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MARKOV'S PRINCIPLE AND SUBSYSTEMS OF INTUITIONISTIC ANALYSIS

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Abstract. Using a technique developed by Coquand and Hofmann [3] we verify that adding the analytical form MP_1: $\forall\alpha(\neg\neg\exists x\alpha(x) = 0 \rightarrow \exists x\alpha(x) = 0)$ of Markov's Principle does not increase the class of Π^0_2 formulas provable in Kleene and Vesley's formal system for intuitionistic analysis, or in subsystems obtained by omitting or restricting various axiom schemas in specified ways.

§1. Introduction. In [6] Kleene proved that Markov's Principle MP_1 is neither provable nor refutable in his formal system **I** for intuitionistic analysis. By the Friedman–Dragalin translation, Markov's Rule is admissible for **I** and many subsystems.

We show that adding MP_1 as an axiom to **I** does not increase consistency strength, in the sense that no additional Π^0_2 formulas become provable. The method, adapted from Coquand and Hofmann's dynamic modification [3] of the Friedman–Dragalin translation, works also for subsystems of **I** with a few interesting exceptions.

§2. Language, logic, and basic mathematical axioms.

2.1. The two-sorted formal language and intuitionistic predicate logic. Kleene and Vesley's language $\mathcal{L}_1$ for two-sorted intuitionistic number theory or "intuitionistic analysis" has variables a,b,c, ... ,x,y,z, ... , intended to range over natural numbers; variables $\alpha, \beta, \gamma, \ldots$, intended to range over one-place number-theoretic functions (choice sequences); finitely many constants $0, ', +, \cdot, \mathrm{f}_4, \ldots, \mathrm{f}_p$, each representing a primitive recursive function or functional, where f_i has k_i places for number arguments and l_i places for type-1 function arguments; parentheses indicating function application; and Church's λ.

The *terms* (of type 0) and *functors* (of type 1) are defined inductively as follows. The number variables and 0 are terms. The function variables and each f_i with $k_i = 1, l_i = 0$ are functors. If $\mathrm{t}_1, \ldots, \mathrm{t}_{k_i}$ are terms and $\mathrm{u}_1, \ldots, \mathrm{u}_{l_i}$ are functors, then $\mathrm{f}_i(\mathrm{t}_1, \ldots, \mathrm{t}_{k_i}, \mathrm{u}_1, \ldots, \mathrm{u}_{l_i})$ is a term. If x is a number variable and t is a term, then λx.t is a functor. And if u is a functor and t is a term, then (u)(t) is a term.

There is one relation symbol = for equality between terms; equality between functors u, v is defined extensionally by $\mathrm{u} = \mathrm{v} \equiv \forall \mathrm{x}(\mathrm{u}(\mathrm{x}) = \mathrm{v}(\mathrm{x}))$ (where x is not free in

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u or v). The atomic formulas of $\mathcal{L}_1$ are the expressions $\mathrm{s} = \mathrm{t}$ where s, t are terms. Composite formulas are defined inductively, using the connectives $\&, \vee, \rightarrow, \neg$, quantifiers $\forall, \exists$ of both sorts, and parentheses (often omitted under the usual conventions on scope). $\mathrm{A} \leftrightarrow \mathrm{B}$ is defined by $(\mathrm{A} \rightarrow \mathrm{B}) \,\&\, (\mathrm{B} \rightarrow \mathrm{A})$.

The logical axioms and rules are those of two-sorted intuitionistic predicate logic, as presented in [6] (building on [4]). If the intuitionistic axiom schema $\neg\mathrm{A} \rightarrow (\mathrm{A} \rightarrow \mathrm{B})$ were replaced by $\neg\neg\mathrm{A} \rightarrow \mathrm{A}$ (of which Markov's Principle MP_1 is a special case), two-sorted classical predicate logic would result.

2.2. Two-sorted intuitionistic arithmetic $\mathbf{IA}_1$. This is a conservative extension, in the language $\mathcal{L}_1$, of the first-order intuitionistic arithmetic $\mathbf{IA}_0$ in [4] based on $=, 0, ', +, \cdot$. The mathematical axioms of $\mathbf{IA}_1$ are:

