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Renormalization group analysis of the magnetohydrodynamic turbulence and dynamo

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The magnetohydrodynamic (MHD) turbulence appears in engineering laboratory flows and is a common phenomenon in natural systems, e.g. stellar and planetary interiors and atmospheres and the interstellar medium. The applications in engineering are particularly interesting due to the recent advancement of tokamak devices, reaching very high plasma temperatures, thus giving hope for the production of thermonuclear fusion power. In the case of astrophysical applications, perhaps the main feature of the MHD turbulence is its ability to generate and sustain large-scale and small-scale magnetic fields. However, a crucial effect of the MHD turbulence is also the enhancement of large-scale diffusion via interactions of small-scale pulsations, i.e. the generation of the so-called turbulent viscosity and turbulent magnetic diffusivity, which typically exceed by orders of magnitude their molecular counterparts. The enhanced resistivity plays an important role in the turbulent dynamo process. Estimates of the turbulent electromotive force (EMF), including the so-called α-effect responsible for amplification of the magnetic energy and the turbulent magnetic diffusion are desired. Here, we apply the renormalization group technique to extract the final expression for the turbulent EMF from the fully nonlinear dynamical equations (Navier–Stokes, induction equation). The simplified renormalized set of dynamical equations, including the equations for the means and fluctuations, is derived and the effective turbulent coefficients such as the viscosity, resistivity, the α-coefficient and the Lorentz-force coefficients are explicitly calculated. The results are also used to demonstrate the influence of magnetic fields on energy and helicity spectra of strongly turbulent flows, in particular the magnetic energy spectrum.

Key words: MHD turbulence, turbulence theory, dynamo theory

1. Introduction

The study of the dynamics and description of the magnetohydrodynamic (MHD) turbulence is of interest from the point of view of engineering and astrophysics.

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The majority of astrophysical magnetic fields, including the terrestrial field, are generated by turbulent flow of a conducting liquid, such as plasma or liquid metal in the interiors of planets, stars, accretion discs, etc. The flow is typically driven thermally, compositionally (cf. e.g. Roberts & Soward 1972; Brandenburg & Subramanian 2005; Dormy & Soward 2007; Roberts & King 2013) or by the development of violent instabilities such as e.g. the magneto-rotational instability (cf.Balbus & Hawley 1991*a*,*b*). The investigation of turbulent magnetized flows involves the description of a very complicated dynamics of small-scale fluctuations (cf. Zeldovich *et al.* 1987; Zeldovich, Ruzmaikin & Sokoloff 1990) and hence it is extremely difficult and requires sophisticated mathematical tools; see Tobias, Cattaneo & Boldyrev (2013) and Tobias (2021) for a review of different approaches and results. To simplify the problem, various assumptions have been put forward and, in particular, a common simplification in the theoretical approaches is the assumption of the so-called weak turbulence, which corresponds to a weak amplitude of turbulent pulsations and linearization of their evolution. Such an approach allows us to calculate the electromotive force, i.e. the α-effect responsible for magnetic energy amplification and the turbulent diffusion (cf. e.g. Steenbeck, Krause & Radler 1966; Roberts 1994; Moffatt & Dormy 2019), but lacks the crucial effect of the nonlinear dynamics of the fluctuations. As argued in Tobias *et al.* (2013), in some cases the regime of weak turbulence can be sustained for long times, nevertheless, it is much more common for natural systems to develop into the strong-turbulence regime, where the evolution of turbulent fluctuations becomes nonlinear. This difficulty is resolved here, by application of the renormalization technique, a statistical closure approximation which is based on systematic, subsequent (iterative) elimination of thin wavenumber bands from the Fourier spectrum of rapidly evolving variables (cf. e.g. Ma & Mazenko 1975). The first to apply this technique in fluid mechanics were Wyld (1961) and Forster, Nelson & Stephen (1977), to study the statistical properties of stationary, homogeneous and isotropic turbulence. However, the first to apply the renormalization group method in magnetohydrodynamics was Moffatt (1981, 1983), who calculated the kinematic α coefficient, in a simplified case of a given stationary, homogeneous and isotropic turbulence flow field, unaffected by the Lorentz force; other interesting numerical and analytic studies of the kinematic dynamo problem by Vincenzi (2002) and Arponen & Horvai (2007) involved the incompressible, stochastic Kraichnan–Kazantsev model of turbulence and focused on determination of the magnetic Prandtl number dependence of the mean field growth rate for that model. A few years after the pioneering works of Moffatt, a comprehensive work on renormalization of the hydrodynamic equations in the absence of the magnetic field was published by Yakhot & Orszag (1986). A number of works on renormalization in non-magnetic stirred, stationary turbulence followed, such as e.g. McComb & Watt (1990, 1992), McComb, Roberts & Watt (1992), Rubinstein & Barton (1991, 1992), Smith & Reynolds (1992), Lam (1992) and Eyink (1994) with seminal reviews by Smith & Woodruff (1998) and McComb (2014). A notable contribution came from Kleeorin & Rogachevskii (1994), who calculated the renormalized Lorentz force for non-helical, stationary, homogeneous and isotropic MHD turbulence, in the absence of dynamo action; the effect of chirality in natural turbulence which leads to the α-effect and the effect of gradients of the mean fields on the dynamics of fluctuations have been excluded from their analysis. A couple of years earlier Adzhemyan, Vasiliev & Gnatich (1987) utilized the renormalization techniques from quantum field theory to comment on the full dynamo problem, with the inclusion of the velocity evolution affected by the Lorentz force, and concluded that the spectral scalings for the turbulent kinetic and magnetic energies are necessarily different. Barbi & Münster (2013) used similar techniques and developed a numerical algorithm to compute renormalized flows, which reproduced standard theoretical scalings. Reviews of approaches to the description of turbulence and MHD turbulence based on the renormalization group technique can be found in Adzhemyan, Antonov & Vasiliev (1999) and Zhou (2010).

A powerful method which allows us to relate the mean electromotive force and turbulent magnetic diffusion to the turbulent energy and helicity tensors is the so-called two-scale direct-interaction approximation dating back to Kraichnan (1959, 1965) and developed e.g. by Yoshizawa (1990) and Yokoi (2013, 2018); see Yoshizawa (1998) and Yokoi (2020) for a review. It is based on the idea of the introduction of the Green's function of turbulence, that is, a tensorial response function to an infinitesimal impulse force and an *ad hoc* introduction of two scales of turbulence in space and time related by the same parameter of expansion. Despite its limitations, it allows us to describe the mean electromotive force in strong turbulence once the statistical properties of the underlying small-scale chaotic flow are known.

Recent investigations of Mizerski (2020, 2021) involved applications of the renormalization group method to study the effect of non-stationarity and anisotropy on the MHD turbulence in what could be called an intermediate regime (cf. also Mizerski (2018*a*,*b*) for the effect of non-stationarity in weak turbulence). Due to the high complexity of the mathematical approach in the case of non-stationary and non-isotropic turbulence, the effect of nonlinear evolution of the fluctuations has been included only at leading order at each step of the renormalization procedure. As a result, although reliable estimates of the electromotive force (EMF) could be made, the wavenumber dependence of all the turbulent coefficients, likewise of the energy and helicity spectra, was not fully resolved. Moreover, the effect of gradients of the mean fields on the dynamics of turbulent pulsations has been neglected.

In this work the renormalization group method is applied to the full system of MHD equations, including the back reaction of the Lorentz force on the flow. Moreover, the effect of gradients of the mean fields on the evolution of turbulent fluctuations is included, which allows us to calculate turbulent diffusivities. The large-scale EMF is calculated for the dynamic nonlinear problem of strong, stationary turbulence, i.e. full renormalization of nonlinear terms is performed including the influence of the turbulent fluctuational diffusivities and of the fluctuational α-effect at each step of the renormalization procedure. Renormalized dynamical equations for the mean and fluctuating fields are obtained, which contain turbulent coefficients describing the net nonlinear effect of short-wavelength fluctuations, such as the turbulent viscosity, EMF (including the turbulent magnetic diffusivity and the α-coefficient) and turbulent coefficients describing the effective Lorentz force at large scales. In numerical approaches to the MHD turbulence there are serious issues with resolution sensitivity, thus an explicit form of the turbulent coefficients is very desirable, however, they are often inserted into the dynamics of mean fields in an arbitrary way, estimated from observations but entirely unrelated to the dynamics of the fluctuations, which is then lost. Here, by application of the renormalization method we derive explicit recursion differential equations for all the turbulent coefficients (mean and fluctuational ones). Analytic solutions are provided for the two cases of non-helical turbulence (which effectively reproduces the results of Kleeorin & Rogachevskii 1994) and helical turbulence. The case of weak turbulence is also presented for comparison with the fully nonlinear case of strong turbulence and the former is used in § 3 to introduce the reader to the undertaken analytic approach.

2. MHD equations and the mean-field dynamo problem

To study the MHD turbulence in an incompressible conducting fluid (plasma) we consider the following dynamical equations describing the evolution of the velocity field of the fluid flow $\boldsymbol{U}(\boldsymbol{x}, t)$ and the magnetic field $\boldsymbol{B}(\boldsymbol{x}.t)$

$$\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{U} = \boldsymbol{f} - \nabla \varPi + \frac{1}{\mu_0\rho}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \nu\nabla^2\boldsymbol{U}, \tag{2.1a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{U} + \eta\nabla^2\boldsymbol{B}, \tag{2.1b}$$

$$\nabla\cdot\boldsymbol{U} = 0 \quad \nabla\cdot\boldsymbol{B} = 0, \tag{2.1c}$$

where ν is the fluid's viscosity, η the magnetic diffusivity and

$$\varPi = \frac{p}{\rho} + \frac{B^2}{2\mu_0\rho}, \tag{2.2}$$

is the total pressure, with p denoting the thermodynamic pressure, ρ the density and μ_0 the vacuum permeability. Without loss of generality we may assume that the forcing is solenoidal

$$\nabla\cdot\boldsymbol{f} = 0. \tag{2.3}$$

The first (2.1a) is the well-known Navier–Stokes equation with the Lorentz force, whereas the second one is the induction equation derived from Maxwell's laws and the material Ohm's law (the latter also utilizes the assumption that the time scale associated with electromagnetic waves L/c is much shorter than any time scale in the fluid flow). The solenoidal constraints for the dynamical fields (2.1c) simply express the law of mass conservation and Gauss' law for magnetism. For the purpose of simplicity we rescale the magnetic field in the following way:

$$\frac{\boldsymbol{B}}{\sqrt{\mu_0\rho}} \rightarrow \boldsymbol{B}, \tag{2.4}$$

so that the prefactor $1/\mu_0\rho$ in the Lorentz-force term in the Navier–Stokes equation is lost.

Next, denoting by angular brackets the ensemble mean,

$$\langle\cdot\rangle - \text{ensemble mean}, \tag{2.5}$$

let us assume

$$\langle\boldsymbol{f}\rangle = 0, \tag{2.6}$$

$$\boldsymbol{U} = \langle\boldsymbol{U}\rangle + \boldsymbol{u}, \quad \boldsymbol{B} = \langle\boldsymbol{B}\rangle + \boldsymbol{b}, \quad p = \langle p\rangle + p', \tag{2.7a–c}$$

and write down separately the equations for the mean fields $\langle\boldsymbol{U}\rangle$ and $\langle\boldsymbol{B}\rangle$ and the turbulent fluctuations $\boldsymbol{u}$ and $\boldsymbol{b}$; this yields

$$\frac{\partial\langle\boldsymbol{U}\rangle}{\partial t} + (\langle\boldsymbol{U}\rangle\cdot\nabla)\langle\boldsymbol{U}\rangle = -\nabla\langle\varPi\rangle + (\langle\boldsymbol{B}\rangle\cdot\nabla)\langle\boldsymbol{B}\rangle + \nu\nabla^2\langle\boldsymbol{U}\rangle - \nabla\cdot(\langle\boldsymbol{uu}\rangle - \langle\boldsymbol{bb}\rangle), \tag{2.8a}$$

$$\frac{\partial\langle\boldsymbol{B}\rangle}{\partial t} = \nabla\times(\langle\boldsymbol{U}\rangle\times\langle\boldsymbol{B}\rangle) + \nabla\times\langle\boldsymbol{u}\times\boldsymbol{b}\rangle + \eta\nabla^2\langle\boldsymbol{B}\rangle, \tag{2.8b}$$

$$\nabla\cdot\langle\boldsymbol{B}\rangle = 0, \quad \nabla\cdot\langle\boldsymbol{U}\rangle = 0, \tag{2.8c}$$

where $\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle$ is the large-scale EMF and

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + (\langle \boldsymbol{U} \rangle \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\langle \boldsymbol{U} \rangle - (\langle \boldsymbol{B} \rangle \cdot \nabla)\boldsymbol{b} - (\boldsymbol{b} \cdot \nabla)\langle \boldsymbol{B} \rangle + \nabla \Pi' \\ = \boldsymbol{f} - \nabla \cdot (\boldsymbol{uu} - \boldsymbol{bb}) + \nabla \cdot (\langle \boldsymbol{uu} \rangle - \langle \boldsymbol{bb} \rangle), \tag{2.9a}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} - \eta \nabla^2 \boldsymbol{b} + (\langle \boldsymbol{U} \rangle \cdot \nabla)\boldsymbol{b} - (\langle \boldsymbol{B} \rangle \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\langle \boldsymbol{B} \rangle - (\boldsymbol{b} \cdot \nabla)\langle \boldsymbol{U} \rangle \\ = \nabla \times (\boldsymbol{u} \times \boldsymbol{b} - \langle \boldsymbol{u} \times \boldsymbol{b} \rangle), \tag{2.9b}$$

$$\nabla \cdot \boldsymbol{b} = 0, \quad \nabla \cdot \boldsymbol{u} = 0. \tag{2.9c}$$

Furthermore, we assume scale separation between the means and the fluctuations, and express the dynamical fields using a Fourier transform defined in the following way:

$$u_i(\boldsymbol{x}, t) = \int_{\Lambda_L}^{\Lambda_\nu} \mathrm{d}^3 k \int_{-\infty}^{\infty} \mathrm{d}\omega \hat{u}_i(\boldsymbol{k}, \omega)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \tag{2.10a}$$

$$b_i(\boldsymbol{x}, t), = \int_{\Lambda_L}^{\Lambda_\eta} \mathrm{d}^3 k \int_{-\infty}^{\infty} \mathrm{d}\omega \hat{b}_i(\boldsymbol{k}, \omega)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \tag{2.10b}$$

$$\langle U \rangle_i(\boldsymbol{x}, t) = \int_0^{\kappa_m} \mathrm{d}^3 \kappa \int_{-\infty}^{\infty} \mathrm{d}\omega \widehat{\langle U \rangle}_i(\boldsymbol{\kappa}, \omega)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}\cdot\boldsymbol{x} - \omega t)}, \tag{2.10c}$$

$$\langle B \rangle_i(\boldsymbol{x}, t) = \int_0^{\kappa_m} \mathrm{d}^3 \kappa \int_{-\infty}^{\infty} \mathrm{d}\omega \widehat{\langle B \rangle}_i(\boldsymbol{\kappa}, \omega)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}\cdot\boldsymbol{x} - \omega t)}, \tag{2.10d}$$

where the upper cutoff for the mean-field Fourier modes κ_m and the lower cutoff for the fluctuations Λ_L satisfy

$$\kappa_m \ll \Lambda_L. \tag{2.11}$$

In the above we have also introduced upper cutoffs for the Fourier modes in the velocity fluctuations Λ_ν and in the magnetic fluctuations Λ_η, which in natural systems appear due to viscous kinetic energy dissipation and resistive dissipation of magnetic energy; for generality the magnetic and kinetic cutoffs are assumed unequal, $\Lambda_\upsilon \neq \Lambda_\eta$. The assumption of separation of spatial scales is standard in mean-field theories, put forward in order to allow for analytic formulation of the dynamics of large-scale fields. Such a clear separation is often absent in natural systems and energy is present at all scales. Still, there are many cases when scale separation is very apparent. Hughes & Tobias (2010) have estimated that a reasonable scale separation of approximately a decade between domain size and the size of the most energetic turbulent eddy, is achieved in numerical simulations with 1000^3 spectral modes, which accurately describe the dynamics at magnetic Reynolds numbers of the order $O(10^3)$. Even more importantly, natural scale separation is introduced by rapid background rotation in convective flows, such as e.g. in the Earth's liquid core, where the large length scale is defined by $E^{-1/3}L$, with $E \ll 1$ being the Ekman number and L the thickness of the convection zone. There are numerous examples of studies which utilize the multiple-scale approach for MHD turbulent convection, such as, Childress & Soward (1972) and Soward (1974), or more recently Mizerski & Tobias (2013), Calkins *et al.* (2015) and Calkins (2018). In the mean-field theory a fluctuational wave vector $k = 2\pi/\mathcal{L}$ is associated with the length scale $\mathscr{L}$, which is regarded as the typical size of the energetic eddies and $\Lambda_L = 2\pi/\mathscr{L}_L$ is defined by the size of the largest, most energetic turbulent eddies, $\mathscr{L}_L$.

The aim of this analysis is to study the effect of the statistically homogeneous, stationary and isotropic MHD turbulence. However, the MHD turbulence may in particular involve the very important process of the generation of large-scale magnetic fields by the complex flow (cf. Moffatt & Dormy 2019). This requires a lack of reflectional symmetry in the flow, thus, for the sake of generality, we will study the general case of helical turbulence. In other words, we consider an MHD system driven by a non-reflectionally symmetric (helical) forcing $\boldsymbol{f}$ in the Navier–Stokes equation. The forcing is assumed Gaussian with zero mean (cf. (2.6)) and is fully defined by the following correlation function:

$$\langle \hat{f}_i(\boldsymbol{k},\omega)\hat{f}_j(\boldsymbol{k}',\omega')\rangle = \left[\frac{D_0}{k^{\sigma_0}}P_{ij}(\boldsymbol{k}) + \mathrm{i}\frac{D_1}{k^{\sigma_1}}\epsilon_{ijk}k_k\right]\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(\omega+\omega'), \tag{2.12}$$

where

$$P_{ij}(\boldsymbol{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}, \tag{2.13}$$

is the projection operator and D_0, D_1 and σ_0, σ_1 are real constants; the correlations function satisfies $k_j\langle \hat{f}_i(\boldsymbol{k},\omega)\hat{f}_j(\boldsymbol{k}',\omega')\rangle = 0$ and $k_i\langle \hat{f}_i(\boldsymbol{k},\omega)\hat{f}_j(\boldsymbol{k}',\omega')\rangle = 0$. The term proportional to D_1 introduces lack of reflectional symmetry (for which helicity can be used as a measure), thus it is crucial for the large-scale dynamo process. The value of $\sigma_0 = -2$ (at $D_1 = 0$) corresponds to fluid in thermal equilibrium (cf. Landau & Lifshitz 1987) whereas $\sigma_0 = 3$ was shown by Yakhot & Orszag (1986) to correspond to the Kolmogorov-type turbulence in the absence of a magnetic field. We will, therefore, assume $\sigma_0 > -2$ and consider non-equilibrium flows. The σ_1 exponent will turn out to influence the turbulent magnetic energy and helicity spectra and we will demonstrate that the value of $\sigma_1 = 5$ corresponds to $k^{-5/3}$ spectral scaling for turbulent helicity, reported for the inertial range of helical isotropic turbulence in a number of works, e.g. Brissaud *et al.* (1973) and Chen *et al.* (2003). Furthermore, we assume that the turbulence is forced only at small scales, i.e. within the wavenumber band $k > \Lambda_L$.

