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A GENERALIZATION OF THE MABINOGION SHEEP PROBLEM OF D. WILLIAMS

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Abstract

In his well-known textbook *Probability with Martingales*, David Williams (1991) introduces the Mabinogion sheep problem in which there is a magical flock of sheep, some black, some white. At each stage $n = 1, 2, \dots$, a sheep (chosen randomly from the entire flock, independently of previous events) bleats; if this bleating sheep is white, one black sheep (if any remain) instantly becomes white; if the bleating sheep is black, one white sheep (if any remain) instantly becomes black. No births or deaths occur. Suppose that one may remove any number of white sheep from the flock at (the end of) each stage $n = 0, 1, \dots$. The object is to maximize the expected final number of black sheep. By applying the martingale optimality principle, Williams showed that the problem is solvable and admits a simple nice solution. In this paper we consider a generalization of the Mabinogion sheep problem with two parameters $0 \le p, q \le 1$, denoted $M(p, q)$, in which at each stage, when the bleating sheep is white (black, respectively), a black (white, respectively) sheep (if any remain) instantly becomes white (black, respectively), with probability p (q, respectively) and nothing changes with probability $1 - p$ ($1 - q$, respectively). Note that the original problem corresponds to $(p, q) = (1, 1)$. Following Williams' approach, we solve the two cases $(p, q) = (1, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ which admit simple solutions.

Keywords: Mabinogion sheep; martingale optimality principle; stochastic control

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1. Introduction

In the tale of Peredur Son of Efrawg in the Welsh folk tales *The Mabinogion*, there is a river valley with a flock of white sheep on one side of the river and a flock of black sheep on the other. Whenever one of the white sheep bleats, one of the black sheep would cross over and become white; and when one of the black sheep bleats, one of the white sheep would cross over and become black. With this story in mind, in his well-known textbook [3] David Williams introduces the Mabinogion problem as follows. There is a magical flock of sheep, some black, some white. At each stage $n = 1, 2, \dots$, a sheep (chosen randomly from the entire flock, independently of previous events) bleats; if this bleating sheep is white, one black sheep (if any remain) instantly becomes white; if the bleating sheep is black, one white sheep (if any remain) instantly becomes black. No births or deaths occur. This transition process continues until all the sheep are of the same color. Suppose that this system can be controlled by removing any number of white sheep at (the end of) each stage $n = 0, 1, \dots$. The object is to maximize the expected final number of black sheep. More precisely, let W_n (B_n, respectively) denote the number of white sheep (black sheep, respectively) at (the beginning of)

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stage $n = 0, 1, \ldots$ with initial state $(W_0, B_0) = (w, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ = \{0, 1, \ldots\} \times \{0, 1, \ldots\}$. Given $(W_n, B_n) = (w_n, b_n)$ (with $w_n > 0$ and $b_n > 0$), if all white sheep are removed, then the next state $(W_{n+1}, B_{n+1}) = (0, b_n)$ with probability 1; if a_n white sheep $(0 \le a_n < w_n)$ are removed, then the next state $(W_{n+1}, B_{n+1}) = (w_n - a_n + 1, b_n - 1)$ (with probability $(w_n - a_n)/(w_n - a_n + b_n)$) or $(w_n - a_n - 1, b_n + 1)$ (with probability $b_n/(w_n - a_n + b_n)$). The object is to maximize $V_\pi(w, b) := \mathbb{E}_{w,b}[B_\infty(\pi)]$ over all policies $\pi \in \mathcal{A}$ where the subscript w and b in $\mathbb{E}_{w,b}$ refers to the initial state $(W_0, B_0) = (w, b)$, $\mathcal{A}$ is the class of all policies $\pi = (a_0, a_1, \ldots)$ (a sequence of actions with $0 \le a_n \le W_n$ white sheep removed at the end of stage n), and $B_\infty(\pi)$ represents the final number of black sheep for the controlled process under policy π. A policy $\pi^* \in \mathcal{A}$ is optimal if

$$V_{\pi^*}(w, b) = \sup_{\pi \in \mathcal{A}} V_\pi(w, b)$$

holds for every initial state (w, b); and $V_{\pi^*}(w, b)$ is the (optimal) value function.

By the usual method of guess-and-verify with some martingale properties of the value function, Williams [3, Section 15.3] showed that the optimal policy π^* takes the following form: if the current state is (w, b) with $w, b > 0$, do nothing if $w < b$; otherwise, immediately reduce the white population to $b - 1$. In other words, it is optimal to remove $\max\{W_n(\pi^*) - B_n(\pi^*) + 1, 0\}$ white sheep at the end of each stage!n until all the sheep are of the same color, where $W_n(\pi^*)$ ($B_n(\pi^*)$, respectively) denotes the number of white sheep (black sheep, respectively) at stage n under policy π^*. More specifically, Williams established that $V_{\pi^*}(w, b) \ge V_{\pi^*}(w - 1, b)$ and $V_{\pi^*}(w, b) \ge (w/(w + b))V_{\pi^*}(w + 1, b - 1) + (b/(w + b))V_{\pi^*}(w - 1, b + 1)$ for $w > 0$ and $b > 0$, thereby implying that for any policy $\pi \in \mathcal{A}$, $\{V_{\pi^*}(W_n(\pi), B_n(\pi))\}_{n \ge 0}$ is a supermartingale with respect to the filtration generated by $\{(W_n(\pi), B_n(\pi)) \colon n \ge 0\}$. Applying the martingale optimality principle yields the optimality of policy π^*. In the literature, Chan [1] considered various diffusion models of the Mabinogion sheep problem and investigated the extent to which the essential results of the original discrete-time problem still hold in the diffusion analogues. Apart from such stochastic control problems, by applying a time-reversal transformation to the classical Ehrenfest urn and using the techniques based on the general principles of analytic combinatorics, Flajolet and Huillet [2] carried out the detailed asymptotic analysis for the distribution of the stopping time (i.e. the time until all the sheep are of the same color without the availability of control to remove white sheep at the end of each stage).

Motivated by the simple optimal policy π^*, in this paper we consider two variants of the Mabinogion sheep problem which are solvable and admit simple solutions. Specifically, we generalize the original problem with two parameters $0 \le p, q \le 1$ (denoted $M(p, q)$), in which, at each stage, when the bleating sheep is white (black, respectively), a black (white, respectively) sheep (if any remain) instantly becomes white (black, respectively) with probability p (q, respectively) and nothing changes with probability $1 - p$ ($1 - q$, respectively). Note that the original problem corresponds to $(p, q) = (1, 1)$.