(a) The axiom-schema of mathematical induction (for all formulas of $\mathcal{L}_1$): $\mathrm{A}(0) \,\&\, \forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \rightarrow \mathrm{A}(\mathrm{x}')) \rightarrow \mathrm{A}(\mathrm{x})$.
(b) The axioms of $\mathbf{IA}_0$ for $=, 0, ', +, \cdot$ (axioms 14–21 on page 82 of [4]) and the axioms expressing the primitive recursive definitions of the additional function constants $\mathrm{f}_4, \dots, \mathrm{f}_{26}$ given in [6] and [5].[1]
(c) The open equality axiom: $\mathrm{x} = \mathrm{y} \rightarrow \alpha(\mathrm{x}) = \alpha(\mathrm{y})$.
(d) The axiom-schema of λ-conversion: $(\lambda \mathrm{x}.\mathrm{t}(\mathrm{x}))(\mathrm{s}) = \mathrm{t}(\mathrm{s})$, where $\mathrm{t}(\mathrm{x})$ is a term and s is free for x in $\mathrm{t}(\mathrm{x})$.

For readers familiar with [6], $\mathbf{IA}_1$ is the subsystem of the "basic system" **B** obtained by omitting the axiom schemas of countable choice and bar induction (${}^x2.1$ and ${}^x26.3$, respectively).

In addition to the open equality axiom (c), the equality axioms

$$\alpha_1 = \beta_1 \& \cdots \& \alpha_{l_i} = \beta_{l_i} \rightarrow \mathrm{f}_i(\mathrm{x}_1, \dots, \mathrm{x}_{k_i}, \alpha_1, \dots, \alpha_{l_i}) = \mathrm{f}_i(\mathrm{x}_1, \dots, \mathrm{x}_{k_i}, \beta_1, \dots, \beta_{l_i}),$$

are provable for all function constants f_i. Thus $\mathbf{IA}_1$ satisfies the replacement property of equality for functors as well as for terms.

$\mathbf{IA}_1$ can only prove the existence of primitive recursive sequences, in the sense that each closed theorem of the form $\exists\alpha \mathrm{A}(\alpha)$ has a primitive recursive witness. The finite list of primitive recursive function constants, with their corresponding

[1] $\mathrm{f}_0 - \mathrm{f}_3$ are $0, ', +, \cdot$ respectively. $\mathrm{f}_4(\mathrm{a}, \mathrm{b}) = \mathrm{a}^{\mathrm{b}}$ (exponentiation), and $\mathrm{f}_5, \dots, \mathrm{f}_{20}$ represent the primitive recursive function(al)s $a!$, $a \dot{-} b$, $pd(a)$, $min(a, b)$, $max(a, b)$, $\overline{sg}(a) = 1 \dot{-} a$, $sg(a) = 1 \dot{-} (1 \dot{-} a)$, $|a - b|$, $rm(a, b)$, $[a/b]$, $\Sigma_{y<b}\alpha(y)$, $\Pi_{y<b}\alpha(y)$, $min_{y \leq b}\alpha(y)$, $max_{y \leq b}\alpha(y)$, p_a (the a^{th} prime, with $p_0 = 2$), and $(a)_i$ (the exponent of p_i in the prime factorization of a) respectively. We write $(\mathrm{a})_\mathrm{i}$ for $\mathrm{f}_{20}(\mathrm{a}, \mathrm{i})$, and similarly for the other function constants. $\mathrm{f}_{21}(\mathrm{a}) = \mathrm{lh}(\mathrm{a}) = \Sigma_{\mathrm{i}<\mathrm{a}}\mathrm{sg}((\mathrm{a})_\mathrm{i})$ represents the number of positive exponents in the prime factorization of a. Bounded quantifiers are defined with the help of bounded sum and product. $\mathrm{Seq}(\mathrm{a})$ is a prime formula equivalent to $\mathrm{a} > 0 \,\&\, \forall \mathrm{i} < \mathrm{lh}(\mathrm{a})\ (\mathrm{a})_\mathrm{i} > 0$, expressing "$a$ codes the finite sequence $((a)_0 - 1, \dots, (a)_{lh(a)-1} - 1)$". $\mathrm{f}_{22}(\mathrm{a}, \mathrm{b}) = \mathrm{a} * \mathrm{b}$ produces a code for the concatenation of two finite sequences from their codes. $\langle\,\rangle = 1$ codes the empty sequence, and $\mathrm{f}_{23}(\mathrm{x}, \alpha) = \overline{\alpha}(\mathrm{x}) = \Pi_{\mathrm{i}<\mathrm{x}}\mathrm{p}_\mathrm{i}^{\alpha(\mathrm{i})+1}$ represents the standard code $\langle \alpha(0) + 1, \dots, \alpha(x - 1) + 1 \rangle$ for the x^{th} initial segment of α. This coding is not onto $\mathbb{N}$, but it satisfies $\langle a_0 + 1, \dots, a_k + 1 \rangle * \langle a_{k+1} + 1, \dots, a_m + 1 \rangle = \langle a_0 + 1, \dots, a_m + 1 \rangle$. In contrast, $\mathrm{f}_{24}(\mathrm{x}, \alpha) = \tilde{\alpha}(\mathrm{x}) = \Pi_{\mathrm{i}<\mathrm{x}}\mathrm{p}_\mathrm{i}^{\alpha(\mathrm{i})}$ cannot code finite sequences directly as $\langle a_0, \dots, a_k \rangle = \langle a_0, \dots, a_k, 0 \rangle$. $\mathrm{f}_{25}(\mathrm{a}, \mathrm{b}) = \mathrm{a} \circ \mathrm{b} = \Pi_{\mathrm{i}<\max(\mathrm{a},\mathrm{b})}\mathrm{p}_\mathrm{i}^{\max((\mathrm{a})_\mathrm{i}, (\mathrm{b})_\mathrm{i})}$, and $\mathrm{f}_{26}(\mathrm{y}) = \mathrm{ccp}(\mathrm{y})$ represents the course-of-values function for the characteristic function of the predicate "y is a computation tree number." These suffice for Kleene's formal treatment ([5] Part I) of recursive partial functionals, including the recursion theorem and a normal form theorem.