We can calculate a positive definite quantity

$$F^2(k,\omega) = \int k^2\,\mathrm{d}\mathring{\Omega}_{\boldsymbol{k}} \int \mathrm{d}^4q'\langle \hat{f}_i(\boldsymbol{k},\omega)\hat{f}_i(\boldsymbol{k}',\omega')\rangle = \frac{8\pi D_0}{k^{\sigma_0-2}} > 0, \tag{2.14}$$

where $\mathring{\Omega}_{\boldsymbol{k}}$ denotes a solid angle associated with the vector $\boldsymbol{k}$ and to simplify notation we have introduced a four-component vector notation

$$\boldsymbol{q}' = (\boldsymbol{k}',\omega'), \quad \int \mathrm{d}^3k' \int_{-\infty}^{\infty} \mathrm{d}\omega'(\cdot) = \int \mathrm{d}^4q'(\cdot). \tag{2.15a,b}$$

This implies

$$D_0 > 0. \tag{2.16}$$

The renormalization group technique is an iterative procedure based on successive elimination of thin wavenumber bands from the Fourier spectrum of the fluctuating fields. In this way, the effect of thin bands of modes with the shortest wavelengths on the remaining modes is calculated at each step of the procedure. The final aim of this approach is to obtain recursion equations for coefficients describing the effective mean Reynolds stress, the mean EMF and the mean Lorentz force (see (4.30*a*,*b*)) as functions of the wavenumber at each step of the procedure. The Reynolds stresses are responsible for the creation of the turbulent viscosity, and the mean EMF for the creation of the turbulent magnetic diffusivity and what is traditionally called the α-effect, which involves the part of the EMF that is linear in the mean magnetic field. The recursion equations (provided in (A91*a*–*e*)) are then solved for $k \to \Lambda_L$ in order to obtain the final forms of the large-scale viscosity, EMF and the Lorentz force which appear in the renormalized mean-field equations and include the effect of nonlinear evolution of turbulent fluctuations. However, for the sake of clarity, we start by considering the simplified limit of weak turbulence, with linearized evolution equations for fluctuations, rather common in the literature on mean-field dynamo theory. This does not involve renormalization, however, it will set the grounds for § 4, where the renormalization method is applied. Moreover, the linear regime, considered in the next section, demonstrates at a simple level the basic ideas of the mathematical approach undertaken to provide a comprehensive picture of the isotropic, homogeneous and stationary MHD turbulence, which may amplify mean magnetic fields.

3. Introductory problem of forced system with very weak turbulent fluctuations – linearization

Utilizing the scale separation assumption and introducing

$$\varepsilon = \frac{\kappa_m}{\Lambda_L}, \quad \boldsymbol{X} = \varepsilon \boldsymbol{x}, \tag{3.1a,b}$$

one can write

$$\langle U\rangle_i \approx \langle U\rangle_{0i} + \varepsilon x_j G_{ij}, \quad \langle B\rangle_i \approx \langle B\rangle_{0i} + \varepsilon x_j \Gamma_{ij}, \tag{3.2a,b}$$

where

$$\langle \boldsymbol{U}\rangle_0 = \langle \boldsymbol{U}\rangle(\varepsilon = 0) = O(1), \quad G_{ij} = \frac{\partial \langle U\rangle_i}{\partial X_j} = O(1), \tag{3.3a,b}$$

$$\langle \boldsymbol{B}\rangle_0 = \langle \boldsymbol{B}\rangle(\varepsilon = 0) = O(1), \quad \Gamma_{ij} = \frac{\partial \langle B\rangle_i}{\partial X_j} = O(1). \tag{3.4a,b}$$

Of course, formally, the gradient matrices are defined at $\varepsilon = 0$ in the expansions, however, we neglect terms of the order $O(\varepsilon^2)$, which allows us to substitute for $G_{ij}(\varepsilon = 0)$ and $\Gamma_{ij}(\varepsilon = 0)$ with G_{ij} and Γ_{ij}; note that, $\langle \boldsymbol{U}\rangle_0$ and $\langle \boldsymbol{B}\rangle_0$ are still allowed to vary on length scales significantly larger than ε^{-1}. The latter expansion allows us to express the terms $(\langle \boldsymbol{U}\rangle \cdot \nabla)\boldsymbol{u}$ and $(\langle \boldsymbol{U}\rangle \cdot \nabla)\boldsymbol{b}$ in the following way:

$$\begin{aligned}
(\langle \boldsymbol{U}\rangle \cdot \nabla)\boldsymbol{u} &= \mathrm{i}\int^{\Lambda_\nu} \mathrm{d}^4 q (\langle U\rangle_{0m} + \varepsilon G_{mj} x_j) k_m \hat{u}_i(\boldsymbol{q})\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \\
&= \mathrm{i}\int^{\Lambda_\nu} \mathrm{d}^4 q \left[\langle U\rangle_{0m} + \mathcal{U} \sin\left(\varepsilon \frac{G_{mj}}{\mathcal{U}} x_j\right)\right] k_m \hat{u}_i(\boldsymbol{q})\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} + O\left(\varepsilon^3 \frac{\mathcal{U}^2}{\mathcal{L}}\right) \\
&= \mathrm{i}\int^{\Lambda_\nu} \mathrm{d}^4 q \langle U\rangle_{0m} k_m \hat{u}_i(\boldsymbol{q})\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \\
&\quad + \frac{\mathcal{U}}{2}\int^{\Lambda_\nu} \mathrm{d}^4 q k_m \hat{u}_i(\boldsymbol{q})\, \mathrm{e}^{-\mathrm{i}\omega t} [\mathrm{e}^{\mathrm{i}(k_j + \varepsilon(G_{mj}/\mathcal{U}))x_j} - \mathrm{e}^{\mathrm{i}(k_j - \varepsilon(G_{mj}/\mathcal{U}))x_j}] + O\left(\varepsilon^3 \frac{\mathcal{U}^2}{\mathcal{L}}\right)
\end{aligned}$$

$$= \int^{\Lambda_\nu} \mathrm{d}^4 q \left\{ (\mathrm{i}\langle U\rangle_{0m} k_m - \varepsilon G_{mm})\hat{u}_i(\boldsymbol{q}) + \frac{\mathcal{U}}{2} k_m \left[\hat{u}_i\left(\boldsymbol{k} - \varepsilon \frac{\boldsymbol{G}_m}{\mathcal{U}}, \omega\right) - \hat{u}_i\left(\boldsymbol{k} + \varepsilon \frac{\boldsymbol{G}_m}{\mathcal{U}}, \omega\right)\right]\right\} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x}-\omega t)} + O\left(\varepsilon^2 \frac{\mathcal{U}^2}{\mathcal{L}}\right), \tag{3.5}$$

where $\mathcal{U}$ is the velocity scale and $\mathcal{L}$ is the length scale of variation of turbulent fluctuations and we have defined the vector $(\boldsymbol{G}_m)_j = G_{mj}$ for $j = 1, 2, 3$, composed of rows of the matrix G_{mj}; of course, $G_{mm} = \bar{\boldsymbol{\nabla}} \boldsymbol{\cdot} \langle \boldsymbol{U}\rangle = 0$. For the purpose of this section, to simplify the notation we will put $\mathcal{U} = 1$.

As explained above, for the sake of clarity, we start by considering the simple problem of weak turbulent fluctuations and the dynamical equations linearized about the means

$$(-\mathrm{i}\omega + \nu k^2)\hat{u}_i(\boldsymbol{q}) - \mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle_0 \hat{b}_i(\boldsymbol{q}) + \mathrm{i}k_i\hat{\Pi} + \mathcal{R}_i^{(u)} = \hat{f}_i(\boldsymbol{q}), \tag{3.6a}$$

$$(-\mathrm{i}\omega + \eta k^2)\hat{b}_i(\boldsymbol{q}) - \mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle_0 \hat{u}_i(\boldsymbol{q}) + \mathcal{R}_i^{(b)} = 0, \tag{3.6b}$$

where

$$\mathcal{R}_i^{(u)} = \varepsilon G_{ij}\hat{u}_j(\boldsymbol{q}) - \varepsilon \Gamma_{ij}\hat{b}_j(\boldsymbol{q}) + \tfrac{1}{2}k_m[\hat{u}_i(\boldsymbol{k} - \varepsilon \boldsymbol{G}_m, \omega) - \hat{u}_i(\boldsymbol{k} + \varepsilon \boldsymbol{G}_m, \omega)] - \tfrac{1}{2}k_m[\hat{b}_i(\boldsymbol{k} - \varepsilon \boldsymbol{\Gamma}_m, \omega) - \hat{b}_i(\boldsymbol{k} + \varepsilon \boldsymbol{\Gamma}_m, \omega)] = O(\varepsilon), \tag{3.7a}$$

$$\mathcal{R}_i^{(b)} = \varepsilon \Gamma_{ij}\hat{u}_j(\boldsymbol{q}) - \varepsilon G_{ij}\hat{b}_j(\boldsymbol{q}) + \tfrac{1}{2}k_m[\hat{b}_i(\boldsymbol{k} - \varepsilon \boldsymbol{G}_m, \omega) - \hat{b}_i(\boldsymbol{k} + \varepsilon \boldsymbol{G}_m, \omega)] - \tfrac{1}{2}k_m[\hat{u}_i(\boldsymbol{k} - \varepsilon \boldsymbol{\Gamma}_m, \omega) - \hat{u}_i(\boldsymbol{k} + \varepsilon \boldsymbol{\Gamma}_m, \omega)] = O(\varepsilon). \tag{3.7b}$$

Since the mean velocity $\langle \boldsymbol{U}\rangle_0$ only creates a shift of the frequency $\omega \to \omega - \boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{U}\rangle_0$, we simply absorb it into the frequency. This is allowed here, since we do not introduce into the problem the time-scale separation and the Fourier frequencies take all real values from $-\infty$ to ∞, whereas the norm of the wave vectors is bounded from above. For the velocity and magnetic fields, likewise the forcings are solenoidal

$$\boldsymbol{k}\boldsymbol{\cdot}\hat{\boldsymbol{u}} = 0, \quad \boldsymbol{k}\boldsymbol{\cdot}\hat{\boldsymbol{b}} = 0, \quad \boldsymbol{k}\boldsymbol{\cdot}\hat{\boldsymbol{f}} = 0. \tag{3.8a–c}$$

Thus, by applying the projection operator to both sides of the equation (2.13) and after some simple algebra we get

$$\hat{u}_i(\boldsymbol{q}) = \frac{\mathrm{i}\hat{f}_i(\boldsymbol{q})}{\sigma(\boldsymbol{q})} - \frac{\mathrm{i}}{\sigma(\boldsymbol{q})}P_{ij}\mathcal{R}_j^{(u)} + \frac{\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle}{(\omega + \mathrm{i}\eta k^2)\sigma(\boldsymbol{q})}\mathcal{R}_i^{(b)}, \tag{3.9a}$$

$$\hat{b}_i(\boldsymbol{q}) = -\frac{\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle_0}{(\omega + \mathrm{i}\eta k^2)\sigma(\boldsymbol{q})}\hat{f}_i(\boldsymbol{q}) - \frac{\mathrm{i}}{\omega + \mathrm{i}\eta k^2}\left[1 + \frac{(\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle)^2}{(\omega + \mathrm{i}\eta k^2)\sigma(\boldsymbol{q})}\right]\mathcal{R}_i^{(b)} + \frac{\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle}{(\omega + \mathrm{i}\eta k^2)\sigma(\boldsymbol{q})}P_{ij}\mathcal{R}_j^{(u)}, \tag{3.9b}$$

with

$$\sigma(\boldsymbol{q}) = \omega + \mathrm{i}\nu k^2 - \frac{(\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle)^2}{\omega + \mathrm{i}\eta k^2}. \tag{3.10}$$

Of course the problem, although linearized in fluctuations, remains nonlinear because of the presence of the forcing $\boldsymbol{f}$, which allows us to calculate the turbulent transport coefficients and the α-effect.

We consider the simplest case of a statistically stationary, helical forcing defined by (2.12). The small corrections from the gradients of means $\boldsymbol{\mathcal{R}}^{(u)}$ and $\boldsymbol{\mathcal{R}}^{(b)}$ will be treated in a perturbational manner within the asymptotic limit $\varepsilon \ll 1$. A similar linearized problem that can be easily compared with the one studied here, has been solved in chapter 12.4.1 of Moffatt & Dormy (2019), although the authors' interests lie predominantly in the α-effect (see also the rest of chapter 12.4 in that book, for other interesting aspects of the linearized regime in stationary turbulence).

In addition, let us also assume that the mean magnetic field is weak enough, so that terms of the order $(\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle)^2$ can be neglected. By the use of (3.9*a*,*b*) and (3.10) and neglecting terms of the order $O(\varepsilon^2, (\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle)^2, \varepsilon\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle)$ we can calculate the mean EMF and the mean Reynolds and Maxwell stresses in the following way:

$$
\begin{aligned}
\mathcal{E}_i = \epsilon_{ijk}\langle u_j b_k\rangle &\approx \epsilon_{ijk}\int \mathrm{d}^4q \int \mathrm{d}^4q'\, \frac{\boldsymbol{k}'\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle\langle\hat{f}_j(\boldsymbol{q})\hat{f}_k(\boldsymbol{q}')\rangle}{(\omega'+\mathrm{i}\eta k'^2)\sigma(\boldsymbol{q}')\sigma(\boldsymbol{q})}\,\mathrm{e}^{\mathrm{i}[(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{\cdot}\boldsymbol{x}-(\omega+\omega')t]} \\
&+\epsilon_{ijk}\int \mathrm{d}^4q \int \mathrm{d}^4q'\, \frac{\langle\hat{f}_j(\boldsymbol{q})\mathcal{R}^{(b)}_k(\boldsymbol{q}')\rangle}{(\omega'+\mathrm{i}\eta k'^2)(\omega+\mathrm{i}\nu k^2)}\,\mathrm{e}^{\mathrm{i}[(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{\cdot}\boldsymbol{x}-(\omega+\omega')t]} \\
&\approx \eta D_1\langle B\rangle_p\epsilon_{ijk}\epsilon_{jkm}\int \frac{\mathrm{d}^4q}{k^{\sigma_1-2}}\,\frac{k_pk_m}{(\omega^2+\eta^2k^4)|\sigma(\boldsymbol{q})|^2} \\
&+\epsilon_{ijk}\int \mathrm{d}^4q \int \mathrm{d}^4q'\, \frac{\mathrm{e}^{\mathrm{i}[(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{\cdot}\boldsymbol{x}-(\omega+\omega')t]}}{(\omega'+\mathrm{i}\eta k'^2)(\omega+\mathrm{i}\nu k^2)}\Big\{\varepsilon\varGamma_{kn}\langle\hat{f}_j(\boldsymbol{q})\hat{u}_n(\boldsymbol{q}')\rangle \\
&-\varepsilon G_{kn}\langle\hat{f}_j(\boldsymbol{q})\hat{b}_n(\boldsymbol{q}')\rangle \\
&+\tfrac{1}{2}k'_m[\langle\hat{f}_j(\boldsymbol{q})\hat{b}_k(\boldsymbol{k}'-\varepsilon\boldsymbol{G}_m,\omega')\rangle-\langle\hat{f}_j(\boldsymbol{q})\hat{b}_k(\boldsymbol{k}'+\varepsilon\boldsymbol{G}_m,\omega')\rangle] \\
&-\tfrac{1}{2}k'_m[\langle\hat{f}_j(\boldsymbol{q})\hat{u}_k(\boldsymbol{k}'-\varepsilon\boldsymbol{\varGamma}_m,\omega')\rangle-\langle\hat{f}_j(\boldsymbol{q})\hat{u}_k(\boldsymbol{k}'+\varepsilon\boldsymbol{\varGamma}_m,\omega')\rangle]\Big\},
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
-\langle u_iu_j\rangle &\approx \int \mathrm{d}^4q \int \mathrm{d}^4q'\, \frac{\mathrm{e}^{\mathrm{i}[(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{\cdot}\boldsymbol{x}-(\omega+\omega')t]}}{\sigma(\boldsymbol{q}')\sigma(\boldsymbol{q})}\Big\{\langle\hat{f}_i(\boldsymbol{q})\hat{f}_j(\boldsymbol{q}')\rangle \\
&\quad-\frac{P_{jk}(\boldsymbol{k}')\langle\hat{f}_i(\boldsymbol{q})\mathcal{R}^{(u)}_k(\boldsymbol{q}')\rangle+P_{ik}(\boldsymbol{k})\langle\hat{f}_j(\boldsymbol{q}')\mathcal{R}^{(u)}_k(\boldsymbol{q})\rangle}{\sigma(\boldsymbol{q}')\sigma(\boldsymbol{q})}\Big\} \\
&\approx -\tfrac{1}{2}D_0\int \frac{\mathrm{d}^4q}{k^{\sigma_0}}\,\frac{P_{ij}(\boldsymbol{k})}{\omega^2+\nu^2k^4} \\
&\quad-\varepsilon G_{kn}\int \mathrm{d}^4q \int \mathrm{d}^4q'\, \frac{P_{jk}(\boldsymbol{k}')\langle\hat{f}_i(\boldsymbol{q})\hat{u}_n(\boldsymbol{q}')\rangle}{(\omega'+\mathrm{i}\nu k'^2)(\omega+\mathrm{i}\nu k^2)}\,\mathrm{e}^{\mathrm{i}[(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{\cdot}\boldsymbol{x}-(\omega+\omega')t]} \\
&\quad-\int \mathrm{d}^4q \int \mathrm{d}^4q'\, \frac{\mathrm{e}^{\mathrm{i}[(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{\cdot}\boldsymbol{x}-(\omega+\omega')t]}P_{jk}(\boldsymbol{k}')k'_m}{2(\omega'+\mathrm{i}\nu k'^2)(\omega+\mathrm{i}\nu k^2)}[\langle\hat{f}_i(\boldsymbol{q})\hat{u}_k(\boldsymbol{k}'-\varepsilon\boldsymbol{G}_m,\omega')\rangle \\
&\quad-\langle\hat{f}_i(\boldsymbol{q})\hat{u}_k(\boldsymbol{k}'+\varepsilon\boldsymbol{G}_m,\omega')\rangle]+(i\leftrightarrow j),
\end{aligned} \tag{3.12}
$$

$$
\langle b_ib_j\rangle \approx 0, \tag{3.13}
$$

where $(i \leftrightarrow j)$ in (3.12) denotes terms of the same structure as all the three previous ones but with exchanged indices i and j. Now we substitute for $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{b}}$ the leading-order terms from (3.9a,b) and make use of (2.12) and $\epsilon_{ijk}P_{jk}=0$ to obtain

$$
\begin{aligned}
\mathcal{E}_i &= 2\eta D_1\langle B\rangle_p \int \frac{\mathrm{d}^4 q}{k^{\sigma_1-2}}\frac{k_p k_i}{(\omega^2+\eta^2k^4)(\omega^2+\nu^2k^4)} \\
&\quad - \eta D_0 \epsilon_{ijk}\frac{\partial\langle B\rangle_k}{\partial x_n}\int \frac{\mathrm{d}^4 q}{k^{\sigma_0-2}}\frac{P_{jn}(\boldsymbol{k})}{(\omega^2+\eta^2k^4)(\omega^2+\nu^2k^4)} \\
&\quad - D_1\int \mathrm{d}^4 q \frac{k_i k_m}{k^{\sigma_1}}\frac{1}{(\omega^2+\nu^2k^4)} \\
&\quad \times\left[\frac{1+\mathrm{i}\varepsilon\boldsymbol{\Gamma}_m\boldsymbol{\cdot}\boldsymbol{x}}{\omega-\mathrm{i}\eta(k^2-2\varepsilon\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{\Gamma}_m)} - \frac{1-\mathrm{i}\varepsilon\boldsymbol{\Gamma}_m\boldsymbol{\cdot}\boldsymbol{x}}{\omega-\mathrm{i}\eta(k^2+2\varepsilon\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{\Gamma}_m)}\right],
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
-\langle u_i u_j\rangle &\approx -\tfrac{1}{2}D_0\int\frac{\mathrm{d}^4 q}{k^{\sigma_0}}\frac{P_{ij}(\boldsymbol{k})}{\omega^2+\nu^2k^4} \\
&\quad + \nu D_0\frac{\partial\langle U\rangle_k}{\partial x_n}\int\frac{\mathrm{d}^4 q}{k^{\sigma_0-2}}\frac{P_{jk}(\boldsymbol{k})P_{in}(\boldsymbol{k})}{(\omega^2+\nu^2k^4)^2} \\
&\quad + \mathrm{i}\tfrac{1}{2}\int \mathrm{d}^4 q\frac{k_m}{\omega^2+\nu^2k^4}\left[\frac{D_0}{k^{\sigma_0}}P_{ik}(\boldsymbol{k}) + \mathrm{i}\frac{D_1}{k^{\sigma_1}}\epsilon_{ikl}k_l\right] \\
&\quad \times\left\{\frac{P_{jk}(\boldsymbol{k}-\varepsilon\boldsymbol{G}_m)(1+\mathrm{i}\varepsilon\boldsymbol{G}_m\boldsymbol{\cdot}\boldsymbol{x})}{\omega-\mathrm{i}\nu(k^2-2\varepsilon\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{G}_m)} - \frac{P_{jk}(\boldsymbol{k}+\varepsilon\boldsymbol{G}_m)(1-\mathrm{i}\varepsilon\boldsymbol{G}_m\boldsymbol{\cdot}\boldsymbol{x})}{\omega-\mathrm{i}\nu(k^2+2\varepsilon\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{G}_m)}\right\} \\
&\quad + (i\leftrightarrow j),
\end{aligned} \tag{3.15}
$$

where we have also made use of the fact that $\omega^2+\eta^2k^4$ and $\omega^2+\nu^2k^4$ are both even functions of ω, which implies

$$\int_{-\infty}^{\infty}\mathrm{d}\omega\frac{\omega}{(\omega^2+\nu^2k^4)(\omega^2+\eta^2k^4)}=0, \quad \int_{-\infty}^{\infty}\mathrm{d}\omega\frac{\omega}{(\omega^2+\nu^2k^4)^2}=0. \tag{3.16a,b}$$

