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A FUNDAMENTAL GROUP TOPOLOGY

C. H. HOUGHTON

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Abstract

The topology of the Čech fundamental group of the one-point compactification of an appropriate space Y induces a topology on the fundamental group of Y. We describe this topology in terms of a topological group introduced by Higman.

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1. Introduction

A free group on a topological space X can be given various topologies, for example the discrete topology or the topology of the free topological group on X. Here we define an intermediate topology on the free group $F = F(X)$ when the space X is 0-dimensional and Hausdorff. In the case where X is the one-point compactification N^* of a countably infinite discrete space N, the group obtained coincides with a topological group $F^{(\omega)}$ studied by Higman (1952).

We are interested in the natural occurrence of the topological group F in the context of Čech homotopy theory. The embedding of an infinite connected locally finite CW-complex Y in its one-point compactification Y^* gives rise to a homomorphism from the fundamental group $\pi(Y)$ to the Čech fundamental group $\check{\pi}(Y^*)$. Our aim is to identify the fundamental group topology thus induced on $\pi(Y)$. Using combinatorial methods, we show that $\pi(Y)$ is the continuous image of an epimorphism from $F^{(\omega)}$. In the 1-dimensional case, where Y is a graph, it follows that $\check{\pi}(Y^*)$ is the completion of a topological free group on the one-point union of N^* and the end space E of Y.

2. A free group topology

Let X be a 0-dimensional Hausdorff space with $x \in X$ chosen as basepoint and let $F = F(X, x)$ be the free group on $X \backslash \{x\}$. Every based map from X to a discrete group extends to a homomorphism of F and we give F the coarsest topology which makes all such extensions continuous. We note some properties of this topological group.

THEOREM 1. *The group $F = F(X, x)$ is 0-dimensional and Hausdorff and the injection from X to F is continuous. Any based map from X to a discrete group G lifts to a continuous homomorphism from F to G.*

PROOF. A subbase for the topology on F is given by the inverse images of elements of discrete groups, with respect to the specified maps. Such images are both open and closed and so F is 0-dimensional. To show that F is Hausdorff, suppose that $v, w \in F$ with $v \neq w$ and that the elements of X appearing in the words v and w are $x_1, \ldots, x_n$. There is a partition of X into open sets $X_0, X_1, \ldots, X_n$ such that $x_i \in X_i$ and $x \in X_0$. Taking G as the free group on $y_1, \ldots, y_n$ and putting $y_0 = 1$, the map $X_i \to y_i$ extends to a homomorphism $F \to G$ under which v and w have distinct images and thus are separated by disjoint open sets. The continuity of the injection follows from the definition of the topology on F, as does the final property.

The constancy sets of a map from X to a discrete group form a partition of X into open sets and, conversely, any such partition can be realised by a map to an appropriate free group. The topology on F is thus given by homomorphisms mapping the parts of such partitions to distinct elements of discrete groups, with the part containing x mapped to 1. For example, let X be the one-point compactification of the countably infinite discrete set $N = \{x_1, x_2, \ldots\}$, the additional point being x. Then the continuous homomorphisms of F are precisely those mapping almost all x_i to 1. This topological group was introduced by Higman (1952) and is denoted by $F^{(\omega)}$.

If X is compact, it is an inverse limit of the finite discrete spaces arising as quotient spaces of finite partitions of X into open sets. We write $X = \varprojlim P_\alpha$, where α runs through all such partitions and P_α denotes the corresponding quotient space. There is an associated inverse system of discrete free groups $F(P_\alpha, p_\alpha)$ of finite rank, where p_α is the projection of x to P_α and the maps are induced by inclusion. We define $C(X, x)$ to be the topological group $\varprojlim F(P_\alpha, p_\alpha)$.

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THEOREM 2. *For compact X, the group $C(X, x)$ is the completion of $F(X, x)$.*

PROOF. For given α, let M_α be the kernel of the epimorphism $F(X, x) \to F(P_\alpha, p_\alpha)$ induced by the projection map $X \to P_\alpha$. The topology on F is the normal subgroup topology given by the M_α and $C(X, x) = \varprojlim F(P_\alpha, p_\alpha) \cong \varprojlim F/M_\alpha$, which is the completion of the Hausdorff group F.

We remark that the paper of Higman (1952) is mainly concerned with properties of the completion of $F^{(\omega)}$.

3. Identifying the fundamental group topology

Suppose that Y^* is the one-point compactification of an infinite connected locally finite CW-complex Y. We establish connections between groups of the type studied above and the Čech fundamental group $\check{\pi}(Y^*)$.

