For values of m greater than 3 the curve which represents Σ_3 is connected with transcendents of Abelian type.

The presence of a multiple chord will restrict the generality of the surface and cause a diminution in the number of simple chords. Let the surface of order $2m-1$ be supposed to possess a multiple chord of order i, then, since two curves of order $m+1$ on π must intersect in $2m-1$ points outwith the F-point system, we must have, if ν be the number of simple F-points,

$$(m+1)^2-i^2-\nu=2m-1,$$

giving

$$\nu=m^2+2-i^2.$$

Thus a chord of order i diminishes the number of simple chords by i^2.

II.

Σ_3 *is degenerate and composed of a conic* σ_2 *and a line* σ_1 *which meets the conic.*

First Series.

12. The representative surface is of order $(2p+1)n-2$, σ_1 being of multiplicity $(p+1)n-2$ and σ_2 of multiplicity $np-1$. There are α_i chords of order $i\,n$, and $2+(2p+1)(n^2-n)$ simple chords.

The plane representation is again of order $(p+1)n$ The curve of order $(p+1)n$ corresponding to a plane section of the surface has a multiple point a of order n, α_i points of order in, and $2+(2p+1)(n^2-n)$ simple points.

The curve $C\sigma_1$ which represents σ_1 is of order $(p+1)n-1$, and $C\sigma_2$ which represents σ_2 is of order $(p+1)n$.

On $C\sigma_1$ the point a is of order $n-1$, and on $C\sigma_2$ of order $n+1$. The i-points are points of order in, and the simple points simple points on each.

To the point a corresponds the residual intersection of the plane of the conic σ_2 with the surface, viz., a curve of order n. It is rational, and has a node of order $n-1$ at the point where σ_1 meets σ_2.

A line through a transforms into a curve of order pn, which meets σ_1 in pn points; it must consequently be a section through σ_1. The curve has a node of order $pn-1$ on σ_2.

The collinear group of $n-1$ points corresponding to a point on σ_2 lies on a line which envelopes a. The group of $(p+1)n-2$ points corresponding to a point of σ_1 lies on a line which envelopes a point b on $C\sigma_1$. The line joining ab meets $C\sigma_1$ and $C\sigma_2$ in $pn-1$ points which are groups for both. These points are fixed, and the line ab corresponds to the section of the surface by a plane through σ_1 tangent to σ_2.

The conic C_2 is now degenerate and composed of two points.

13. *Pinch points on* σ_1 *and* σ_2.

The number of pinch points on σ_1 is equal to the number of tangents which can be drawn from b to $C\sigma_1$. It is

$$(2p+1)n^2-3n-2.$$

Similarly, the number of pinch points on σ_2 is the number of tangents which can be drawn from a to $C\sigma_2$. This number is

$$(2p+1)n^2+2pn-7n-2.$$

The number of triple tangent planes through simple chords is

$$[2+(2p+1)(n^2-n)]\,[(6p+3)n^2-10n-pn+\Sigma\alpha_i-2].*$$

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* There is a reduction of $pn-1$ to be made from the number derived from the formula of §7. This arises from the presence of a degenerate curve in the pencil.

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Second Series.

14. The representative surface derived, as in § 10, is of order $(2p+1)n+2p-2$. It has α_i chords of order $i(n+1)$ and $5+(n-1)[2p(n+1)+n+2]$ simple chords. σ_1 is of multiplicity $(p+1)n+p-2$, and σ_2 of multiplicity $np+p-1$.

The plane representation is of order $(p+1)n+p$. The curves of order $(p+1)n+p$ have, as before, a multiple point a of order n.

$C\sigma_1$ is of order $(p+1)n+p-1$, and has a for a point of order $n-1$. $C\sigma_2$ is of order $(p+1)n+p$, and has a for a point of order $n+1$.

The pinch points on σ_1 number $(2p+1)n^2+n(2p-1)-4$. On σ_2 the number is $(2p+1)n^2+n(4p-5)+2p-4$.