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axioms, is intended to be expanded as needed. Here we use the λ notation to explicitly define termwise multiplication of sequences: $(\alpha \cdot \beta)$ will abbreviate $\lambda x(\alpha(x) \cdot \beta(x))$. We also define $sg(\alpha) = \lambda x.sg(\alpha(x))$, in effect adding binary sequence variables to $\mathcal{L}_1$.

2.3. Intuitionistic recursive analysis IRA. The principle of countable choice for numbers is expressed in $\mathcal{L}_1$ by the schema (*2.2 in [6]):

$$AC_{00}: \quad \forall x \exists y A(x, y) \rightarrow \exists \alpha \forall x A(x, \alpha(x)),$$

where α, x must be free for y in $A(x, y)$. Intuitionistic recursive analysis **IRA** can be axiomatized, as a subsystem of Kleene and Vesley's **B**, by $\mathbf{IA}_1$ + qf-AC_{00}, where qf-AC_{00} is the restriction of AC_{00} to formulas $A(x, y)$ without sequence quantifiers and with only bounded number quantifiers. **IRA** ensures that the range of the type-1 variables contains all general recursive sequences and is closed under general recursive processes. Troelstra's **EL** and Veldman's **BIM** are alternative axiomatizations of **IRA**, cf. [7], [8].

In the two-sorted language, **IRA** + MP_1 + CT_1 formalizes Russian recursive analysis (**RUSS** in [2]), where MP_1 is the functional form of Markov's Principle

$$MP_1: \quad \forall \alpha[\neg\neg\exists x \alpha(x) = 0 \rightarrow \exists x \alpha(x) = 0]$$

and CT_1 expresses Church's Thesis:

$$CT_1: \quad \forall \alpha \exists e \forall x \exists y[T_0(e, x, y) \ \& \ U(y) = \alpha(x)].$$

The general recursive functions form a classical ω-model of **RUSS** and hence of **IRA**, but **RUSS** + AC_{00} (unlike **IRA** + AC_{00}) is inconsistent with classical logic.

§3. Definition of the translation, and properties proved in $\mathbf{IA}_1$.

3.1. Definition. Let $Z(\alpha)$ abbreviate $\exists x\, \alpha(x) = 0$. To each formula E of $\mathcal{L}_1$ and each sequence variable α not occurring in E, we associate another formula E^α with the same free variables plus α, by induction on the logical form of E as follows. For cases 4 and 5, β should be distinct from α, and $A^{sg(\beta)}$ is the result of substituting $sg(\beta)$ for γ in the definition of A^γ. Similarly for $B^{\alpha \cdot \beta}$ in Case 4.

1. P^α is $P \vee Z(\alpha)$ if P is prime.
2. $(A \ \& \ B)^\alpha$ is $A^\alpha \ \& \ B^\alpha$.
3. $(A \vee B)^\alpha$ is $A^\alpha \vee B^\alpha$.
4. $(A \rightarrow B)^\alpha$ is $\forall \beta(A^{sg(\beta)} \rightarrow B^{\alpha \cdot \beta})$.
5. $(\neg A)^\alpha$ is $\forall \beta(A^{sg(\beta)} \rightarrow Z(\alpha \cdot \beta))$.
6. $(\forall x A(x))^\alpha$ is $\forall x A^\alpha(x)$.
7. $(\exists x A(x))^\alpha$ is $\exists x A^\alpha(x)$.
8. $(\forall \gamma A(\gamma))^\alpha$ is $\forall \gamma A^\alpha(\gamma)$.
9. $(\exists \gamma A(\gamma))^\alpha$ is $\exists \gamma A^\alpha(\gamma)$.

