A novel method to solve the quartic equation

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1. *Introduction*

The first solution to the quartic equation is attributed to Lodovico Ferrari, a student of Geralamo Cardano. The solution was published alongside the solution of the cubic in Cardano's book *Ars Magna* [1]. In this Article, we introduce a new canonical form to simplify the derivation of the roots of the equation

$$z^4 + z^3 + fz^2 + g = 0 \qquad \text{with} \qquad f, g \in \mathbb{R}. \tag{1}$$

Consider the general quartic equation with real coefficients:

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

Ferrari depresses this equation by using a change of variable $x = y - \frac{a}{4}$ to remove the cubic term. So, without loss of generality, let us start with the traditional reduced equation

$$y^4 + py^2 + qy + r = 0 \tag{2}$$

where

$$p = -\frac{3a^2}{8} + b, \qquad q = \frac{a^3}{8} - \frac{ab}{2} + c, \qquad r = -\frac{3a^4}{256} + \frac{a^2b}{16} - \frac{ac}{4} + d.$$

When $r = 0$, we have a trivial root $y_0 = 0$ and we end up with a cubic. When $q = 0$, we have a biquadratic equation. Therefore, let us assume from here on that $r \neq 0$ and $q \neq 0$.

Any quartic equation can be reduced to the canonical form (1) using one of the following two transformations:

Transformation 1:

We start from the reduced equation (2). A change of variable

$$z = \frac{r}{yq}$$

leads to

$$f = \frac{pr}{q^2} \qquad \text{and} \qquad g = \frac{r^3}{q^4}.$$

Transformation 2:

For simplicity of computation, we will start again from the reduced equation (2). We note this is not a requirement for this transformation, as the same algebraic steps described below can be applied directly to the general quartic equation.

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Let us define $y = \xi + m$. Then

$$\xi^4 + 4m\xi^3 + (6m^2 + p)\xi^2 + (4m^3 + 2pm + q)\xi + (m^4 + pm^2 + qm + r) = 0. \quad (3)$$

Choose a real m (a cubic has at least one real root) that satisfies

$$4m^3 + 2pm + q = 0.$$

This reduces (3) to

$$\xi^4 + 4m\xi^3 + (6m^2 + p)\xi^2 + (m^4 + pm^2 + qm + r) = 0.$$

We then use a change of variable

$$z = \frac{\xi}{4m}.$$

Notice $m \neq 0$ since $q \neq 0$. This further simplifies the equation to the canonical form (1) where

$$f = \frac{6m^2 + p}{16m^2}, \qquad g = \frac{m^4 + pm^2 + qm + r}{256m^4}.$$

2. *Derivation*

We start with the canonical equation (1)

$$z^4 + z^3 + fz^2 + g = 0.$$

We assume $g \neq 0$ otherwise the equation is trivial.

We look for a set of parameters u, v and w so that the equation above is equivalent to

$$(uz^2 + wz)^2 = (z^2 + v)^2. \quad (4)$$

To avoid the collapse of (4), we further impose that $u^2 \neq 1$ and $u \neq 0$.

Expanding the equation leads to the following quartic equation

$$z^4 + \frac{2uw}{u^2 - 1}z^3 + \frac{w^2 - 2v}{u^2 - 1}z^2 + \frac{v^2}{1 - u^2} = 0. \quad (5)$$

Equations (1) and (4) are equivalent if, and only if, the coefficients of (1) and (5) are equal. Starting with the constant and cubic terms

$$\frac{v^2}{1 - u^2} = g \qquad \text{or} \qquad u^2 = 1 - \frac{v^2}{g}, \quad (6)$$

$$\frac{2uw}{u^2 - 1} = 1 \qquad \text{or} \qquad w = \frac{u^2 - 1}{2u}, \quad (7)$$

equating the quadratic terms leads to

$$\frac{w^2 - 2v}{u^2 - 1} = f.$$

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Using (7) to substitute w,

$$\frac{u^2 - 1}{4u^2} + \frac{2v}{1 - u^2} = f. \tag{8}$$

Thus

$$8u^2v - (u^2 - 1)^2 = 4fu^2(u^2 - 1).$$

Substituting for u^2 from condition (6), we obtain

$$8v\left(1 - \frac{v^2}{g}\right) - \frac{v^4}{g^2} = 4f\left(1 - \frac{v^2}{g}\right)\frac{v^2}{g}.$$

So, multiplying both sides by g^2 and bringing the terms to one side, we have

$$4f(g - v^2)v^2 - 8gv(g - v^2) + v^4 = 0.$$

Thus

$$v\left((1 - 4f)v^3 + 8gv^2 + 4fgv - 8g^2\right) = 0.$$

Using (6), $v \neq 0$ since $g \neq 0$. Therefore

$$h(v) = (1 - 4f)v^3 + 8gv^2 + 4fgv - 8g^2 = 0. \tag{9}$$

As a reminder, Ferrari's method requires solving a cubic equation, which is traditionally called the resolvent cubic. Similarly in our case, (9) is the resolvent cubic that is necessary to solve canonical equation (1).

