A CORRESPONDENCE BETWEEN INVERSE SUBSEMIGROUPS, OPEN WIDE SUBGROUPOIDS AND CARTAN INTERMEDIATE C*-SUBALGEBRAS

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Abstract   For a given inverse semigroup action on a topological space, one can associate an étale groupoid. We prove that there exists a correspondence between the certain subsemigroups and the open wide subgroupoids in case that the action is strongly tight. Combining with the recent result of Brown et al., we obtain a correspondence between the certain subsemigroups of an inverse semigroup and the Cartan intermediate subalgebras of a groupoid C*-algebra.

Keywords: C*-algebras; étale groupoids; inverse semigroups

2020 *Mathematics subject classification:* Primary 20M18; 22A22; 46L05

0. Introduction

Given an action of an inverse semigroup on a topological space, one can associate an étale groupoid by taking a germ (see Subsection 1.3). For a given étale groupoid, one can construct groupoid C*-algebras, which are initiated by Renault [9]. It is a natural task to investigate the relation among them and actually many researchers have been doing this (for example, see [2, 3]). In this paper, we establish a correspondence between the set of certain subsemigroups and the set of wide open wide subgroupoids of the associated groupoids. We consider inverse semigroups acting on topological spaces in the "strongly tight" way (see Definition 2.1.1). Our main theorem, Theorem 2.1.10, states that wide open subgroupoids of associated groupoids with strongly tight actions corresponds to certain subsemigroups of the inverse semigroups. Combining with the work in [1], we obtain a correspondence between Cartan intermediate subalgebras in groupoid C*-algebras and certain subsemigroups of inverse semigroups. As an application, we compute all Cartan intermediate subalgebras of the Cuntz algebras, which contains the fixed point algebras.

This paper is organized as follows. Section 1 is devoted for preliminaries. In § 2, we investigate open subgroupoids of étale groupoids associated with strongly tight

actions. Then we establish a correspondence between open wide subgroupoids and certain subsemigroups (Theorem 2.1.10).

In § 3, we give applications of our correspondence. The first application is regarding to inverse semigroups which consist of compact bisections of étale groupoids. We show that a class of open wide subgroupoids of an ample groupoid is described by an inverse semigroup of compact bisections (Corollary 3.1.3). As the second application, we study certain subsemigroups of the polycyclic monoids. This study is applied to the computation of Cartan intermediate subalgebras between the Cuntz algebras and the fixed point algebras.

In § 4, we summarize the relation between Cartan intermediate subalgebras of C*-algebras and certain subsemigroups of inverse semigroups. Then we compute Cartan intermediate subalgebras of the Cuntz algebras which contains the fixed point algebras.

In § 5, we mention the relation between strongly tight actions and tight groupoids. We give a characterization of a tight groupoid with the compact unit space in Corollary 5.2.5.

1. Preliminaries

1.1. Inverse semigroups

We recall the basic notions about inverse semigroups. See [4] or [8] for more details. An inverse semigroup S is a semigroup where for every $s \in S$ there exists a unique $s^* \in S$ such that $s = ss^*s$ and $s^* = s^*ss^*$. We denote the set of all idempotents in S by $E(S) := \{e \in S \mid e^2 = 2\}$. It is known that $E(S)$ is a commutative subsemigroup of S. An inverse semigroup which consists of idempotents is called a (meet) semilattice. A zero element is a unique element $0 \in S$ such that $0s = s0 = 0$ holds for all $s \in S$. A unit is a unique element $1 \in S$ such that $1s = s1 = s$ holds for all $s \in S$. In this paper, we assume that every inverse semigroup always has a zero element, although it does not necessarily have a unit. An inverse semigroup with a unit is called an inverse monoid. By a subsemigroup of S, we mean a subset of S which is closed under the product and inverse of S. For $s, t \in S$, we write $s \leq t$ if $ts^*s = s$ holds. Then this defines a partial order on S. Note that $e \leq f$ holds if and only if $ef = e$ holds for $e, f \in E(S)$. A pair $s, t \in S$ is said to be compatible if $s^*t, st^* \in E(S)$ holds. Notice that s, t are compatible if there exists $u \in S$ such that $s, t \leq u$. A subsemigroup of an inverse semigroup S is said to be wide if it contains $E(S)$. A subset $I \subset E(S)$ is called an ideal if $e \in I$ and $f \leq e$ implies $f \in I$. A subset $C \subset I$ of an ideal $I \subset E(S)$ is called a cover if for every $e \in I \setminus \{0\}$ there exists $c \in C$ such that $ec \neq 0$.

For a topological space X, we denote by I_X the set of all homeomorphisms between open sets in X. Then I_X is an inverse semigroup with respect to the product defined by the composition of maps. For $f, g \in I_X$, note that $f \leq g$ holds if and only if $\operatorname{dom} f \subset \operatorname{dom} g$ and $f(x) = g(x)$ hold for all $x \in \operatorname{dom} f$.

1.2. Étale groupoids

We recall the basic notions on étale groupoids. See [8, 11] for more details.

A groupoid is a set G together with a distinguished subset $G^{(0)} \subset G$, source and range maps $d, r\colon G \to G^{(0)}$ and a multiplication

$$G^{(2)}\colon = \{(\alpha, \beta) \in G \times G \mid d(\alpha) = r(\beta)\} \ni (\alpha, \beta) \mapsto \alpha\beta \in G$$

such that

(1) for all $x \in G^{(0)}$, $d(x) = x$ and $r(x) = x$,

(2) for all $\alpha \in G$, $\alpha d(\alpha) = r(\alpha)\alpha = \alpha$,

(3) for all $(\alpha, \beta) \in G^{(2)}$, $d(\alpha\beta) = d(\beta)$ and $r(\alpha\beta) = r(\alpha)$,

(4) if (α, β), $(\beta, \gamma) \in G^{(2)}$, we have $(\alpha\beta)\gamma = \alpha(\beta\gamma)$,

(5) every $\gamma \in G$, there exists $\gamma' \in G$ which satisfies (γ', γ), $(\gamma, \gamma') \in G^{(2)}$ and $d(\gamma) = \gamma'\gamma$ and $r(\gamma) = \gamma\gamma'$.

Since the element γ' in (5) is uniquely determined by γ, γ' is called the inverse of γ and denoted by γ^{-1}. We call $G^{(0)}$ the unit space of G. A subgroupoid of G is a subset of G which is closed under the inversion and multiplication. A subgroupoid of G is said to be wide if it contains $G^{(0)}$.

A topological groupoid is a groupoid equipped with a topology where the multiplication and the inverse are continuous. A topological groupoid is said to be étale if the source map is a local homeomorphism. Note that the range map of an étale groupoid is also a local homeomorphism. An étale groupoid is said to be ample if it has an open basis which consists of compact sets. In this paper, we mainly treat ample groupoids.

A topological groupoid G is said to be topologically principal if the set

$$\{x \in G^{(0)} \mid G(x) = \{x\}\}$$

is dense in $G^{(0)}$, where $G(x)$ is the isotropy group at $x \in G^{(0)}$:

$$G(x)\colon = \{\alpha \in G \mid d(\alpha) = r(\alpha) = x\}.$$

1.3. Étale groupoids associated with inverse semigroup actions

An étale groupoid arises from an action of an inverse semigroup to a topological space. We recall how to construct an étale groupoid from an inverse semigroup action. We begin with the definition of an inverse semigroup action.

Let X be a topological space. Recall that I_X is an inverse semigroup of homeomorphisms between open sets in X. An action $\alpha\colon S \curvearrowright X$ is a semigroup homomorphism $S \ni s \mapsto \alpha_s \in I_X$. In this paper, we always assume that every action α satisfies $\bigcup_{e \in E(S)} \operatorname{dom}(\alpha_e) = X$ and $\operatorname{dom}(\alpha_0) = \emptyset$. For $e \in E(S)$, we denote the domain of α_e by D_e^α. Then α_s is a homeomorphism from $D_{s^*s}^\alpha$ to $D_{ss^*}^\alpha$. We often omit α of D_e^α if there is no chance to confuse.