The number of triple tangent planes is

$$[(6p+3)n^2+5pn-4n-p+\Sigma\alpha_i-5]\,[5+(n-1)\{2p\,(n+1)+n+2\}].$$

15. *Particular cases.*

If we put $p=1$, the orders of the above series become $3n-2$ and $3n$ respectively.

For $n=2$ we get from the first a quartic with a nodal line. From the above formulae the results obtained by Clebsch follow readily.

Putting $n=2$ in the second series, we get a sextic surface having σ_1 as triple line and σ_2 as double conic. The surface contains 15 simple chords. A plane section is represented by a quintic curve having a double point at a and 15 simple points. $C\sigma_1$ is of order 4, and has a simple point at a. $C\sigma_2$ is of order 5, and has a triple point at a. The latter is hyperelliptic, and being of deficiency 3, we should expect that σ_2 would have 8 pinch points on it. There are, in fact, 8 pinch points on σ_2 and 10 on σ_1. The lines joining a to each of the 15 simple points give rise to conics on the surface. The correspondent of a itself is a conic, so that there are 16 simple conics on the surface.

The number of triple tangent planes is 15×32 or 480. The planes of the conics are quadruple tangent planes.

Third Series.

16. For this type of degenerate cubic a third series of surfaces exists. A conicoid through Σ_2 transforms into a surface

of order $(2p+1)n-1$, having Σ_2 as curve of multiplicity $(p+1)n-2$, together with α_i chords of multiplicity in and $2+(n-1)[(2p+1)n+2]$ simple chords. The plane representation of sections of the surface is still of order $(p+1)n$. The curves of this order have a node of order $(n-1)$ at a, and $2+(n-1)[(2p+1)n+2]$ simple points.

$C\sigma_1$ is of order $(p+1)n-1$ with a of order $n-2$. $C\sigma_2$ is of order $(p+1)n$ with a of order n. The number of pinch points on σ_1 is $(2p+1)n^2-n-6$, and the number on Σ_2 is $(2p+1)n^2+n(2p-5)$. The number of triple tangent planes is

$$[(6p+3)n^2-4n-pn+\Sigma\alpha_i-9][2+(n-1)\{(2p+1)n+2\}].$$

As particular cases we have the following: $p=2$, $n=1$ gives a quartic with a double conic. This surface was also discussed by Clebsch (Crelle, Bd. 68). $p=1$, $n=2$ gives a quintic with a double line and double conic, the degenerate case of the quintic of § 11.

III.

Σ_3 is degenerate and composed of two non-intersecting lines σ_1 and σ_2, which are met by a third σ_3.

First Series.

17. The representative surfaces of order $2np-1$ having σ_1 and σ_2 as lines of multiplicity $np-1$ and σ_3 of multiplicity $n(p-1)$. It possesses also α_i chords (of σ_1 σ_2) of multiplicity in and $1+2p(n^2-n)$ simple chords. The plane representation is of order $(p+1)n$, but there are two cases according as $p=1$ or $p>1$. For $p>1$, $(p+1)n$ is of the lowest possible order. The curves of this order on π have two multiple points a and a' of order n, α_i F-points of order in and $1+2p(n^2-n)$ simple points. Corresponding to each of the points a, a', we get a plane curve on the surface which is the residual intersection of the planes $\sigma_1\sigma_3$, $\sigma_2\sigma_3$ with the surface. These curves are rational and have each nodes of order $n-1$ at the point in which their plane intersects σ_1 or σ_2. σ_1 is represented by a curve of order $(p+1)n-1$, having a as node of order $n-1$ and a' as node of order n. σ_2 is represented by a curve of the same order, having a as node of order n and a' as node of order $n-1$. The lines of the collinear groups of $np-1$ points corresponding to points on σ_1 envelope a', and similarly the lines of the groups for points on σ_2 envelope a. The number of pinch points on σ_1 and σ_2 is $2pn^2-4n$.

The number of triple tangent planes through a simple chord is $[6pn^2-2pn-8n+\Sigma\alpha_i+2]\,[1+2p(n^2-n)]$, excluding the planes through σ_1 and σ_2, which are $(pn-1)$-ple tangent planes.

The line $a\,a'$ represents the chord σ_3.