We summarise the results of Houghton (1981) which lead to a determination of $\check{\pi}(Y^*)$. Choosing a chain $Y = Y_0 \supseteq Y_1 \supseteq \cdots$ of cofinite subcomplexes of Y with $\bigcap Y_i = \varnothing$, we have $Y^* = \varprojlim Y/Y_i$ and then $\check{\pi}(Y^*) = \varprojlim \pi(Y/Y_i, y_i)$, where y_i is the projection of a chosen basepoint y of Y. We put $\pi(Y, y) = H$ and choose an element e of the end space $E = \varprojlim \pi_0(Y_i)$ of Y. Standard methods show that $\pi(Y/Y_i, y_i)$ has the form $(H/H_i) * F_i$, where H_i is the kernel of the homomorphism induced by the map $Y \to Y/Y_i$ and F_i is the free group on the based set $\pi_0(Y_i)$, with the projection of e as basepoint. The homomorphism from $(H/H_{i+1}) * F_{i+1}$ to $(H/H_i) * F_i$ is given by the natural map $H/H_{i+1} \to H/H_i$ together with the map $F_{i+1} \to F_i$ corresponding to $\pi_0(Y_{i+1}) \to \pi_0(Y_i)$. If M denotes $\bigcap H_i$ then, in the case $M = H$, we have $\check{\pi}(Y^*, y) = \varprojlim F_i$, which is the completion $C(E, e)$ of $F(E, e)$. In the general case, $\check{\pi}(Y^*)$ is the completion of the Hausdorff group $(H/M) * F(E, e)$, where H/M has the normal subgroup topology given by the family $\{H_i/M\}$.

Our main aim in this section is to investigate the topology on H given by the normal subgroups H_i. From Houghton (1981), the subgroup H_i of $H = \pi(Y, y)$ is generated by all path classes $[\beta][\delta][\beta]^{-1}$ for which the image of δ lies in Y_i. We shall first consider the case where Y is 1-dimensional and we begin with a combinatorial lemma.

LEMMA 3. *Let Γ be an infinite connected locally finite graph. One can choose a chain $\Gamma = \Gamma_0 \supsetneqq \Gamma_1 \supsetneqq \cdots$ of cofinite subgraphs and a spanning tree T of Γ such that $\bigcap \Gamma_i = \varnothing$ and, for all i, each intersection of T with a component of Γ_i is connected.*

PROOF. We use induction and begin by choosing y in the vertex set V of Γ and taking $T_1 = \{y\}$. For the general step, suppose that T_i is a finite tree in Γ with vertex set S_i and let Γ_i be the full subgraph of Γ based on $V \backslash S_i$. In each component of Γ_i choose a finite tree visiting each vertex adjacent to T_i and join this tree by a single edge to T_i. Applying this procedure to all components of Γ_i, we obtain a tree T_{i+1}. The tree T is defined as the union of the T_i. At each stage of the construction we have incorporated all vertices adjacent to the existing tree and hence T spans Γ and $\bigcap \Gamma_i = \varnothing$. For each component U of Γ_i there is a single edge of T joining U to T_i and thus $T \cap U$ is connected.

THEOREM 4. *Let Γ be an infinite connected locally finite graph. With respect to the topology induced by the map $\pi(\Gamma) \to \check{\pi}(\Gamma^*)$, the group $\pi(\Gamma)$ is topologically isomorphic to $F^{(\omega)}$. Furthermore, $\check{\pi}(\Gamma^*)$ is topologically isomorphic to $C(X, x)$, where X is the one-point union $E \vee N^*$ of the end space E of Γ and the one-point compactification of a countably infinite discrete set N.*

PROOF. The fundamental group $F = \pi(\Gamma, y)$ is freely generated by a set bijective with the set of edges of Γ not in T. For such an edge from u to v, the corresponding path class is obtained by going from y to u via T, traversing the edge, and returning from v via T. For given i, the lemma implies that almost all such classes will be of the form $[\beta][\delta][\beta]^{-1}$, with the image of δ lying in Y_i. If the free generators of F are denoted arbitrarily by $x_1, x_2, \ldots$ then a homomorphism of F will be continuous with respect to our topology if and only if almost all x_i are mapped trivially. Thus F is topologically isomorphic to $F^{(\omega)}$. Now $\check{\pi}(\Gamma^*)$ is the completion of $F^{(\omega)} * F(E, e)$, where the topology on the free product is precisely that given by its interpretation as $F(E \vee N^*)$. Thus $\pi(\Gamma^*)$ is $C(E \vee N^*)$.

COROLLARY 5. *Let Y be an infinite connected locally finite CW-complex with 1-skeleton Γ. The topology induced on $H = \pi(Y)$ by the homomorphism to $\check{\pi}(Y^*)$ is identical with the topology defined by the epimorphism from $F^{(\omega)}$ to H, where $F^{(\omega)}$ is realised as $\pi(\Gamma)$.*

PROOF. Choosing the Γ_i as in Lemma 3, we put $Y_i^{(1)} = \Gamma_i$ and define Y_i inductively by attaching to $Y_i^{(n)}$ all $(n+1)$-cells with boundary in $Y_i^{(n)}$. The sets are cofinite subcomplexes of Y with $\bigcap Y_i = \varnothing$ and the result follows from the previous description of the topology on H.

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Department of Pure Mathematics
University College
P.O. Box 78
Cardiff, CF1 1XL
Wales, U.K.