The last line of (3.14) and the last two lines of (3.15), obtained by integration of the Dirac functions of the type $\delta(\boldsymbol{k}+\boldsymbol{k}'-\varepsilon\boldsymbol{G}_m)$ over the $\boldsymbol{k}'$ domain have a non-symmetric integration domain for the wave vector $\boldsymbol{k}$. Therefore, to simplify the calculation we make a substitution $\boldsymbol{k}\mapsto\boldsymbol{k}+\frac{1}{2}\varepsilon\boldsymbol{G}_m$. Hence, making use of

$$\int \mathrm{d}\mathring{\Omega}\underbrace{k_m\ldots k_n k_k}_{N}=0, \quad \text{for any odd } N \text{ and all } m,\ldots,n,k, \tag{3.17a}$$

$$\int k^2\,\mathrm{d}\mathring{\Omega}\frac{k_m k_i}{k^4}=\frac{4\pi}{3}\delta_{mi}, \tag{3.17b}$$

$$\int k^2\,\mathrm{d}\mathring{\Omega}\frac{k_m k_n k_p k_q}{k^6}=\frac{4\pi}{15}(\delta_{mn}\delta_{pq}+\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np}), \tag{3.17c}$$

$$\int_{-\infty}^{+\infty}\mathrm{d}\omega\frac{1}{(\omega'^2+\nu^2k'^4)(\omega'^2+\eta^2k'^4)}=\frac{\pi}{\nu\eta(\nu+\eta)k'^6}, \tag{3.17d}$$

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{(\omega'^2+\nu^2k'^4)^2}=\frac{\pi}{2\nu^3k'^6}, \tag{3.17e}$$

$$\frac{1}{(k'^2 + \varepsilon\, \boldsymbol{k}' \boldsymbol{\cdot} \boldsymbol{G}_m)^{\sigma_0/2}} = \frac{1}{k'^{\sigma_0}} - \frac{\sigma_0 \varepsilon\, \boldsymbol{k}' \boldsymbol{\cdot} \boldsymbol{G}_m}{2k'^{\sigma_0+2}}, \tag{3.17f}$$

$$\frac{1}{\omega^2 + \eta^2 k^4 - 2\varepsilon\eta^2 k^2 \boldsymbol{k} \boldsymbol{\cdot} \boldsymbol{\Gamma}_m} \approx \frac{1}{\omega^2 + \eta^2 k^4} + \frac{2\varepsilon\eta^2 k^2 \boldsymbol{k} \boldsymbol{\cdot} \boldsymbol{\Gamma}_m}{(\omega^2 + \eta^2 k^4)^2}, \tag{3.17g}$$

where $\mathring{\Omega}$ denotes a solid angle and neglecting terms of order $O(\varepsilon^2)$ we get

$$\begin{aligned}
\boldsymbol{\mathcal{E}} &= \frac{8\pi^2 D_1}{3\nu(\nu+\eta)}[\langle \boldsymbol{B}\rangle_0 + (\boldsymbol{x} \boldsymbol{\cdot} \boldsymbol{\nabla})\langle \boldsymbol{B}\rangle] \int_{\Lambda_L}^{\Lambda} \frac{\mathrm{d}k}{k^{\sigma_1}} - \frac{8\pi^2 D_0}{3\nu(\nu+\eta)} \boldsymbol{\nabla} \times \langle \boldsymbol{B}\rangle \int_{\Lambda_L}^{\Lambda} \frac{\mathrm{d}k}{k^{\sigma_0+2}} \\
&= \frac{8\pi^2 D_1}{3\nu(\nu+\eta)} \langle \boldsymbol{B}\rangle \int_{\Lambda_L}^{\Lambda} \frac{\mathrm{d}k}{k^{\sigma_1}} - \frac{8\pi^2 D_0}{3\nu(\nu+\eta)} \boldsymbol{\nabla} \times \langle \boldsymbol{B}\rangle \int_{\Lambda_L}^{\Lambda} \frac{\mathrm{d}k}{k^{\sigma_0+2}} \\
&= \frac{8\pi^2 D_1}{3(\sigma_1 - 1)\nu(\nu+\eta)} \left(\frac{1}{\Lambda_L^{\sigma_1-1}} - \frac{1}{\Lambda^{\sigma_1-1}} \right) \langle \boldsymbol{B}\rangle \\
&\quad - \frac{8\pi^2 D_0}{3(\sigma_0 + 1)\nu(\nu+\eta)} \left(\frac{1}{\Lambda_L^{\sigma_0+1}} - \frac{1}{\Lambda^{\sigma_0+1}} \right) \boldsymbol{\nabla} \times \langle \boldsymbol{B}\rangle \\
&\overset{\Lambda_L \ll \Lambda}{\approx} \frac{8\pi^2 D_1}{3(\sigma_1 - 1)\nu(\nu+\eta)\Lambda_L^{\sigma_1-1}} \langle \boldsymbol{B}\rangle - \frac{8\pi^2 D_0}{3(\sigma_0+1)\nu(\nu+\eta)\Lambda_L^{\sigma_0+1}} \boldsymbol{\nabla} \times \langle \boldsymbol{B}\rangle,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
-\langle u_i u_j\rangle &\approx \frac{14\pi^2 D_0}{15\nu^2} \left(\frac{\partial \langle U\rangle_j}{\partial x_i} + \frac{\partial \langle U\rangle_i}{\partial x_j} \right) \int_{\Lambda_L}^{\Lambda_\nu} \frac{\mathrm{d}k}{k^{\sigma_0+2}} \\
&\quad - \frac{2\pi^2 D_0}{15\nu^2}(2+\sigma_0) \left(\frac{\partial \langle U\rangle_i}{\partial x_j} + \frac{\partial \langle U\rangle_j}{\partial x_i} \right) \int_{\Lambda_L}^{\Lambda_\nu} \frac{\mathrm{d}k}{k^{\sigma_0+2}} + \text{const.} \\
&\quad + \frac{2\pi^2 D_1}{3\nu^2} (\epsilon_{ijm} + \epsilon_{jim}) x_j \frac{\partial \langle U\rangle_m}{\partial x_j} \int \frac{\mathrm{d}k}{k^{\sigma_1}} \\
&\approx \frac{2\pi^2(5-\sigma_0)}{15\nu^2(\sigma_0+1)} D_0 \left(\frac{1}{\Lambda_L^{\sigma_0+1}} - \frac{1}{\Lambda_\nu^{\sigma_0+1}} \right) \left(\frac{\partial \langle U\rangle_i}{\partial x_j} + \frac{\partial \langle U\rangle_j}{\partial x_i} \right) + \text{const.} \\
&\overset{\Lambda_L \ll \Lambda}{\approx} \frac{2\pi^2(5-\sigma_0)}{15\nu^2(\sigma_0+1)} \frac{D_0}{\Lambda_L^{\sigma_0+1}} \left(\frac{\partial \langle U\rangle_i}{\partial x_j} + \frac{\partial \langle U\rangle_j}{\partial x_i} \right) + \text{const.}
\end{aligned} \tag{3.19}$$

In the above we have also used

$$P_{jk}\left(\boldsymbol{k} - \tfrac{1}{2}\varepsilon \boldsymbol{G}_m\right) = P_{jk}(\boldsymbol{k}) - \varepsilon \left(\frac{G_{mp} k_p k_j k_k}{k^4} - \frac{k_j G_{mk} + k_k G_{mj}}{2k^2} \right), \tag{3.20}$$

and $\Lambda = \min(\Lambda_\nu, \Lambda_\eta)$; in principle, both cutoffs can be infinite. Therefore, in a weak turbulence with a weak mean magnetic field and in the limit $\Lambda_L \ll \Lambda$ the turbulent magnetic diffusivity and viscosity and the so-called $\bar{\alpha}$-coefficient defined as

$$\boldsymbol{\mathcal{E}} = \bar{\alpha} \langle \boldsymbol{B}\rangle - (\bar{\eta} - \eta) \boldsymbol{\nabla} \times \langle \boldsymbol{B}\rangle, \tag{3.21}$$

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take the form

$$\bar{\eta} = \eta \left[1 + \frac{8\pi^2 D_0}{3(\sigma_0 + 1)\nu\eta(\nu + \eta)\Lambda_L^{\sigma_0+1}} \right] \approx \frac{8\pi^2 D_0}{3(\sigma_0 + 1)\nu(\nu + \eta)\Lambda_L^{\sigma_0+1}}, \tag{3.22}$$

$$\bar{\nu} = \nu \left[1 + \frac{2\pi^2(5 - \sigma_0)D_0}{15\nu^3(\sigma_0 + 1)\Lambda_L^{\sigma_0+1}} \right] \approx \frac{2\pi^2(5 - \sigma_0)D_0}{15\nu^2(\sigma_0 + 1)\Lambda_L^{\sigma_0+1}}, \tag{3.23}$$

$$\bar{\alpha} = \frac{8\pi^2 D_1}{3(\sigma_1 - 1)\nu(\nu + \eta)\Lambda_L^{\sigma_1-1}}. \tag{3.24}$$

It follows that models of weak turbulence with a forcing for which the correlations are defined with $\sigma_0 > 5$ are unphysical, since they lead to negative turbulent diffusion.

4. Renormalization procedure of the MHD equations

Let us introduce the non-dimensional variables

$$\boldsymbol{f} \mapsto \mathcal{F}\boldsymbol{f}, \quad \boldsymbol{U} \mapsto \mathcal{U}\boldsymbol{U}, \quad \boldsymbol{b} \mapsto \mathcal{B}\boldsymbol{b}, \quad \langle \boldsymbol{B} \rangle \mapsto B_0 \langle \boldsymbol{B} \rangle, \quad \Pi \mapsto \mathcal{L}\mathcal{F}\Pi, \tag{4.1a}$$

$$\boldsymbol{x} \mapsto \mathcal{L}\boldsymbol{x}, \quad t \mapsto \frac{\mathcal{U}}{\mathcal{F}} t, \tag{4.1b}$$

$$D_0 \mapsto \frac{\mathcal{F}\mathcal{U}}{\mathcal{L}^{\sigma_0-3}} D_0, \quad D_1 \mapsto \frac{\mathcal{F}\mathcal{U}}{\mathcal{L}^{\sigma_1-4}} D_1, \tag{4.1c}$$

where $\mathcal{L}$ is some length scale of variation of the fluctuations,

$$\frac{2\pi}{\Lambda_L} \leq \mathcal{L} \leq \frac{2\pi}{\Lambda}, \quad \Lambda = min(\Lambda_{\nu,} \Lambda_\eta), \tag{4.2}$$

and the scales $\mathcal{U}$ and $\mathcal{B}$ are defined by the norm of the initial velocity fluctuation, i.e. $\mathcal{B} \sim \mathcal{U} = \|\boldsymbol{u}(t = 0)\|_{L^2}$; in an analogous way we define $B_0 = \|\langle \boldsymbol{B} \rangle (t = 0)\|_{L^2}$ and the characteristic length scale of variation of the mean fields will be denoted by

$$L \lesssim \frac{2\pi}{\kappa_m}. \tag{4.3}$$

Physically, we can associate the scale of the stirring force $\mathcal{F} = \sqrt{\omega_s D_0 \mathcal{L}^{\sigma_0-3}}$, where ω_s is the scale of the fluctuational frequencies, with the magnitude of the driving force, thus, for example, the buoyancy force. On the other hand, the helical part proportional to D_1 can be associated with the Coriolis force, which in natural systems is typically responsible for introducing a lack of reflectional symmetry, thus $2\Omega\mathcal{U} = \sqrt{\omega_s D_1 \mathcal{L}^{\sigma_1-4}}$, where Ω denotes the magnitude of the background rotation. The aim is to try to mimic in the simplest way the turbulence in astrophysical systems. Of course, this is a great simplification, because the Coriolis force introduces anisotropy (one distinguished axis of rotation), which is neglected here in order to obtain analytic results for strong isotropic turbulence. A study of the α-effect in non-isotropic rapidly rotating turbulence via the renormalization theory has been done in Mizerski (2021), in an effectively weakly nonlinear limit.

We introduce the following non-dimensional parameters:

$$Ro = \frac{\mathcal{U}^2}{\mathcal{L}\mathcal{F}}, \quad E_\nu = \frac{\nu\mathcal{U}}{\mathcal{L}^2\mathcal{F}}, \quad E_\eta = \frac{\eta\mathcal{U}}{\mathcal{L}^2\mathcal{F}}, \quad \beta = \frac{B_0}{\mathcal{B}}, \tag{4.4a–d}$$

and

$$H = \frac{\mathcal{B}^2}{\mathcal{U}^2}, \quad \varepsilon = \frac{\mathcal{L}}{L}. \tag{4.5a,b}$$

Defining

$$\boldsymbol{X} = \varepsilon \boldsymbol{x}, \quad \bar{\nabla} = \nabla_X, \tag{4.6a,b}$$

and assuming

$$\bar{\nabla}\langle \boldsymbol{U}\rangle \sim \varepsilon\frac{\mathcal{U}}{\mathcal{L}}, \quad \bar{\nabla}\langle \boldsymbol{B}\rangle \sim \varepsilon\frac{\mathcal{B}}{\mathcal{L}} = \frac{\varepsilon}{\beta}\frac{B_0}{\mathcal{L}}, \tag{4.7a,b}$$

we can rewrite the dynamical equations in non-dimensional form

$$\frac{\partial\langle \boldsymbol{U}\rangle}{\partial t} + \varepsilon Ro(\langle \boldsymbol{U}\rangle \boldsymbol{\cdot} \bar{\nabla})\langle \boldsymbol{U}\rangle = -\varepsilon\bar{\nabla}\langle \Pi\rangle + \varepsilon Ro\beta H(\langle \boldsymbol{B}\rangle \boldsymbol{\cdot} \bar{\nabla})\langle \boldsymbol{B}\rangle + \varepsilon^2 E_\nu \bar{\nabla}^2\langle \boldsymbol{U}\rangle \\ - \varepsilon Ro\bar{\nabla} \boldsymbol{\cdot} (\langle \boldsymbol{uu}\rangle - H\langle \boldsymbol{bb}\rangle), \tag{4.8a}$$

$$\frac{\partial\langle \boldsymbol{B}\rangle}{\partial t} = \varepsilon Ro(\langle \boldsymbol{B}\rangle \boldsymbol{\cdot} \bar{\nabla})\langle \boldsymbol{U}\rangle - \varepsilon\frac{Ro}{\beta}(\langle \boldsymbol{U}\rangle \boldsymbol{\cdot} \bar{\nabla})\langle \boldsymbol{B}\rangle + \varepsilon Ro\beta^{-1}\bar{\nabla} \times \langle \boldsymbol{u} \times \boldsymbol{b}\rangle + \frac{\varepsilon^2}{\beta}E_\eta\bar{\nabla}^2\langle \boldsymbol{B}\rangle, \tag{4.8b}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} - E_\nu\nabla^2\boldsymbol{u} + Ro(\langle \boldsymbol{U}\rangle \boldsymbol{\cdot} \nabla)\boldsymbol{u} - Ro\beta H(\langle \boldsymbol{B}\rangle \boldsymbol{\cdot} \nabla)\boldsymbol{b} + \nabla\Pi' = \boldsymbol{f} - Ro\nabla \boldsymbol{\cdot} (\boldsymbol{uu} - H\boldsymbol{bb}) \\ + Ro\varepsilon\bar{\nabla} \boldsymbol{\cdot} (\langle \boldsymbol{uu}\rangle - H\langle \boldsymbol{bb}\rangle) - Ro\varepsilon(\boldsymbol{u} \boldsymbol{\cdot} \bar{\nabla})\langle \boldsymbol{U}\rangle + RoH\varepsilon(\boldsymbol{b} \boldsymbol{\cdot} \bar{\nabla})\langle \boldsymbol{B}\rangle, \tag{4.8c}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} - E_\eta\nabla^2\boldsymbol{b} + Ro(\langle \boldsymbol{U}\rangle \boldsymbol{\cdot} \nabla)\boldsymbol{b} = Ro\beta(\langle \boldsymbol{B}\rangle \boldsymbol{\cdot} \nabla)\boldsymbol{u} + Ro\nabla \times (\boldsymbol{u} \times \boldsymbol{b}) - Ro\varepsilon\bar{\nabla} \times \langle \boldsymbol{u} \times \boldsymbol{b}\rangle \\ + Ro\varepsilon(\boldsymbol{b} \boldsymbol{\cdot} \bar{\nabla})\langle \boldsymbol{U}\rangle - Ro\varepsilon(\boldsymbol{u} \boldsymbol{\cdot} \bar{\nabla})\langle \boldsymbol{B}\rangle, \tag{4.8d}$$

$$\nabla \boldsymbol{\cdot} \langle \boldsymbol{B}\rangle = 0, \quad \nabla \boldsymbol{\cdot} \langle \boldsymbol{U}\rangle = 0, \quad \nabla \boldsymbol{\cdot} \boldsymbol{b} = 0, \quad \nabla \boldsymbol{\cdot} \boldsymbol{u} = 0. \tag{4.8e}$$

Note that $H \neq 0$ is associated with the presence of the Lorentz force, i.e. when $H = 0$ the dynamo problem becomes kinematic. We stress here that the turbulent fluctuations are defined in such a way that they do not depend on the slow variable $\boldsymbol{X}$ since spatial scale separation has been assumed. Of course, the Reynolds and Maxwell stresses as nonlinear quantities do depend on the slow variable. For the formal two-scale direct-interaction approach see Yoshizawa (1998) and Yokoi (2020), where the dependencies on slow and fast variables are formally treated with the multiple-scale asymptotic method. We now introduce the following assumptions:

$$Ro \ll 1, \quad 1 \ll \beta \ll Ro^{-1}, \tag{4.9a}$$

$$\frac{\kappa_m}{\Lambda_L} = O(Ro^2), \Rightarrow \varepsilon = O(Ro^2), \tag{4.9b}$$

and

$$H = O(1), \tag{4.10}$$

where the assumption (4.9*b*) allows us to retain the effect of weak gradients of means $\nabla \langle \boldsymbol{U} \rangle$ and $\nabla \langle \boldsymbol{B} \rangle$ on the fluctuations at the highest order. Introducing a new shorter notation

$$\boldsymbol{q} = (\boldsymbol{k}, \omega), \quad \int_{\Lambda_L}^{\Lambda_i} \mathrm{d}^3 k \int_{-\infty}^{\infty} \mathrm{d}\omega\,(\boldsymbol{\cdot}) = \int^{\Lambda_i} \mathrm{d}^4 q(\boldsymbol{\cdot}), \tag{4.11a,b}$$

so that e.g.