For an action $\alpha\colon S \curvearrowright X$, we associate an étale groupoid $S \ltimes_\alpha X$ as the following. First we put the set $S * X\colon = \{(s, x) \in S \times X \mid x \in D_{s^*s}^\alpha\}$. Then we define an equivalence

relation $\sim$ on $S * X$ by declaring that $(s, x) \sim (t, y)$ holds if

$$x = y \text{ and there exists } e \in E(S) \text{ such that } x \in D_e^\alpha \text{ and } se = te.$$

Set $S \ltimes_\alpha X := S * X/\sim$ and denote the equivalence class of $(s, x) \in S * X$ by $[s, x]$. The unit space of $S \ltimes_\alpha X$ is X, where X is identified with the subset of $S \ltimes_\alpha X$ via the injection

$$X \ni x \mapsto [e, x] \in S \ltimes_\alpha X, x \in D_e^\alpha.$$

The source map and range maps are defined by

$$d([s, x]) = x, r([s, x]) = \alpha_s(x)$$

for $[s, x] \in S \ltimes_\alpha X$. The product of $[s, \alpha_t(x)], [t, x] \in S \ltimes_\alpha X$ is $[st, x]$. The inverse should be $[s, x]^{-1} = [s^*, \alpha_s(x)]$. Then $S \ltimes_\alpha X$ is a groupoid in these operations. For $s \in S$ and an open set $U \subset D_{s^*s}^\alpha$, define

$$[s, U] := \{[s, x] \in S \ltimes_\alpha X \mid x \in U\}.$$

These sets form an open basis of $S \ltimes_\alpha X$. In these structures, $S \ltimes_\alpha X$ is an étale groupoid.

2. Correspondence between subsemigroups and subgroupoids

In this section, we consider strong tight actions of inverse semigroups (Definition 2.1.1). Then we establish a correspondence between certain subsemigroups of an inverse semigroup and open wide subgroupoids of an étale groupoid associated with a strongly tight action (Theorem 2.1.10). Then we observe a condition for an open wide subgroupoid to be closed in terms of an inverse semigroup.

2.1. Correspondence between subsemigroups and subgroupoids

We begin with the definition of a strongly tight action.

Definition 2.1.1. Let S be an inverse semigroup and X be a locally compact Hausdorff space. An action $\alpha\colon S \curvearrowright X$ is said to be ample if $D_e^\alpha \subset X$ is a compact set for all $e \in E(S)$. We say that an ample action $\alpha\colon S \curvearrowright X$ is strongly tight if $\{D_e^\alpha\}_{e \in E(S)}$ is a basis of X.

We remark that if there exists a strongly tight action $\alpha\colon S \curvearrowright X$, then X is totally disconnected and $S \ltimes_\alpha X$ is an ample groupoid.

Strongly tight actions are related with the actions on tight spectrums of inverse semigroups, which are investigated in [3]. We will see a relation between strongly tight actions and tight groupoids in § 5.

We construct subsemigroups from wide groupoids.

Definition 2.1.2. Let S be an inverse semigroup, X be a locally compact Hausdorff space and $\alpha\colon S \curvearrowright X$ be an action. Put $G := S \ltimes_\alpha X$. For a wide subgroupoid $H \subset G$, we define

$$T_H := \{s \in S \mid [s, D_{s^*s}] \subset H\}.$$

Proposition 2.1.3. *In the above notation, T_H is a wide subsemigroup of S.*

Proof. For $e \in E(S)$, $[e, D_e] \subset G^{(0)} \subset H$ holds. Hence T_H contains $E(S)$.

Next, we show that T_H is a subsemigroup of S. We show $st \in T_H$ for s, $t \in T_H$. For $x \in D_{(st)^*st}$, it follows that $[s, \alpha_t(x)]$, $[t, x] \in H$ from s, $t \in T_H$. Thus, we obtain

$$[st, x] = [s, \alpha_t(x)][t, x] \in H.$$

Therefore, we have $[st, D_{(st)^*st}] \subset H$ and $st \in T_H$.

It is clear that T_H is closed under the inverse. Hence T_H is a wide subsemigroup of S. □

We define a class of subsemigroups which corresponds to open wide subgroupoids (c.f. Theorem 2.1.10).

Definition 2.1.4. Let S be an inverse semigroup, X be a locally compact Hausdorff space and $\alpha \colon S \curvearrowright X$ be an action. A wide subsemigroup $T \subset S$ is said to be α-join closed if T has the next property:

'For every $s \in S$, s belongs to T if and only if there exists a finite set $F \subset E(S)$ such that $sf \in T$ holds for all $f \in F$ and $D_{s^*s} \subset \bigcup_{f \in F} D_f$ holds.'

Remark 2.1.5. The "only if" part in the previous definition always holds for all wide subsemigroups T. Indeed, if $s \in T$, then $F := \{s^*s\}$ satisfies $sf \in T$ for all $f \in F$ and $D_{s^*s} \subset \bigcup_{f \in F} D_{s^*s}$.

Proposition 2.1.6. *Let S be an inverse semigroup, X be a locally compact Hausdorff space and $\alpha \colon S \curvearrowright X$ be an action. For a wide subgroupoid $H \subset S \ltimes_\alpha X$, the wide subsemigroup $T_H \subset S$ is α-join closed.*

Proof. Take $s \in S$ and assume that there exists a finite set $F \subset E(S)$ such that $sf \in T_H$ for all $f \in F$ and $D_{s^*s} \subset \bigcup_{f \in F} D_f$. It suffices to show $s \in T_H$. For $x \in D_{s^*s}$, there exists $f \in F$ with $x \in D_f$. Since we have $sf \in T_H$, it follows

$$[s, x] = [sf, x] \in H.$$

Thus, we obtain $[s, D_{s^*s}] \subset H$ and therefore $s \in T_H$. □

The proof of the next proposition is left to the reader.

Proposition 2.1.7. *Let S be an inverse semigroup, X be a locally compact Hausdorff space and $\alpha \colon S \curvearrowright X$ be an action. For a wide subsemigroup $T \subset S$, the map*

$$T \ltimes_\alpha X \ni [t, x] \mapsto [t, x] \in S \ltimes_\alpha X$$

is an open map and an isomorphism onto its image.

Via the map in the previous proposition, $T \ltimes_\alpha X$ is identified with the wide open subgroupoid of $S \ltimes_\alpha X$.

Lemma 2.1.8. *Let S be an inverse semigroup, X be a locally compact Hausdorff space and $\alpha\colon S \curvearrowright X$ be an action. For a wide subsemigroup $T \subset S$, $T_{T \ltimes_\alpha X} \supset T$ holds. Moreover, if T is α-join closed and $\alpha\colon S \curvearrowright X$ is ample, then $T_{T \ltimes_\alpha X} = T$ holds.*

Proof. The inclusion $T_{T \ltimes_\alpha X} \supset T$ is clear. Assuming that T is α-join closed and $\alpha\colon S \curvearrowright X$ is ample, we show $T_{T \ltimes_\alpha X} \subset T$. Take $s \in T_{T \ltimes_\alpha X}$ and fix $x \in D_{s^*s}$. Since we have $[s, x] \in T \ltimes_\alpha X$, there exists $e_x \in E(S)$ such that $se_x \in T$ and $x \in D_{e_x}$. Since we assume that D_{s^*s} is compact, there exists a finite set $P \subset D_{s^*s}$ with $D_{t^*t} \subset \bigcup_{x \in P} D_{e_x}$. Using the condition that T is α-join closed, we obtain $t \in T$. Now we have shown $T_{T \ltimes_\alpha X} \subset T$. $\qquad\square$

