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AN EXISTENCE THEOREM FOR ABSTRACT STABLE SETS

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Abstract

We provide an existence theorem for stable sets which is equivalent to Zorn's lemma and study the connections between the unique stable set for majorization and the stable sets for the dominance relation.

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I. Introduction

One of the main concerns of cooperative game theory is to find von Neumann-Morgenstern $(vN - M)$ stable sets, that is, subsets of the set of imputations which are stable with respect to the binary relation "dominance". Lucas [5] suggested that one can abstract these concepts and attempt to find subsets of arbitrary sets of points which are "stable" with respect to any irreflexive binary relation on this set. This approach was begun in von Neumann and Morgenstern [7].

The purpose of this note is to provide an existence theorem of stable sets under a general framework. We will use the partial order R and a set X to replace "dominance" and imputation set X, respectively. It turns out that the existence theorem is equivalent to Zorn's lemma.

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2. Definitions and main theorem

Let R be a partial order and X be an arbitrary set. A subset V of X is said to be a stable set if V satisfies the following two conditions:

(i) for any $x \in X - V$, there exists a $y \in V$ such that yRx;

(ii) for any x, $y \in V$, both xRy and yRx fail.

We call (i) internally stable and (ii) externally stable. For any $M \subseteq X$, we denote

$$R(M) = \{x \in X : \text{ there exists a } y \in M \text{ such that } yRx\}.$$

DEFINITION. $\{x_i\}_{i=1}^{\infty}$ is a chain of relation R if $x_{i+1}Rx_i$ for all $i \in Z^+$. We say that the chain $\{x_i\}_{i=1}^{\infty}$ has an upper bound in X if there exists $x \in X$ such that xRx_i for all $i \in Z^+$.

THEOREM. *Let R be a partial order on a set X. Assume that for each chain $\{x_i\}_{i=1}^{\infty}$ of relation R in X with $x_{i+1}Rx_i$ for each $i \in Z^+$ has an upper bound in X. Then the set of maximal elements is the unique stable set.*

Before we prove the theorem, let us state Zorn's lemma [6, p. 248] first.

ZORN'S LEMMA. *Let R be a partial order on a set X. Assume that for each chain $\{x_i\}_{i=1}^{\infty}$ of relation R in X with $x_{i+1}Rx_i$ for each $i \in Z^+$ has an upper bound in X. Then X has a maximal element.*

We observe that the hypothesis of our main result is exactly the same as the hypothesis of Zorn's lemma. Hence, if one accepts the conclusion of Zorn's lemma then one may conclude that the set M of maximal elements is not empty. Of course, the other direction also holds. We will show that the set M is the unique stable set.

PROOF OF THE MAIN THEOREM. Define $M = \{x \in X : x$ is a maximal element with respect to R on $X\}$. Then $M \neq \phi$ by Zorn's Lemma.

Let $W = X - [R(M) \cup M]$. If we can show that $W = \phi$, then $X = R(M) \cup M$. Since M is the set of maximal elements, M is internally stable. By the definition of $R(M)$, we know that M also satisfies external stability. We conclude that M is a stable set.

We are going to show that $W = \phi$. Since R is transitive, $R(M \cup R(M)) = R(M)$. Let $\{x_i\}_{i=1}^{\infty} \subset W \subset X$ with $x_{i+1}Rx_i$ for all i. By the assumption that every chain of X has an upper bound in X, there exists an $x \in X$ such that xRx_i for all $i \in Z^+$. If $x \notin W$, then $x \in M$ or $x \in R(M)$. This implies $\{x_i\}_{i=1}^{\infty} \subset R(M)$ by transitivity of R. This violates the assumption

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that $W = X - [R(M) \cup M]$. Hence, W inherits the main property of X that every chain in W has an upper bound in W. Suppose that $W \neq \phi$. Then W has a nonempty set of maximal elements by Zorn's Lemma. Since $R(M \cup R(M)) = R(M)$, each maximal element of W is in M, contradicting $W \cap M = \phi$. Therefore, $W = \phi$.

It remains to show that M is the only possible stable set. Clearly M is a subset of any stable set V. On the other hand, suppose that V is stable and $x \in V - M$. Then there exists a $y \in X$ with yRx. Since V is internally stable, $y \notin V$. Then there exists a $z \in V$ with zRy. By transitivity of R, zRx, contradicting the internal stability of V. We obtain the desired result. □

The following example shows that the converse of this theorem is not true. That is, the set M of maximal elements for a partial order can be the unique stable set even when the hypothesis of Zorn's lemma does not hold.