$$u_i(\boldsymbol{x}, t) = \int^{\Lambda_\nu} \mathrm{d}^4 q \hat{u}_i(\boldsymbol{q})\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x} - \omega t)}, \tag{4.12}$$

and utilizing ((3.2*a*,*b*)–(3.5)) the equations for the fluctuations take the form

$$\begin{aligned}(-\mathrm{i}\omega + E_\nu k^2)\hat{u}_i(\boldsymbol{q}) - \mathrm{i}Ro\beta H(\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle_0)\hat{b}_i(\boldsymbol{q}) + \mathrm{i}k_i\hat{\varPi} &= \hat{f}_i(\boldsymbol{q}) - \mathrm{i}Rok_j[\mathbb{I}_{ij}^{(u)} - H\mathbb{I}_{ij}^{(b)}] \\ &+ \mathrm{i}Rok_j[\langle \mathbb{I}_{ij}^{(u)}\rangle - H\langle \mathbb{I}_{ij}^{(b)}\rangle] - Ro\varepsilon G_{ij}\hat{u}_j(\boldsymbol{q}) + RoH\varepsilon \varGamma_{ij}\hat{b}_j(\boldsymbol{q}) \\ &- \frac{Ro}{2}k_m[\hat{u}_i(\boldsymbol{k} - \varepsilon \boldsymbol{G}_m, \omega) - \hat{u}_i(\boldsymbol{k} + \varepsilon \boldsymbol{G}_m, \omega)] \\ &+ \frac{Ro}{2}k_m[\hat{b}_i(\boldsymbol{k} - \varepsilon \boldsymbol{\varGamma}_m, \omega) - \hat{b}_i(\boldsymbol{k} + \varepsilon \boldsymbol{\varGamma}_m, \omega)],\end{aligned} \tag{4.13a}$$

$$\begin{aligned}(-\mathrm{i}\omega + E_\eta k^2)\hat{b}_i(\boldsymbol{q}) &= \mathrm{i}Ro\beta(\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle_0)\hat{u}_i(\boldsymbol{q}) + \mathrm{i}Rok_j\mathbb{I}_{ij}^{(ub)} - \mathrm{i}Rok_j\langle \mathbb{I}_{ij}^{(ub)}\rangle \\ &+ Ro\varepsilon G_{ij}\hat{b}_j(\boldsymbol{q}) - Ro\varepsilon \varGamma_{ij}\hat{u}_j(\boldsymbol{q}) \\ &- \frac{Ro}{2}k_m[\hat{b}_i(\boldsymbol{k} - \varepsilon \boldsymbol{G}_m, \omega) - \hat{b}_i(\boldsymbol{k} + \varepsilon \boldsymbol{G}_m, \omega)] \\ &+ \frac{Ro}{2}k_m[\hat{u}_i(\boldsymbol{k} - \varepsilon \boldsymbol{\varGamma}_m, \omega) - \hat{u}_i(\boldsymbol{k} + \varepsilon \boldsymbol{\varGamma}_m, \omega)],\end{aligned} \tag{4.13b}$$

$$\boldsymbol{k}\boldsymbol{\cdot}\hat{\boldsymbol{u}}(\boldsymbol{q}) = 0, \quad \boldsymbol{k}\boldsymbol{\cdot}\hat{\boldsymbol{b}}(\boldsymbol{q}) = 0, \quad \boldsymbol{k}\boldsymbol{\cdot}\hat{\boldsymbol{f}}(\boldsymbol{q}) = 0, \tag{4.13c}$$

where the constant term $-Ro\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{U}\rangle_0$ has been absorbed into the frequency ω (as in the previous section) and

$$\mathbb{I}_{ij}^{(u)} = \theta_{\Lambda_\nu}\int^{\Lambda_\nu} \mathrm{d}^4 q' \hat{u}_i(\boldsymbol{q} - \boldsymbol{q}')\hat{u}_j(\boldsymbol{q}'), \quad \mathbb{I}_{ij}^{(b)} = \theta_{\Lambda_\eta}\int^{\Lambda_\eta} \mathrm{d}^4 q' \hat{b}_i(\boldsymbol{q} - \boldsymbol{q}')\hat{b}_j(\boldsymbol{q}'), \tag{4.14a,b}$$

$$\mathbb{I}_{ij}^{(ub)} = \theta_\Lambda \epsilon_{ijk}\epsilon_{kmn}\int^{\Lambda} \mathrm{d}^4 q' \hat{u}_m(\boldsymbol{q} - \boldsymbol{q}')\hat{b}_n(\boldsymbol{q}'), \tag{4.15}$$

with

$$\Lambda = min(\Lambda_\nu, \Lambda_\eta), \quad \theta_{\Lambda_i} = \theta(\Lambda_i - k). \tag{4.16a,b}$$

The convolution integrals possess the following symmetry properties:

$$\mathbb{I}_{ij}^{(u)} = \mathbb{I}_{ji}^{(u)}, \quad \mathbb{I}_{ij}^{(b)} = \mathbb{I}_{ji}^{(b)}, \quad \mathbb{I}_{ij}^{(ub)} = -\mathbb{I}_{ji}^{(ub)}. \tag{4.17a–c}$$

Factors θ were added for clarity in the following calculations. It should be noted at this stage that the convolution integrals, which represent the nonlinear interactions between fluctuating turbulent fields, are not neglected in the evolution equations for the fluctuations (4.13a,b), and their role is crucial. In fact, it is the aim of the entire renormalization procedure to go beyond the weak turbulence regime and quantitatively express the effect of those nonlinearities on the dynamics of the mean fields. Therefore, contrary to the findings of the previous section, where the MHD turbulence was studied in the weak regime (i.e. under neglection of $\mathbb{I}_{ij}^{(u)}$, $\mathbb{I}_{ij}^{(b)}$ and $\mathbb{I}_{ij}^{(ub)}$ in (4.13a,b)), we will now investigate the nonlinear evolution of the turbulent fluctuations and its effect on the dynamics of mean fields. This corresponds to a regime when the turbulence is no longer weak and, although we have assumed $Ro \ll 1$, we will refer to the studied regime as the 'strong-turbulence' regime, to clearly distinguish from the simplest case of a weak, linear turbulence.

In order to eliminate pressure we apply the projection operator (2.13) to both sides of the Navier–Stokes equation (4.13a) and, after some simple algebra, we get more explicit expressions for $\hat{u}_i(\boldsymbol{q})$ and $\hat{b}_i(\boldsymbol{q})$

$$
\begin{aligned}
\hat{u}_i(\boldsymbol{q}) = {} & \frac{1}{\gamma_u}\hat{f}_i(\boldsymbol{q}) - \tfrac{1}{2}\mathrm{i}Ro\frac{P_{imn}(\boldsymbol{k})}{\gamma_u}[\mathbb{I}_{mn}^{(u)} - H\mathbb{I}_{mn}^{(b)} - \langle\mathbb{I}_{mn}^{(u)}\rangle + H\langle\mathbb{I}_{mn}^{(b)}\rangle] \\
& - Ro(Ro\beta)H\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle_0}{\gamma_u\gamma_\eta}k_j[\mathbb{I}_{ij}^{(ub)} - \langle\mathbb{I}_{ij}^{(ub)}\rangle] \\
& - Ro\varepsilon\frac{P_{ij}(\boldsymbol{k})}{\gamma_u}G_{jk}\hat{u}_k(\boldsymbol{q}) + RoH\varepsilon\frac{P_{ij}(\boldsymbol{k})}{\gamma_u}\varGamma_{jk}\hat{b}_k(\boldsymbol{q}) \\
& + \mathrm{i}Ro(Ro\beta)H\varepsilon\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle_0}{\gamma_u\gamma_\eta}G_{ij}\hat{b}_j(\boldsymbol{q}) - \mathrm{i}Ro(Ro\beta)H\varepsilon\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle_0}{\gamma_u\gamma_\eta}\varGamma_{ij}\hat{u}_j(\boldsymbol{q}) \\
& - \frac{Ro}{2\gamma_u}k_mP_{ij}(\boldsymbol{k})[\hat{u}_j(\boldsymbol{k}-\varepsilon\boldsymbol{G}_m,\omega) - \hat{u}_j(\boldsymbol{k}+\varepsilon\boldsymbol{G}_m,\omega)] \\
& + \frac{Ro}{2\gamma_u}k_mP_{ij}(\boldsymbol{k})[\hat{b}_j(\boldsymbol{k}-\varepsilon\boldsymbol{\varGamma}_m,\omega) - \hat{b}_j(\boldsymbol{k}+\varepsilon\boldsymbol{\varGamma}_m,\omega)] \\
& - \mathrm{i}Ro(Ro\beta)H\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle_0}{2\gamma_u\gamma_\eta}k_m[\hat{b}_i(\boldsymbol{k}-\varepsilon\boldsymbol{G}_m,\omega) - \hat{b}_i(\boldsymbol{k}+\varepsilon\boldsymbol{G}_m,\omega)] \\
& + \mathrm{i}Ro(Ro\beta)H\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle_0}{2\gamma_u\gamma_\eta}k_m[\hat{u}_i(\boldsymbol{k}-\varepsilon\boldsymbol{\varGamma}_m,\omega) - \hat{u}_i(\boldsymbol{k}+\varepsilon\boldsymbol{\varGamma}_m,\omega)],
\end{aligned} \tag{4.18a}
$$

$$
\begin{aligned}
\hat{b}_i(\boldsymbol{q}) = {} & \mathrm{i}Ro\beta\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle_0}{\gamma_\eta}\hat{u}_i(\boldsymbol{q}) + \mathrm{i}\frac{Ro}{\gamma_\eta}k_j[\mathbb{I}_{ij}^{(ub)} - \langle\mathbb{I}_{ij}^{(ub)}\rangle] \\
& + \frac{Ro}{\gamma_\eta}\varepsilon G_{ij}\hat{b}_j(\boldsymbol{q}) - \frac{Ro}{\gamma_\eta}\varepsilon\varGamma_{ij}\hat{u}_j(\boldsymbol{q}) \\
& - \frac{Ro}{2\gamma_\eta}k_m[\hat{b}_i(\boldsymbol{k}-\varepsilon\boldsymbol{G}_m,\omega) - \hat{b}_i(\boldsymbol{k}+\varepsilon\boldsymbol{G}_m,\omega)] \\
& + \frac{Ro}{2\gamma_\eta}k_m[\hat{u}_i(\boldsymbol{k}-\varepsilon\boldsymbol{\varGamma}_m,\omega) - \hat{u}_i(\boldsymbol{k}+\varepsilon\boldsymbol{\varGamma}_m,\omega)],
\end{aligned} \tag{4.18b}
$$

where

$$
P_{imn}(\boldsymbol{k}) = k_mP_{in}(\boldsymbol{k}) + k_nP_{im}(\boldsymbol{k}), \tag{4.19}
$$

$$\gamma_u = \gamma_\nu + H(Ro\beta)^2 \frac{(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2}{\gamma_\eta}, \quad \gamma_\nu = -\mathrm{i}\omega + E_\nu k^2, \quad \gamma_\eta = -\mathrm{i}\omega + E_\eta k^2. \tag{4.20}$$

The following simple transformation allows us to return to original dimensional variables

$$E_\nu \to \nu, \quad E_\eta \to \eta, \quad H \to 1, \quad Ro \to 1, \quad \beta \to 1. \tag{4.21a–e}$$

In the non-dimensional variables the isotropic, homogeneous and stationary forcing is still defined by the same formula

$$\langle \hat{f}_i(\boldsymbol{k},\omega)\hat{f}_j(\boldsymbol{k}',\omega')\rangle = \left[\frac{D_0}{k^{\sigma_0}}P_{ij}(\boldsymbol{k}) + \mathrm{i}\frac{D_1}{k^{\sigma_1}}\epsilon_{ijk}k_k\right]\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(\omega+\omega'). \tag{4.22}$$

(Of course when we return to dimensional variables $D_0 \to D_0$, $D_1 \to D_1$.)

We can now comment on the second assumption in (4.9*a*), which states that the mean field is much stronger than the fluctuating one, but $Ro\beta \ll 1$. The former allows for proper formulation of the problem, since, for a strong mean field, the fluctuating magnetic field (4.18*b*) is expressed at leading order by the stirring force and the nonlinearities, which are of the order $O(Ro)$, and can be treated in a perturbational sense. The iterational procedure of renormalization is then applicable. On the other hand, the assumption $Ro\beta \ll 1$ allows for expansion of factors such as γ_u^{-1} and therefore explicit calculation of Fourier integrals in the nonlinear $\mathbb{I}$-terms. In turn, the final recursion differential relations for the coefficients describing the renormalized EMF, the Reynolds stresses and the Lorentz force can be solved analytically; thus, in particular, the full leading-order form of the turbulent diffusivities and the α-coefficient can be determined.

We now start the renormalization procedure of taking successive little bites off the Fourier spectrum from the short-wavelength side in order to obtain the final nonlinear effect of the fluctuations on the means. At the first step of the procedure we introduce a parameter λ_1, which satisfies

$$\delta\lambda = \Lambda_{max} - \lambda_1 \ll 1, \quad \text{where } \Lambda_{max} = \max(\Lambda_\nu, \Lambda_\eta), \tag{4.23}$$

and divide the Fourier spectrum into two parts

$$\hat{u}_i^{>}(\boldsymbol{k},\omega) = \theta(k-\lambda_1)\hat{u}_i(\boldsymbol{k},\omega), \text{ or } \hat{u}_i^{>}(\boldsymbol{k},\omega) = \hat{u}_i(\boldsymbol{k}^{>},\omega), \quad \lambda_1 < |\boldsymbol{k}^{>}| < \Lambda_{max}, \tag{4.24}$$

$$\hat{u}_i^{<}(\boldsymbol{k},\omega) = \theta(\lambda_1-k)\hat{u}_i(\boldsymbol{k},\omega), \text{ or } \hat{u}_i^{<}(\boldsymbol{k},\omega) = \hat{u}_i(\boldsymbol{k}^{<},\omega), \quad |\boldsymbol{k}^{<}| < \lambda_1, \tag{4.25}$$

and the same for $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{f}}$. The equations for the fields $\hat{u}_i^{<}(\boldsymbol{k},\omega)$ and $\hat{b}_i^{<}(\boldsymbol{k},\omega)$ are obtained by projecting (4.13*a*) onto the direction perpendicular to $\boldsymbol{k}$ (with the use of (2.13)), and averaging both $\boldsymbol{P}(\boldsymbol{k})\cdot$ (4.13*a*) and (4.13*b*) over the first shell ($\lambda_1 < k < \Lambda_{max}$)

$$\begin{aligned}
(-\mathrm{i}\omega + E_\nu k^2)\hat{u}_i^{<}(\boldsymbol{q}) = {} & \hat{f}_i^{<}(\boldsymbol{q}) + \mathrm{i}Ro\beta H(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)\hat{b}_i^{<}(\boldsymbol{q}) \\
& - \tfrac{1}{2}\mathrm{i}RoP_{imn}(\boldsymbol{k})[\mathbb{I}_{mn}^{(u<)} - H\mathbb{I}_{mn}^{(b<)} - \langle\mathbb{I}_{mn}^{(u<)}\rangle + H\langle\mathbb{I}_{mn}^{(b<)}\rangle] \\
& - \tfrac{1}{2}\mathrm{i}RoP_{imn}(\boldsymbol{k})\left\{\theta_{\Lambda_\nu}\int^{\Lambda_\nu} \mathrm{d}^4q'[\langle\hat{u}_m^{>}(\boldsymbol{q}')\hat{u}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c\right. \\
& -\langle\hat{u}_m^{>}(\boldsymbol{q}')\hat{u}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle] \\
& \left. - H\theta_{\Lambda_\eta}\int^{\Lambda_\eta} \mathrm{d}^4q'[\langle\hat{b}_m^{>}(\boldsymbol{q}')\hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle\hat{b}_m^{>}(\boldsymbol{q}')\hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle]\right\}
\end{aligned}$$

$$
\begin{aligned}
&- Ro\varepsilon P_{ij}(\boldsymbol{k}) G_{jk}\hat{u}_k^<(\boldsymbol{q}) + RoH\varepsilon P_{ij}(\boldsymbol{k})\Gamma_{jk}\hat{b}_k^<(\boldsymbol{q}) \\
&- \frac{Ro}{2} P_{ij}(\boldsymbol{k}) k_m [\hat{u}_j(\boldsymbol{k}^< - \varepsilon \boldsymbol{G}_m, \omega) - \hat{u}_j(\boldsymbol{k}^< + \varepsilon \boldsymbol{G}_m, \omega)] \\
&+ \frac{Ro}{2} P_{ij}(\boldsymbol{k}) k_m [\hat{b}_j(\boldsymbol{k}^< - \varepsilon \boldsymbol{\Gamma}_m, \omega) - \hat{b}_j(\boldsymbol{k}^< + \varepsilon \boldsymbol{\Gamma}_m, \omega)]
\end{aligned} \tag{4.26a}
$$

$$
\begin{aligned}
(-\mathrm{i}\omega + E_\eta k^2)\hat{b}_i^<(\boldsymbol{q}) &= \mathrm{i}Ro\beta(\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle_0)\hat{u}_i^<(\boldsymbol{q}) + \mathrm{i}Rok_j[\mathbb{I}_{ij}^{(u^<b^<)} - \langle \mathbb{I}_{ij}^{(u^<b^<)}\rangle] \\
&+ \mathrm{i}Rok_j\epsilon_{ijk}\epsilon_{kmn}\int^{\Lambda} \mathrm{d}^4q'[\langle \hat{u}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle_c \\
&- \langle \hat{u}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle] + Ro\varepsilon G_{ij}\hat{b}_j^<(\boldsymbol{q}) - Ro\varepsilon\Gamma_{ij}\hat{u}_j^<(\boldsymbol{q}) \\
&- \frac{Ro}{2} k_m [\hat{b}_i(\boldsymbol{k}^< - \varepsilon \boldsymbol{G}_m, \omega) - \hat{b}_i(\boldsymbol{k}^< + \varepsilon \boldsymbol{G}_m, \omega)] \\
&+ \frac{Ro}{2} k_m [\hat{u}_i(\boldsymbol{k}^< - \varepsilon \boldsymbol{\Gamma}_m, \omega) - \hat{u}_i(\boldsymbol{k}^< + \varepsilon \boldsymbol{\Gamma}_m, \omega)].
\end{aligned} \tag{4.26b}
$$

On the other hand, to get equations for $\hat{u}_i^>(\boldsymbol{k},\omega)$ and $\hat{b}_i^>(\boldsymbol{k},\omega)$ we utilize (4.18*a*,*b*), however, we substitute explicitly for $\hat{u}_i(\boldsymbol{q})$ from (4.18*a*) into the equation for the fluctuating magnetic field (4.18*b*) but neglect terms of the order $o(Ro^3)$ and $O(Ro^2(Ro\beta)^2)$

$$
\begin{aligned}
\hat{u}_i^>(\boldsymbol{q}) &= \frac{1}{\gamma_u}\hat{f}_i^>(\boldsymbol{q}) - \tfrac{1}{2}\mathrm{i}Ro\frac{P_{imn}(\boldsymbol{k})}{\gamma_u}[\mathbb{I}_{mn}^{(u^<)} - H\mathbb{I}_{mn}^{(b^<)} - \langle \mathbb{I}_{mn}^{(u^<)}\rangle + H\langle \mathbb{I}_{mn}^{(b^<)}\rangle] \\
&- HRo(Ro\beta)\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle_0}{\gamma_u\gamma_\eta}k_j[\mathbb{I}_{ij}^{(u^<b^<)} - \langle \mathbb{I}_{ij}^{(u^<b^<)}\rangle] \\
&- Ro\varepsilon\frac{P_{ij}(\boldsymbol{k})}{\gamma_u^2}G_{jk}\hat{f}_k^>(\boldsymbol{q}) \\
&- \frac{Ro}{2\gamma_u}k_m P_{ij}(\boldsymbol{k})\left[\frac{\hat{f}_j(\boldsymbol{k}^> - \varepsilon\boldsymbol{G}_m,\omega)}{\gamma_u(\boldsymbol{k}^> - \varepsilon\boldsymbol{G}_m,\omega)} - \frac{\hat{f}_j(\boldsymbol{k}^> + \varepsilon\boldsymbol{G}_m,\omega)}{\gamma_u(\boldsymbol{k}^> + \varepsilon\boldsymbol{G}_m,\omega)}\right] \\
&- \mathrm{i}\frac{Ro}{\gamma_u}P_{imn}[\mathbb{J}_{mn}^{(u)} - H\mathbb{J}_{mn}^{(b)}] - HRo(Ro\beta)\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle_0}{\gamma_u\gamma_\eta}k_j\mathbb{J}_{ij}^{(ub)} + R_i^{(u)},
\end{aligned} \tag{4.27a}
$$

$$
\begin{aligned}
\hat{b}_i^>(\boldsymbol{q}) &= \mathrm{i}Ro\beta\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle_0}{\gamma_u\gamma_\eta}\hat{f}_i^>(\boldsymbol{q}) + \mathrm{i}\frac{Ro}{\gamma_\eta}k_j[\mathbb{I}_{ij}^{(u^<b^<)} - \langle \mathbb{I}_{ij}^{(u^<b^<)}\rangle] \\
&+ \tfrac{1}{2}Ro(Ro\beta)\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle_0}{\gamma_u\gamma_\eta}P_{imn}(\boldsymbol{k})[\mathbb{I}_{mn}^{(u^<)} - H\mathbb{I}_{mn}^{(b^<)} - \langle \mathbb{I}_{mn}^{(u^<)}\rangle + H\langle \mathbb{I}_{mn}^{(b^<)}\rangle] \\
&- \frac{Ro}{\gamma_u\gamma_\eta}\varepsilon\Gamma_{ij}\hat{f}_j^>(\boldsymbol{q}) + \frac{Ro}{2\gamma_\eta}k_m\left[\frac{\hat{f}_i(\boldsymbol{k}^> - \varepsilon\boldsymbol{\Gamma}_m,\omega)}{\gamma_u(\boldsymbol{k}^> - \varepsilon\boldsymbol{\Gamma}_m,\omega)} - \frac{\hat{f}_i(\boldsymbol{k}^> + \varepsilon\boldsymbol{\Gamma}_m,\omega)}{\gamma_u(\boldsymbol{k}^> + \varepsilon\boldsymbol{\Gamma}_m,\omega)}\right] \\
&+ \mathrm{i}\frac{Ro}{\gamma_\eta}k_j\mathbb{J}_{ij}^{(ub)} + Ro(Ro\beta)\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle \boldsymbol{B}\rangle_0}{\gamma_u\gamma_\eta}P_{imn}[\mathbb{J}_{mn}^{(u)} - H\mathbb{J}_{mn}^{(b)}] + R_i^{(b)},
\end{aligned} \tag{4.27b}
$$

where $\langle\boldsymbol{\cdot}\rangle_c$ denotes the conditional average over the first shell ($\lambda_1 < k < \Lambda_{max}$) statistical subensemble, described at the beginning of Appendix A (cf. McComb *et al.* 1992 and