For an arbitrary policy $\tilde{\pi} \in \mathcal{A}$, under model $M(p, q)$, let $\widetilde{V}(w, b) = V_{\tilde{\pi}}(w, b)$, the expected final number of black sheep when $\tilde{\pi}$ is employed. If the policy is to do nothing for state (w, b) with $w > 0$ and $b > 0$, then we have, by the first-step analysis,

$$\begin{aligned}\widetilde{V}(w, b) = \frac{w}{w + b}(p\widetilde{V}(w + 1, b - 1) + (1 - p)\widetilde{V}(w, b)) \\ + \frac{b}{w + b}(q\widetilde{V}(w - 1, b + 1) + (1 - q)\widetilde{V}(w, b)),\end{aligned}$$

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which reduces to

$$\widetilde{V}(w,b) = \frac{pw}{pw+qb}\widetilde{V}(w+1,b-1) + \frac{qb}{pw+qb}\widetilde{V}(w-1,b+1). \tag{1.1}$$

Inspired by the policy π^* for model $M(1,1)$ and in view of (1.1), we propose and investigate the optimality of the following policy for model $M(p,q)$.

- Policy $\pi_{p,q}$: if the current state is (w,b) with $w,b > 0$, do nothing if $pw < qb$; otherwise immediately reduce the white population to $\lceil qb/p \rceil - 1$, where $\lceil x \rceil$ denotes the smallest integer not less than x. In other words, at the end of each stage n, if the transition process does not stop then keep $\min\{W_n^*(\pi_{p,q}), W_n(\pi_{p,q})\}$ white sheep, where $W_n^*(\pi_{p,q}) = \max\{k \in \mathbb{Z}^+ \colon pk < qB_n(\pi_{p,q})\}$.

Note that $\pi^* = \pi_{1,1}$. We shall show that $\pi_{p,q}$ is optimal for $M(p,q)$ when $(p,q) = (1,\frac{1}{2})$ and $(\frac{1}{2},1)$. However, $\pi_{p,q}$ is in general not optimal for $M(p,q)$. Consider model $M(\frac{13}{14},1)$ and let $\widehat{V}(w,b)$ be the (optimal) value function. Obviously, $\widehat{V}(0,2) = 2$ and $\widehat{V}(2,0) = 0$. To determine the value of $\widehat{V}(1,1)$, we need to consider two options for state $(1,1)$: either to remove the single white sheep or to do nothing. For the former option, the system ends with one black sheep. For the latter option, by (1.1), the expected final number of black sheep is

$$\frac{13/14}{13/14+1}\widehat{V}(2,0) + \frac{1}{13/14+1}\widehat{V}(0,2) = \frac{28}{27} > 1,$$

implying that it is optimal to do nothing in state $(1,1)$ and $\widehat{V}(1,1) = \frac{28}{27}$. Thus, we obtain the values of $\widehat{V}(w,b)$ for (w,b) with $w+b=2$. Continuing in this manner, we obtain the values of $\widehat{V}(w,b)$ for (w,b) with $w+b = 3, 4$, i.e.

$$\widehat{V}(2,2) = \widehat{V}(1,2) = \tfrac{2632}{1107},$$
$$\widehat{V}(3,1) = \widehat{V}(2,1) = \widehat{V}(1,1), \quad \text{and} \quad \widehat{V}(1,3) = \tfrac{220\,192}{60\,885} > 3 = \widehat{V}(0,3).$$

In particular, $\widehat{V}(2,2) = \widehat{V}(1,2) = \frac{2632}{1107}$, implying that it is optimal to remove one white sheep in state $(2,2)$. However, the policy $\pi_{13/14,1}$ is to do nothing in this state since $pw = \frac{13}{14}\cdot 2 < 1\cdot 2 = qb$, so that the expected final number of black sheep under $\pi_{13/14,1}$ is equal to

$$\frac{13/7}{13/7+2}\widehat{V}(3,1) + \frac{2}{13/7+2}\widehat{V}(1,3) = \frac{3\,903\,508}{1\,643\,895} < \frac{2632}{1107} = \widehat{V}(2,2).$$

Therefore, $\pi_{13/14,1}$ is not optimal for $M(\frac{13}{14},1)$. Moreover, we also find that $\pi_{p,q}$ is optimal for $M(\frac{1}{3},1)$ provided that $w+b \le 34$, but it is not optimal when $(w,b) = (26,9)$.

The rest of this paper is organized as follows. In Section 2 we present our main results, Theorems 2.1 and 2.2, showing that $\pi_{1,1/2}$ and $\pi_{1/2,1}$ are optimal for $M(1,\frac{1}{2})$ and $M(\frac{1}{2},1)$, respectively. Since their proofs are similar, we prove Theorem 2.1 only. Following Williams' approach, we need to verify conditions (S1) and (S2) (see Section 2) in order to prove Theorem 2.1, which requires tedious calculations and is done in Sections 3 and 4. In Section 3 we present several key lemmas and then use them to verify (S1) and (S2). The key lemmas are proved in Section 4. Finally, Section 5 contains concluding remarks and asymptotic results for the value functions under $M(1,\frac{1}{2})$ and $M(\frac{1}{2},1)$.

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2. Main results

For model $M(p, q)$, if the current state is (w, b) with $w, b > 0$ (possibly with some white sheep just removed), then the next state is either $(w + 1, b - 1)$, $(w - 1, b + 1)$, or (w, b) with respective probabilities $pw/(w + b)$, $qb/(w + b)$, $1 - (pw + qb)/(w + b)$. Thus, given that a transition of the state has occurred, the next state is either $(w + 1, b - 1)$ or $(w - 1, b + 1)$ with respective (conditional) probabilities $pw/(pw + qb)$, $qb/(pw + qb)$ (see (1.1)). In light of this observation, it is readily seen that model $M(p, q)$ is equivalent to $M(1, q/p)$ for $p \geq q$ and equivalent to $M(p/q, 1)$ for $p \leq q$. So it suffices to consider $M(1, q)$ and $M(p, 1)$ with $0 < p, q < 1$. (The $M(1, 1)$ case is the original Mabinogion problem, while the $M(1, 0)$ and $M(0, 1)$ cases are trivial.)

We now present our main results.

Theorem 2.1. *The following policy $\pi_1 = \pi_{1,1/2}$ is optimal for $M(1, \frac{1}{2})$.*

- *Policy π_1: if the current state is (w, b) with $w, b > 0$, do nothing if $w < b/2$; otherwise, immediately reduce the white population to just less than half the black population.*

Theorem 2.2. *The following policy $\pi_2 = \pi_{\frac{1}{2},1}$ is optimal for $M(\frac{1}{2}, 1)$.*

- *Policy π_2: if the current state is (w, b) with $w, b > 0$, do nothing if $w < 2b$; otherwise, immediately reduce the white population to just less than twice the black population.*

As the proof of Theorem 2.2 is similar to that of Theorem 2.1, we shall deal only with the $M(1, \frac{1}{2})$ case. For $w, b \in \mathbb{Z}^+$, let $V(w, b) = V_{\pi_1}(w, b)$, the expected final number of black sheep when π_1 is employed and there are w white and b black sheep at time 0. Following [3], to prove Theorem 2.1, it suffices to show that

(S1) $V(w, b) \geq V(w - 1, b)$ for $w > 0$ and $b > 0$;

(S2) $V(w, b) \geq (2w/(2w + b))V(w + 1, b - 1) + (b/(2w + b))V(w - 1, b + 1)$ for $w > 0$ and $b > 0$.