Lemma 2.1.9. *Let S be an inverse semigroup, X be a locally compact Hausdorff space and $\alpha\colon S \curvearrowright X$ be an action. Put $G := S \ltimes_\alpha X$. For a wide groupoid $H \subset G$, $T_H \ltimes_\alpha X \subset H$ holds. Moreover, if $H \subset G$ is open and $\alpha\colon S \curvearrowright X$ is strongly tight, $T_H \ltimes_\alpha X = H$ also holds.*

Proof. Assume that $[s, x] \in T_H \ltimes_\alpha X$. Then there exists $t \in T_H$ such that $[s, x] = [t, x]$. Now we have

$$[s, x] = [t, x] \in [t, D_{t^*t}] \subset H.$$

Next, we show the other inclusion $T_H \ltimes_\alpha X \supset H$ under the assumption that α is strongly tight and H is open. Take $[s, x] \in H$. Since H is open and α is strongly tight, there exists $e \in E(S)$ such that $x \in D_e \subset D_{s^*s}$ and $[s, D_e] \subset H$. One can see $[se, D_{(se)^*se}] \subset H$, so we have $se \in T_H$. Therefore, it follows $[s, x] = [se, x] \in T_H \ltimes_\alpha X$. $\qquad\square$

The next theorem follows from Lemmas 2.1.8 and 2.1.9.

Theorem 2.1.10. *Let S be an inverse semigroup, X be a locally compact Hausdorff space and $\alpha\colon S \curvearrowright X$ be an action. Assume that $\alpha\colon S \curvearrowright X$ is strongly tight and put $G := S \ltimes_\alpha X$. Let $\mathcal{T}$ denote the set of all wide α-join closed subsemigroups of S. In addition, let $\mathcal{H}$ denote the set of all wide open subgroupoids of G. Then maps*

$$\mathcal{T} \ni T \to T \ltimes_\alpha X \in \mathcal{H}$$

and

$$\mathcal{H} \ni H \to T_H \in \mathcal{T}$$

are inverse maps of each other.

2.2. Closedness of subgroupoid

We give conditions where $T \ltimes_\alpha X$ is closed in $S \ltimes_\alpha X$.

Definition 2.2.1. *Let S be an inverse semigroup and $T \subset S$ be a wide subsemigroup. For $s \in S$, we define $\mathcal{J}_s^T \subset E(S)$ as*

$$\mathcal{J}_s^T := \{e \in E(S) \mid se \in T \text{ and } e \leq s^*s\}.$$

Proposition 2.2.2. *In the above notation, $\mathcal{J}_s^T$ is an ideal of $E(S)$.*

Proof. Assume $e \in \mathcal{J}_s^T$ and $f \leq e$. Then we have $sf = sef \in T$ and $f \leq e \leq s^*s$. Hence we obtain $f \in \mathcal{J}_s^T$. $\qquad\square$

We remark that

$$\mathcal{J}_s^{E(S)} = \{e \in E(S) \mid e \leq s\}$$

holds. This ideal appears in [3, Definition 3.11].

Assume that an action $\alpha\colon S \curvearrowright X$ is given. For an ideal $\mathcal{J} \subset E(S)$, we define $D(\mathcal{J}) := \bigcup_{e \in \mathcal{J}} D_e$. The next lemma is a slight generalization of [3, Proposition 3.14].

Lemma 2.2.3. *Let S be an inverse semigroup, X be a locally compact Hausdorff space and $\alpha\colon S \curvearrowright X$ be an action. Assume that we are given a wide subsemigroup $T \subset S$. Then the formula*

$$[s, D(\mathcal{J}_s^T)] = [s, D_{s^*s}] \cap (T \ltimes_\alpha X)$$

holds for all $s \in S$.

Proof. Take $[s, x] \in [s, D(\mathcal{J}_s^T)]$. Then there exists $e \in \mathcal{J}_s^T$ with $x \in D_e$. By the definition of $\mathcal{J}_s^T$, we have $se \in T$ and $e \leq s^*s$. Hence we obtain

$$[s, x] = [se, x] \in [s, D_{s^*s}] \cap (T \ltimes_\alpha X).$$

Now we have shown $[s, D(\mathcal{J}_s^T)] \subset [s, D_{s^*s}] \cap (T \ltimes_\alpha X)$. To show the reverse inclusion, take $[s, x] \in [s, D_{s^*s}] \cap (T \ltimes_\alpha X)$. Since $[s, x]$ belongs to $T \ltimes_\alpha X$, there exists $t \in T$ and $f \in E(S)$ such that $sf = tf$ and $x \in D_f$ hold. Since we have $ss^*sf = sf = tf \in T$, s^*sf belongs to $\mathcal{J}_s^T$. Since we also have $x \in D_{s^*sf} \subset D(\mathcal{J}_s^T)$, we obtain $[s, x] \in [s, D(\mathcal{J}_s^T)]$. $\quad\square$

Proposition 2.2.4. *Let S be an inverse semigroup, X be a locally compact Hausdorff space and $\alpha\colon S \curvearrowright X$ be an action. Assume that we are given a wide subsemigroup $T \subset S$. The following conditions are equivalent:*

(1) *$T \ltimes_\alpha X$ is a closed subset of $S \ltimes_\alpha X$,*

(2) *for every $s \in S$, $D(\mathcal{J}_s^T)$ is a closed subset of D_{s^*s} with respect to the relative topology of D_{s^*s}.*

Proof. First, we show that (1) implies (2). By Lemma 2.2.3 and (1), $[s, D(\mathcal{J}_s^T)]$ is a closed subset of $[s, D_{s^*s}]$. Since the restriction of the domain map $d\colon [s, D_{s^*s}] \to D_{s^*s}$ is a homeomorphism, $d([s, D(\mathcal{J}_s^T)]) = D(\mathcal{J}_s^T)$ is closed in D_{s^*s}. Next, we show that (2) implies (1). It follows that $[s, D(\mathcal{J}_s^T)]$ is a closed subset of $[s, D_{s^*s}]$ from the same argument in the above and (2). We have that $[s, D_{s^*s}] \setminus [s, D(\mathcal{J}_s^T)]$ is open in $S \ltimes_\alpha X$ since $[s, D_{s^*s}]$ is open in $S \ltimes_\alpha X$ and $[s, D(\mathcal{J}_s^T)]$ is closed in $[s, D_{s^*s}]$. One can see that

the formula

$$S \ltimes_\alpha X \backslash T \ltimes_\alpha X = \bigcup_{s \in S} ([s, D_{s^*s}] \backslash [s, D(\mathcal{J}_s^T)])$$

holds. Hence $S \ltimes_\alpha X \backslash T \ltimes_\alpha X$ is open in $S \ltimes_\alpha X$, which implies $T \ltimes_\alpha X$ is closed in $S \ltimes_\alpha X$. $\hspace{2cm}\square$

The next Lemma is essentially same as the [3, Proposition 3.7]. We give a proof for the reader's convenience.

Lemma 2.2.5 (c.f. [3, Proposition 3.7]). *Let S be an inverse semigroup, X be a locally compact Hausdorff space and $\alpha \colon S \curvearrowright X$ be a strongly tight action. Assume that $D_e \neq \emptyset$ holds for every $e \in E(S) \backslash \{0\}$. For a ideal $\mathcal{J} \subset E(S)$ and a finite subset $C \subset \mathcal{J}$, the followings are equivalent:*

(1) *C is a cover of $\mathcal{J}$,*

(2) *$\bigcup_{c \in C} D_c = D(\mathcal{J})$.*

Proof. First, we show (1) implies (2). The inclusion $\bigcup_{c \in C} D_c \subset D(\mathcal{J})$ follows from $C \subset \mathcal{J}$. We show the reverse inclusion. Take $x \in D(\mathcal{J})$. Then there exists $q \in \mathcal{J}$ such that $x \in D_q$. Assume that $x \notin D_c$ holds for all $c \in C$. For each $c \in C$, there exists $e_c \in E(S)$ such that $x \in D_{e_c}$ and $D_{e_c} \cap D_c = \emptyset$ since each D_c is closed in X and $\{D_e\}_{e \in E(S)}$ is a basis of X. Since $D_{ce_c} = D_c \cap D_{e_c} = \emptyset$ and we assume $D_e \neq \emptyset$ for all $e \in E(S) \backslash \{0\}$, we have $ce_c = 0$. Putting $p \colon = q \prod_{c \in C} e_c$, we have $p \in \mathcal{J} \backslash \{0\}$ since $\mathcal{J}$ is ideal and $x \in D_p$. However, we also have $cp = 0$ for each $c \in C$, which contradicts to the condition that C is a cover.