McComb & Watt 1990, 1992); furthermore, we have defined

$$\mathbb{J}_{mn}^{(u)}(\boldsymbol{q}) = \theta_{\Lambda_\nu} \int^{\Lambda_\nu} \mathrm{d}^4 q' \hat{u}_m^{<}(\boldsymbol{q}') \hat{u}_n^{>}(\boldsymbol{q}-\boldsymbol{q}'), \tag{4.28a}$$

$$\mathbb{J}_{mn}^{(b)}(\boldsymbol{q}) = \theta_{\Lambda_\eta} \int^{\Lambda_\eta} \mathrm{d}^4 q' \hat{b}_m^{<}(\boldsymbol{q}') \hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}'), \tag{4.28b}$$

$$\mathbb{J}_{ij}^{(ub)}(\boldsymbol{q}) = \epsilon_{ijk}\epsilon_{kmn} \int^{\Lambda} \mathrm{d}^4 q' [\hat{u}_m^{<}(\boldsymbol{q}') \hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}') + \hat{u}_m^{>}(\boldsymbol{q}') \hat{b}_n^{<}(\boldsymbol{q}-\boldsymbol{q}')], \tag{4.28c}$$

and the rests in (4.27*a*,*b*) are given by

$$\begin{aligned} R_i^{(u)} &= -\frac{Ro}{\gamma_u}\left\{\tfrac{1}{2}\mathrm{i}P_{imn}[\mathbb{I}_{mn}^{(u^>)} - H\mathbb{I}_{mn}^{(b^>)}] + HRo\beta\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle}{\gamma_\eta}k_j\mathbb{I}_{ij}^{(u^>b^>)}\right\} \\ &\quad + \frac{Ro}{\gamma_u}\left\{\tfrac{1}{2}\mathrm{i}P_{imn}[\langle\mathbb{I}_{mn}^{(u^>)}\rangle - H\langle\mathbb{I}_{mn}^{(b^>)}\rangle] + HRo\beta\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle}{\gamma_\eta}k_j\langle\mathbb{I}_{ij}^{(u^>b^>)}\rangle\right\}, \end{aligned} \tag{4.29a}$$

$$\begin{aligned} R_i^{(b)} &= \mathrm{i}\frac{Ro}{\gamma_\eta}k_j[\mathbb{I}_{ij}^{(u^>b^>)} - \langle\mathbb{I}_{ij}^{(u^>b^>)}\rangle] \\ &\quad + \tfrac{1}{2}Ro(Ro\beta)\frac{\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle_0}{\gamma_u\gamma_\eta}P_{imn}[\mathbb{I}_{mn}^{(u^>)} - H\mathbb{I}_{mn}^{(b^>)} - \langle\mathbb{I}_{mn}^{(u^>)}\rangle + H\langle\mathbb{I}_{mn}^{(b^>)}\rangle] \\ &\quad - \mathrm{i}HRo(Ro\beta)^2\frac{(\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle)^2}{\gamma_u\gamma_\eta^2}k_j[\mathbb{I}_{ij}^{(u^<b^<)} + \mathbb{J}_{ij}^{(ub)} + \mathbb{I}_{ij}^{(u^>b^>)} - \langle\mathbb{I}_{ij}^{(u^<b^<)}\rangle - \langle\mathbb{I}_{ij}^{(u^>b^>)}\rangle]. \end{aligned} \tag{4.29b}$$

The rests will be neglected either on the basis of generating only third-order statistical correlations, as in the case of all the terms of second order in $\boldsymbol{u}^>$ or $\boldsymbol{b}^>$, or because of the kept order of accuracy in the asymptotic limit $Ro \ll Ro\beta \ll 1$, which will allow us to neglect terms of order $O(Ro(Ro\beta)^2)$. For details the reader is referred to Appendix A.

In what follows we provide a short description of the asymptotic renormalization procedure, described in detail in Appendix A. First, we introduce (4.27*a*,*b*) into (4.26*a*,*b*) and calculate the dynamical effect of short-wavelength components $\hat{u}_i^>(\boldsymbol{k},\omega)$ and $\hat{b}_i^>(\boldsymbol{k},\omega)$ on the evolution of $\hat{u}_i^<(\boldsymbol{k},\omega)$ and $\hat{b}_i^<(\boldsymbol{k},\omega)$ (long-wavelength modes). This results in corrections to some of the terms in (4.26*a*,*b*), but also generates terms with a new structure. Therefore, a next step is necessary, involving calculation of the effect of the next shell $\lambda_2 = \lambda_1 - \delta\lambda < k < \lambda_1$ (new short-wavelength modes) on the modes with $k < \lambda_1 - \delta\lambda$ (new long-wavelength modes). We can then take the limit of infinitesimally narrow wavenumber bands $\delta\lambda \to 0$, which leads to differential recursion relations for all the coupling constants introduced into the equations for long-wavelength modes by couplings of the short-wavelength ones. Moffatt (1983) obtained such equations but in the kinematic case with the turbulence flow given beforehand, unaffected by the Lorentz force. In the non-magnetic case, Yakhot & Orszag (1986) calculated the leading-order correction from short-wavelength modes in the Navier–Stokes equation which was proportional to $k^2\hat{u}_i^<\delta\lambda$, thus creating a viscosity correction; the turbulent viscosity was then obtained from an equation of the form $\mathrm{d}\nu_{turb}/\mathrm{d}\lambda = f(\lambda)$ with an ‘ initial’ condition $\nu_{eff}(\lambda = \Lambda) = \nu$. In the case at hand under the assumptions $Ro \ll Ro\beta \ll 1$, the explicit calculation of two initial steps of the renormalization procedure is enough to derive the final differential recursion relations with satisfactory accuracy. The details of the procedure are provided in Appendix A.

4.1. *Dynamics of mean fields*

The renormalized mean-field equations take the following form:

$$\frac{\partial \langle \boldsymbol{U}\rangle}{\partial t} + (\langle \boldsymbol{U}\rangle \cdot \nabla)\langle \boldsymbol{U}\rangle = -\nabla\left(\frac{\langle p\rangle}{\rho} + \bar{Q}_p\frac{\langle \boldsymbol{B}\rangle^2}{2}\right) + \bar{Q}(\langle \boldsymbol{B}\rangle \cdot \nabla)\langle \boldsymbol{B}\rangle + \bar{\nu}\nabla^2\langle \boldsymbol{U}\rangle, \quad (4.30a)$$

$$\frac{\partial \langle \boldsymbol{B}\rangle}{\partial t} = \nabla \times (\bar{\alpha}\langle \boldsymbol{B}\rangle) + \nabla \times (\langle \boldsymbol{U}\rangle \times \langle \boldsymbol{B}\rangle) + \bar{\eta}\nabla^2\langle \boldsymbol{B}\rangle, \quad (4.30b)$$

where the coefficients $\bar{\nu}$, $\bar{\eta}$, $\bar{\alpha}$, $\bar{Q}$ and $\bar{Q}_p$ include the effect of the turbulent fluctuations on the means. We note that, in the absence of chirality (in non-helical turbulence), $D_1 = 0$, the α-effect vanishes, i.e. $\bar{\alpha} = 0$. In such a case, unless the mean flow is capable of generating the mean magnetic field through advection and stretching, that is via the term $\nabla \times (\langle \boldsymbol{U}\rangle \times \langle \boldsymbol{B}\rangle)$, the mean magnetic field can only decay from some initial state through the effective diffusivity $\bar{\eta}$ and the transfer of the magnetic energy to the kinetic one via the Lorentz force. In other words, unless the energy transfer through $\nabla \times (\langle \boldsymbol{U}\rangle \times \langle \boldsymbol{B}\rangle)$ and the Lorentz force can account for magnetic energy gain, in the reflectionally symmetric case we are dealing with the problem of magnetic energy relaxation. On the other hand, in chiral turbulence, the α-effect can account for rapid amplification of the energy of the mean magnetic field. The saturation of the energy occurs via the effect of the turbulent magnetic diffusivity and the effect of the Lorentz force present in the $\bar{\alpha}$ coefficient. The two coefficients $\bar{Q}$ and $\bar{Q}_P$ in the mean-field Navier–Stokes equation describe the renormalized Lorentz force. The general differential recursion relations for all the renormalized coefficients $\bar{\nu}$, $\bar{\eta}$, $\bar{\alpha}$, $\bar{Q}$ and $\bar{Q}_P$ are solved in Appendix A, see (A91*a*–*d*) and below.

4.2. *Dynamics of turbulent fluctuations in the limit $k \to \Lambda_L$*

The evolution of the fluctuating fields in the limit $k \to \Lambda_L$ is governed by the following set of equations in the Fourier space:

$$\begin{aligned}[-\mathrm{i}\omega + \breve{\nu}(k)k^2]\hat{u}_i(\boldsymbol{q}) = {} & \hat{f}_i(\boldsymbol{q}) + \mathrm{i}(\boldsymbol{k}\cdot\langle \boldsymbol{B}\rangle_0)\hat{b}_i(\boldsymbol{q}) \\ & - \tfrac{1}{2}\mathrm{i}P_{imn}(\boldsymbol{k})[\mathbb{I}^{(u)}_{mn} - \mathbb{I}^{(b)}_{mn} - \langle\mathbb{I}^{(u)}_{mn}\rangle + \langle\mathbb{I}^{(b)}_{mn}\rangle] \\ & - P_{ij}(\boldsymbol{k})\frac{\partial\langle U\rangle_j}{\partial x_k}\hat{u}_k(\boldsymbol{q}) + P_{ij}(\boldsymbol{k})\frac{\partial\langle B\rangle_j}{\partial x_k}\hat{b}_k(\boldsymbol{q}) \\ & - \tfrac{1}{2}P_{ij}(\boldsymbol{k})k_m[\hat{u}_j(\boldsymbol{k} - \nabla\langle U\rangle_m, \omega) - \hat{u}_j(\boldsymbol{k} + \nabla\langle U\rangle_m, \omega)] \\ & + \tfrac{1}{2}P_{ij}(\boldsymbol{k})k_m[\hat{b}_j(\boldsymbol{k} - \nabla\langle B\rangle_m, \omega) - \hat{b}_j(\boldsymbol{k} + \nabla\langle B\rangle_m, \omega)], \quad (4.31a)\end{aligned}$$

$$\begin{aligned}&[-\mathrm{i}\omega + \breve{\eta}(k)k^2]\hat{b}_i(\boldsymbol{q}) - \mathrm{i}\breve{\alpha}(k)\epsilon_{ijk}k_j\hat{b}_k(\boldsymbol{q}) \\ &\quad = \mathrm{i}(\boldsymbol{k}\cdot\langle \boldsymbol{B}\rangle_0)\hat{u}_i(\boldsymbol{q}) + \mathrm{i}Ro k_j[\mathbb{I}^{(ub)}_{ij} - \langle\mathbb{I}^{(ub)}_{ij}\rangle] \\ &\qquad + \frac{\partial\langle U\rangle_i}{\partial x_j}\hat{b}_j(\boldsymbol{q}) - \frac{\partial\langle B\rangle_i}{\partial x_j}\hat{u}_j(\boldsymbol{q})\end{aligned}$$

$$- \tfrac{1}{2}k_m[\hat{b}_i(\boldsymbol{k} - \nabla\langle U\rangle_m, \omega) - \hat{b}_i(\boldsymbol{k} + \nabla\langle U\rangle_m, \omega)] \\ + \tfrac{1}{2}k_m[\hat{u}_i(\boldsymbol{k} - \nabla\langle B\rangle_m, \omega) - \hat{u}_i(\boldsymbol{k} + \nabla\langle B\rangle_m, \omega)]. \tag{4.31b}$$

The general formulae for all the renormalized coefficients $\breve{\nu}(k)$, $\breve{\eta}(k)$ and $\breve{\alpha}(k)$ are provided in Appendix A, see (A54*a,b*) and below. We now turn to the problem of turbulent energy and helicity spectra and their scaling laws with respect to the wavenumber.

4.2.1. *Energy and helicity spectra in the limit $k \to \varLambda_L$*

It is of interest to determine the scaling exponents of the energy and helicity spectra and show how the scaling laws change when the turbulence shifts from the weak regime to the strong one; the former is characterized by weak fluctuations and thus the dynamical equations for fluctuations are linearized about the means as in (3.6*a,b*), whereas, in the strong regime, defined by (4.8*c,d*), the dynamics is predominantly determined by the nonlinearities $\nabla \cdot (\boldsymbol{uu} - \boldsymbol{bb})$ and $\nabla \times (\boldsymbol{u} \times \boldsymbol{b})$ in the fluctuational equations. Naturally, the spectral scaling laws for energies and helicity will be determined by the exponents σ_0 and σ_1, which define the correlation function of the stirring force, cf. (2.12). Unfortunately, those are not well-established quantities. The only clue is provided by comparison with the Kolmogorov-like spectral scaling laws in the non-magnetic case. Such a comparison allows us to determine the spectral structure of the forcing correlations (i.e. the values of the exponents σ_0 and σ_1) which lead to the expected kinetic energy and helicity spectra in the absence of a magnetic field. Yakhot & Orszag (1986) have demonstrated that the purely hydrodynamic (non-MHD), viscously controlled renormalized scaling of the kinetic energy spectrum agrees with the Kolmogorov $k^{-5/3}$ law for $\sigma_0 = 3$. Below, we will demonstrate that the value of $\sigma_1 = 5$ corresponds to $k^{-5/3}$ spectral scaling for turbulent helicity (expected in helical isotropic turbulence, cf. Brissaud *et al.* 1973 or Chen *et al.* 2003).

Within the considered asymptotic limit $Ro \ll Ro\beta \ll 1$ the fluctuating velocity and magnetic fields at each step of the renormalization procedure at the leading order are given by

$$\hat{u}_i(\boldsymbol{q}) \approx \frac{1}{\gamma_\nu(\boldsymbol{q})}\hat{f}_i(\boldsymbol{q}), \quad \hat{b}_i(\boldsymbol{q}) \approx \mathrm{i}\boldsymbol{k} \cdot \langle \boldsymbol{B}\rangle \frac{K_{ij}}{\gamma_\nu}\hat{f}_j(\boldsymbol{q}), \tag{4.32a,b}$$

$$K_{ik} = \frac{1}{\gamma_\alpha^2}(\gamma_\eta \delta_{ik} + \mathrm{i}\breve{\alpha}\epsilon_{ijk}k_j), \quad \gamma_\alpha^2 = \gamma_\eta^2 - k^2\breve{\alpha}^2, \tag{4.33}$$

where the γ-factors are now defined using the renormalized fluctuational diffusivities for small wavenumbers

$$\gamma_\nu = -\mathrm{i}\omega + \breve{\nu}(k)k^2, \quad \gamma_\eta = -\mathrm{i}\omega + \breve{\eta}(k)k^2. \tag{4.34a,b}$$

It follows that the fluctuating vorticity at the leading order takes the simple form

$$\hat{w}_i(\boldsymbol{q}) = \mathrm{i}\epsilon_{ijk}k_j\hat{u}_k(\boldsymbol{q}) = \frac{\mathrm{i}}{\gamma_\nu(\boldsymbol{q})}\epsilon_{ijk}k_j\hat{f}_k(\boldsymbol{q}). \tag{4.35}$$

With the aid of (2.12) and (3.17*a,b*), it is now possible to write down explicit general formulae for the turbulent spectra of the fluctuating kinetic and magnetic energies and

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helicity in the following way:

$$\langle e_k\rangle = \tfrac{1}{2}\langle u_i(\boldsymbol{x},t)u_i(\boldsymbol{x},t)\rangle = \tfrac{1}{2}\int \mathrm{d}^4q\int \mathrm{d}^4q'\frac{\mathrm{e}^{\mathrm{i}[(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{\cdot}\boldsymbol{x}-(\omega+\omega')t]}}{\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q})}\langle \hat{f}_i(\boldsymbol{q})\hat{f}_i(\boldsymbol{q}')\rangle$$
$$= 4\pi D_0\int_{\Lambda_L}^{\Lambda}\frac{\mathrm{d}k}{k^{\sigma_0-2}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\omega^2+\breve{\nu}(k)^2k^4}$$
$$= 4\pi^2 D_0\int_{\Lambda_L}^{\Lambda}\frac{\mathrm{d}k}{\breve{\nu}(k)k^{\sigma_0}}, \tag{4.36a}$$

$$\langle e_m\rangle = \tfrac{1}{2}\langle b_i(\boldsymbol{x},t)b_i(\boldsymbol{x},t)\rangle$$
$$= -\tfrac{1}{2}\langle B\rangle_m\langle B\rangle_n\int \mathrm{d}^4q\int \mathrm{d}^4q'\frac{k_mk'_nK_{ij}(\boldsymbol{q})K_{ik}(\boldsymbol{q}')}{\gamma_\nu(\boldsymbol{q})\gamma_\nu(\boldsymbol{q}')}\langle \hat{f}_j(\boldsymbol{q})\hat{f}_k(\boldsymbol{q}')\rangle\,\mathrm{e}^{\mathrm{i}[(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{\cdot}\boldsymbol{x}-(\omega+\omega')t]}$$
$$= \tfrac{1}{2}\langle B\rangle_m\langle B\rangle_n\int \mathrm{d}^4q\frac{k_mk_n}{(\omega^2+\breve{\nu}^2k^4)\gamma_\alpha^2(\boldsymbol{q})\gamma_\alpha^2(-\boldsymbol{q})}\left[|\gamma_\eta(\boldsymbol{q})|^2\frac{D_0}{k^{\sigma_0}}P_{jk}(\boldsymbol{k})\delta_{jk}\right.$$
$$\left.+\breve{\alpha}^2k^2\frac{D_0}{k^{\sigma_0}}P_{jk}(\boldsymbol{k})P_{jk}(\boldsymbol{k})+\breve{\alpha}\frac{D_1}{k^{\sigma_1}}\epsilon_{jkl}\epsilon_{jtk}k_lk_t(\gamma_\eta(\boldsymbol{q})+\gamma_\eta(-\boldsymbol{q}))\right]$$
$$= \frac{4\pi}{3}\langle B\rangle^2\int_{\Lambda_L}^{\Lambda}\mathrm{d}k\left[\mathcal{I}_{e1}(k)\frac{D_0}{k^{\sigma_0-4}}-2\breve{\alpha}\breve{\eta}\mathcal{I}_{e2}(k)\frac{D_1}{k^{\sigma_1-8}}\right], \tag{4.36b}$$

$$\langle h_k\rangle = \langle u_i(\boldsymbol{x},t)w_i(\boldsymbol{x},t)\rangle = \mathrm{i}\epsilon_{ijk}\int \mathrm{d}^4q\int \mathrm{d}^4q'\frac{k'_j\,\mathrm{e}^{\mathrm{i}[(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{\cdot}\boldsymbol{x}-(\omega+\omega')t]}}{\gamma_\nu(\boldsymbol{q})\gamma_\nu(\boldsymbol{q}')}\langle \hat{f}_i(\boldsymbol{q})\hat{f}_k(\boldsymbol{q}')\rangle$$
$$= -8\pi D_1\int_{\Lambda_L}^{\Lambda}\frac{\mathrm{d}k}{k^{\sigma_1-4}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\omega^2+\breve{\nu}(k)^2k^4}$$
$$= -8\pi^2 D_1\int_{\Lambda_L}^{\Lambda}\mathrm{d}k\frac{1}{\breve{\nu}(k)k^{\sigma_1-2}}, \tag{4.36c}$$

where

$$\mathcal{I}_{e1}(k) = \int_{-\infty}^{\infty}\mathrm{d}\omega\frac{\breve{\eta}^2k^4+\breve{\alpha}^2k^2+\omega^2}{(\omega^2+\breve{\nu}^2k^4)[(\breve{\eta}^2k^4-\breve{\alpha}^2k^2-\omega^2)^2+4\breve{\eta}^2k^4\omega^2]}, \tag{4.37a}$$
$$\mathcal{I}_{e2}(k) = \int_{-\infty}^{\infty}\mathrm{d}\omega\frac{1}{(\omega^2+\breve{\nu}^2k^4)[(\breve{\eta}^2k^4-\breve{\alpha}^2k^2-\omega^2)^2+4\breve{\eta}^2k^4\omega^2]}. \tag{4.37b}$$

Note that h_k denotes the kinetic helicity (in contrast to the magnetic one). Thus the spectral densities are given by the integrands in the final expressions for each quantity

$$\hat{e}_k(k) = \frac{4\pi^2D_0}{\breve{\nu}(k)k^{\sigma_0}},\quad \hat{e}_m(k) = \frac{4\pi}{3}\langle B\rangle^2\left[\mathcal{I}_{e1}(k)\frac{D_0}{k^{\sigma_0-4}}-2\breve{\alpha}\breve{\eta}\mathcal{I}_{e2}(k)\frac{D_1}{k^{\sigma_1-8}}\right], \tag{4.38a}$$
$$\hat{h}_k(k) = -\frac{8\pi^2D_1}{\breve{\nu}(k)k^{\sigma_1-2}}. \tag{4.38b}$$

Recall that, in non-dimensional units, the total magnetic energy is $\langle B\rangle^2+\beta^{-2}\langle b^2\rangle$ where by assumption $\beta \gg 1$. Furthermore, since by the use of (4.18b) the fluctuating magnetic field satisfies $\boldsymbol{b} = O(Ro\beta)$, the fluctuating magnetic energy constitutes a $O(Ro^2)$ correction to the total magnetic energy. Note also that the fluctuating magnetic energy spectrum is in general strongly influenced by the helical component of the forcing.