Here, (S2) comes from rearranging the inequality

$$V(w, b) \geq \frac{w}{w + b} V(w + 1, b - 1) + \frac{b}{w + b}\left(\frac{1}{2} V(w - 1, b + 1) + \frac{1}{2} V(w, b)\right).$$

The task of verifying (S1) and (S2) is much more involved than that for $M(1, 1)$ in [3]. In Section 3 we prove (S1) and (S2) with the help of several lemmas, whose proofs are given in Section 4.

3. Proofs of (S1) and (S2)

For $w, b \in \mathbb{Z}^+$, let $V(w, b) = V_{\pi_1}(w, b)$. Then the following hold:

(a1) $V(0, b) = b$ and $V(w, 0) = 0$;

(a2) $V(w, b) = V(w - 1, b)$ whenever $2w \geq b$ and $w > 0$;

(a3) $V(w, b) = (2w/(2w + b))V(w + 1, b - 1) + (b/(2w + b))V(w - 1, b + 1)$ whenever $0 < 2w < b$.

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Define $v_k = V(k, 2k)$. Referring to (a1), (a2), and (a3), we have, for $k \geq 1$,

$$V(k-i, 2k+i) = v_k + (3k - v_k)\biggl(\sum_{j=k-i}^{k-1} \frac{\binom{3k-1}{j}}{2^j} \bigg/ \sum_{j=0}^{k-1} \frac{\binom{3k-1}{j}}{2^j}\biggr), \qquad i = 1, 2, \dots, k, \tag{3.1}$$

$$V(k+1-i, 2k+i) = v_k + (3k+1 - v_k)\biggl(\sum_{j=k-i}^{k-1} \frac{\binom{3k}{j+1}}{2^{j+1}} \bigg/ \sum_{j=0}^{k} \frac{\binom{3k}{j}}{2^j}\biggr), \qquad i = 1, 2, \dots, k+1, \tag{3.2}$$

$$V(k+2-i, 2k+i) = V(k, 2k+1) + (3k+2 - V(k, 2k+1))\biggl(\sum_{j=k-i}^{k-2} \frac{\binom{3k+1}{j+2}}{2^{j+2}} \bigg/ \sum_{j=0}^{k} \frac{\binom{3k+1}{j}}{2^j}\biggr), \qquad i = 1, 2, \dots, k+2, \tag{3.3}$$

For $k \geq 1$, let

$$A_k = \frac{3}{2} \sum_{j=0}^{k-1} \frac{\binom{3k-1}{j}}{2^j} \bigg/ \frac{\binom{3k-1}{k-1}}{2^{k-1}}.$$

Then we see that $A_k \geq \frac{3}{2}$ and $A_k \leq 3k/2$ (since $\binom{3k-1}{j}/2^j$ is increasing in $0 \leq j \leq k-1$) for all $k \geq 1$.

To prove (S1) and (S2), we need some useful identities in terms of A_k, as stated in Lemma 3.1 below.

Lemma 3.1. *For each $k = 1, 2, \dots,$ the following relations hold:*

$$\sum_{j=0}^{k} \frac{\binom{3k}{j}}{2^j} = \frac{\binom{3k}{k}}{2^k}\biggl[\frac{2}{3}(A_k + 1)\biggr], \qquad \sum_{j=0}^{k} \frac{\binom{3k+1}{j}}{2^j} = \frac{\binom{3k}{k}}{2^k}\biggl(A_k + \frac{1}{2}\biggr), \tag{3.4}$$

$$\begin{aligned} V(k, 2k+1) &= v_k + (3k+1 - v_k)\frac{3}{2(A_k+1)}, \\ V(k, 2k+3) &= v_{k+1} + (3k+3 - v_{k+1})\frac{3}{2A_{k+1}}, \end{aligned} \tag{3.5}$$

$$v_{k+1} = \biggl(1 - \frac{(3k+1)/(2k+1)}{A_k + 1/2}\biggr)\biggl[\biggl(1 - \frac{3}{2(A_k+1)}\biggr)v_k + \frac{3}{2(A_k+1)}(3k+1)\biggr] + \frac{(3k+1)/(2k+1)}{A_k + 1/2}(3k+2), \tag{3.6}$$

$$A_{k+1} = \frac{k+1}{(3k+1)(3k+2)}\biggl[\frac{9}{2}(2k+1)A_k + \frac{3}{4}\biggr]. \tag{3.7}$$

In the next lemma we give bounds for v_k, which play a significant role in the proofs of (S1) and (S2).

Lemma 3.2. *For each* $k = 1, 2, \dots,$

$$3k - A_k \le v_k \le \frac{(3k+1)(6k+1)}{2A_k + 6k + 3}. \tag{3.8}$$

With Lemmas 3.1 and 3.2, we are now ready to prove (S1).

Proof of (S1). By (a2), (S1) holds when $2w \ge b > 0$. Hence, we need only to establish (S1) when $0 < 2w < b$. Now if $0 < 2w < b$ and $w+b \equiv 1 \pmod 3$, then $(w, b) = (k+1-i, 2k+i)$ for some $k \ge 1$ and i with $i = 1, 2, \dots, k$. In view of (3.1) and (3.2), it is enough to show that, for $\ell = k-i, k-i+1, \dots, k-1$,

$$(3k+1-v_k)\biggl(\frac{\binom{3k}{\ell+1}}{2^{\ell+1}} \bigg/ \sum_{j=0}^{k} \frac{\binom{3k}{j}}{2^j}\biggr) \ge (3k - v_k)\biggl(\frac{\binom{3k-1}{\ell}}{2^{\ell}} \bigg/ \sum_{j=0}^{k-1} \frac{\binom{3k-1}{j}}{2^j}\biggr);$$

moreover, since $(\binom{3k}{\ell+1}/2^{\ell+1})/(\binom{3k-1}{\ell}/2^{\ell}) = 3k/2(\ell+1)$ is decreasing in ℓ, we only need to establish the case $\ell = k-1$, i.e.

$$(3k+1-v_k)\biggl(\frac{\binom{3k}{k}}{2^{k}} \bigg/ \sum_{j=0}^{k} \frac{\binom{3k}{j}}{2^j}\biggr) \ge (3k - v_k)\biggl(\frac{\binom{3k-1}{k-1}}{2^{k-1}} \bigg/ \sum_{j=0}^{k-1} \frac{\binom{3k-1}{j}}{2^j}\biggr),$$

or, equivalently, in terms of A_k and using (3.4),

$$(3k+1-v_k)\frac{1}{A_k+1} \ge (3k - v_k)\frac{1}{A_k},$$

which reduces to

$$v_k \ge 3k - A_k.$$

This is just a part of Lemma 3.2, so (S1) is verified for the $w + b \equiv 1 \pmod 3$ case.