Example. Let $x = \{x_i\}_{i=1}^{\infty} \cup \{y_j\}_{j=1}^{\infty}$. Define the relation R as follows: $y_j R x_i$ for each $j \geq i$ and $x_{i+1} R x_i$ for each $i \in Z^+$. Then R is a partial order. Since y_h and y_k are not comparable with respect to R, for $h \neq k$, $\{y_j\}_{j=1}^{\infty}$ is internally stable. For each $x_l \in \{x_i\}_{i=1}^{\infty}$, there exists $y_{l+1} \in \{y_j\}_{j=1}^{\infty}$ such that $y_{l+1} R x_l$. Hence, we obtain that $\{y_j\}_{j=1}^{\infty}$ is a stable set with respect to R in X. However, $\{x_i\}_{i=1}^{\infty}$ has no upper bound.

3. The dominance relation and majorization

An n-person game is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a set of n players and v is a real-valued function on 2^N with $v(\phi) = 0$. The set of imputations is

$$A = \left\{ x \in R^n : \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}),\ \text{ for } i \in N \right\}.$$

For any x and $y \in A$ and nonempty $S \subseteq N$, we say x dominates y via S, denote by $x \operatorname{dom}_s y$, if $x_i > y_i$ for all $i \in S$ and $\sum_{i \in S} x_i \leq v(S)$. We say x dominates y, denoted by $x \operatorname{dom} y$, if there is some $S \subseteq N$ such that $x \operatorname{dom}_s y$. A subset V of A is said to be a $vN-M$ stable set if V satisfies both internal stability and external stability under the binary relation dom.

Definition. Let $a, b \in A$. Then b majorizes a if $b \operatorname{dom} a$ and for all $c \in A$, if $c \operatorname{dom} b$ then $c \operatorname{dom} a$. This is written as $b \to a$.

An imputation a is called unmajorized if there does not exist b in A such that $b \to a$. It is clear that the binary relation dom is not a partial order. Gillies [2] showed that the relation $\to$ is a partial order and preserved in the

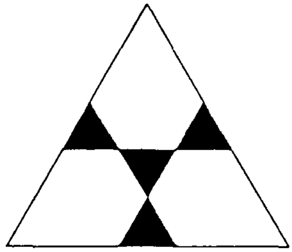

FIGURE 1

limit. That is, if $\{a_i\}_{i=1}^{\infty}$ is a sequence for which $a_{i+1} \to a_i$ holds for each i, and if a is any limit of this sequence, then $a \to a_i$ for every i.

We know that the imputation set A is compact and the fact that $\to$ is a partial order and preserved in the limit, hence, for any sequence $\{a_i\}_{i=1}^{\infty}$ with $a_{i+1} \to a_i$ in A, there exists a limit a of $\{a_i\}_{i=1}^{\infty}$ in A such that $a \to a_i$ for all i. In other words, every chain $\{x_i\}_{i=1}^{\infty}$ of relation $\to$ in A has an upper bound. Zorn's lemma and Gillies' results show that M, the set of all unmajorized elements, is nonempty.

It is well known that if an imputation which is majorized will not be in any $vN - M$ stable sets. Therefore the union of all $vN - M$ stable sets contains in M. It is clear that M is not a $vN - M$ stable set. For instance, we consider a 3-person game as follows:

$$v(123) = 1, \; v(12) = v(13) = v(23) = 0.6,$$
$$v(i) = 0, \qquad \text{for } i \in \{1, 2, 3\}.$$

The game has nonempty core. The set of all unmajorized elements is the set of shaded area in Figure 1.

Hence, in this 3-person case, M is equal to union of all $vN - M$ stable sets. However, this is not always the case, Lucas [3, 4] construct a ten-person game which has no $vN - M$ stable set and we know that $M \neq \phi$ for all games.

Besides nonexistence of $vN - M$ stable set, there are some pathologies which have been found for $vN - M$ stable set. Although there are many undesirable results for $vN - M$ stable set, Chang [1] wrote as follow: "these need not imply that the basic idea of von Neumann and Morgenstern is not a good one. It may only indicate that their particular condition should be altered."

From previous arguments about the relationship between $vN - M$ stable sets and unmajorized elements, we will guess that unmajorized elements might provide us a way to modify von Neumann and Morgenstern's idea.

References

[1] C. Chang, 'Absorbing set for n-Person Games,' *Z. Oper. Res. Ser. A* **31** (1987), 15–29.
[2] D. B. Gillies, *Solutions to general non-zero sum games*, Ann. of Math. Studies 40, Princeton University Press, Princeton, N. J., 1959, 47–80.
[3] W. F. Lucas, 'A game with no solution', *Bull. Amer. Math. Soc.* (*N.S.*) **74** (1968), 237–239.
[4] W. F. Lucas, 'The proof that a game may not have solution', *Trans. Amer. Math. Soc.* **137** (1969), 219–229.
[5] W. F. Lucas, 'Some recent developments in n-person game theory' *SIAM Rev.* **13** (1971), 491–523.
[6] P. Suppes, *Axiomatic set theory*, (Dover, 1960).
[7] J. von Neumann and O. Morgenstern, *Games and economic behavior*, (3rd. edition, Princeton Press, 1953).

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