From now on, let $\alpha \in 2^{\mathbb{N}}$ abbreviate $\alpha = sg(\alpha)$.

3.2. Proposition.

(a) $\mathbf{IA}_1 \vdash \forall \alpha \forall \beta(Z(\alpha \cdot \beta) \leftrightarrow Z(\alpha) \vee Z(\beta))$.
(b) $\mathbf{IA}_1 \vdash \forall \alpha(E^\alpha \leftrightarrow E^{(sg(\alpha))})$ for all formulas E.
(c) $\mathbf{IA}_1 \vdash \forall \alpha \in 2^{\mathbb{N}}(E(\alpha) \leftrightarrow E(sg(\alpha)))$.

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PROOF. (a) holds by intuitionistic logic, (b) is proved by formula induction, and the replacement property of equality for functors guarantees (c). ⊣

3.3. Lemma. $\mathbf{IA}_1 \vdash \forall\alpha\forall\beta\forall\gamma(\mathrm{E}^\alpha \,\&\, \gamma = \alpha\cdot\beta \to \mathrm{E}^\gamma)$.

PROOF. Only Cases 4 and 5 require attention. If E is $\mathrm{A}\to\mathrm{B}$ where A, B both satisfy the lemma, assume $(\mathrm{A}\to\mathrm{B})^\alpha \,\&\, \gamma=\alpha\cdot\beta$. If $\mathrm{A}^{\mathrm{sg}(\delta)}$ then $\mathrm{B}^{\alpha\cdot\delta}$ by definition of $(\mathrm{A}\to\mathrm{B})^\alpha$, and $\delta\cdot\gamma = (\alpha\cdot\delta)\cdot\beta$ so $\mathrm{B}^{\delta\cdot\gamma}$ by the induction hypothesis on B. So $(\mathrm{A}\to\mathrm{B})^\gamma$.

If E is $\neg$A where A satisfies the lemma, assume $(\neg\mathrm{A})^\alpha \,\&\, \gamma=\alpha\cdot\beta$. If $\mathrm{A}^{\mathrm{sg}(\delta)}$, then $\mathrm{Z}(\alpha\cdot\delta)$ by definition of $(\neg\mathrm{A})^\alpha$, so $\mathrm{Z}(\gamma\cdot\delta)$ by Proposition 3.2(a). So $(\neg\mathrm{A})^\gamma$. ⊣

3.4. Lemma. $\mathbf{IA}_1 \vdash \forall\alpha(\mathrm{Z}(\alpha)\to\mathrm{E}^\alpha)$ for all formulas E.

3.5. Lemma.

(a) $\mathbf{IA}_1 \vdash \forall\alpha\in 2^{\mathbb{N}}((\mathrm{A}\to\mathrm{B})^\alpha \to (\mathrm{A}^\alpha\to\mathrm{B}^\alpha))$.
(b) $\mathbf{IA}_1 \vdash \forall\alpha\in 2^{\mathbb{N}}(\mathrm{A}\to\mathrm{B})^\alpha \leftrightarrow \forall\alpha\in 2^{\mathbb{N}}(\mathrm{A}^\alpha\to\mathrm{B}^\alpha)$.

PROOF. (a) follows immediately from the definition and Proposition 3.2(b) with the fact that $\alpha\cdot\alpha=\alpha$ for all $\alpha\in 2^{\mathbb{N}}$.