The proof of (S1) for the $w+b \equiv 2 \pmod 3$ and $w+b \equiv 0 \pmod 3$ cases follow in a similar manner and we omit the details. □

Next, we state a few more lemmas, in order to prove (S2).

Lemma 3.3. *For* $w, b \in \mathbb{Z}^+$ *with* $0 \le 2w \le b$,

$$V(w, b+1) - V(w, b) \ge 1.$$

Lemma 3.4. *For each* $k = 0, 1, \dots,$

(i) $((k+1+\ell)/(2k+2+\ell))V(k, 2k+1) + ((k+1)/(2k+2+\ell))V(k+\ell, 2k+3)$ *is decreasing in* $\ell \ge 1$;

(ii) $(2(k+1+\ell)/(4k+2\ell+3))V(k, 2k) + ((2k+1)/(4k+2\ell+3))V(k+\ell, 2k+2)$ *is decreasing in* $\ell \ge 1$.

Lemma 3.5. *For each* $k = 1, 2, \dots,$

(i) *we have*

$$V(k, 2k+2) \ge \tfrac{1}{2}V(k, 2k+1) + \tfrac{1}{2}V(k, 2k+3) \tag{3.9}$$

is equivalent to

$$v_k \le \frac{(3k+1)(6k+1)}{2A_k + 6k + 3}; \tag{3.10}$$

(ii) *and*

$$V(k,2k+1)\geq\frac{2k+2}{4k+3}V(k,2k)+\frac{2k+1}{4k+3}V(k,2k+2)$$

is equivalent to

$$v_k\leq\frac{(3k+1)(A_k+3k+1)}{2A_k+3k+2}.$$

Proof of (S2). Because of (a3), (S2) automatically holds when $0<2w<b$, so we need to establish it only when $2w\geq b>0$, i.e. for k and ℓ in $\mathbb{Z}^+$,

$$V(k+\ell+1,2k+2)\geq\frac{k+1+\ell}{2k+2+\ell}V(k+2+\ell,2k+1)+\frac{k+1}{2k+2+\ell}V(k+\ell,2k+3)$$

and

$$V(k+\ell+1,2k+1)\geq\frac{2(k+1+\ell)}{4k+2\ell+3}V(k+2+\ell,2k)+\frac{2k+1}{4k+2\ell+3}V(k+\ell,2k+2),$$

which, via (a2), reduce, respectively, to

$$V(k,2k+2)\geq\frac{k+1+\ell}{2k+2+\ell}V(k,2k+1)+\frac{k+1}{2k+2+\ell}V(k+\ell,2k+3)\tag{3.11}$$

and

$$V(k,2k+1)\geq\frac{2(k+1+\ell)}{4k+2\ell+3}V(k,2k)+\frac{2k+1}{4k+2\ell+3}V(k+\ell,2k+2).\tag{3.12}$$

By Lemma 3.4, it suffices to show that (3.11) and (3.12) hold for all $k\geq 0$ with $\ell=0$ and $\ell=1$, i.e. for all $k\geq 0$,

$$V(k,2k+2)\geq\tfrac{1}{2}V(k,2k+1)+\tfrac{1}{2}V(k,2k+3),\tag{3.13}$$

$$V(k,2k+2)\geq\frac{k+2}{2k+3}V(k,2k+1)+\frac{k+1}{2k+3}V(k+1,2k+3),\tag{3.14}$$

$$V(k,2k+1)\geq\frac{2k+2}{4k+3}V(k,2k)+\frac{2k+1}{4k+3}V(k,2k+2),\tag{3.15}$$

$$V(k,2k+1)\geq\frac{2k+4}{4k+5}V(k,2k)+\frac{2k+1}{4k+5}V(k,2k+2).\tag{3.16}$$

Clearly, by using (a1), (a2), and (a3), it can be verified that (3.13)–(3.16) hold for $k=0$. It remains to show that (3.13)–(3.16) hold for all $k\geq 1$. We prove this by the following four steps. We shall first show in step 1 that (3.13) holds for all $k\geq 1$ and then prove in steps 2–4 that (3.13) implies (3.14), (3.13) implies (3.15), and (3.15) implies (3.16).

Step 1. By Lemma 3.5(i), (3.13) is equivalent to

$$v_k\leq\frac{(3k+1)(6k+1)}{2A_k+6k+3}.\tag{3.17}$$

This is just a part of Lemma 3.2, so (3.13) holds for all $k\geq 1$.

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Step 2. We now prove that (3.13) implies (3.14). Obviously, (3.14) holds if we can show that

$$\begin{aligned} F_k &:= \frac{1}{2}V(k, 2k+1) + \frac{1}{2}V(k, 2k+3) \\ &\geq \frac{k+2}{2k+3}V(k, 2k+1) + \frac{k+1}{2k+3}V(k+1, 2k+3) \\ &:= T_k. \end{aligned}$$

Now,

$$\begin{aligned} F_k - T_k &= \frac{-1}{2(2k+3)}V(k, 2k+1) + \frac{1}{2}V(k, 2k+3) - \frac{k+1}{2k+3}V(k+1, 2k+3) \\ &= \frac{1}{2k+3}\bigg\{\frac{V(k, 2k+3) - V(k, 2k+1)}{2} \\ &\qquad - (k+1)[V(k+1, 2k+3) - V(k, 2k+3)]\bigg\} \\ &= \frac{1}{2k+3}\bigg\{\frac{V(k, 2k+3) - V(k, 2k+1)}{2} - \frac{3(k+1)(A_{k+1} + v_{k+1} - 3k - 3)}{2A_{k+1}(A_{k+1}+1)}\bigg\} \\ &\geq \frac{1}{2k+3}\bigg\{1 - \frac{3(k+1)(A_{k+1} + v_{k+1} - 3k - 3)}{2A_{k+1}(A_{k+1}+1)}\bigg\}. \end{aligned}$$