Next, we show (2) implies (1). Take $e \in \mathcal{J} \backslash \{0\}$. Then there exists $c \in C$ such that $D_e \cap D_c \neq \emptyset$, which implies $ec \neq 0$. Hence C is a cover of $\mathcal{J}$. $\hspace{1cm}\square$

Now we obtain the characterization about the closedness of open wide subgroupoids.

Theorem 2.2.6. *Let S be an inverse semigroup, X be a locally compact Hausdorff space and $\alpha \colon S \curvearrowright X$ be a strongly tight action. Assume that $D_e \neq \emptyset$ holds for every $e \in E(S) \backslash \{0\}$. For a wide subsemigroup $T \subset S$, the following conditions are equivalent:*

(1) *$T \ltimes_\alpha X$ is closed in $S \ltimes_\alpha X$,*

(2) *for every $s \in S$, $D(\mathcal{J}_s^T)$ is relatively closed in D_{s^*s},*

(3) *for every $s \in S$, $\mathcal{J}_s^T$ has a finite cover.*

Proof. Now it suffices to show that (2) and (3) are equivalent, since Proposition 2.2.4 states that (1) and (2) are equivalent. First, we show that (2) implies (3). Since we assume that the action α is ample, D_{s^*s} is compact. Then $D(\mathcal{J}_s^T)$ is also compact by (2). Hence there exists a finite set $C \subset \mathcal{J}_s^T$ such that $\bigcup_{c \in C} D_c = D(\mathcal{J}_s^T)$. By Lemma 2.2.5, C is a finite cover of $\mathcal{J}_s^T$.

Next, we show that (3) implies (2). Take $s \in S$ and a finite cover C of $\mathcal{J}_s^T$. By Lemma 2.2.5 again, we have $D(\mathcal{J}_s^T) = \bigcup_{c \in C} D_c$. Hence we have $D(\mathcal{J}_s^T)$ is compact and therefore closed in D_{s^*s} since each D_c is compact. $\qquad\square$

Wide clopen subgroupoids arise from partial group homomorphisms. We observe this fact in the remainder of this subsection.

Let S be an inverse semigroup and Γ be a group. Put $S^\times := S \setminus \{0\}$. A map $\sigma \colon S^\times \to \Gamma$ is called a partial homomorphism if $\sigma(st) = \sigma(s)\sigma(t)$ holds for any pair $s, t \in S^\times$ with $st \neq 0$. A partial homomorphism gives us a suitable subsemigroup as follows.

Proposition 2.2.7. *Let S be an inverse semigroup, Γ be a group and $\sigma \colon S^\times \to \Gamma$ be a partial homomorphism. Assume that we are given a locally compact space X and an action $\alpha \colon S \curvearrowright X$ where $D_e \neq \emptyset$ holds for each $e \in E(S) \setminus \{0\}$. Then the following statements hold*:

(1) $\ker \sigma := \sigma^{-1}(e) \cup \{0\}$ *is a α-join closed wide subsemigroup of S,*

(2) $\ker \sigma \ltimes_\alpha X$ *is closed in $S \ltimes_\alpha X$.*

Proof. First, we show (1). One can see that $\ker \sigma$ is a wide subsemigroup of S in a straightforward way. We show $\ker \sigma$ is α-join closed. Take $s \in S$ and assume that there exists a finite set $F \subset E(S)$ such that $sF \subset \ker \sigma$ and $D_{s^*s} \subset \bigcup_{f \in F} D_f$. It suffices to show $s \in \ker \sigma$. We may assume that $s \neq 0$. Then there exists $f \in F$ such that $D_{s^*s} \cap D_f \neq \emptyset$, which implies $sf \neq 0$. Since we have $sf \in \ker \sigma$, it follows

$$\sigma(s) = \sigma(s)e = \sigma(s)\sigma(f) = \sigma(sf) = e.$$

Hence $s \in \ker \sigma$.

Next, we show (2). Although it is possible to apply Proposition 2.2.4, we show (2) using a cocycle[*] on a groupoid. We define the map $c_\sigma \colon S \ltimes_\alpha X \to \Gamma$ by

$$c_\sigma([s, x]) = \sigma(s), \quad [s, x] \in S \ltimes_\alpha X.$$

Then c_σ is a continuous cocycle. One can see that

$$\ker \sigma \ltimes_\alpha X = c_\sigma^{-1}(e)$$

holds. Hence $\ker \sigma \ltimes_\alpha X$ is closed in $S \ltimes_\alpha X$. $\qquad\square$

3. Applications and examples

3.1. Inverse semigroups of compact bisections

Let G be an ample étale groupoid. Recall that an open set $U \subset G$ is called a bisection if the restrictions $d|_U$ and $r|_U$ are homeomorphisms onto the images. Let $I(G)$ denote the

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[*] A map from a groupoid to a group is called a cocycle if it preserves the products.

set of all compact bisections of G. For $U, V \in I(G)$, their product UV is defined by

$$UV := \{\alpha\beta \in G \mid \alpha \in U, \beta \in V, d(\alpha) = r(\beta)\}.$$

Then UV belongs to $I(G)$. It is known that $I(G)$ becomes an inverse semigroup. Note that the inverse of $U \in I(G)$ is given by

$$U^{-1} = \{\alpha^{-1} \in G \mid \alpha \in U\}.$$

The order of $I(G)$ as an inverse semigroup coincides with the order defined by inclusion. A pair $U, V \in I(G)$ is said to be compatible if $U^{-1}V$ and UV^{-1} belong to $E(I(G))$. If $U, V \in I(G)$ are compatible, $U \cup V$ is an element of $I(G)$. Note that $U \cup V$ is the least upper bound of $\{U, V\}$. Thus, $I(G)$ admits joins of compatible pairs in $I(G)$. A subsemigroup $T \subset I(G)$ is said to be join closed if all joins of compatible pair of T also belongs to T.

For $U \in I(G)$, we have a homeomorphism $\rho_U \colon d(U) \to r(U)$ defined by

$$\rho_U(d(\alpha)) = r(\alpha), \alpha \in U.$$

Then the map $U \mapsto \rho_U$ defines an action $\rho \colon I(G) \curvearrowright G^{(0)}$. One can see that ρ is strongly tight. The following theorem is essentially same as [7, Theorem 2.8].

Theorem 3.1.1 (c.f. [7, Theorem 2.8]). *Let G be an ample étale groupoid. Then G is isomorphic to $I(G) \ltimes_\rho G^{(0)}$.*

Proof. For $\alpha \in G$, there exists $U_\alpha \in I(G)$ such that $\alpha \in U_\alpha$ since G is ample. Then $[U_\alpha, d(\alpha)] \in I(G) \ltimes_\rho G^{(0)}$ is independent of the choice of U_α. Thus, we obtain the map

$$\Phi \colon G \ni \alpha \mapsto [U_\alpha, d(\alpha)] \in I(G) \ltimes_\rho G^{(0)}.$$

One can see that Φ is an isomorphism as a morphism between étale groupoids. Indeed, the map $\Psi \colon I(G) \ltimes_\rho G^{(0)} \to G$ defined by $\Psi([U, x]) = d_U^{-1}(x)$ is the inverse map of Φ. □

Lemma 3.1.2. *Let G be an ample étale groupoid. Then a wide subsemigroup $T \subset I(G)$ is ρ-join closed if and only if T is join closed.*

Proof. Assume that $T \subset I(G)$ is join closed. Take $U \in I(G)$ and there exists a finite set $\mathcal{F} \subset E(I(G))$ such that $U\mathcal{F} \in T$ and $D_{U^*U}^\rho \subset \bigcup_{O \in \mathcal{F}} D_O^\rho$. Observe that elements in $U\mathcal{F}$ are pairwisely compatible and $\bigvee_{O \in \mathcal{F}} UO = U$ holds. Since T is join closed, U belongs to T.