In the following sections the results for all the renormalized coefficients $\bar{\nu}$, $\bar{\eta}$, $\bar{\alpha}$, $\bar{Q}$ and $\bar{Q}_p$, $\breve{\nu}(k)$, $\breve{\eta}(k)$, $\breve{\alpha}(k)$ and the spectral densities $\hat{e}_k(k)$, $\hat{e}_m(k)$, $\hat{h}_k(k)$ will be provided. We start with the simplest case of reflectionally symmetric turbulence, $D_1 = 0$, when the α-effect is eliminated. This will allow for a smooth introduction to the more complicated problem of helical turbulence considered in § 6. Moreover, in the next section we will explicitly consider two distinguished regimes of the MHD turbulence – the strong and weak regimes.

5. Non-helical MHD turbulence

The MHD turbulence in the absence of the flow chirality has been studied via the renormalization group method by Kleeorin & Rogachevskii (1994). They have concluded that the MHD turbulence creates a strong negative contribution to the effective mean Lorentz force so that, in particular, the effective magnetic pressure can in fact become negative. Here, we will recall some of the results of Kleeorin & Rogachevskii (1994), at least in a qualitative way, and study the reflectionally symmetric MHD turbulence in two distinguished regimes: weak and strong. Moreover, by inclusion of the gradients of the means in the evolution equations for the fluctuations, the correspondence between the renormalized fluctuation diffusivities $\breve{\nu}(\Lambda_L)$, $\breve{\eta}(\Lambda_L)$ and the mean transport coefficients $\bar{\nu}$ and $\bar{\eta}$ will be clearly demonstrated.

5.1. *Strong turbulence*

In strong turbulence the evolution of pulsations is nonlinear and to take proper account of the nonlinearities $\nabla \cdot (\boldsymbol{uu} - \boldsymbol{bb})$ and $\nabla \times (\boldsymbol{u} \times \boldsymbol{b})$ we apply the renormalization group technique. The details of the procedure are postponed until Appendix A and, here, we only provide the final results. First, the recursions for the fluctuational diffusivities (A57) and (A61) have to be resolved; the general solution of the latter is obtained via the method of characteristics in (A63*a,b*) and also (A67), (A68). Including the dominant terms in the limit $\Lambda_L \leq k \ll \Lambda$ yields

$$\breve{\nu}(k) = \frac{1}{a}\breve{\eta}(k) = \left[\frac{2\pi^2 D_0(5-\sigma_0)}{5(\sigma_0+1)}\right]^{1/3} \frac{1}{k^{(\sigma_0+1)/3}}, \quad \text{where } a = \tfrac{1}{2}\left(\sqrt{1+\frac{240}{3(5-\sigma_0)}}-1\right). \tag{5.1}$$

Let us first examine the Fourier spectra of turbulent kinetic energy and helicity and the magnetic energy which are obtained from (4.38*a,b*) by substitution for $\breve{\nu}(k)$ and $\breve{\eta}(k)$ from (5.1) and $D_1 = 0 \Rightarrow \breve{\alpha} = 0$,

$$\hat{e}_k(k) = 4\left[\frac{5(\sigma_0+1)}{2(5-\sigma_0)}\right]^{1/3} \frac{(\pi^2 D_0)^{2/3}}{k^{2\sigma_0/3-1/3}}, \tag{5.2a}$$

$$\hat{e}_m(k) = \frac{4\pi}{3}\langle B\rangle^2 \frac{D_0}{k^{\sigma_0-4}} \frac{\pi}{\breve{\nu}\breve{\eta}(\breve{\nu}+\breve{\eta})k^6} = \frac{10(\sigma_0+1)}{3a(1+a)(5-\sigma_0)}\frac{\langle B\rangle^2}{k}, \tag{5.2b}$$

where we have used

$$\mathcal{I}_{e1}(k) = \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{1}{(\omega^2 + \breve{\nu}^2 k^4)(\omega^2 + \breve{\eta}^2 k^4)} = \frac{\pi}{\breve{\nu}\breve{\eta}(\breve{\nu} + \breve{\eta})k^6}. \tag{5.3}$$

Observe that we have reproduced the result of Yakhot & Orszag (1986) for the kinetic energy spectral scaling law, namely, that it corresponds to the Kolmogorov spectral law $k^{-5/3}$ for $\sigma_0 = 3$. The k^{-1} spectrum for the magnetic energy, independent of the value of σ_0, is in agreement with the findings of Kleeorin & Rogachevskii (1994) and an earlier dimensional analysis of Ruzmaikin & Shukurov (1982). Since the parameter σ_0 describes the forcing which generates the turbulence and we have determined that, in the non-helical case, $\sigma_0 = 3$ is consistent with the expected turbulent energy spectrum, in the following we will only consider the case with $\sigma_0 = 3$. The strong-turbulence energy spectra in the non-helical case are therefore given by

$$\hat{e}_k(k) = \frac{4(\sqrt{5}\pi^2 D_0)^{2/3}}{k^{5/3}}, \quad \hat{e}_m(k) = \frac{2}{3}\frac{\langle B\rangle^2}{k}. \tag{5.4a,b}$$

and the fluctuational diffusivities take the form

$$\breve{\nu}(k) = \left(\frac{\pi^2 D_0}{5}\right)^{1/3} \frac{1}{k^{4/3}}, \quad \breve{\eta}(k) = a\left(\frac{\pi^2 D_0}{5}\right)^{1/3} \frac{1}{k^{4/3}}, \quad \Lambda_L \leq k \ll \Lambda, \tag{5.5a,b}$$

where

$$a = \frac{\sqrt{41} - 1}{2} \approx 2.7. \tag{5.6}$$

Having found $\breve{\nu}(k)$ and $\breve{\eta}(k)$ we can introduce them into the recursion relations (A81) and (A91*a*–*c*) for the mean-field coefficients, which can be solved to obtain in the limit $k \to \Lambda_L \ll \Lambda$

$$\bar{Q}_p \approx -\frac{4(4a-1)}{15} \ln \frac{\Lambda}{\Lambda_L} \approx -2.6 \ln \frac{\Lambda}{\Lambda_L} \tag{5.7a}$$

$$\bar{Q} \approx -\frac{8}{3a} \ln \frac{\Lambda}{\Lambda_L} \approx -\ln \frac{\Lambda}{\Lambda_L} \tag{5.7b}$$

$$\bar{\nu} = \frac{1}{a}\bar{\eta} = \left(\frac{\pi^2 D_0}{5}\right)^{1/3} \frac{1}{\Lambda_L^{4/3}} = \left(\frac{D_0}{80\pi^2}\right)^{1/3} \mathcal{L}_L^{4/3}, \tag{5.7c}$$

where $\mathcal{L}_L$ is the largest fluctuational length scale and thus could be interpreted as the size of the largest eddies in the system. It follows that

$$\bar{\nu} = \breve{\nu}(\Lambda_L), \quad \bar{\eta} = \breve{\eta}(\Lambda_L), \tag{5.8a,b}$$

i.e. the fluctuational diffusivities at the largest turbulent scales and the mean diffusivities are equal. Note that, in agreement with the results of Kleeorin & Rogachevskii (1994) in the strong-turbulence limit, the contribution to the Lorentz force from interactions of turbulent pulsations, which becomes dominant when $\Lambda_L \ll \Lambda$, is negative, cf. (5.7*a*,*b*).

5.2. *Weak turbulence*

The weak turbulence corresponds to the simplest case when the amplitude of the turbulent pulsations is small enough so that their evolution can be considered linear, as in (3.6*a*,*b*); this means that the terms $\nabla \cdot (\boldsymbol{uu} - \boldsymbol{bb})$ and $\nabla \times (\boldsymbol{u} \times \boldsymbol{b})$ are neglected in the evolution equations for the fluctuations. Tobias *et al.* (2013) argued that, under certain conditions, the state of weak turbulence can survive for long periods of time without being destroyed by nonlinear interactions of wave packets, which nevertheless eventually still lead to transition into the strong-turbulence regime.

The case at hand in fact does not require renormalization, since the nonlinearities are neglected. The turbulent diffusivities can be taken from the introductory § 3 with $D_1 = 0$ and the coefficients $\bar{Q}$ and $\bar{Q}_p$ can be calculated in a similar way. However, we can alternatively utilize the recursion differential equations for the turbulent diffusivities and the Lorentz force coefficients obtained via the renormalization, which can be found in (A81) and (A81*a*–*c*) and, for the current case of weak turbulence, replace the fluctuational diffusivities $\breve{\nu}(k)$ and $\breve{\eta}(k)$ by the molecular ones, ν and η. The solutions in the limit $k \to \varLambda_L \ll \varLambda$ are provided only for the case $\sigma_0 = 3$

$$\bar{Q}_p = 1 - \frac{2\pi^2(4\eta - \nu)}{15\nu^2\eta(\nu + \eta)} \frac{D_0}{\varLambda_L^4}, \tag{5.9a}$$

$$\bar{Q} = 1 - \frac{2\pi^2}{15\nu^2\eta} \frac{D_0}{\varLambda_L^4}, \tag{5.9b}$$

$$\bar{\nu} = \nu + \frac{\pi^2 D_0}{15\nu^2 \varLambda_L^4} \approx \frac{\pi^2 D_0}{15\nu^2 \varLambda_L^4}, \tag{5.9c}$$

$$\bar{\eta} = \eta + \frac{2\pi^2 D_0}{3\nu(\nu + \eta)\varLambda_L^4} \approx \frac{2\pi^2 D_0}{3\nu(\nu + \eta)\varLambda_L^4}. \tag{5.9d}$$

The Fourier spectra of turbulent kinetic energy and helicity and the magnetic energy are obtained from (4.38*a*,*b*) by substitution $\breve{\nu} = \nu$ and $\breve{\eta} = \eta$ and $D_1 = 0$, which yields

$$\hat{e}_k(k) = \frac{4\pi^2 D_0}{\nu k^3}, \quad \hat{e}_m(k) = \frac{4\pi^2 D_0}{3\nu\eta(\nu + \eta)} \langle B \rangle^2 \frac{1}{k^5}. \tag{5.10a,b}$$

6. Helical MHD turbulence

We now turn to the more complicated problem of MHD turbulence with chirality. In natural systems such as e.g. stellar and planetary interiors, chirality is typically introduced into the flow by rotation of the system; the latter creates the Coriolis force in the rotating frame which breaks the reflectional symmetry. However, since it also introduces significant complexity through anisotropy, we get around this difficulty by considering a non-reflectionally symmetric but statistically isotropic forcing; this allows us to perform the renormalization of the MHD equations in the strong-turbulence regime.

6.1. *Strong turbulence*

The case of strong turbulence is nonlinear and thus we apply the renormalization technique. As remarked below (A45*d*), at the leading order, the turbulent fluctuational viscosity remains unchanged with respect to the non-helical case (cf. (5.5*a*,*b*)), and thus takes the form

$$\breve{\nu}(k) = \left(\frac{\pi^2 D_0}{5}\right)^{1/3} \frac{1}{k^{4/3}}, \quad \Lambda_L \leq k \ll \Lambda, \tag{6.1}$$

and we recall that this result is obtained for $\sigma_0 = 3$.

Let us also recall that, in helical, isotropic, homogeneous and stationary turbulence, the magnitude of the helicity spectrum (4.38*b*) is bounded by the energy spectrum (cf. e.g. Moffatt & Dormy 2019, p. 198, (7.54))

$$|\hat{h}_k(k)| \leq 2k\hat{e}_k(k), \quad \text{for all } k. \tag{6.2}$$

This implies $D_1 \leq k^{\sigma_1-\sigma_0-1} D_0$. The recursion differential equations for the turbulent fluctuating magnetic diffusivity (A54*a*) and the turbulent fluctuating EMF ($\breve{\alpha}$ coefficient) (A54*b*) can be solved explicitly in the limit

$$\frac{D_0 \Lambda_L^{\sigma_1-1-\sigma_0}}{D_1} \gg 1, \tag{6.3}$$

which, for $\sigma_1 - \sigma_0 - 1 > 0$, determines that the helical component of the driving force is weaker than the non-helical one at all fluctuating wavelengths $\Lambda_L \leq k \leq \Lambda$ (cf. (2.12)), consistently with (6.2). In other words, in such a limit the non-helical driving dominates the helical one and, roughly speaking, this could correspond to a situation expected in many natural systems where thermal/compositional buoyancy is stronger than the Coriolis force. Of course, the total effect of the helical driving remains non-negligible and significantly affects the dynamics of the mean fields through creation of the α-effect. The solutions for $\breve{\eta}$ and $\breve{\alpha}$ in this limit take the form

$$\breve{\eta}(k) = a\breve{\nu}(k) = a\left(\frac{\pi^2 D_0}{5}\right)^{1/3} \frac{1}{k^{4/3}}, \quad \Lambda_L \leq k \ll \Lambda, \tag{6.4a}$$

$$\breve{\alpha}(k) = \frac{10 D_1}{(2a+1)D_0}\breve{\nu}(k) = \frac{10 D_1}{(2a+1)D_0}\left(\frac{\pi^2 D_0}{5}\right)^{1/3} \frac{1}{k^{4/3}}, \quad \Lambda_L \leq k \ll \Lambda, \tag{6.4b}$$

with the constant $a \approx 2.7$ given in (5.6).

Substitution of $\breve{\nu}(k)$, $\breve{\eta}(k)$ and $\breve{\alpha}(k)$ from (6.1) and (6.4*a*,*b*) into the formulae (4.38*a*,*b*) leads to the following form of the energy spectra in helical MHD turbulence, in the limit (6.3):

$$\hat{e}_k(k) \approx \frac{4\pi^2 D_0}{\breve{\nu}(k)k^3} \approx \frac{4(\sqrt{5}\pi^2 D_0)^{2/3}}{k^{5/3}}, \tag{6.5a}$$

$$\begin{aligned}\hat{e}_m(k) &\approx \frac{4\pi^2}{3}\langle B\rangle^2 \left[\frac{D_0}{\breve{\nu}\breve{\eta}(\breve{\nu}+\breve{\eta})k^5} - \frac{D_1(\breve{\nu}+2\breve{\eta})\breve{\alpha}}{\breve{\nu}\breve{\eta}^2(\breve{\nu}+\breve{\eta})^2 k^{\sigma_1+2}}\right] \\ &\approx \frac{2}{3}\langle B\rangle^2 \left(\frac{1}{k} - \frac{D_1^2}{D_0^2 k^{\sigma_1-2}}\right),\end{aligned} \tag{6.5b}$$

$$\hat{h}_k(k) \approx -\left(\frac{5\pi}{D_0}\right)^{1/3} \frac{8D_1}{k^{\sigma_1-10/3}}, \tag{6.5c}$$

where we have used

$$\mathcal{I}_{e1}(k) \approx \frac{\pi}{\breve{\nu}\breve{\eta}(\breve{\nu}+\breve{\eta})k^6}, \quad \mathcal{I}_{e2}(k) \approx \frac{\pi(\breve{\nu}+2\breve{\eta})}{2\breve{\nu}\breve{\eta}^3(\breve{\nu}+\breve{\eta})^2 k^{10}}. \tag{6.6a,b}$$

Similarly to what we have done for the exponent σ_0, we can now compare the obtained helicity spectrum (6.5*c*) with the Kolmogorov-type scaling for an isotropic, homogeneous and stationary turbulence $\hat{h}_k(k) \sim k^{-5/3}$ (cf. Brissaud *et al.* 1973; Chen *et al.* 2003) which leads to

$$\sigma_1 - \tfrac{10}{3} = \tfrac{5}{3} \Rightarrow \sigma_1 = 5. \tag{6.7}$$

The parameters σ_0 and σ_1 describe the statistical properties of the forcing which generates the turbulence and since the values of $\sigma_0 = 3$ and $\sigma_1 = 5$ correspond to the expected turbulent energy and helicity spectra, in the following we will only consider the case with $\sigma_0 = 3$ and $\sigma_1 = 5$. Finally, the turbulent energy and helicity spectra are given by

$$\hat{e}_k(k) \approx \frac{4(\sqrt{5}\pi^2 D_0)^{2/3}}{k^{5/3}}, \quad \hat{e}_m(k) \approx \frac{2}{3}\langle B\rangle^2 \frac{1}{k}, \quad \hat{h}_k(k) \approx -\left(\frac{5\pi}{D_0}\right)^{1/3} \frac{8D_1}{k^{5/3}}. \tag{6.8a–c}$$

Furthermore, in the currently considered limit

$$D_0 \Lambda_L \gg D_1, \tag{6.9}$$

(cf. (6.3) with $\sigma_0 = 3$ and $\sigma_1 = 5$) the mean renormalized coefficients $\bar{\nu}$, $\bar{\eta}$, $\bar{Q}$, $\bar{Q}_p$ at leading order remain unaltered with respect to the non-helical case (cf. equations (A91*a*–*c*) and the discussion below)

$$\bar{\nu} = \breve{\nu}(\Lambda_L) = \left(\frac{\pi^2 D_0}{5}\right)^{1/3} \frac{1}{\Lambda_L^{4/3}}, \quad \bar{\eta} = \breve{\eta}(\Lambda_L) = a\left(\frac{\pi^2 D_0}{5}\right)^{1/3} \frac{1}{\Lambda_L^{4/3}}, \tag{6.10a,b}$$

$$\bar{Q}(\Lambda_L) \approx -\frac{8\pi^2 D_0}{15a} \int_{\Lambda_L}^{\Lambda} \frac{d\lambda}{\lambda^5} \frac{1}{\breve{\nu}^3} \approx -\frac{8}{3a} \ln \frac{\Lambda}{\Lambda_L}, \tag{6.11}$$

$$\bar{Q}_p(\Lambda_L) = -\frac{8(4a-1)\pi^2 D_0}{150} \int_{\Lambda_L}^{\Lambda} \frac{d\lambda}{\lambda^5} \frac{1}{\breve{\nu}^3} \approx -\frac{4(4a-1)}{15} \ln \frac{\Lambda}{\Lambda_L}, \tag{6.12}$$

and the effect of chirality is expressed by the α-effect

$$\bar{\alpha} = \breve{\alpha}(\Lambda_L) = \frac{10 D_1}{(2a+1) D_0} \left(\frac{\pi^2 D_0}{5}\right)^{1/3} \frac{1}{\Lambda_L^{4/3}} + O(Ro^2 (Ro\beta)^4 \mathcal{U}) + o(Ro^4 \mathcal{U}), \tag{6.13}$$

Note that, in the latter expression, the quenching effect of the Lorentz force vanishes at the order $Ro^2(Ro\beta)^2\langle B\rangle^2$ (cf. the expansion of the γ^{-1} factors (A8)) and thus saturation effects for the mean magnetic field evolution can only be present at the unexplored order $Ro^2(Ro\beta)^4\langle B\rangle^4$.