Here the third equality follows by performing some calculations using the fact that $V(k+1, 2k+3) = v_{k+1} + (3k+4-v_{k+1})(3/2(A_{k+1}+1))$ (see (3.5)) and $V(k, 2k+3) = v_{k+1} + (3k+3-v_{k+1})(3/2A_{k+1})$ (from (3.1) and the definition of A_{k+1}). The inequality follows from $V(k, 2k+3) - V(k, 2k+1) \geq 2$, which is a consequence of Lemma 3.3. Therefore, $F_k \geq T_k$ holds if we can show that

$$\frac{3(k+1)(A_{k+1} + v_{k+1} - 3k - 3)}{2A_{k+1}(A_{k+1}+1)} \leq 1,$$

which is equivalent to

$$v_{k+1} \leq 3k + 3 - A_{k+1} + \frac{2A_{k+1}(A_{k+1}+1)}{3(k+1)}. \tag{3.18}$$

By Lemma 3.2, the following holds for all $k \geq 1$:

$$v_{k+1} \leq \frac{(3k+4)(6k+7)}{2A_{k+1} + 6k + 9} = 3k + 3 - A_{k+1} + \frac{(2A_{k+1}+1)(A_{k+1}+1)}{2A_{k+1} + 6k + 9},$$

which implies that (3.18) holds if

$$\frac{(2A_{k+1}+1)(A_{k+1}+1)}{2A_{k+1} + 6k + 9} \bigg/ \frac{2A_{k+1}(A_{k+1}+1)}{3(k+1)} \leq 1.$$

To show that this latter inequality holds, we have

$$\begin{aligned} \frac{(2A_{k+1}+1)(A_{k+1}+1)}{2A_{k+1} + 6k + 9} \bigg/ \frac{2A_{k+1}(A_{k+1}+1)}{3(k+1)} &= \left(1 + \frac{1}{2A_{k+1}}\right)\frac{3k+3}{2A_{k+1} + 6k + 9} \\ &\leq \frac{4(3k+3)}{3(6k+12)} \\ &< 1, \end{aligned}$$

where the first inequality follows from the fact that $A_{k+1} \geq \frac{3}{2}$. Hence, (3.14) holds.

Step 3. We prove that (3.13) (which is equivalent to (3.17)) implies (3.15). By Lemma 3.5(ii), (3.15) is equivalent to

$$v_k \leq \frac{(3k+1)(A_k+3k+1)}{2A_k+3k+2}. \tag{3.19}$$

In view of (3.17), (3.19) holds if we can prove that

$$\frac{6k+1}{2A_k+6k+3} \Big/ \frac{A_k+3k+1}{2A_k+3k+2} < 1 \quad \text{for all } k \geq 1.$$

In fact, we have

$$\begin{aligned}\frac{6k+1}{2A_k+6k+3} \Big/ \frac{A_k+3k+1}{2A_k+3k+2} &= \frac{6k+1}{2A_k+6k+3}\left(1+\frac{A_k+1}{A_k+3k+1}\right)\\ &\leq \frac{6k+1}{2A_k+6k+3}\left(1+\frac{A_k+1}{3k+5/2}\right)\\ &= \frac{6k+1}{2A_k+6k+3}\left(\frac{2A_k+6k+7}{6k+5}\right)\\ &= \frac{6k+1}{6k+5}\left(1+\frac{4}{2A_k+6k+3}\right)\\ &\leq \frac{6k+1}{6k+5}\left(1+\frac{4}{6k+6}\right)\\ &= \frac{18k^2+33k+5}{18k^2+33k+15}\\ &< 1,\end{aligned}$$

where the first and second inequalities follow since $A_k \geq \frac{3}{2}$. Thus, (3.15) holds.

Step 4. We show that (3.15) implies (3.16). In fact, this implication holds if we can show that

$$\frac{2k+2}{4k+3}V(k,2k) + \frac{2k+1}{4k+3}V(k,2k+2) \geq \frac{2k+4}{4k+5}V(k,2k) + \frac{2k+1}{4k+5}V(k,2k+2),$$

which holds since $V(k,2k+2) \geq 2 + V(k,2k) > V(k,2k)$, by Lemma 3.3. This completes the proof of (S2). □

4. Proofs of the key lemmas

In this section, we prove Lemmas 3.1–3.5. We first prove Lemma 3.1,which is based on Pascal's formula and (3.1)–(3.3).

Proof of Lemma 3.1. (i) Recalling that

$$A_k = \frac{3}{2}\sum_{j=0}^{k-1}\frac{\binom{3k-1}{j}}{2^j} \Big/ \frac{\binom{3k-1}{k-1}}{2^{k-1}},$$

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we have, by Pascal's formula,

$$\begin{aligned}\sum_{j=0}^{k}\frac{\binom{3k}{j}}{2^j} &= \sum_{j=0}^{k}\frac{\binom{3k-1}{j}}{2^j}+\sum_{j=1}^{k}\frac{\binom{3k-1}{j-1}}{2^j}\\ &= \frac{3}{2}\sum_{j=0}^{k-1}\frac{\binom{3k-1}{j}}{2^j}+\frac{\binom{3k-1}{k}}{2^k}\\ &= \frac{\binom{3k-1}{k-1}}{2^{k-1}}(A_k+1)\\ &= \frac{\binom{3k}{k}}{2^k}\Big[\frac{2}{3}(A_k+1)\Big].\end{aligned}$$

Similarly,

$$\sum_{j=0}^{k}\frac{\binom{3k+1}{j}}{2^j}=\frac{\binom{3k}{k}}{2^k}\Big(A_k+\frac{1}{2}\Big). \tag{4.1}$$

(ii) It follows from (3.2) with $i=1$,

$$\begin{aligned}V(k,2k+1) &= v_k+(3k+1-v_k)\bigg(\frac{\binom{3k}{k}}{2^k}\bigg/\sum_{j=0}^{k}\frac{\binom{3k}{j}}{2^j}\bigg)\\ &= v_k+(3k+1-v_k)\frac{3}{2(A_k+1)},\end{aligned} \tag{4.2}$$

where the second equality is by (i). Similarly, from (3.1) and the definition of A_{k+1}, we have

$$\begin{aligned}V(k,2k+3) &= v_{k+1}+(3k+3-v_{k+1})\bigg(\frac{\binom{3k+2}{k}}{2^k}\bigg/\sum_{j=0}^{k}\frac{\binom{3k+2}{j}}{2^j}\bigg)\\ &= v_{k+1}+(3k+3-v_{k+1})\frac{3}{2A_{k+1}}.\end{aligned}$$

(iii) Applying (3.3) with $i=2$ and noting that $v_{k+1}=V(k+1,2k+2)=V(k,2k+2)$, we have

$$v_{k+1}=V(k,2k+1)+(3k+2-V(k,2k+1))\bigg(\frac{\binom{3k+1}{k}}{2^k}\bigg/\sum_{j=0}^{k}\frac{\binom{3k+1}{j}}{2^j}\bigg) \tag{4.3}$$

$$=V(k,2k+1)+(3k+2-V(k,2k+1))\bigg(\frac{3k+1}{2k+1}\bigg/A_k+\frac{1}{2}\bigg), \tag{4.4}$$

where the second equality follows by substituting (4.1) into (4.3). After substituting (4.2) into (4.4) and performing a bit of algebra, we see that the claimed expression holds. The proof of (iv) is similar to (i) and is thus omitted. □

We are now in a position to prove Lemma 3.2.