To show the converse, assume that $T \subset I(G)$ is ρ-join closed. Let $U, V \in T$ be compatible. Put $\mathcal{F} := \{U^{-1}U, V^{-1}V\} \subset E(I(G))$. Then one can see that $(U \cup V)\mathcal{F} = \{U, V\} \subset T$ hold. Since we have

$$\operatorname{dom}(\rho_{U \cup V}) = d(U) \cup d(V) = \operatorname{dom}\rho_U \cup \operatorname{dom}\rho_V,$$

we obtain $U \cup V \in T$ by the ρ-closedness of T. □

Theorem 2.1.10, Theorem 3.1.1 and Lemma 3.1.2 yield the next corollary.

Corollary 3.1.3. *Let G be an ample étale groupoid. Then there is a correspondence between the set of all open wide subgroupoids of G and the set of all wide join closed subsemigroups of $I(G)$.*

3.2. Polycyclic monoids

We apply Theorem 2.1.10 to the polycyclic monoids P_n. See [5] or [8, Example 3 in Chapter 4.2] for details on the polycyclic monoids. Remark that the polycyclic monoids are called the Cuntz semigroups in [8].

Definition 3.2.1. Let $n \geq 2$ be a natural number. The polycyclic monoid P_n is an inverse monoid defined by

$$P_n := \, _1< s_1, s_2, \ldots, s_n \mid s_i^* s_j = \delta_{i,j} 1 > .$$

Set $\Sigma_n := \{1, 2, \ldots, n\}$ and

$$\Sigma_n^{\mathbb{N}} := \{(x_i)_{i=1}^{\infty} \mid x_i \in \Sigma_n \text{ for all } i \in \mathbb{N}\}.$$

It follows that $\Sigma_n^{\mathbb{N}}$ is a compact Hausdorff space from Tychonoff's theorem. We write a finite sequence on Σ_n like $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$, where each μ_j is an element of Σ_n. Here, $l \in \mathbb{N}$ is called the length of μ, which we denote by $|\mu|$. The only element of length zero is denoted by ε, which is called the empty word. We denote the set of all finite sequence on Σ_n by Σ_n^*. For $\mu \in \Sigma_n^*$, we define a cylinder set $C(\mu) \subset \Sigma_n^{\mathbb{N}}$ as the set of all infinite sequences which begin with μ :

$$C(\mu) := \{(x_i)_{i=1}^{\infty} \in \Sigma_n^{\mathbb{N}} \mid x_i = \mu_i \text{ for all } i = 1, 2, \ldots, |\mu|\}.$$

We represent an element of $C(\mu)$ as μx with $x \in \Sigma_n^{\mathbb{N}}$. Each $C(\mu)$ is a compact open set of $\Sigma_n^{\mathbb{N}}$ and the family of all $C(\mu)$ is a basis of $\Sigma_n^{\mathbb{N}}$. For $\mu \in \Sigma_n^*$, we define $s_\mu \in P_n$ as

$$s_\mu := s_{\mu_1} s_{\mu_2} \cdots s_{\mu_{|\mu|}}.$$

For the empty word $\varepsilon \in \Sigma_n^{\mathbb{N}}$, we define $s_\varepsilon = 1$. It is known that an element of $P_n \setminus \{0\}$ is represented as the form $s_\mu s_\nu^*$ for unique $\mu, \nu \in \Sigma_n^*$.

Now we define an action $\beta \colon P_n \curvearrowright \Sigma_n^{\mathbb{N}}$. For $s_\mu s_\nu^* \in P_n$, define $\beta_{s_\mu s_\nu^*} \colon C(\mu) \to C(\nu)$ by

$$\beta_{s_\mu s_\nu^*}(\nu x) = \mu x, x \in \Sigma_n^{\mathbb{N}}.$$

Then the map $s_\mu s_\nu^* \mapsto \beta_{s_\mu s_\nu^*}$ defines an action $\beta \colon P_n \curvearrowright \Sigma_n^{\mathbb{N}}$. Since the domain of $s_\mu s_\mu^*$ coincides with $C(\mu)$, the action β is strongly tight.

For $k, l \in \mathbb{N}$, we define

$$P_n^{k,l} := \{s_\mu s_\nu^* \in P_n \mid |\mu| = k, |\nu| = l\}.$$

Observe that

$$M_n := \bigcup_{k \in \mathbb{N}} P_n^{k,k} \cup \{0\} = \{s_\mu s_\nu^* \in P_n \mid |\mu| = |\nu|\} \cup \{0\}$$

is a wide subsemigroup of P_n.

We investigate β-join closed subsemigroups $T \subset P_n$ such that $M_n \subset T$. For $m \in \mathbb{N}$, define

$$P_n^m := \bigcup_{k-l \in m\mathbb{Z}} P_n^{k,l} \cup \{0\}.$$

Then one can see that P_n^m is a β-join closed subsemigroup which contains M_n. Notice that $P_n^0 = M_n$. Conversely, we obtain the following proposition.

Proposition 3.2.2. *Assume that $T \subsetneq P_n$ is a β-join closed subsemigroup which contains M_n. Then $T = P_n^m$ holds for some $m \in \mathbb{N}$.*

In order to prove this proposition, we prepare a few lemmas. The next lemma follows from straightforward calculations.

Lemma 3.2.3. *For i, j, k, $l \in \mathbb{N}$, we have*

$$P_n^{i,j} P_n^{k,l} = \begin{cases} P_n^{i+k-j,l} \cup \{0\} & (k \geq j), \\ P_n^{i,j-k+l} \cup \{0\} & (k \leq j). \end{cases}$$

Lemma 3.2.4. *Let $T \subset P_n$ be a wide subsemigroup which contains M_n. Then the following statements hold :*

(1) *If $s_\mu s_\nu^* \in T$ holds, then $P_n^{|\mu|,|\nu|} \subset T$ holds.*

(2) *$P_n^{k,l} \subset T$ implies $P_n^{l,k} \subset T$.*

(3) *$P_n^{k,l} \subset T$ implies $P_n^{k+1,l+1} \subset T$.*

Moreover, if T is β-join closed, then the following holds:

(4) *If $P_n^{k,l}$ holds for k, $l \in \mathbb{Z}_{>0}$, then $P_n^{k-1,l-1} \subset T$ holds.*

Proof. (1) Assume $|\mu| = |\mu'|$ and $|\nu| = |\nu'|$ hold for μ', $\nu' \in \Sigma_n^{\mathbb{N}}$. Then we have $s_{\mu'} s_\mu^*$, $s_\nu s_{\nu'}^* \in M_n \subset T$. Since we assume $s_\mu s_\nu^* \in T$, it follows

$$s_{\mu'} s_{\nu'} = s_{\mu'} s_\mu^* s_\mu s_\nu^* s_\nu s_{\nu'}^* \in T.$$

Hence we have $P_n^{|\mu|,|\nu|} \subset T$.

(2) is clear, so we show (3) next. Take $s_\mu s_\nu^* \in P_n^{k,l}$ and x, $y \in \Sigma_n$ arbitrarily. Then we have

$$s_{\mu x} s_{\nu x}^* = s_\mu s_\nu^* s_{\nu x} s_{\nu x}^* \in T,$$

where we use the fact $s_{\nu x} s_{\nu x}^* \in M_n \subset T$. Using (1) in the above, we obtain $P_n^{k+1,l+1} \subset T$.