For $p=1$ a further quadratic transformation is necessary, having AB and one of the simple F-points as F-points. Applying this to the curve of order $(p+1)n$, we get a curve of order $2np-1$, having two nodes of order $pn-1$ and $2p(n^2-n)$ simple points.

Putting $p=1$, we see that plane sections of the surface are represented by curves of order $2n-1$, having 2 points of order $n-1$ and $2n(n-1)$ simple points.

σ_1 (σ_2) is represented by a curve of order $2n-2$, having $a\,(a')$ as node of order $n-2$, $(n-1)$.

The line $a\,a'$ represents the simple chord corresponding to the point chosen in the quadratic transformation. The number of pinch points on σ_1 (σ_2) is $2n^2-4n$. The number of triple tangent planes is $[1+2n(n-1)]\,[6n^2-10n+1]$.

Second Series.

18. The surfaces are of order $2np+2p-1$. σ_1 and σ_2 are lines of multiplicity $np+p-1$, and σ_3 is of multiplicity $n(p-1)+p$.

There are α_i chords of order $i(n+1)$ and $4+(n-1)[2p(n+1)+2]$ simple chords.

The plane representation of lowest order for all cases is of order $(p+1)n+p$.

A plane section of the surface transforms into a curve of order $(p+1)n+p$, having two multiple points of order n $(a,\ a')$, α_i points of order $i(n+1)$, and $4+(n-1)\{2p(n+1)+2\}$ simple points.

σ_1 and σ_2 are represented by curves of order $(p+1)n+p-1$, having $a\,(a')$ as nodes of order $n-1$ (n). σ_3 is represented by the line $a\,a'$. The pinch points on σ_1 or σ_2 number $2pn^2+2n(p-1)-2$. The number of triple tangent planes is

$$[6pn^2+4pn-2n-2p+\Sigma\alpha_i-1]\,[4+(n-1)\{2p(n+1)+2\}].$$

For $p=1$ the series is identical with the preceding, and may be obtained by putting in place of n in it $n+1$.

19. *Particular cases.*

Putting $p=1$, $n=3$ in first series or $p=1$, $n=2$ in second, we obtain a quintic with two non-intersecting nodal lines. This

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surface has 13 lines and 26 conics. There are 6 pinch points on each of σ_1 and σ_2. There are also $13 \times 25 = 325$ triple tangent planes through simple chords.

$\sigma_1 \sigma_2$ are represented by quartics having a (a') as points of order 1 (2), and *vice-versa*. These quartics are of deficiency 2, and hence we should expect 6 pinch points on each of σ_1 and σ_2.

These results agree with those obtained analytically by Clebsch.

Third Series.

20. As in the preceding case (II.), a third series is obtained by transforming a conicoid through σ_1 and σ_3. The surfaces thus obtained are of order $2np$ and have σ_1 as line of multiplicity $np - 1$ and σ_2 as line of multiplicity np. σ_3 is of multiplicity $n(p-1)+1$.

There are, as in the previous cases, α_i chords of order in; the number of simple chords is $1 + (n-1)(2pn+2)$. The curves representing plane sections are of order $(p+1)n$ with nodes of order n, $n-1$ at a, a', together with α_i multiple points of order in, and $1 + (n-1)(2pn+2)$ simple points.

σ_2 is represented by a curve of order $(p+1)n-1$, having nodes of order $n-1$ at a and a. σ_1 is represented by a curve of order $(p+1)n-1$, having nodes of order $n-2$, n at a and a' respectively. σ_3 is represented by the line aa'. The number of pinch points on σ_2 is $2pn^2 - 2n$, and the number on σ_1 is $2pn^2 - 2n - 4$.

The triple tangent planes number

$$[1 + (n-1)(2pn+2)]\,[6pn^2 - 2pn - 2n + \Sigma\alpha_i - 4].$$

The following particular cases are of interest: $p = 2$, $n = 1$ gives quartic with two intersecting double lines and 16 simple lines, treated by Körndorfer (*Math. Ann.* III.). $p = 1$, $n = 3$ gives a sextic with a triple line, a double line, and 18 simple lines. $p = 1$ gives the singly infinite series of order $2n$, with a line of multiplicity n and one of multiplicity $n-1$.