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Proof of Lemma 3.2. We proceed to prove by induction on k. Since $A_1 = \frac{3}{2}$ and $v_1 = 2$, it can be seen that (3.8) holds for $k = 1$. Now, making the induction hypothesis on (3.8), we have to prove that

$$3(k+1) - A_{k+1} \le v_{k+1} \le \frac{(3k+4)(6k+7)}{2A_{k+1} + 6(k+1) + 3}. \tag{4.5}$$

By (3.6), we have

$$\begin{aligned} v_{k+1} - 3(k+1) &= \left(1 - \frac{(3k+1)/(2k+1)}{(A_k + 1/2}\right)\left[\left(1 - \frac{3/2}{A_k+1}\right)(v_k - 3k - 1) - 1\right] - 1 \\ &= \frac{A_k - 1/2 - k/(2k+1)}{A_k + 1/2}\left[\frac{A_k - 1/2}{A_k + 1}(v_k - 3k - 1) - 1\right] - 1. \end{aligned} \tag{4.6}$$

Applying the induction hypothesis that $v_k - 3k \ge -A_k$ to (4.6), it follows that

$$v_{k+1} - 3(k+1) \ge -A_k - \left(\frac{1}{2} - \frac{k}{2k+1}\right)$$

and so

$$v_{k+1} - 3(k+1) + A_{k+1} \ge -A_k - \left(\frac{1}{2} - \frac{k}{2k+1}\right) + A_{k+1}. \tag{4.7}$$

By (3.7), we also have

$$A_{k+1} = \frac{k+1}{(3k+1)(3k+2)}\left[\frac{9}{2}(2k+1)A_k + \frac{3}{4}\right],$$

which together with (4.7) implies that

$$\begin{aligned} &v_{k+1} - 3(k+1) + A_{k+1} \\ &\quad \ge -A_k - \left(\frac{1}{2} - \frac{k}{2k+1}\right) + \frac{k+1}{(3k+1)(3k+2)}\left[\frac{9}{2}(2k+1)A_k + \frac{3}{4}\right] \\ &\quad > -A_k - \frac{1}{2(2k+1)} + \frac{1}{3(3k+1)}\left[\frac{9}{2}(2k+1)A_k + \frac{3}{4}\right] \\ &\quad = \frac{1}{2(3k+1)}\left[A_k + \frac{1}{2} - \frac{3k+1}{2k+1}\right] \\ &\quad > 0, \end{aligned}$$

where the second inequality follows from the fact that $(k+1)/(3k+2) > \frac{1}{3}$ and the last inequality follows since $A_k \ge \frac{3}{2} > (3k+1)/(2k+1)$. Therefore, $v_{k+1} - 3(k+1) + A_{k+1} > 0$, i.e. $v_{k+1} \ge 3(k+1) - A_{k+1}$. By induction, this proves that $v_k \ge 3k - A_k$ for all $k \ge 1$.

Now, it remains to prove the second inequality of (4.5). We have, by (3.6),

$$\begin{aligned} v_{k+1} = &\frac{A_k - 1/2 - k/(2k+1)}{A_k + 1/2}\left[A_k - \frac{1/2}{A_k+1}v_k + \frac{3(3k+1)/2}{A_k+1}\right] \\ &+ \frac{(3k+1)(3k+2)/(2k+1)}{A_k + 1/2}. \end{aligned}$$

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This together with the induction hypothesis $v_k \le (3k+1)(6k+1)/(2A_k+6k+3)$ implies that

$$v_{k+1} \le \frac{A_k - 1/2 - k/(2k+1)}{A_k + 1/2}\left[\frac{A_k - 1/2}{A_k+1}\left(\frac{(3k+1)(6k+1)}{2A_k+6k+3}\right) + \frac{3(3k+1)/2}{A_k+1}\right] + \frac{(3k+1)(3k+2)/(2k+1)}{A_k+1/2},$$

which reduces to

$$v_{k+1} \le \frac{4(k+1)(3k+1)(3k+2)}{(2k+1)(2A_k+6k+3)} := G_k. \tag{4.8}$$

By (3.7), we also have

$$2A_{k+1} + 6(k+1) + 3 = \frac{3(k+1)}{(3k+1)(3k+2)}\left[3(2k+1)A_k + \frac{1}{2} + \frac{(2k+3)(3k+1)(3k+2)}{k+1}\right],$$

which implies that

$$\begin{aligned}\frac{(3k+4)(6k+7)}{2A_{k+1}+6(k+1)+3} &= (3k+4)(6k+7) \\ &\quad\times\left(\frac{3(k+1)}{(3k+1)(3k+2)}\left[3(2k+1)A_k + \frac{1}{2} + \frac{(2k+3)(3k+1)(3k+2)}{k+1}\right]\right)^{-1} \\ &:= H_k. \end{aligned} \tag{4.9}$$

In view of (4.8) and (4.9), the second inequality of (4.5) is verified if we can prove that $G_k \le H_k$ for all $k \ge 1$. Note that

$$\begin{aligned}\frac{G_k}{H_k} &= \frac{18(k+1)^2}{(3k+4)(6k+7)}\left[\frac{2A_k + 6k + 6 - 4/3(k+1) - 1/3(2k+1)}{2A_k+6k+3}\right] \\ &= \frac{18(k+1)^2}{(3k+4)(6k+7)}\left[1 + \frac{3 - 4/3(k+1) - 1/3(2k+1)}{2A_k+6k+3}\right] \\ &\le \frac{18(k+1)^2}{(3k+4)(6k+7)}\left(1 + \frac{3}{6(k+1)}\right) \\ &= \frac{18k^2+45k+27}{18k^2+45k+28} \\ &< 1,\end{aligned}$$

where the inequality follows from the fact that $A_k \ge 3/2$ and $3 - 4/3(k+1) - 1/3(2k+1) < 3$. Hence, $G_k \le H_k$ for all $k \ge 1$ and the proof is completed. □

To prove Lemma 3.3 we shall make use of the following lemma.