It is also of interest to point out that, in strongly helical strong turbulence, when the helical component of the driving force (2.12) is of comparable magnitude to the non-helical one at all wavelengths (and (6.3) is not satisfied), the k-dependencies of the fluctuation coefficients $\breve{\eta}(k)$ and $\breve{\alpha}(k)$ and hence also of the magnetic energy and helicity spectra are likely not to take the form of simple scaling laws. The general system of equations for the turbulent renormalized fluctuational coefficients $\breve{\eta}(k)$ and $\breve{\alpha}(k)$ obtained in the asymptotic limit $Ro \ll Ro\beta \ll 1$ is provided in (A54*a*,*b*) with the integrals $\mathcal{I}_1(k)$ and $\mathcal{I}_2(k)$ defined in (A50*a*,*b*); at the leading order of the limit $Ro \ll Ro\beta \ll 1$, the turbulent fluctuational viscosity $\breve{\nu}(k)$ is still given by (A58) in the general strongly helical case, which reduces to (6.1) for $\Lambda_L \leq k \ll \Lambda$. Once $\breve{\nu}(k)$, $\breve{\eta}(k)$ and $\breve{\alpha}(k)$ are known, the mean coefficients $\bar{\eta}$, $\bar{\alpha}$, $\bar{Q}$ and $\bar{Q}_p$ can be computed from (A91*a*–*d*), with $\bar{\nu}$ unaltered with respect to (6.10*a*,*b*). In principle, this general case which consists of (A54*a*,*b*) and (A91*a*–*d*) with the aid of (A58), (A50*a*,*b*) and (A85*a*–*d*) and the 'boundary' conditions

$$\breve{\eta}(\Lambda) = \eta, \quad \breve{\alpha}(\Lambda) = 0, \quad \bar{\eta}(\Lambda) = \eta, \quad \bar{\alpha}(\Lambda) = 0, \quad \bar{Q}(\lambda) = 1, \quad \bar{Q}(\Lambda) = 1, \tag{6.14a–f}$$

can be solved numerically.

6.2. *Weak turbulence*

The case of weak turbulence is fairly simple and, as already remarked, does not require renormalization, because the nonlinearities $\nabla \cdot (\boldsymbol{uu} - \boldsymbol{bb})$ and $\nabla \times (\boldsymbol{u} \times \boldsymbol{b})$ in the equations for the fluctuations are neglected. Therefore, the evolution of the fluctuations is not influenced by the effects of nonlinearities and thus there is no effect of turbulence in the fluctuational turbulent coefficients, i.e. $\breve{\nu} = \nu$, $\breve{\eta} = \eta$ and $\breve{\alpha} = 0$. It follows that the mean turbulent diffusivities $\bar{\nu}$ and $\bar{\eta}$ can be taken from the introductory § 3. The Lorentz-force coefficients $\bar{Q}_p$, $\bar{Q}$ and $\bar{\alpha}$ can be calculated with the use of the recursion differential equations (A91*a*,*b*,*d*) obtained via the renormalization, where for the current case of weak turbulence we replace the fluctuational diffusivities $\breve{\nu}(k)$ and $\breve{\eta}(k)$ by the molecular ones ν and η and substitute $\breve{\alpha}(k) = 0$. The solutions in the limit $k \to \Lambda_L \ll \Lambda$ are provided for $\sigma_0 = 3$ and $\sigma_1 = 5$

$$\bar{Q}_p = 1 - \frac{2\pi^2(4\eta - \nu)}{15\nu^2\eta(\nu+\eta)}\frac{D_0}{\Lambda_L^4}, \quad \bar{Q} = 1 - \frac{2\pi^2}{15\nu^2\eta}\frac{D_0}{\Lambda_L^4}, \tag{6.15a}$$

$$\bar{\nu} = \frac{\pi^2 D_0}{15\nu^2\Lambda_L^4}, \quad \bar{\eta} = \frac{2\pi^2 D_0}{3\nu(\nu+\eta)\Lambda_L^4}, \quad \bar{\alpha} = \frac{2\pi^2 D_1}{3\nu(\nu+\eta)\Lambda_L^4} - \frac{4\pi^2 D_1 \langle B\rangle^2}{15\nu^2\eta(\nu+\eta)\Lambda_L^6}. \tag{6.15b}$$

It is also notable that the effect of the Lorentz force is now present in the EMF at the order $\langle B\rangle^2$, contrary to (6.13); hence, strong turbulence tends to suppress the saturation effects for the mean magnetic field, and the magnetic energy $\langle B\rangle^2$ can saturate only above a certain threshold, when the saturation effects can enter the dynamics. We also emphasize that, in the current case of weak turbulence, there is no need to invoke the asymptotic limit (6.9) and the results are valid for $D_0\Lambda_L/D_1 \leq 1$. Furthermore, although it may seem obvious it is perhaps worth mentioning that the role of turbulent magnetic diffusivity in the large-scale dynamo process is non-trivial and naive substitution for ν and η of their turbulent counterparts in the expression for $\bar{\alpha}$ in (6.15*b*) does not lead to anything similar in form to the strong-turbulence value (6.13).

The Fourier spectra of the turbulent kinetic and magnetic energies and the helicity can be obtained from (4.38*a*,*b*) again, by substitution $\breve{\nu} = \nu$ and $\breve{\eta} = \eta$ and $\breve{\alpha} = 0$, which yields

$$\hat{e}_k(k) = \frac{4\pi^2 D_0}{\nu k^3}, \quad \hat{e}_m(k) = \frac{4\pi^2 D_0}{3\nu\eta(\nu+\eta)}\langle B\rangle^2\frac{1}{k^5}, \quad \hat{h}_k(k) = -\frac{8\pi^2 D_1}{\nu k^3}. \tag{6.16a–c}$$

Note that the energy spectra are the same as in the case of non-helical weak turbulence (5.10*a*,*b*).

Summarizing, the physical pictures of the helical weak and strong turbulence considered here are rather simple, i.e. the pictures differ from their relevant non-helical cases only by the presence of helicity and non-zero EMF: $\bar{\alpha}$ in the case of weak turbulence and both $\breve{\alpha}$ and $\bar{\alpha}$ in the case of weakly helical strong turbulence. Note, however, that, as remarked at the end of the previous subsection, the helical turbulence outside the asymptotic limit $D_0 \Lambda_L \gg D_1$ is much more complex than the non-helical case even in the isotropic case.

7. Role of diffusivities

It is very important to understand the role of viscosity and magnetic diffusivity in the presented approach. First of all we recall that the assumption

$$Ro\beta \ll 1, \tag{7.1}$$

is utilized to facilitate the Fourier integrals by expanding the integrands in Taylor series in $Ro\beta$, such as e.g. in (A8). One must realize that this implies we assume (cf. (4.20))

$$|-\mathrm{i}\omega + E_\nu k^2| \gg \frac{H(Ro\beta)^2(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2}{|-\mathrm{i}\omega + E_\eta k^2|}, \tag{7.2}$$

for all admissible values of ω, thus, in particular, for ω close (and equal to) zero. Consequently, we require

$$E_\nu E_\eta \Lambda_L^2 \gg H(Ro\beta)^2 \left(\frac{\boldsymbol{\Lambda}_L}{\Lambda_L}\cdot\langle\boldsymbol{B}\rangle\right)^2 \sim (Ro\beta)^2, \tag{7.3}$$

hence, the approach is valid only when both the diffusivities are non-zero, thus for example the limit $E_\nu \ll 1$ and/or $E_\eta \ll 1$ must be taken with care, incorporating (7.3). In other words, setting $\nu = 0$ makes the problem ill posed, because the ω-integrals cease to converge. The case of $\eta = 0$ is similar and at least some of the ω-integrals diverge in this limit. Hence, we must require

$$\nu \neq 0, \quad \text{and} \quad \eta \neq 0, \tag{7.4a,b}$$

for soundness and consistency of the mathematical approach. However, we can easily see from the analysis in § 3 that, for weak turbulence, setting $\eta = 0$ leads to a vanishing α-effect (vanishing of the part of the EMF which is not proportional to $\nabla\langle\boldsymbol{B}\rangle$) for all values of $Ro\beta$, despite irregularity of the ω-integral in the perturbation expansion for $Ro\beta \ll 1$. It is clear from the first line in (3.11) and $\langle\hat{f}_j(\boldsymbol{q})\hat{f}_k(\boldsymbol{q}')\rangle \sim \delta(\omega+\omega')$ that, since the integrand is an odd function of ω, the entire integral over $-\infty < \omega < \infty$ must vanish, even without expanding in Taylor series in $Ro\beta$. Consequently, we arrive at the well-known result that, in a weak turbulence, the mean-field dynamo does not operate when the magnetic diffusivity is too weak and negligible (cf. Moffatt & Dormy 2019).

Equation (7.3) involves non-dimensional Λ_L and $\langle\boldsymbol{B}\rangle$, which by definition are order-unity quantities. It follows that the validity of the current approach can be expressed in a more straightforward manner

$$\frac{\nu\eta}{\mathcal{U}^2\mathcal{L}^2} \gg \beta^2, \quad \text{or equivalently } \frac{Pm}{Rm_L^2\varepsilon^2} \gg \beta^2, \tag{7.5}$$

where $Rm_L = \mathcal{U}L/\eta$ is the magnetic Reynolds number based on the global length scale L, a useful parameter in the description of the MHD turbulence in natural systems, in particular the theory of natural dynamos. From the point of view of applications to real astrophysical systems, the condition (7.5) is quite restrictive, since the astrophysical length scales are very large and the condition (7.5) demands that the magnetic Reynolds number Rm_L, which is proportional to L, is 'not too large.' Nevertheless, the upper bound for Rm_L can be very large in particular for high-Pm systems, and then the magnetic Reynolds number is allowed to be very high.

8. Estimates of the turbulent coefficients in natural systems

For the sake of providing sensible estimates of the turbulent coefficients $\breve{\nu}$, $\breve{\eta}$, $\breve{\alpha}$ $\bar{\nu}$, $\bar{\eta}$ and $\bar{\alpha}$ we associate the non-helical part of the forcing $\sim D_0$ with the turbulence driving which, e.g. in natural systems, could be realized by thermal/compositional buoyancy and the helical part $\sim D_1$ with the Coriolis force, which in natural systems is typically responsible for lack of reflexional symmetry in turbulence (cf. discussion below (4.3)). We recall here the expression for force correlations

$$\langle \hat{f}_i(\boldsymbol{k},\omega)\hat{f}_j(\boldsymbol{k}',\omega')\rangle = \left[\frac{D_0}{k^3}P_{ij}(\boldsymbol{k}) + \mathrm{i}\frac{D_1}{k^4}\epsilon_{ijk}\frac{k_k}{k}\right]\delta(\boldsymbol{k}+\boldsymbol{k}')\delta(\omega+\omega'). \tag{8.1}$$

Since

$$\langle \hat{f}_i(\boldsymbol{k},\omega)\hat{f}_j(\boldsymbol{k}',\omega')\rangle \sim \frac{\mathcal{F}^2\mathcal{L}^6}{\omega_s^2}, \tag{8.2}$$

in accordance with the above statement we assume the following for the non-helical driving and the helical part of the forcing:

$$\mathcal{F}^2 = D_0\omega_s, \quad 4\varOmega^2\mathcal{U}^2 = \mathcal{F}^2_{helical} = D_1\omega_s\mathcal{L}, \tag{8.3a,b}$$

where the latter has been estimated with the magnitude of the Coriolis force $\mathcal{F} = 2\varOmega\mathcal{U}$. Taking $\omega_s = \varOmega$ and fixing the length scale at the maximal fluctuational length scale $\mathcal{L}_L$ one obtains

$$D_1 = \frac{4\varOmega\mathcal{U}^2}{\mathcal{L}_L}, \quad D_0 = \frac{\mathcal{F}^2}{\varOmega} = 16\varOmega^3\mathcal{L}_L^2\widetilde{Ra}^2E^4, \tag{8.4a,b}$$

where

$$\widetilde{Ra} = \frac{\mathcal{F}\mathcal{L}_L^3}{\nu^2}, \quad E = \frac{\nu}{2\varOmega\mathcal{L}_L^2}. \tag{8.5a,b}$$

Hence, for example for thermal buoyancy driving $\mathcal{F} = g\beta\Delta T$, where β denotes the thermal expansion coefficient and ΔT the temperature jump across the fluid layer, the parameter $\widetilde{Ra} = g\beta\Delta T\mathcal{L}_L^3/\nu^2$ is proportional to the standard Rayleigh number; E is the Ekman number based on the maximal fluctuational length scale and molecular viscosity. Let us also introduce the new Rossby number

$$\widetilde{Ro} = \frac{\mathcal{U}}{2\varOmega\mathcal{L}_L}. \tag{8.6}$$

Note that, if we assume the magnitudes of the driving $\mathcal{F}$ and the Coriolis force $2\varOmega\mathcal{U}$ to be comparable, then $\widetilde{Ra}E^2 = O(Ro)$ and $\widetilde{Ro} = O(Ro)$.

8.1. *Estimates for strong turbulence in the limit $\mathcal{F} \gg 2\varOmega\mathcal{U}$*

When driving (say thermal) is strong and its magnitude significantly exceeds the magnitude of the Coriolis force the situation corresponds to that defined by (6.9); the latter is equivalent to $\mathcal{F} \gg 2\varOmega\mathcal{U}$ by (8.4*a,b*). Thus, in the current case, the turbulent coefficients can be estimated using (6.10*a,b*) and (6.13), which yields

$$\bar{\nu} = \breve{\nu}(\varLambda_L) = \left(\frac{\pi^2 D_0}{5}\right)^{1/3} \frac{1}{\varLambda_L^{4/3}} \approx \left(\frac{\mathcal{F}^2\mathcal{L}_L^4}{80\pi^2\varOmega}\right)^{1/3}$$
$$\approx 0.11\varOmega\mathcal{L}_L^2\left(\frac{\mathcal{F}}{\varOmega^2\mathcal{L}_L}\right)^{2/3} \approx 0.27\varOmega\mathcal{L}_L^2(\widetilde{Ra}E^2)^{2/3}, \qquad (8.7a)$$

$$\bar{\eta} = \breve{\eta}(\varLambda_L) = a\bar{\nu} \approx 0.74\varOmega\mathcal{L}_L^2(\widetilde{Ra}E^2)^{2/3}, \qquad (8.7b)$$

$$\bar{\alpha} = \breve{\alpha}(\varLambda_L) = \frac{10D_1}{(2a+1)D_0}\bar{\nu} \approx 1.56\frac{4\varOmega^2\mathcal{U}^2}{\mathcal{L}_L}\left(\frac{\mathcal{L}_L^4}{80\pi^2\varOmega\mathcal{F}^4}\right)^{1/3}$$
$$\approx \varOmega\mathcal{L}_L\frac{5.8}{\pi^{2/3}}\widetilde{Ro}^2\left(\frac{\varOmega^2\mathcal{L}_L}{\mathcal{F}}\right)^{4/3} = 0.43\varOmega\mathcal{L}_L\widetilde{Ro}^2(\widetilde{Ra}E^2)^{-4/3}. \qquad (8.7c)$$

Such expressions may be useful in particular applications of the theory to the MHD turbulence. For example, in natural systems such as stellar and planetary interiors or accretion disks, the observational data allow us to estimate the parameters $\varOmega$, $\mathcal{L}_L$, $\widetilde{Ro}$, $\widetilde{Ra}$ and E for a particular system and in turn obtain estimates of the turbulent diffusivities and the magnitude of the α-effect with the use of (8.7*a–c*).

It is also of interest to point out that similar estimates can be made for the case of weak turbulence under the assumptions $Ro \ll Ro\beta \ll 1$, in which case $\bar{\nu} \approx 1.7 \times 10^{-3}\varOmega\mathcal{L}_L^2\widetilde{Ra}E^2$, $\bar{\eta} = \bar{\nu}10Pm/(1+Pm)$ and $\bar{\alpha} = -(\bar{\eta}/\mathcal{L}_L)(\widetilde{Ro}^2/E^4\widetilde{Ra}^2)$. However, the case of strong MHD turbulence seems to be much more common in natural systems.

8.2. *The α-effect in weak turbulence with very weak molecular diffusivities ν and η*

It is evident from § 7 that the presented theory is valid only for non-zero molecular viscosity and molecular magnetic resistivity and, moreover, there exists a lower bound for their product (7.3). However, there are known and important cases of vigorous plasma flow when the molecular diffusivities of the plasma are extremely small, such as for example in the interstellar galactic medium, intracluster medium and some accretion disks. A consistent description of the MHD turbulence in such objects requires estimates of the magnitude of the α-effect when both E_ν and E_η are very small and (7.3) is not satisfied in order to properly incorporate the dynamo effect. Therefore, it seems useful to provide such an estimate at least for the simplest case of weak turbulence, when we can neglect the effect of turbulence on the diffusivities, to clearly demonstrate the differences between such a case and the case of finite diffusivities characterized by (7.3). Neglecting of the mean-field gradients, the mean EMF can be calculated as follows (cf. § 3 or Moffatt &

Dormy 2019; Mizerski 2020):

$$
\begin{aligned}
\mathcal{E}_i &= \epsilon_{ijk}\langle u_j b_k\rangle \\
&= \epsilon_{ijk}\int \mathrm{d}^4q \int \mathrm{d}^4q' \frac{\boldsymbol{k}'\cdot\langle\boldsymbol{B}\rangle}{(\omega'+\mathrm{i}\eta k'^2)\sigma(\boldsymbol{q}')\sigma(\boldsymbol{q})}\langle \hat{f}_j(\boldsymbol{q})\hat{f}_k(\boldsymbol{q}')\rangle\, \mathrm{e}^{\mathrm{i}[(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}-(\omega+\omega')t]} \\
&= -\mathrm{i}\epsilon_{ijk}\epsilon_{jkl}D_1\int \mathrm{d}^4q \frac{\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle}{k^5|\sigma(\boldsymbol{q})|^2(\omega-\mathrm{i}\eta k^2)}k_l \\
&= 2\eta D_1\langle B\rangle_m\int \mathrm{d}^4q\frac{k_mk_i}{k^3|\sigma(\boldsymbol{q})|^2(\omega^2+\eta^2k^4)},
\end{aligned}
\tag{8.8}
$$

where we recall

$$
\sigma(\boldsymbol{q}) = \omega + \mathrm{i}\nu k^2 - \frac{(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2}{\omega+\mathrm{i}\eta k^2}, \tag{8.9}
$$

and since $|\sigma(\boldsymbol{q})|^2(\omega^2+\eta^2k^4)$ is an even function of ω we have used

$$
\int_{-\infty}^{\infty}\mathrm{d}\omega\omega/|\sigma(\boldsymbol{q})|^2(\omega^2+\eta^2k^4) = 0. \tag{8.10}
$$

Next, we express $|\sigma(\boldsymbol{q})|^2$ in the following form:

$$
\begin{aligned}
|\sigma(\boldsymbol{q})|^2 &= \left|\omega\left(1-\frac{(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2}{\omega^2+\eta^2k^4}\right)+\mathrm{i}k^2\left[\nu+\eta\frac{(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2}{\omega^2+\eta^2k^4}\right]\right|^2 \\
&= \frac{\omega^2(\omega^2+\eta^2k^4-(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2)^2+k^4[\nu(\omega^2+\eta^2k^4)+\eta(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2]^2}{(\omega^2+\eta^2k^4)^2} \\
&= \frac{\omega^4+\omega^2[k^4(\nu^2+\eta^2)-2(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2]+(\nu\eta k^4+(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2)^2}{\omega^2+\eta^2k^4}.
\end{aligned}
\tag{8.11}
$$

Using

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\mathrm{d}\omega\frac{1}{\omega^4+\omega^2[k^4(\nu^2+\eta^2)-2(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2]+(\nu\eta k^4+(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2)^2} \\
&\quad = \frac{\pi}{k^2(\nu+\eta)[\nu\eta k^4+(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2]},
\end{aligned}
\tag{8.12}
$$

and introducing spherical variables (k,θ,φ) with $\theta=0$ on the axis of the mean magnetic field, one obtains

$$
\begin{aligned}
\mathcal{E}_i &= 2\eta D_1\langle B\rangle_m\int\frac{\mathrm{d}^4q}{k^3}\frac{k_mk_i}{\omega^4+\omega^2[k^4(\nu^2+\eta^2)-2(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2]+(\nu\eta k^4+(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle)^2)^2} \\
&= \frac{2\pi\eta D_1}{(\nu+\eta)}\langle B\rangle_m\int_{\Lambda_L}^{\Lambda}\frac{\mathrm{d}k}{k}\int_{-1}^{1}\mathrm{d}(\cos\theta)\int_0^{2\pi}\mathrm{d}\varphi\frac{k_mk_i/k^2}{\nu\eta k^4+k^2\langle B\rangle^2\cos^2\theta}, \\
&= \frac{8\pi^2\eta D_1}{(\nu+\eta)}\frac{\langle B\rangle_i}{\langle B\rangle^2}\int\frac{\mathrm{d}k}{k^3}\left[1-\sqrt{\frac{\nu\eta k^2}{\langle B\rangle^2}}\arctan\sqrt{\frac{\langle B\rangle^2}{\nu\eta k^2}}\right].
\end{aligned}
\tag{8.13}
$$