For (b), the implication from left to right follows from (a) by logic. For the converse assume $\forall\alpha\in 2^{\mathbb{N}}(\mathrm{A}^\alpha\to\mathrm{B}^\alpha)$ and $\alpha\in 2^{\mathbb{N}}$ and $\mathrm{A}^{\mathrm{sg}(\beta)}$; then $\mathrm{B}^{\mathrm{sg}(\beta)}$ by the assumption, so B^β by Proposition 3.2(b), so $\mathrm{B}^{\alpha\cdot\beta}$ by Lemma 3.3. So $(\mathrm{A}\to\mathrm{B})^\alpha$. ⊣

3.6. Lemma. If E is $\exists x\,\alpha(x)=0$ (i.e., $\mathrm{Z}(\alpha)$), then $\mathbf{IA}_1$ proves:

(a) $\forall\beta\in 2^{\mathbb{N}}(\mathrm{E}^\beta \leftrightarrow \mathrm{E}\vee\mathrm{Z}(\beta))$.
(b) $\forall\beta\in 2^{\mathbb{N}}((\neg\mathrm{E})^\beta \leftrightarrow (\mathrm{E}\to\mathrm{Z}(\beta)))$.
(c) $\forall\beta\in 2^{\mathbb{N}}((\neg\neg\mathrm{E})^\beta \leftrightarrow \mathrm{E}\vee\mathrm{Z}(\beta))$.
(d) $\forall\beta\in 2^{\mathbb{N}}(\neg\neg\mathrm{E}\leftrightarrow\mathrm{E})^\beta$.

PROOF. (a) is immediate by Definition 3.1 with intuitionistic logic. For (b), under the assumption $\beta\in 2^{\mathbb{N}}$ and using (a), Proposition 3.2, intuitionistic logic and the fact that $\beta\cdot\beta=\beta$, we have the following chain of equivalences:

$$
\begin{aligned}
(\neg\mathrm{E})^\beta &\leftrightarrow \forall\gamma\in 2^{\mathbb{N}}(\mathrm{E}^\gamma\to\mathrm{Z}(\beta\cdot\gamma))\\
&\leftrightarrow \forall\gamma\in 2^{\mathbb{N}}(\mathrm{E}\vee\mathrm{Z}(\gamma)\to\mathrm{Z}(\beta\cdot\gamma))\\
&\leftrightarrow \forall\gamma\in 2^{\mathbb{N}}(\mathrm{E}\to\mathrm{Z}(\beta\cdot\gamma)) \leftrightarrow (\mathrm{E}\to\mathrm{Z}(\beta)).
\end{aligned}
$$

For (c), under the assumption $\beta\in 2^{\mathbb{N}}$, by (b) we have

$$(\neg\neg\mathrm{E})^\beta \leftrightarrow \forall\gamma\in 2^{\mathbb{N}}((\neg\mathrm{E})^\gamma\to\mathrm{Z}(\beta\cdot\gamma)) \leftrightarrow \forall\gamma\in 2^{\mathbb{N}}((\mathrm{E}\to\mathrm{Z}(\gamma))\to\mathrm{Z}(\beta\cdot\gamma)).$$

If $\forall\gamma\in 2^{\mathbb{N}}((\mathrm{E}\to\mathrm{Z}(\gamma))\to\mathrm{Z}(\beta\cdot\gamma))$, let $\gamma=\mathrm{sg}(\alpha)$; then $\mathrm{Z}(\gamma)\leftrightarrow\mathrm{Z}(\alpha)$ and $\gamma\in 2^{\mathbb{N}}$. Then $(\mathrm{Z}(\gamma)\to\mathrm{Z}(\gamma))\to\mathrm{Z}(\beta\cdot\gamma)$ since E is $\mathrm{Z}(\alpha)$, so $\mathrm{Z}(\beta\cdot\gamma)$, so $\mathrm{Z}(\beta)\vee\mathrm{Z}(\gamma)$ by Proposition 3.2(a), so $\mathrm{Z}(\beta)\vee\mathrm{E}$, so $\mathrm{E}\vee\mathrm{Z}(\beta)$. For the converse use Proposition 3.2(a). Then (d) follows from (a) and (c) with Lemma 3.5(b). ⊣

3.7. Lemma.

(a) If E has no $\to$ or $\neg$ then $\mathbf{IA}_1 \vdash (\mathrm{E}^{\lambda z.1}\leftrightarrow\mathrm{E})$.
(b) $\mathbf{IA}_1 \vdash (\neg\mathrm{A})^{\lambda z.1}\to\neg(\mathrm{A}^{\lambda z.1})$ for all formulas A.
(c) If E is constructed from prime formulas and their negations using only & and $\vee$, then $\mathbf{IA}_1 \vdash (\mathrm{E}^{\lambda z.1}\to\mathrm{E})$.

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§4. Applications to subsystems of Kleene's formal system I for intuitionistic analysis.