If $f = \frac{1}{4}$, (1) is simply $\left(z^2 + \frac{z}{2}\right)^2 + g = 0$ which is trivial.

We assume that $1 - 4f \neq 0$. (9) is a cubic equation with at least one real root v. Let us further show that we can choose a positive real value for u. Indeed, if $g < 0$, $1 - \frac{v^2}{g} > 0$ so (6) admits a strictly positive solution u. If $g > 0$, since $h(-\sqrt{g}) = -g\sqrt{g} < 0$ and $h(\sqrt{g}) = g\sqrt{g} > 0$, there is a real root v such that $-\sqrt{g} < v < \sqrt{g}$; therefore $1 - \frac{v^2}{g} > 0$ and (6) admits a strictly positive solution u

$$u = \sqrt{1 - \frac{v^2}{g}} > 0. \tag{10}$$

Notice that since $g \neq 0$, $v \neq 0$ therefore $u^2 \neq 1$. This satisfies the conditions imposed above ($u^2 \neq 1$ and $u \neq 0$).

Roots formulae

We have found parameters u, v and w so that (1) and (4) are equivalent. Therefore the solutions of (1) are the roots of the two quadratic equations resulting from (4), namely

$$uz^2 + \frac{u^2 - 1}{2u}z = \varepsilon_k(z^2 + v) \qquad (k = 1, 2)$$

where $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$

$$uz^2 + \frac{u^2 - 1}{2u}z - \varepsilon_k\left(z^2 + v\right) = 0.$$

Thus

$$\left(u - \varepsilon_k\right)z^2 + \frac{u^2 - 1}{2u}z - \varepsilon_k v = 0$$

and

$$z^2 + \frac{u^2 - 1}{2u\left(u - \varepsilon_k\right)}z - \varepsilon_k\frac{v}{u - \varepsilon_k} = 0.$$

Since $u^2 - 1 = \left(u - \varepsilon_k\right)\left(u + \varepsilon_k\right)$, we can simplify the linear term, giving

$$z^2 + \frac{u + \varepsilon_k}{2u}z - \varepsilon_k\frac{v}{u - \varepsilon_k} = 0.$$

Separating the two cases of ε_k,

$$z^2 + \frac{u - 1}{2u}z + \frac{v}{1 + u} = 0 \qquad \text{and} \qquad z^2 + \frac{1 + u}{2u}z + \frac{v}{1 - u} = 0.$$

The discriminants are

$$\Delta = \left(\frac{u - 1}{2u}\right)^2 - 4\frac{v}{1 + u} \qquad \text{and} \qquad \Delta' = \left(\frac{u + 1}{2u}\right)^2 - 4\frac{v}{1 - u}.$$

Let us express Δ and Δ' in terms of u. Recall from (8):

$$\frac{u^2 - 1}{4u^2} + \frac{2v}{1 - u^2} = f.$$

Multiplying both sides by $2(1 - u)$,

$$\frac{(u^2 - 1)(1 - u)}{2u^2} + \frac{4v}{1 + u} = 2f(1 - u).$$

So

$$\frac{4v}{1 + u} = 2f(1 - u) - \frac{(u^2 - 1)(1 - u)}{2u^2}.$$

Therefore

$$\Delta = \left(\frac{u - 1}{2u}\right)^2 + \frac{(u^2 - 1)(1 - u)}{2u^2} - 2f(1 - u),$$

which simplifies to

$$\Delta = \frac{(8f - 2)u^3 + (3 - 8f)u^2 - 1}{4u^2}.$$

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The working for Δ' is similar and leads to

$$\Delta' = \frac{(2 - 8f)u^3 + (3 - 8f)u^2 - 1}{4u^2}.$$

Therefore the four roots are

$$z_{1,2} = \frac{(1 - u) \pm \sqrt{(8f - 2)u^3 + (3 - 8f)u^2 - 1}}{4u} \tag{11}$$

and

$$z_{3,4} = \frac{-(1 + u) \pm \sqrt{(2 - 8f)u^3 + (3 - 8f)u^2 - 1}}{4u}. \tag{12}$$

Summary

Outside trivial cases, general quartic equations can be transformed into the canonical form (1)

$$z^4 + z^3 + fz^2 + g = 0$$

which can be solved using the steps below (assuming $g \neq 0$ and $f \neq \frac{1}{4}$, which are simple cases):

1. First, find the real root(s) of the resolvent cubic:

$$(1 - 4f)v^3 + 8gv^2 + 4fgv - 8g^2 = 0.$$

2. Second, select a root v such as $u = \sqrt{1 - \frac{v^2}{g}}$ is real (this is always possible, as proved above).
3. Finally, use (11) and (12) to obtain the four roots for the canonical equation.