IV.

Rational ruled surfaces having Σ_3 as multiple curve.

21. We shall treat briefly the plane representation of such surfaces.

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They are obtained by transforming a conicoid through Σ_3 by the transformation T'_{2p-1} (loc. cit.). The order is $2p$. Σ_3 is a curve of multiplicity p, and there are α_i chords of multiplicity i.

Any plane section of the surface is unicursal. The plane representation is obtained by transforming the conicoid into a plane in a space (1′).

A plane section of the surface transforms into a curve of order $p+1$, having a multiple point of order p and one of order 1 (a, b).

Σ_3' is represented by a curve of order $p+2$, having a as multiple point of order p and b as a double point. The deficiency of this curve is $(p-1)$. Corresponding to a multiple chord of order i we get i straight lines through a.

Any generator of the surface is represented by a line through a. In particular, the line ab corresponds to a fixed point B on the surface. To a corresponds a curve of order p on the surface, and to b a fixed generator through B.

The group of points of order p which corresponds to a point of Σ_3' is not a collinear group. The curves representing Σ_3' are all hyperelliptic in character, and in the general case $2p$ tangents can be drawn from a' to the curve. There are consequently $2p$ tangents to Σ_3', which are generators of the surface.

22. *Synthetic investigation for the number of pinch points on the cubic Σ_3 when the latter is a proper curve.*

Consider the pencil of curves of order $(p+1)n$ which have a particular simple F-point on π as double point. This pencil corresponds to the plane sections of the surface through a simple chord. The number of base points of the pencil is $(p+1)^2 n^2$. Of these the F-points account for $4+[2+(2p+2)(n^2-n)]+\Sigma\alpha_i i^2 n^2$, leaving $(2p+2)n-6$. These must be fixed points on C_{Σ_3}, and together with the double point they constitute two groups of $(p+1)n-2$ points corresponding to the two intersections of the simple chord with Σ_3.

Any curve of the pencil meets C_{Σ_3} in a group of $(p+1)n-2$ variable collinear points. When the cutting plane through the simple chord meets Σ_3 in a pinch point, two of the above points of intersection will coincide. We have consequently to find the number of curves of the pencil which touch C_{Σ_3}. By applying the method of correspondence I find that the points of contact lie on a curve of order $4(p+1)n-4$, having a node of order 3 at each of the $2+(2p+2)(n^2-n)$ simple F-points, a node of order 5 at the double F-point, a node of order 2 at each of the $(2p+2)n-6$ fixed points of C_{Σ_3}, and a node of order $4in-1$ at each of the i-points. The number of proper intersections of this locus with C_{Σ_3} is therefore given by

$$4[(p+1)n-1][2(p+1)n-1]-10-2[(2p+2)n-6]$$
$$-6[2+(2p+2)(n^2-n)]-\Sigma\alpha_i\, 2in(4in-1),$$

giving, on reduction, the expression already found, viz.

$$(4p+4)n^2+2np-10n-6.$$

This is the number of curves which touch C_{Σ_3}, and hence the number of pinch points on Σ_3.

For the second series of surfaces we obtain similarly the number $(4p+4)n^2+n(6p-6)+2p-10$.

By putting $p=1$ in these expressions and combining, we find that for a surface of order $2m-1$ without multiple chords the number of pinch points is $2m^2-8$. This method is of very general application.

For the ruled surfaces we find that a pencil of planes through a generator is represented on π by a pencil of curves of order p, having as base points a node of order p at the F-point of order p, a simple point at the F-point of order 1, and $2(p-1)$ fixed points on C_{Σ_3}. While the determination of the number of pinch points by the above method presents no difficulty, the number is in this case more easily determined by applying to the pencil and C_{Σ_3} a de Jonquiere's transformation of order p, having the p-node and the $2(p-1)$ fixed points as F-system.

The pencil of curves transforms into a pencil of lines through the F-points of order 1, and C_{Σ_3} into a curve similar to itself. The number of tangents which can be drawn from the simple F-point to C_{Σ_3}' is $4p-4$, and this the number of pinch points on Σ_3.