4.1. Theorem. If **T** is a theory extending $\mathbf{IA}_1$ by axioms and axiom schemas $F_1, \ldots, F_n$ such that $\mathbf{T} \vdash \forall\beta \in 2^{\mathbb{N}} (F_i)^\beta$ for $i = 1, \ldots, n$, and if E is derivable in **T** from assumptions $A_1, \ldots, A_m$ with all free variables held constant in the deduction, then E^β is derivable in **T** from the assumptions $\beta \in 2^{\mathbb{N}}$, $(A_1)^\beta, \ldots, (A_m)^\beta$ with all free variables held constant.

PROOF. $\mathbf{IA}_1 \vdash \forall\alpha \in 2^{\mathbb{N}} E^\alpha$ when E is any axiom of $\mathbf{IA}_1$, using the lemmas in the previous section with $\forall\alpha \in 2^{\mathbb{N}}(\alpha \cdot \alpha = \alpha)$ as appropriate (e.g., for the mathematical induction schema). If B^β and $(B \to C)^\beta$ are derivable in $\mathbf{IA}_1$ from $\beta = sg(\beta)$, $(A_1)^\beta, \ldots, (A_m)^\beta$ with the free variables held constant, then by Lemma 3.5(a) so is $B^\beta \to C^\beta$, and therefore also C^β. Similarly for the other rules of inference. $\dashv$

4.2. Lemma. $\mathbf{IA}_1 + AC_{00} \vdash \forall\beta \in 2^{\mathbb{N}}(AC_{00})^\beta$, and similarly for qf-$AC_{00}$ and for Kleene's stronger countable choice principle (axiom schema ^{x}2.1 in [6]):

$$AC_{01}: \quad \forall x \exists\alpha A(x, \alpha) \to \exists\beta\forall x A(x, \lambda y.\beta(\langle x, y\rangle)).$$

PROOF. By the definition with Lemma 3.5(b). $\dashv$

4.3. Lemma. $\mathbf{IA}_1 + BI_1 \vdash \forall\beta \in 2^{\mathbb{N}}(BI_1)^\beta$ where Kleene's version of Brouwer's bar induction principle ("the bar theorem," axiom schema ^{x}26.3b in [6]) is

$$\begin{aligned} BI_1: \quad & \forall\alpha\exists x\rho(\overline{\alpha}(x)) = 0 \ \&\ \forall w(Seq(w)\ \&\ \rho(w) = 0 \to A(w)) \\ & \&\ \forall w(Seq(w)\ \&\ \forall s A(w * \langle s+1\rangle) \to A(w)) \to A(\langle\ \rangle). \end{aligned}$$

PROOF. Assume $\beta \in 2^{\mathbb{N}}$ and

(i) $(\forall\alpha\exists x\rho(\overline{\alpha}(x)) = 0)^\beta$,
(ii) $(\forall w(Seq(w)\ \&\ \rho(w) = 0 \to A(w)))^\beta$,
(iii) $(\forall w(Seq(w)\ \&\ \forall s A(w * \langle s+1\rangle) \to A(w)))^\beta$.

By Lemma 3.5 it will be enough to prove $A^\beta(\langle\ \rangle)$. By the definition and the lemmas in the previous section, over $\mathbf{IA}_1$ the numbered assumptions are equivalent respectively to

(i') $\forall\alpha\exists x(\rho(\overline{\alpha}(x)) = 0 \vee Z(\beta))$,
(ii') $\forall w\forall\gamma \in 2^{\mathbb{N}}((Seq(w)\ \&\ \rho(w) = 0) \vee Z(\gamma) \to A^{\beta\cdot\gamma}(w)))$,
(iii') $\forall w\forall\gamma \in 2^{\mathbb{N}}((Seq(w) \vee Z(\gamma))\ \&\ \forall s A^\gamma(w * \langle s+1\rangle) \to A^{\beta\cdot\gamma}(w))$.

In $\mathbf{IA}_1$ we may define $\sigma \in 2^{\mathbb{N}}$ so that

$$\sigma(w) = 0 \leftrightarrow \rho(w) = 0 \vee \exists x \le w\beta(x) = 0.$$

From (i') it follows immediately that $\forall\alpha\exists x\sigma(\overline{\alpha}(x)) = 0$. From (ii') with $\gamma = \beta$ and the fact that $\beta = \beta \cdot \beta$ we have $\forall w(Seq(w)\ \&\ \sigma(w) = 0 \to A^\beta(w))$. From (iii') similarly, $\forall w(Seq(w)\ \&\ \forall s A^\beta(w * \langle s+1\rangle) \to A^\beta(w))$, so $A^\beta(\langle\ \rangle)$ follows by BI_1. $\dashv$