Finally we show (4) under the assumption that T is β-join closed. Take $s_\mu s_\nu^* \in P_n^{k-1,l-1}$. For each $x \in \Sigma_n$, we have

$$s_\mu s_\nu^* s_{\nu x} s_{\nu x}^* = s_{\mu x} s_{\nu x}^* \in T,$$

since we assume $P_n^{k,l} \subset T$. Observe that

$$D_{(s_\mu s_\nu^*)^* s_\mu s_\nu^*}^\beta = D_{s_\nu s_\nu^*}^\beta = \bigcup_{x \in \Sigma_n} D_{s_{\nu x} s_{\nu x}^*}^\beta (=C(\nu)).$$

Since T is β-join closed, we have $s_\mu s_\nu^* \in T$. Hence we have shown $P_n^{k-1,l-1} \subset T$. $\qquad\square$

Proof of Proposition 3.2.2. We may assume that $M_n \subsetneq T$. We define

$$m := \min\{\|\mu| - |\nu|\| \in \mathbb{N} \mid s_\mu s_\nu^* \in T \backslash M_n\}(>0).$$

We show $T = P_n^m$. By the definition of m, there exists $s_\mu s_\nu^* \in T$ such that $\||\mu| - |\nu|\| = m$. Since T is closed under the inverse, we may assume $|\mu| - |\nu| = m$. Using (1) of Lemma 3.2.4, we have $P_m^{|\mu|,|\nu|} \subset T$. Applying (4) of Lemma 3.2.4 repeatedly, we obtain $P_n^{m,0} \subset T$ and it follows $P_n^{0,m} \subset T$ from (2) of Lemma 3.2.4. Now one can see that $P_n^{k,l} \subset T$ holds for k, l with $k - l \in m\mathbb{Z}$. Hence we obtain $P_n^m \subset T$.

Next, we show $T \subset P_n^m$. Assume that there exists $s_\mu s_\nu^* \in T$ such that $s_\mu s_\nu^* \notin P_n^m$. We may assume that $|\mu| > |\nu|$. Take $a, b \in \mathbb{N}$ such that $|\mu| - |\nu| = am + b$ and $1 \le b \le m - 1$. We have $P_n^{|\mu|,|\nu|} \subset T$ by (1) of Lemma 3.2.4. Using (4) of Lemma 3.2.4 repeatedly, we have $P_n^{am+b,0} \subset T$. Since we have $P_n^{m,0} \subset T$, it follows

$$P_n^{(a-1)m+b,0} \subset P_n^{am+b,0} P_n^{m,0} \subset T,$$

where we used Lemma 3.2.3. Repeating this argument inductively, we obtain $P_n^{b,0} \subset T$. This contradicts to the minimality of m. Now we have shown $T = P_n^m$. $\qquad\square$

By Theorem 2.1.10 and Proposition 3.2.2, an open proper intermediate subgroupoid between $P_n \ltimes_\alpha \Sigma_n^{\mathbb{N}}$ and $M_n \ltimes_\beta \Sigma_n^{\mathbb{N}}$ is given by the form $P_n^m \ltimes_\beta \Sigma_n^{\mathbb{N}}$ for some $m \in \mathbb{N}$. Now we see $P_n^m \ltimes_\beta \Sigma_n^{\mathbb{N}}$ is closed. Observe that $P_n \ltimes_\beta \Sigma_n^{\mathbb{N}}$ has a continuous cocycle $c \colon P_n \ltimes_\beta \Sigma_n^{\mathbb{N}} \to \mathbb{Z}$ defined by $c([s_\mu s_\nu^*, x]) = |\mu| - |\nu|$. Since we have $P_n^m \ltimes_\beta \Sigma_n^{\mathbb{N}} = c^{-1}(m\mathbb{Z})$, it follows that $P_n^m \ltimes_\beta \Sigma_n^{\mathbb{N}}$ is a closed subset of $P_n \ltimes_\beta \Sigma_n^{\mathbb{N}}$. Hence we obtain the next proposition.

Proposition 3.2.5. *Every open wide normal subgroupoid of $P_n \ltimes_\beta \Sigma_n^{\mathbb{N}}$ which contains $M_n \ltimes_\beta \Sigma_m^{\mathbb{N}}$ is closed.*

It follows from Corollary 5.2.5 that $P_n \ltimes_\beta \Sigma_n^{\mathbb{N}}$ is isomorphic to the tight groupoid of P_n. See § 5 for more details.

4. Applications to the theory of C*-algebras

4.1. Analysis of cartan intermediate subalgebras by using inverse semigroups

In this section, we explain a correspondence between Cartan intermediate subalgebras and certain subsemigroups of an inverse semigroup.

Definition 4.1.1. Let A be a C*-algebra. A commutative subalgebra $D \subset A$ is called a Cartan subalgebra if the following conditions hold :

(1) The inclusion $D \subset A$ is non-degenerate (i.e. D contains an approximate unit for A).

(2) The set of normalizers generates A, where $n \in A$ is called a normalizer if $nDn^* \cup n^*Dn \subset D$ holds.

(3) There is a faithful conditional expectation $E \colon A \to D$.

(4) The commutant D' coincides with D, where $D' \colon = \bigcap_{d \in D} \{a \in A \mid da = ad\}$.

In this case, we call (A, D) a Cartan pair.

We investigate a certain class of intermediate C*-subalgebras between Cartan pairs defined as follows.

Definition 4.1.2. Let (A, D) be a Cartan pair. Then an intermediate C*-subalgebra $D \subset B \subset A$ is called a Cartan intermediate subalgebra if $(B.D)$ is a Cartan pair.

Renault's cerebrated theorem states that a Cartan pair arises from a twisted groupoid. We refer to [1, 10, 11] for twists of étale groupoids. A twisted groupoid over G is a topological groupoid Σ with the central extension

$$G^{(0)} \times \mathbb{T} \hookrightarrow \Sigma \xrightarrow{q} G,$$

where $\mathbb{T}$ is the circle group. In this paper, this twist is abbreviated to $q \colon \Sigma \to G$. We denote the reduced C*-algebra of the twist $q \colon \Sigma \to G$ by $C_\lambda^*(\Sigma)$. Recall that $C_\lambda^*(\Sigma)$ contains $C_0(G^{(0)})$ as a subalgebra. We denote the reduced C*-algebra of G by $C_\lambda^*(G)$, which is isomorphic to the reduced C*-algebra of the trivial twist $G \times \mathbb{T} \to G$.

Theorem 4.1.3 (Renault [10, Theorem 5.9]). *Let (A, D) be a Cartan pair where A is separable. Then there exists a twist $q \colon \Sigma \to G$ such that A is isomorphic to $C_\lambda^*(\Sigma)$ via an isomorphism which maps D to $C_0(G^{(0)})$, where G is second countable topologically principal locally compact Hausdorff étale groupoid. This twist $q \colon \Sigma \to G$ is unique up to isomorphism.*

Remark 4.1.4. We shall remark that étale groupoids arising from Cartan pairs are Hausdorff, while étale groupoids arising from inverse semigroup actions are not necessarily Hausdorff. In [3, Theorem 3.15], the authors obtained a necessary and sufficient condition where étale groupoids arising from inverse semigroup actions are Hausdorff.

From now on, we identify $C_\lambda^*(\Sigma)$ and $C_0(G^{(0)})$ with A and D respectively for a Cartan pair (A, D).

Let $q \colon \Sigma \to G$ be a twist and $H \subset G$ be a wide open subgroupoid. [1, Lemma 3.2] states that $\Sigma_H \colon = q^{-1}(H)$ naturally becomes a twist over H and there exists a natural inclusion $C_\lambda^*(\Sigma_H) \subset C_\lambda^*(\Sigma)$. The authors in [1] showed this map $H \mapsto C_\lambda^*(\Sigma_H)$ gives a certain correspondence as follows.