Lemma 4.1. *For $w, b \in \mathbb{Z}^+$ with $0 \le 2w < b$,*

$$V(w,b) > V(w+1, b-1).$$

Proof. We shall treat only the $w+b \equiv 0 \pmod 3$ case. The other two cases $w+b \equiv 1 \pmod 3$ and $w+b \equiv 2 \pmod 3$ can be proved in a similar manner. Suppose that $w+b = 3k$ for some $k \in \mathbb{N}$. Clearly, $V(0, 3k) = 3k > V(1, 3k-1)$. It follows from (a3) that, for $1 \le i \le k-1$,

$$V(i, 3k-i) = \frac{2i}{3k+i} V(i+1, 3k-i-1) + \frac{3k-i}{3k+i} V(i-1, 3k-i+1),$$

so

$$V(i, 3k-i) - V(i+1, 3k-i-1) = \frac{3k-i}{2i}[V(i-1, 3k-i+1) - V(i, 3k-i)].$$

Since $V(0, 3k) - V(1, 3k-1) > 0$, it follows that $V(i, 3k-i) - V(i+1, 3k-i-1) > 0$ for $i = 1, 2, \ldots, k-1$, completing the proof. □

We are now ready to prove Lemma 3.3.

Proof of Lemma 3.3. We shall treat only the $w+b \equiv 0 \pmod 3$. The other two cases $w+b \equiv 1 \pmod 3$ and $w+b \equiv 2 \pmod 3$ can be proved in a similar manner. Suppose that $w+b = 3k$ for some $k \in \mathbb{N}$. Let

$$d = \min\{V(i, 3k-i+1) - V(i, 3k-i) \colon i = 0, 1, \ldots, k\}.$$

Our goal is to prove that $d = 1$, which surely implies the desired result. Letting $0 < i < k$, $u = V(i+1, 3k-i-1)$, and $w = V(i-1, 3k-i+1)$, we have, from (a3),

$$V(i, 3k-i+1) = \frac{2i}{3k+i+1}(u+\varepsilon_1) + \frac{3k-i+1}{3k+i+1}(w+\varepsilon_2),$$
$$V(i, 3k-i) = \frac{2i}{3k+i}u + \frac{3k-i}{3k+i}w,$$

where

$$\varepsilon_1 = V(i+1, 3k-i) - V(i+1, 3k-i-1),$$
$$\varepsilon_2 = V(i-1, 3k-i+2) - V(i-1, 3k-i+1).$$

Note that $\varepsilon_1, \varepsilon_2 \ge d$, by the definition of d. Then

$$\begin{aligned} &V(i, 3k-i+1) - V(i, 3k-i) \\ &\quad = \frac{2i}{(3k+i)(3k+i+1)}(w-u) + \frac{2i}{3k+i+1}\varepsilon_1 + \frac{3k-i+1}{3k+i+1}\varepsilon_2 \\ &\quad \ge \frac{2i}{(3k+i)(3k+i+1)}(w-u) + d \\ &\quad > d, \end{aligned}$$

where the first inequality follows since $\varepsilon_1, \varepsilon_2 \ge d$ and the second inequality follows from Lemma 4.1, which gives $w - u = V(i-1, 3k-i+1) - V(i+1, 3k-i-1) > 0$.

So far we have proved that $V(i, 3k-i+1) - V(i+1, 3k-i-1) > d$ for $0 < i < k$. Since $V(0, 3k+1) - V(0, 3k) = 1$, it suffices to show that $V(k, 2k+1) - V(k, 2k) \ge 1$. By (3.5),

$$V(k, 2k+1) - V(k, 2k) = V(k, 2k+1) - v_k = (3k+1-v_k)\frac{3}{2(A_k+1)}.$$

Thus, we need to show that $(3k+1-v_k)(3/2(A_k+1)) \geq 1$, which reduces to

$$v_k \leq 3k + \frac{1}{3} - \frac{2A_k}{3}. \tag{4.10}$$

By Lemma 3.2, we have

$$\begin{aligned} v_k - \left(3k + \frac{1}{3} - \frac{2A_k}{3}\right) &\leq \frac{(3k+1)(6k+1)}{2A_k + 6k + 3} - \left(3k + \frac{1}{3} - \frac{2A_k}{3}\right) \\ &= \frac{2(2A_k - 3k)(A_k+1)}{2A_k + 6k + 3} \\ &\leq 0, \end{aligned}$$

where the last inequality follows from the fact that $A_k \leq 3k/2$ for all $k \geq 1$. Hence, (4.10) holds and the proof is complete. □

Finally, we prove Lemmas 3.4 and 3.5.

Proof of Lemma 3.4. The proofs for (i) and (ii) are similar, so we prove (i) only. For $k \geq 0$ and $\ell \geq 1$, define

$$f(k, \ell) = \frac{k+1+\ell}{2k+2+\ell} V(k, 2k+1) + \frac{k+1}{2k+2+\ell} V(k+\ell, 2k+3).$$

Note that

$$\begin{aligned} f(k,\ell) - f(k,\ell+1) &= \left(\frac{k+1+\ell}{2k+2+\ell} - \frac{k+2+\ell}{2k+3+\ell}\right) V(k, 2k+1) \\ &\quad + (k+1)\left(\frac{1}{2k+2+\ell} - \frac{1}{2k+3+\ell}\right) V(k+\ell, 2k+3) \\ &= \frac{k+1}{(2k+2+\ell)(2k+3+\ell)} \Big[V(k+\ell, 2k+3) - V(k, 2k+1)\Big] \\ &\geq \frac{k+1}{(2k+2+\ell)(2k+3+\ell)} \Big[V(k, 2k+3) - V(k, 2k+1)\Big] \\ &> 0, \end{aligned}$$

where the first inequality is due to $V(k+\ell, 2k+3) \geq V(k, 2k+3)$, by (S1); and the last inequality follows from Lemma 3.3 that $V(k, 2k+3) - V(k, 2k+1) \geq 2 > 0$. The proof is complete. □

Proof of Lemma 3.5. As the proofs for (i) and (ii) are similar, we prove (i) only. By the definition of v_k and (a2), we have

$$V(k, 2k+2) = V(k+1, 2k+2) = v_{k+1}. \tag{4.11}$$

From (a2) and (3.5),

$$V(k, 2k+1) = v_k + (3k+1-v_k)\frac{3}{2(A_k+1)}, \tag{4.12}$$

and

$$V(k, 2k+3) = v_{k+1} + (3(k+1) - v_{k+1})\frac{3}{2A_{k+1}}. \tag{4.13}$$

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We also have, by (3.7),

$$A_{k+1} = \frac{k+1}{(3k+1)(3k+2)}\left[\frac{9}{2}(2k+1)A_k + \frac{3}{4}\right]. \tag{4.14}$$