4.4. Lemma. $\mathbf{IA}_1 + CC_{10} \vdash \forall\beta \in 2^{\mathbb{N}} (CC_{10})^\beta$ where CC_{10} is

$$\forall\alpha\exists x A(\alpha, x) \to \exists\sigma\forall\alpha(\exists y\sigma(\overline{\alpha}(y)) > 0\ \&\ \forall y(\sigma(\overline{\alpha}(y)) > 0 \to A(\alpha, \sigma(\overline{\alpha}(y)) \dot{-} 1))).$$

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CC_{10} is a minor variation of, and is equivalent over $\mathbf{IA}_1$ + qf-AC_{00} to, Kleene and Vesley's continuous choice schema *27.2 ("Brouwer's Principle for numbers").

Proof. Assume $\beta \in 2^{\mathbb{N}}$ and $\forall\alpha\exists x A^{\beta}(\alpha, x)$. By Lemma 3.5(b) it will be enough to find a σ such that for all α:

(i) $\exists y(\sigma(\overline{\alpha}(y)) > 0 \vee Z(\beta))$ and
(ii) $\forall y \forall \gamma \in 2^{\mathbb{N}}(\sigma(\overline{\alpha}(y)) > 0 \vee Z(\gamma) \to A^{\beta\cdot\gamma}(\alpha, \sigma(\overline{\alpha}(y)) \dot{-} 1))$.

CC_{10} provides a σ such that for all α:

(i') $\exists y\, \sigma(\overline{\alpha}(y)) > 0$ and
(ii') $\forall y(\sigma(\overline{\alpha}(y)) > 0 \to A^{\beta}(\alpha, \sigma(\overline{\alpha}(y)) \dot{-} 1))$.

Obviously (i') entails (i). To prove (ii), let $y \in \mathbb{N}$ and $\gamma \in 2^{\mathbb{N}}$. If $\sigma(\overline{\alpha}(y)) > 0$ then $A^{\beta\cdot\gamma}(\alpha, \sigma(\overline{\alpha}(y)) \dot{-} 1)$ by (ii') with Lemma 3.3, and if $Z(\gamma)$ then $A^{\beta\cdot\gamma}(\alpha, \sigma(\overline{\alpha}(y)) \dot{-} 1)$ by Lemmas 3.4 and 3.3, so $\sigma(\overline{\alpha}(y)) > 0 \vee Z(\gamma) \to A^{\beta\cdot\gamma}(\alpha, \sigma(\overline{\alpha}(y)) \dot{-} 1)$. ⊣

4.5. Lemma. $\mathbf{IA}_1$ + qf-AC_{00} + $CC_{11} \vdash \forall\gamma \in 2^{\mathbb{N}}\, (CC_{11})^{\gamma}$ where CC_{11} is

$$\forall\alpha\exists\beta A(\alpha,\beta) \to \exists\sigma\forall\alpha\exists\beta[\forall x \exists y(\sigma(\langle x+1\rangle * \overline{\alpha}(y)) = \beta(x) + 1 \\ \& \forall z < y\, \sigma(\langle x+1\rangle * \overline{\alpha}(z)) = 0) \,\&\, A(\alpha,\beta)],$$

which is equivalent over $\mathbf{IA}_1$ + qf-AC_{00} to Kleene's strongest continuous choice principle, "Brouwer's Principle for functions" (axiom schema ^{x}27.1 in [6]).

Proof. Assume $\gamma \in 2^{\mathbb{N}}$ and $\forall\alpha\exists\beta A^{\gamma}(\alpha,\beta)$. By Lemma 3.5(b) it will be enough to find a σ such that

$$\forall\alpha\exists\beta[\forall x\exists y((\sigma(\langle x+1\rangle * \overline{\alpha}(y)) = \beta(x) + 1 \\ \& \forall z < y\, \sigma(\langle x+1\rangle * \overline{\alpha}(z)) = 0) \vee Z(\gamma)) \,\&\, A^{\gamma}(\alpha,\beta)].$$

CC_{11} provides a σ which suffices. ⊣

4.6. Corollary. If $\mathbf{T}$ is $\mathbf{IA}_1$, Kleene's neutral theory $\mathbf{B} = \mathbf{IA}_1 + AC_{01} + BI_1$, Kleene's intuitionistic analysis $\mathbf{I} = \mathbf{B} + CC_{11}$ or any subsystem of $\mathbf{I}$ obtained by adding to $\mathbf{IA}_1$ any of the schemas qf-AC_{00}, AC_{00}, AC_{01}, BI_1 and/or CC_{10}, then $\mathbf{T}$ + MP_1 and $\mathbf{T}$ prove the same Π^0_2 statements.