Theorem 4.1.5 (Brown *et al.* [1, Theorem 3.3, Lemma 3.4]). *Let (A, D) be a Cartan pair with a separable A and $q \colon \Sigma \to G$ be an associated twist. Then the above map $H \mapsto C_\lambda^*(\Sigma_H)$ gives a one-to-one correspondence between the set of open wide subgroupoids of G and the set of Cartan intermediate subalgebras $D \subset B \subset A$ in the sence of Definition 4.1.2. In addition, there exists a conditional expectation from $C_\lambda^*(\Sigma)$ to $C_\lambda^*(\Sigma_H)$ if and only if $H \subset G$ is closed.*

Combining Theorem 4.1.5 with Theorem 2.1.10, we obtain the next Corollary.

Corollary 4.1.6. *Let (A, D) be a Cartan pair with separable A and $q \colon \Sigma \to G$ be an associated twist. Assume that $G = S \ltimes_\alpha X$ holds for some strongly tight action $\alpha \colon S \curvearrowright X$. Then there exists a one-to-one correspondence between the set of α-join closed wide subsemigroups of S and the set of Cartan intermediate subalgebras $D \subset B \subset A$. More precisely, the map $T \mapsto C_\lambda^*(\Sigma_{T \ltimes_\alpha X})$ gives the above correspondence.*

Example 4.1.7. We investigate certain subalgebras of the Cuntz algebras by using the polycyclic monoids here. For $n \in \mathbb{N}$ with $n \geq 2$, the Cuntz algebra O_n is the universal unital C*-algebra generated by isometries $S_1, \ldots, S_n$ which satisfy Cuntz relation as follows:

$$S_i^* S_j = \delta_{i,j} 1, \sum_{i=1}^{n} S_i S_i^* = 1.$$

For a finite sequence $\mu = (\mu_1, \ldots, \mu_l)$ on $\{1, \ldots, n\}$, we define

$$S_\mu \colon = S_{\mu_1} S_{\mu_2} \cdots S_{\mu_l}.$$

Then O_n is the closure of the linear span of $\{S_\mu S_\nu^*\}_{\mu,\nu}$, where μ and ν are taken over the all finite sequences on $\{1, \ldots, n\}$. Let D_n be the subalgebra of O_n generated by $\{S_\mu S_\mu^*\}_\mu$, where μ is taken over the all finite sequences on $\{1, \ldots, n\}$. We denote the gauge action by $\tau \colon \mathbb{T} \curvearrowright O_n$. Note that the gauge action satisfies $\tau_z(S_i) = z S_i$ for all $z \in \mathbb{T}$ and $i = 1, 2, \ldots, n$. We denote the fixed point algebra of τ by

$$O_n^\tau \colon = \bigcap_{z \in \mathbb{T}} \{x \in O_n \mid \tau_z(x) = x\}.$$

Then O_n^τ is the closure of the linear span of

$$\{S_\mu S_\nu^* \in O_n \mid |\mu| = |\nu|\},$$

where $|\mu|$ denotes the length of μ.

The polycyclic monoids have strongly tight actions $\beta \colon P_n \curvearrowright \Sigma_n^{\mathbb{N}}$, described in § 3.2. Put $G_n \colon = P_n \ltimes_\beta \Sigma_n^{\mathbb{N}}$. Then G_n is a topologically principal locally compact Hausdorff second countable ample groupoid. For $s_i \in P_n$, let $\chi_{[s_i, D_{s_i^* s_i}]}$ denote the characteristic function on $[s_i, D_{s_i^* s_i}] \subset G_n$. Then $\{\chi_{[s_i, D_{s_i^* s_i}]}\}_{i=1}^n$ are elements of $C_\lambda^*(G_n)$ and generate $C_\lambda^*(G_n)$. Since $\{\chi_{[s_i, D_{s_i^* s_i}]}\}_{i=1}^n$ satisfies the Cuntz relation, O_n and $C_\lambda^*(G_n)$ are isomorphic

via the unique isomorphism $\Phi\colon O_n \to C_\lambda^*(G_n)$ such that $\Phi(S_i) = \chi_{[s_i, D_{s_i^* s_i}]}$ holds for all $i = 1, \ldots, n$. One can see that

$$\Phi(D_n) = C(\Sigma_n^{\mathbb{N}}) \text{ and } \Phi(O_n^\tau) = C_\lambda^*(M_n \ltimes_\beta \Sigma_n^{\mathbb{N}})$$

hold. Define $O_n^m \subset O_n$ to be the subalgebra generated by

$$\{S_\mu S_\nu^* \in O_n \mid |\mu| - |\nu| \in m\mathbb{Z}\}.$$

One can see that

$$\Phi(O_n^m) = C_\lambda^*(P_n^m \ltimes_\beta \Sigma_n^{\mathbb{N}})$$

holds. Therefore, it follows from Proposition 3.2.2 that a Cartan intermediate subalgebra $O_n^\tau \subset B \subset O_n$ coincides with O_n^m for some $m \in \mathbb{N}$. Moreover, every Cartan intermediate subalgebra between O_n^τ and O_n admits a conditional expectation from O_n by Proposition 3.2.5 and Theorem 4.1.5.

We note that O_n^m is isomorphic to O_{n^m}. Indeed, $\{S_\mu\}_{|\mu|=m}$ generates O_n^m and satisfies the Cuntz relation.

5. Relation between strongly tight actions and tight groupoids

In this section, we observe that tight groupoids, which are investigated in [3], are related with strongly tight actions.

5.1. Tight groupoids

First, we recall the definition of tight groupoids. Refer to [2] or [3] for more details. Let S be an inverse semigroup. A character on $E(S)$ is a non-zero semigroup homomorphism from $E(S)$ to $\{0, 1\}$, where $\{0, 1\}$ is equipped with the usual multiplication. We denote the set of all characters on $E(S)$ by $\widehat{E}(S)$. Letting $\widehat{E}(S)$ be equipped with the pointwise convergence topology, $\widehat{E}(S)$ is a locally compact Hausdorff space. For a $\xi \in \widehat{E}(S)$, $\xi^{-1}(\{1\}) \subset E(S)$ is a proper filter in the following sense :

(1) $\xi^{-1}(\{1\})$ does not contain 0,

(2) if e and f belongs to $\xi^{-1}(\{1\})$, then ef also belongs to $\xi^{-1}(\{1\})$,

(3) if $e \in \xi^{-1}(\{1\})$ and $f \geq e$ hold, then f belongs to $\xi^{-1}(\{1\})$.

A character $\xi \in \widehat{E}(S)$ is called an ultracharacter if $\xi^{-1}(\{1\})$ is a maximal proper filter. A character $\xi \in \widehat{E}(S)$ is an ultracharacter if and only if there is no character $\eta \in \widehat{E}(S)$ such that $\xi < \eta$ holds. The set of all ultracharacters on $E(S)$ is denoted by $\widehat{E}_\infty(S)$. The closure of $\widehat{E}_\infty(S)$ in $\widehat{E}(S)$ is denoted by $\widehat{E}_{\text{tight}}(S)$. An element in $\widehat{E}_{\text{tight}}(S)$ is called a tight character.