3. *Examples*

3.1 Example 1

$$x^4 + 2x^3 + 12x^2 + 96 = 0.$$

To obtain the canonical form, substitute $z = \frac{1}{2}x$; then

$$z^4 + z^3 + 3z^2 + 6 = 0.$$

So

$$f = 3 \qquad \text{and} \qquad g = 6.$$

The resolvent cubic (9) becomes

$$-11v^3 + 48v^2 + 72v - 288 = 0.$$

This equation has three real solutions. We choose the one with the smallest absolute value

$$v \approx 2.3197,$$

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giving

$$u = \sqrt{1 - \frac{v^2}{g}} \approx 0.3212.$$

The solutions of the quartic equation are $x_i = 2z_i$ for $i = 1, 2, 3, 4$ where

$$z_{1,2} = \frac{(1-u) \pm \sqrt{(8f-2)u^3 + (3-8f)u^2 - 1}}{4u} \approx 0.5283 \pm i1.2152,$$

$$z_{3,4} = \frac{-(1+u) \pm \sqrt{(2-8f)u^3 + (3-8f)u^2 - 1}}{4u} \approx -1.0283 \pm i1.5362.$$

3.2 Example 2

$$x^4 + 2x^3 + 5x^2 - 7x + 6 = 0$$

From (2): $p = 3.5$, $q = -11$ and $r \approx 10.5625$.

Using transformation 1:

$$h = \frac{pr}{q^2} \approx 0.3055, \qquad g = \frac{r^3}{q^4} \approx 0.0805.$$

This leads to a resolvent cubic:

$$(1 - 4f)v^3 + 8gv^2 + 4fgv - 8g^2 = 0.$$

This equation has three real solutions. We choose the one with the smallest absolute value:

$$v \approx 0.2239,$$

giving

$$u = \sqrt{1 - \frac{v^2}{g}} \approx 0.6139.$$

The solutions of the canonical equation are $x_i = \dfrac{r}{qz_i} - \dfrac{b}{4a}$ where

$$z_{1,2} = \frac{(1-u) \pm \sqrt{(8f-2)u^3 + (3-8f)u^2 - 1}}{4u} \approx 0.1572 \pm i0.3377,$$

$$z_{3,4} = \frac{-(1-u) \pm \sqrt{(2-8f)u^3 + (3-8f)u^2 - 1}}{4u} \approx -0.6572 \pm i0.3849.$$

The solutions of the original equation are thus

$$x_{1,2} \approx -1.5879 \pm i2.3369,$$
$$x_{3,4} \approx 0.5879 \pm i0.6371.$$

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Using transformation 2:

Solving the equation

$$4m^3 + 2pm + q = 0$$

leads to $m = 1$ and to

$$f = \frac{6m^2 + p}{16m^2} \approx 0.5938, \qquad g = \frac{m^4 + pm^2 + qm + r}{256m^4} \approx 0.0159.$$

This leads to a resolvent cubic

$$(1 - 4f)v^3 + 8gv^2 + 4fgv - 8g^2 = 0.$$

This equation has three real solutions. We choose the one with the smallest absolute value

$$v \approx 0.0496,$$

giving

$$u = \sqrt{1 - \frac{v^2}{g}} \approx 0.9192.$$

The solutions of the canonical equation are:

$$z_{1,2} = \frac{(1 - u) \pm \sqrt{(2 - 8f)u^3 + (3 - 8f)u^2 - 1}}{4u} \approx 0.0220 \pm i0.1593,$$

$$z_{3,4} = \frac{-(1 + u) \pm \sqrt{(8f - 2)u^3 + (3 - 8f)u^2 - 1}}{4u} \approx -0.5220 \pm i0.5842.$$

Since

$$x_i = 4mz_i + m - \frac{b}{4a},$$

the solutions of the original equation are:

$$x_{1,2} \approx 0.5879 \pm i0.6371,$$

$$x_{3,4} \approx -1.5879 \pm i2.3369.$$

Reference

1. Girolamo Cardano, *Ars Magna* (1545).

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