Proof. By Lemma 3.6(d), $\mathbf{T} \vdash \forall\beta \in 2^{\mathbb{N}}(MP_1)^{\beta}$. Hence by Theorem 4.1 with Lemmas 4.2–4.5, if $\mathbf{T} + MP_1 \vdash E$ then $\mathbf{T} \vdash \forall\beta \in 2^{\mathbb{N}}\, E^{\beta}$.

If E is $\forall x\exists y A(x,y)$ where $A(x,y)$ has only bounded numerical quantifiers, then $A(x,y)$ is equivalent over $\mathbf{IA}_1$ to a formula of the type described in Lemma 3.7(c), so by Theorem 4.1: if $\mathbf{T} + MP_1 \vdash E$ then $\mathbf{T} \vdash E^{\lambda z.1}$ so $\mathbf{T} \vdash E$. ⊣

4.7. Remarks. Lemma 3.7(c) holds also for formulas E constructed from prime formulas and their negations using only &, $\vee$, $\forall$ and $\exists$, in particular for all prenex formulas. It follows, for each subsystem $\mathbf{T}$ of Kleene's $\mathbf{I}$ described in the statement of Corollary 4.6, that any prenex formula provable in $\mathbf{T} + MP_1$ is provable in $\mathbf{T}$.

Kleene's original versions of the continuous choice principles would also satisfy Lemmas 4.4 and 4.5 over $\mathbf{IA}_1$ + qf-AC_{00}. By Theorem 4.1 and Lemma 3.5, the equivalences between our versions and Kleene's persist under the translation, and the proofs for CC_{10} and CC_{11} are simpler.

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The question whether or not the "minimal" system $\mathbf{M} = \mathbf{IA}_1 + \mathrm{AC}_{00}!$ proves the same Π^0_2 formulas as $\mathbf{M} + \mathrm{MP}_1$ is still open, as far as we know, because $(\forall x \exists! y A(x, y))^\alpha$ does not entail $\forall x \exists! y A^\alpha(x, y)$ unless $\alpha = \lambda x.1$. However, if

$$\mathrm{AC}^\vee_{00}: \quad \forall x(A(x) \vee B(x)) \to \exists \alpha \forall x[(\alpha(x) = 0 \;\&\; A(x)) \vee (\alpha(x) \neq 0 \;\&\; B(x))]$$

is the axiom of countable choice for two alternatives, then $\mathbf{IRA} + \mathrm{AC}^\vee_{00} + \mathrm{MP}_1$ is Π^0_2-conservative over $\mathbf{IRA} + \mathrm{AC}^\vee_{00}$ by Theorem 4.1. Since $\mathrm{AC}_{00}!$ is equivalent over $\mathbf{IRA}$ to

$$\forall x(A(x) \vee \neg A(x)) \to \exists \alpha \forall x(\alpha(x) = 0 \leftrightarrow A(x))$$

by [8], any prenex formula provable in $\mathbf{M} + \mathrm{MP}_1$ is provable in $\mathbf{IRA} + \mathrm{AC}^\vee_{00}$.

Because the translation $E \mapsto E^\beta$ essentially involves binary sequence quantifiers, it does not appear to solve the corresponding problem for $\mathbf{IA}_1 + \mathrm{AC}^{Ar}_{00}$ or for Solovay's system $\mathbf{S} = \mathbf{IA}_1 + \mathrm{AC}^{Ar}_{00} + \mathrm{BI}_1$, where AC^{Ar}_{00} is the restriction of AC_{00} to arithmetical formulas $A(x, y)$ (with sequence parameters allowed). In the presence of bar induction, arithmetical countable choice interacts strongly with MP_1; e.g., Solovay showed that the classical version $\mathbf{S} + (\neg\neg A \to A)$ of $\mathbf{S}$ can be interpreted negatively in $\mathbf{IRA} + \mathrm{BI}_1 + \mathrm{MP}_1$.[2]

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[2]In fact, his proof justifies a stronger result: $\mathbf{S} + (\neg\neg A \to A)$ can be interpreted negatively in $\mathbf{IRA} + \mathrm{BI}_1 + \mathrm{DNS}_1$, where DNS_1 is the schema $\forall \alpha \neg\neg \exists x A(\overline{\alpha}(x)) \to \neg\neg \forall \alpha \exists x A(\overline{\alpha}(x))$ for quantifier-free formulas $A(w)$. Another note with this and related results is in progress.

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