After substituting (4.14) into (4.13), (4.13) can be written as

$$V(k, 2k+3) = v_{k+1} + (3(k+1) - v_{k+1})\frac{3(3k+1)(3k+2)}{(k+1)[9(2k+1)A_k + 3/2]}. \tag{4.15}$$

Substituting (4.11), (4.12), and (4.15) into (3.9) and performing some calculations, we see that (3.9) becomes

$$\begin{aligned}&\left(1 + \frac{3(3k+1)(3k+2)}{(k+1)[9(2k+1)A_k + 3/2]}\right)v_{k+1}\\ &\quad \geq \left(1 - \frac{3}{2(A_k+1)}\right)v_k + \frac{3(3k+1)}{2(A_k+1)} + \frac{9(3k+1)(3k+2)}{9(2k+1)A_k + 3/2}.\end{aligned} \tag{4.16}$$

By using the recurrence relation for v_k (see (3.6)), (4.16) can be expressed as

$$\begin{aligned}&\left(1 + \frac{(3k+1)(3k+2)}{(k+1)(3(2k+1)A_k + 1/2)}\right)\\ &\quad \times \left(\frac{A_k - 1/2 - k/(2k+1)}{A_k + 1/2}\left[\frac{A_k - 1/2}{A_k+1}v_k + \frac{3(3k+1)/2}{A_k+1}\right]\right.\\ &\qquad \left. + \frac{(3k+1)(3k+2)/(2k+1)}{A_k + 1/2}\right)\\ &\qquad \geq \frac{(2A_k - 1)}{2(A_k+1)}v_k + \frac{3(3k+1)}{2(A_k+1)} + \frac{3(3k+1)(3k+2)}{3(2k+1)A_k + 1/2}.\end{aligned}$$

Collecting together the terms in v_k, transposing terms, and after simplifying, we obtain

$$\begin{aligned}&\frac{(2A_k - 1)}{2(A_k+1)}v_k\\ &\quad \leq -\frac{3(3k+1)}{2(A_k+1)}\\ &\qquad + \left[\left(1 + \frac{(3k+1)(3k+2)}{(k+1)(3(2k+1)A_k + 1/2)}\right)\frac{(3k+1)(3k+2)/(2k+1)}{A_k + 1/2}\right.\\ &\qquad\quad \left. - \frac{3(3k+1)(3k+2)}{3(2k+1)A_k + 1/2}\right] \qquad (4.17)\\ &\qquad \times \left[1 - \left(1 + \frac{(3k+1)(3k+2)}{(k+1)(3(2k+1)A_k + 1/2)}\right)\left(\frac{A_k - 1/2 - k/(2k+1)}{A_k + 1/2}\right)\right]^{-1}.\end{aligned} \tag{4.18}$$

After some manipulation, the last term on the right-hand side of (4.17) can be expressed as

$$\frac{2(3k+1)(3k+2)}{2A_k + 6k + 3}. \tag{4.19}$$

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In view of (4.17) and (4.19), we have

$$\frac{(2A_k - 1)}{2(A_k + 1)} v_k \le -\frac{3(3k+1)}{2(A_k+1)} + \frac{2(3k+1)(3k+2)}{2A_k + 6k + 3},$$

which reduces to

$$v_k \le \frac{(3k+1)(6k+1)}{2A_k + 6k + 3},$$

as claimed in (3.10). This completes the proof. □

5. Concluding remarks

In the stochastic control literature, the Mabinogion sheep problem was first proposed by Williams [3], who showed that the simple policy π^* is optimal. Inspired by his work, we have proposed a more general model $M(p, q)$ and corresponding policy $\pi_{p,q}$ which reduces to Williams' model and policy for $p = q = 1$. Following Williams' method of proof, we showed that $\pi_{p,q}$ is optimal for $M(p, q)$ in the two cases $(p, q) = (1, \frac{1}{2})$ and $(\frac{1}{2}, 1)$. We also pointed out by counterexample that $\pi_{p,q}$ is in general not optimal for $M(p, q)$. It should be remarked that both the model $M(p, q)$ and the policy $\pi_{p,q}$ depend on (p, q) only through their ratio p/q. In particular, when $p = q$, $M(p, q)$ is equivalent to $M(1, 1)$ (Williams' model) and $\pi_{p,q}$ coincides with π^* (Williams' policy). Thus, π^* is optimal for $M(p, q)$ when $p = q$. In view of the main results of this paper together with Williams' result, model $M(p, q)$ is solvable and admits a simple solution when $p/q \in \{1, \frac{1}{2}, 2\}$. On the other hand, it appears to be difficult to solve the general (p, q)-problem whose solution is likely to be complicated.

It may be of interest to consider the following (seemingly 'intuitively clear') statements for general $M(p, q)$ concerning threshold-structure properties of the optimal policy under $M(p, q)$:

(A) $V(w, b) = V(w - 1, b)$ implies $V(w + 1, b) = V(w, b)$;

(B) $V(w, b) = V(w - 1, b)$ implies $V(w, b - 1) = V(w - 1, b - 1)$;

(C) $V(w, b) = V(w - 1, b)$ implies $V(w + 1, b - 1) = V(w, b - 1)$,

where $V(w, b)$ denotes the optimal value at state (w, b) under $M(p, q)$. While (C) can be established, the proofs of (A) and (B) remain elusive. Note that (A) and (B) together imply (C). Note also that (A), (B), and (C) are equivalent to saying that if to do nothing is not optimal at state (w, b), then to do nothing is not optimal at state $(w + 1, b)$, $(w, b - 1)$, and $(w + 1, b - 1)$, respectively.

Finally, we state (without proof) some asymptotic results for the optimal value functions under $M(1, \frac{1}{2})$ and $M(\frac{1}{2}, 1)$.

Theorem 5.1. (i) *Let V be the value function under the $M(1, \frac{1}{2})$ model. Then*

$$V(k, 2k) = 3k - \frac{\sqrt{3\pi k}}{2} + \frac{1}{6} + \frac{\pi}{8} + o(1) \quad \text{as } k \to \infty.$$

(ii) *Let U be the value function under the $M(\frac{1}{2}, 1)$ model. Then*

$$U(2k, k) = 3k - \sqrt{3\pi k} - \frac{1}{3} + \frac{\pi}{2} + o(1) \quad \text{as } k \to \infty.$$

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References

[1] Chan, T. (1996). Some diffusion models for the Mabinogion sheep problem of Williams. *Adv. Appl. Prob.* **28,** 763–783.

[2] Flajolet, P. and Huillet, T. (2008). Analytic combinatorics of the Mabinogion urn. In *Fifth Colloquium on Mathematics and Computer Science*, Association of Discrete Mathematics and Theoretical Computer Science, Nancy, pp. 549–572.

[3] Williams, D. (1991). *Probability with Martingales*. Cambridge University Press.