We define the spectral action $\beta\colon S \curvearrowright \widehat{E}(S)$. For $e \in E(S)$, put

$$D_e^\beta := \{\xi \in \widehat{E}(S) \mid \xi(e) = 1\}.$$

Note that D_e^β is a compact open set of $\widehat{E}(S)$. For $s \in S$ and $\xi \in D_{s^*s}^\beta$, define $\beta_s(\xi) \in D_{ss^*}^\beta$ by

$$\beta_s(\xi)(e) := \xi(s^*es), e \in E(S).$$

Then $\beta_s\colon D_{s^*s}^\beta \to D_{ss^*}^\beta$ is a homeomorphism. The map $s \mapsto \beta_s$ defines an action $\beta\colon S \curvearrowright \widehat{E}(S)$. It is known that $\widehat{E}_\infty(S)$ and $\widehat{E}_{\mathrm{tight}}(S)$ are β-invariant (see [2, Proposition 12.11]). The restrictions of β to $\widehat{E}_\infty(S)$ and $\widehat{E}_{\mathrm{tight}}(S)$ are denoted by

$$\theta_\infty\colon S \curvearrowright \widehat{E}_\infty(S) \text{ and } \theta\colon S \curvearrowright \widehat{E}_{\mathrm{tight}}(S)$$

respectively. The tight groupoid of S is defined as $G_{\mathrm{tight}}(S) := S \ltimes_\theta \widehat{E}_{\mathrm{tight}}(S)$.

5.2. Characterization of tight groupoids

In this subsection, we characterize strongly tight actions with non-empty domains. The proof of the next proposition is left to the reader.

Proposition 5.2.1. *Let S be an inverse semigroup, X be a locally compact Hausdorff space and $\alpha\colon S \curvearrowright X$ be an action. For $x \in X$, we define $\xi_x\colon E(S) \to \{0, 1\}$ by*

$$\xi_x(e) := \begin{cases} 1 & (x \in D_e^\alpha), \\ 0 & (x \notin D_e^\alpha). \end{cases}$$

Then $\xi_x \in \widehat{E}(S)$.

Strongly tight actions with non-empty domains are characterized as the following theorem.

Theorem 5.2.2. *Let S be an inverse semigroup, X be a locally compact Hausdorff space and $\alpha\colon S \curvearrowright X$ be a strongly tight action such that $D_e \neq \emptyset$ holds for each $e \in E(S)\backslash\{0\}$. Then the map $X \ni x \mapsto \xi_x \in \widehat{E}(S)$ in Proposition 5.2.1 gives a homeomorphism*

$$X \ni x \mapsto \xi_x \in \widehat{E}_\infty(S)$$

which induces an isomorphism

$$S \ltimes_\alpha X \ni [s, x] \mapsto [s, \xi_x] \in S \ltimes_{\theta_\infty} \widehat{E}_\infty(S).$$

Proof. This is a simple modification of [12, Proposition 5.5]. We give a proof for the reader's convenience.

For $x \in X$, we show $\xi_x \in \widehat{E}_\infty(S)$. Assume that there exists $\eta \in \widehat{E}(S)$ such that $\xi_x < \eta$. Then there exists $f \in E(S)$ such that $\xi_x(f) = 0$ and $\eta(f) = 1$. Since we assume that

$\{D_e^\alpha\}_{e \in E(S)}$ is a basis of X, there exists $e \in E(S)$ such that $x \in D_e^\alpha$ and $D_e^\alpha \cap D_f^\alpha = \emptyset$. By the assumption $\xi_x < \eta$, we have $\eta(e) = 1$. By $D_{ef}^\alpha = D_e^\alpha \cap D_f^\alpha = \emptyset$, we have $ef = 0$ and therefore, $\eta(ef) = 0$. This contradicts to $\eta(ef) = \eta(e)\eta(f) = 1$. Hence ξ_x is an ultracharacter.

We define the map $\Phi \colon X \ni x \mapsto \xi_x \in \widehat{E}_\infty(S)$. We show that Φ is a homeomorphism. It is easy to show that Φ is continuous. To show that Φ is injective, take $x, y \in X$ with $x \neq y$. Since a family $\{D_e^\alpha\}_{e \in E(S)}$ is a basis of X, there exists $e \in E(S)$ such that $x \in D_e^\alpha$ and $y \notin D_e^\alpha$. Then $\xi_x(e) = 1$ and $\xi_y(e) = 0$. Therefore, we have $\xi_x \neq \xi_y$ and Φ is injective.

Next, take $\xi \in \widehat{E}_\infty(S)$ to show that Φ is surjective. Because a family $\{D_e^\alpha \mid \xi(e) = 1\}$ has the finite intersection property, $\bigcap_{\xi(e)=1} D_e^\alpha$ is not empty. Take $x \in \bigcap_{\xi(e)=1} D_e^\alpha$. Then we have $\xi \leq \xi_x$. By the maximality of ξ, we obtain $\xi = \xi_x$. Therefore, the map $x \mapsto \xi_x$ is surjective.

Now one can check that $\Phi(D_e^\alpha) = D_e^\beta$ holds. Using this, it follows that Φ is a homeomorphism.

It is straightforward to check that there exists a (unique) isomorphism which maps $[s, x] \in S \ltimes_\alpha X$ to $[s, \xi_x] \in S \ltimes_{\theta_\infty} \widehat{E}_\infty(S)$. $\qquad\square$

Remark 5.2.3. It seems difficult to drop the assumption that $D_e \neq \emptyset$ for $e \in E(S) \setminus \{0\}$. Define matrices

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $E \colon = \{0, p, q, 1\}$ is a semilattice with respect to the usual multiplication. Let $X = \{x\}$ be a singleton. Define an action $\alpha \colon E \curvearrowright X$ by declaring $D_1 = X$ and $D_p = D_q = D_0 = \emptyset$. Then α is strongly tight, although $\widehat{E}_\infty$ is not homeomorphic to X. Note that ξ_x, which is defined in the proof of Theorem 5.2.2, is not an ultracharacter. Therefore, it seems difficult to find a natural map between X and $\widehat{E}_\infty$.

The author in [6] showed the following theorem.

Theorem 5.2.4 (Lawson [6, Theorem 2.5]). *Let E be a semilattice with zero and unit elements. Then $\widehat{E}_\infty = \widehat{E}_{\text{tight}}$ holds if and only if $\widehat{E}_\infty$ is compact.*

Theorem 5.2.2 and Theorem 5.2.4 yield the following characterization of tight groupoids.

Corollary 5.2.5. *Let S be an inverse semigroup. Consider the following conditions.*

(1) *S has a strongly tight action on a compact Hausdorff space X.*

(2) *$\widehat{E}(S)_\infty$ is compact,*

(3) *$\widehat{E}(S)_{\text{tight}} = \widehat{E}(S)_\infty$.*

Then $(1) \Leftrightarrow (2)$ and $(2) \Rightarrow (3)$ hold. If S has a unit element, $(3) \Rightarrow (2)$ also holds. Moreover, if (1) holds, then $S \ltimes X$ is isomorphic to $G_{\text{tight}}(S)$.

Remark 5.2.6. The implication $(3) \Rightarrow (2)$ in Corollary 5.2.5 dose not necessarily hold in general. Let E be a semilattice generated by 0 and $\{p_i\}_{i \in \mathbb{N}}$ with the relation

$$p_i p_j = \begin{cases} p_i & (i = j), \\ 0 & (i \neq j). \end{cases}$$

Then $\widehat{E}_\infty = \widehat{E}_{\text{tight}}$ holds, although $\widehat{E}_\infty$ is not compact. Indeed $\widehat{E}_\infty$ is homeomorphic to $\mathbb{N}$.

Remark 5.2.7. There exists a semilattice E such that $\widehat{E}_\infty$ is a locally compact although $\widehat{E}_\infty \subsetneq \widehat{E}_{\text{tight}}$ holds. Let E be the semilattice in Remark 5.2.6. Put $E^1 :=$ $E \cup \{1\}$. Then $\widehat{E^1}_\infty$ is locally compact. In addition, we have $\widehat{E^1}_\infty \subsetneq \widehat{E^1}_{\text{tight}}$. Indeed, $\widehat{E^1}_\infty$ and $\widehat{E^1}_{\text{tight}}$ are isomorphic to $\mathbb{N}$ and $\mathbb{N} \cup \{\infty\}$ respectively. Therefore, we can not relax the condition (2) in Corollary 5.2.5.

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