So far, we have simply calculated the EMF for the weak-turbulence case in a straightforward manner, without making any simplifying assumptions about the magnitudes of the diffusivities or the magnitude of the mean magnetic field. In particular, we have avoided the expansion in small $Ro\beta$ such as in (A8) which would require (7.3). We now make a somewhat opposite assumption to $Ro\beta \ll 1$ that the field is strong enough and/or the diffusivities are weak enough so that

$$\Lambda_L \ll \frac{\langle B \rangle}{\sqrt{\nu\eta}}. \tag{8.14}$$

This allows us to express the EMF in the following way:

$$\begin{aligned}\mathcal{E}_i &\approx \frac{8\pi^2\eta D_1}{(\nu+\eta)}\frac{\langle B\rangle_i}{\langle B\rangle^2}\left\{\frac{1}{2\Lambda_L^2} - \frac{\pi\sqrt{\nu\eta}}{2\langle B\rangle\Lambda_L} + \frac{\nu\eta}{\langle B\rangle^2}\ln\frac{\Lambda}{\Lambda_L}\right\} \\ &\approx \frac{4\pi^2\eta D_1}{(\nu+\eta)\Lambda_L^2}\frac{\langle B\rangle_i}{\langle B\rangle^2}\left\{1 - \frac{\pi\sqrt{\nu\eta}\Lambda_L}{\langle B\rangle}\right\} \\ &\approx \frac{4\pi^2\eta D_1}{(\nu+\eta)\Lambda_L^2}\frac{\langle B\rangle_i}{\langle B\rangle^2}.\end{aligned} \tag{8.15}$$

Utilizing the same approach as in (8.4*a*,*b*) we get the following estimate for the mean $\bar{\alpha}$ coefficient

$$\bar{\alpha} \approx \frac{16\widetilde{Ro}^2}{(1+Pm)}\frac{\Omega^3\mathcal{L}_L^3}{\langle B\rangle^2}. \tag{8.16}$$

Such an estimate is valid in the limit of extremely weak diffusivities which enter the expression only through the magnetic Prandtl number based on molecular diffusivities $Pm = \nu/\eta$. It is noteworthy that the EMF is inversely proportional to $\langle B\rangle^2$ at leading order. Hence, the coefficient $\bar{\alpha}$ is singular at $\langle \boldsymbol{B}\rangle = 0$ but, according to our assumption (8.14), $\langle \boldsymbol{B}\rangle = 0$ is excluded in the considered regime. Still, it is of some interest to observe that the relation (8.14) could be satisfied by a relatively weak mean magnetic field and extremely weak diffusivities, which would make the magnitude of the EMF rather large and thus vivid amplification of the magnetic energy $\langle B\rangle^2$ would be expected.

9. Limitations of the current theory and relations to some previous nonlinear approaches

The main limitations of the current theory result from the assumptions of isotropy, homogeneity and stationarity of the forcing (cf. (2.12)). The forcing has been assumed to act only at small scales and its non-helical and helical components to be related to buoyancy driving and the Coriolis force, respectively. In other words, the latter effects can only be considered at small scales, which seems right for the buoyancy force, but, as shown in Yokoi & Yoshizawa (1993), the large-scale Coriolis force in inhomogeneous turbulence can contribute to the so-called vortex-motive force and the creation of complex large-scale flows; such flows can then significantly alter the large-scale hydromagnetic dynamo process.

Furthermore, the gradients of the mean fields have been assumed weak in the presented theory, according to (4.6*a*,*b*) and (4.9*b*). This is just enough to calculate the leading-order form of the turbulent diffusion, but, in conjunction with the assumed statistical isotropy of the driving force, eliminates some effects which are based on the inhomogeneity of turbulence, in particular the effect of strong gradients of means on the small-scale turbulence, which can contribute to mean-field dynamo. For example, the effect of magnetic pumping or the shear-current effect (Krause & Rädler 1980; Rogachevskii & Kleeorin 2004), the so-called cross-helicity dynamo (Yokoi 2013) and the cross-helicity effect coupled with the mean magnetic strain (Yoshizawa 1990; Yokoi 2013) have been excluded in the presented theory. In addition, stationarity of the forcing has recently been shown by Mizerski (2018*a*,*b*, 2020) to be significantly limiting, as interactions between waves with distinct frequencies provide a powerful mechanism of mean-field dynamo generation, operating even in the absence of magnetic diffusion.

However, the advantage of the presented theory lies in the clarity of the considered model, with a given driving force that conceptually can be attributed to common physical effects. Such an approach allows for clear estimates of transport coefficients in real systems, in contrast to approaches which only relate statistical properties of the turbulent velocity and magnetic fields. The estimates provided here include the nonlinear dynamics of the MHD turbulence and in this sense are more accurate than commonly used estimates of the EMF based on weak (linear) turbulence.

10. Conclusions

The presented analysis was focused on the derivation of the full set of MHD equations describing the dynamics of strong (fully nonlinear) stirred turbulence and the derivation of its statistical characteristics such as the energy and helicity spectra. This includes the evolution equations for the mean velocity and mean magnetic field, likewise the equations for the turbulent fluctuations. We have considered the more general case of helical turbulence, since in many natural systems the helicity plays a crucial role of inducing the large-scale magnetic dynamo effect, thus substantially modifying the physical picture of the MHD turbulence. An important feature of the analysis performed was the inclusion of the effect of the Lorentz force on the flow, hitherto scarcely considered in the literature. We have applied the renormalization technique in the spirit of Moffatt (1983), Yakhot & Orszag (1986) and McComb *et al.* (1992) (see also Smith & Woodruff (1998) for a review of the method), which allowed us to incorporate the effect of the nonlinear terms in the dynamical equations for turbulent fluctuations, in the limit of a small 'Rossby' number Ro, defined as a relative measure of the fluid's inertia with respect to the stirring force. The main results are listed below.

- The full set of the renormalized mean-field MHD equations was derived, including the effective form of the large-scale Lorentz force in the mean Navier–Stokes equation for strong helical turbulence; the Lorentz force was reported to be strongly influenced by the turbulent diffusion.
- Moreover, the complete form of the mean EMF, including the effect of gradients of the mean fields was obtained for the studied parameter regime; both its linear and nonlinear dependences on the large-scale magnetic field, the latter resulting from the action of the Lorentz force, were described within the considered asymptotic limit $Ro \ll Ro\beta \ll 1$.
- The effect of nonlinearities $\nabla \cdot (\boldsymbol{uu} - \boldsymbol{bb})$ and $\nabla \times (\boldsymbol{u} \times \boldsymbol{b})$ and the gradients of the mean fields $\nabla \langle \boldsymbol{U} \rangle$ and $\nabla \langle \boldsymbol{B} \rangle$ on the dynamics of the turbulent fluctuations has been included. This allowed us to obtain all the turbulent renormalized coefficients, such as the mean turbulent diffusivities $\bar{\nu}$ and $\bar{\eta}$, the mean turbulent $\bar{\alpha}$ coefficient, the Lorentz force coefficients $\bar{Q}_p$ and $\bar{Q}$ and the fluctuational diffusivities $\breve{\nu}$ and $\breve{\eta}$ together with the fluctuational $\breve{\alpha}$ coefficient for the two cases of strong non-helical turbulence and strong helical turbulence in the limit $D_0 \Lambda_L \gg D_1$, cf. §§ 5 and 6.

- Furthermore, the turbulent kinetic and magnetic energy spectra and the turbulent helicity spectrum were obtained for strong helical turbulence.
- We have reported that the mean turbulent diffusivities and fluctuational turbulent diffusivities are the same in the case of strong turbulence, i.e. $\bar{\nu} = \breve{\nu}(\Lambda_L)$, $\bar{\eta} = \breve{\eta}(\Lambda_L)$.
- The general recursion differential equations for all the turbulent coefficients are provided in (A54*a*,*b*) and (A91*a*–*d*) with the aid of (A58), (A50*a*,*b*) and (A85*a*–*d*) and with the 'boundary' conditions (6.14*a*–*f*). In particular, it is noteworthy that (A54*a*,*b*) and (A91*d*) contain the influence of the fluctuational magnetic diffusivity on the α-effect. The aforementioned set of recursion differential equations can be used to resolve the subgrid dynamics in numerical simulations, i.e. to provide the effective turbulent coefficients for simulations with a given spatial resolution, incapable of fully resolving the dynamics of small-scale turbulent fluctuations. Assuming the highest wavenumber achievable in a simulation is λ_M, the recursions (A54*a*,*b*), (A91*a*–*d*) with (A58), (A50*a*,*b*) and (A85*a*–*d*) and with the boundary conditions (6.14*a*–*f*) can be solved numerically on the interval $\lambda_M \leq k \leq \Lambda$ (or even for $\Lambda = \infty$), which grasps the effect of subgrid fluctuational dynamics and provides reasonable estimates of effective diffusivities and the α coefficients for the dynamical (4.30*a*,*b*) and (4.31*a*,*b*).
- From the point of view of applications to the MHD turbulence in particular physical systems it is desirable to provide estimates of the turbulent coefficients that can be utilized for the theoretical description of the phenomenon, e.g. in numerical simulations. Such estimates have been provided in § 8 based on an assumed correspondence between the helical component of the driving and the Coriolis force in natural systems.
- It is to be emphasized that the presented analysis is valid only for non-zero diffusivities, therefore no conclusions can be drawn for the limiting case of $\nu = 0$ and/or $\eta = 0$. However, the limit of weak viscosity and/or magnetic diffusivity is accessible within the range of validity of the constraint (7.3).
- Of particular astrophysical interest is the limit when both the molecular viscosity and resistivity are extremely low, unattainable within the presented approach based on renormalization in strong turbulence. However, we have considered such a limit in § 8.2 for the case of weak turbulence defined by the linear evolution of fluctuations (i.e. neglecting the nonlinear terms $\nabla \cdot (\boldsymbol{uu} - \boldsymbol{bb})$ and $\nabla \times (\boldsymbol{u} \times \boldsymbol{b})$ in the evolution equations for the fluctuations). The full form of the EMF for wave packets has been calculated and the $\bar{\alpha}$ coefficient was shown to be proportional to $\langle B \rangle^{-2}$ in the limit $\langle B \rangle / \sqrt{\nu\eta} \gg \Lambda_L$; in this limit the small diffusivities enter the expression for the mean EMF only through the magnetic Prandtl number, $\bar{\alpha} \sim (1 + Pm)^{-1} \langle B \rangle^{-2}$, where $Pm = \nu/\eta$.
- For the sake of clear comparison, all the turbulent coefficients and turbulent energy and helicity spectra were also calculated for the case of weak helical turbulence in the limit $Ro \ll Ro\beta \ll 1$ under the constraint (7.3).
- Naturally, all the renormalized $\bar{\alpha}$ and $\breve{\alpha}$ coefficients are proportional to D_1, that is, the magnitude of the non-reflectionally symmetric component of the stirring force. It follows that the α-effect is impossible if the turbulence is not helical: $(D_1 = 0) \Rightarrow (\bar{\alpha} = 0)$.

A significant limitation of the presented analysis is that the turbulence, i.e. the stirring force, was assumed to be isotropic. This is a great simplification since natural astrophysical and geophysical flows, in the majority of cases, possess at least one distinguished axis associated with the background rotation, which introduces anisotropy. Application of the renormalization group technique to rapidly rotating MHD flows has been considered in Mizerski (2021), within what might be called an 'intermediate turbulence' limit when the effect of nonlinearities is included only at leading order at every step of the renormalization procedure.

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Appendix A

The details of the renormalization procedure applied in order to obtain the mean-field equations are given in here. First of all, we clarify how the ensemble averaging should be understood and explain the concept of a conditional average over a statistical subensemble for short-wavelength modes. We adopt the approach of McComb *et al.* (1992) (cf. also McComb & Watt 1990, 1992). The essential idea of this approach is the introduction of a subensemble of flow realizations including near-chaotic statistical properties for the short-wavelength shell $\lambda_1 < k \leq \Lambda$, but remaining quasi-deterministic for $k \leq \lambda_1$. The subensemble average can be precisely defined and then, utilizing the assumption that in the turbulent cascade the energy transfer in the Fourier space is local (i.e. the assumption of ergodicity of the system), the following crucial properties can be proved:

$$\langle \hat{\boldsymbol{u}}^{<}(\boldsymbol{q})\rangle_c = \hat{\boldsymbol{u}}^{<}(\boldsymbol{q}), \quad \langle \hat{\boldsymbol{u}}^{<}(\boldsymbol{q})\hat{\boldsymbol{u}}^{<}(\boldsymbol{q}')\rangle_c \approx \hat{\boldsymbol{u}}^{<}(\boldsymbol{q})\hat{\boldsymbol{u}}^{<}(\boldsymbol{q}'), \tag{A1a}$$

$$\langle \hat{\boldsymbol{u}}^{>}(\boldsymbol{q}')\rangle_c \approx \langle \hat{\boldsymbol{u}}^{>}(\boldsymbol{q}')\rangle = 0, \quad \langle \hat{\boldsymbol{u}}^{<}(\boldsymbol{q})\hat{\boldsymbol{u}}^{>}(\boldsymbol{q}')\rangle_c \approx \hat{\boldsymbol{u}}^{<}(\boldsymbol{q})\langle \hat{\boldsymbol{u}}^{>}(\boldsymbol{q}')\rangle_c \approx 0, \tag{A1b}$$

$$\langle \hat{\boldsymbol{u}}^{>}(\boldsymbol{q})\hat{\boldsymbol{u}}^{>}(\boldsymbol{q}')\rangle_c \approx \langle \hat{\boldsymbol{u}}^{>}(\boldsymbol{q})\hat{\boldsymbol{u}}^{>}(\boldsymbol{q}')\rangle. \tag{A1c}$$

For details see particularly section IV and the beginning of § V in McComb *et al.* (1992).

We now substitute the expressions for short-wavelength modes from (4.27*a*,*b*) into the conditional averages in the equations for long-wavelength modes in (4.26*a*,*b*). Neglecting higher-order correlations of the type $\langle \hat{u}_i^{>}\hat{u}_j^{>}\hat{f}_k^{>}\rangle_c$ etc. (which eliminates the rests in (4.27*a*,*b*)) and using $\langle \hat{f}_i^{>}\rangle_c = 0$, $\langle \hat{u}_i^{<}\rangle_c = \hat{u}_i^{<}$ and $\langle \hat{b}_i^{<}\rangle_c = \hat{b}_i^{<}$ one obtains

$$\begin{aligned}
&\langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{u}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{u}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle \\
&\quad = -\frac{\mathrm{i}Ro}{\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')} P_{nqp}(\boldsymbol{k}-\boldsymbol{k}')[\langle \hat{f}_m^{>}(\boldsymbol{q}')\mathbb{J}_{qp}^{(u)}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - H\langle \hat{f}_m^{>}(\boldsymbol{q}')\mathbb{J}_{qp}^{(b)}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c] \\
&\qquad - \frac{\mathrm{i}Ro}{\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')} P_{mqp}(\boldsymbol{k}')[\langle \hat{f}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\mathbb{J}_{qp}^{(u)}(\boldsymbol{q}')\rangle_c - H\langle \hat{f}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\mathbb{J}_{qp}^{(b)}(\boldsymbol{q}')\rangle_c] \\
&\qquad + O(Ro(Ro\beta))
\end{aligned} \tag{A2a}$$

$$\begin{aligned}
\langle \hat{u}_i^{>}(\boldsymbol{q}')\hat{u}_j^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle &= \frac{1}{\gamma_u(\boldsymbol{q}')\gamma_u(\boldsymbol{q}-\boldsymbol{q}')}\langle \hat{f}_i^{>}(\boldsymbol{q}')\hat{f}_j^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle \\
&\quad - Ro\varepsilon\frac{P_{jp}(\boldsymbol{k}-\boldsymbol{k}')}{\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')^2} G_{pk}\langle \hat{f}_i^{>}(\boldsymbol{q}')\hat{f}_k^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle
\end{aligned}$$

$$
\begin{aligned}
&- Ro\varepsilon \frac{P_{ip}(\boldsymbol{k}')}{\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}')^2} G_{pk} \langle \hat{f}_j^>(\boldsymbol{q}-\boldsymbol{q}')\hat{f}_k^>(\boldsymbol{q}')\rangle \\
&- \frac{Ro(k_p-k_p')P_{jq}(\boldsymbol{k}-\boldsymbol{k}')}{2\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')} \left[\frac{\langle \hat{f}_i^>(\boldsymbol{q}')\hat{f}_q((\boldsymbol{k}-\boldsymbol{k}')^> - \varepsilon \boldsymbol{G}_p, \omega-\omega')\rangle}{\gamma_\nu((\boldsymbol{k}-\boldsymbol{k}')^> - \varepsilon \boldsymbol{G}_p, \omega-\omega')} \right. \\
&\left. - \frac{\langle \hat{f}_i^>(\boldsymbol{q}')\hat{f}_q((\boldsymbol{k}-\boldsymbol{k}')^> + \varepsilon \boldsymbol{G}_p, \omega-\omega')\rangle}{\gamma_\nu((\boldsymbol{k}-\boldsymbol{k}')^> + \varepsilon \boldsymbol{G}_p, \omega-\omega')} \right] \\
&- \frac{Ro k_p' P_{iq}(\boldsymbol{k}')}{2\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')} \left[\frac{\langle \hat{f}_j^>(\boldsymbol{q}-\boldsymbol{q}')\hat{f}_q(\boldsymbol{k}'^> - \varepsilon \boldsymbol{G}_p, \omega')\rangle}{\gamma_\nu(\boldsymbol{k}'^> - \varepsilon \boldsymbol{G}_p, \omega')} \right. \\
&\left. - \frac{\langle \hat{f}_j^>(\boldsymbol{q}-\boldsymbol{q}')\hat{f}_q(\boldsymbol{k}'^> + \varepsilon \boldsymbol{G}_p, \omega')\rangle}{\gamma_\nu(\boldsymbol{k}'^> + \varepsilon \boldsymbol{G}_p, \omega')} \right] + o(Ro^3), \qquad \text{(A2}b\text{)}
\end{aligned}
$$

$$
\langle \hat{b}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle \hat{b}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle = O(Ro(Ro\beta)), \qquad \text{(A2}c\text{)}
$$

$$
\begin{aligned}
&\langle \hat{b}_i^>(\boldsymbol{q}')\hat{b}_j^>(\boldsymbol{q}-\boldsymbol{q}')\rangle \\
&= -(Ro\beta)^2 \frac{(\boldsymbol{k}'\cdot\langle \boldsymbol{B}\rangle)((\boldsymbol{k}-\boldsymbol{k}')\cdot\langle \boldsymbol{B}\rangle)}{\gamma_u(\boldsymbol{q}')\gamma_\eta(\boldsymbol{q}')\gamma_u(\boldsymbol{q}-\boldsymbol{q}')\gamma_\eta(\boldsymbol{q}-\boldsymbol{q}')} \langle \hat{f}_i^>(\boldsymbol{q}')\hat{f}_j^>(\boldsymbol{q}-\boldsymbol{q}')\rangle + O(Ro^2(Ro\beta)^2), \qquad \text{(A2}d\text{)}
\end{aligned}
$$

$$
\begin{aligned}
&\langle \hat{u}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle \hat{u}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle \\
&= \frac{\mathrm{i}Ro}{\gamma_\nu(\boldsymbol{q}')\gamma_\eta(\boldsymbol{q}-\boldsymbol{q}')}(k_s-k_s')\langle \hat{f}_m^>(\boldsymbol{q}')\mathbb{J}_{ns}^{(ub)}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c + O(Ro(Ro\beta)), \qquad \text{(A2}e\text{)}
\end{aligned}
$$

$$
\begin{aligned}
&\epsilon_{kmn}\langle \hat{u}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle = \frac{\mathrm{i}Ro\beta\epsilon_{kmn}(\boldsymbol{k}-\boldsymbol{k}')\cdot\langle \boldsymbol{B}\rangle_0}{\gamma_u(\boldsymbol{q}')\gamma_u(\boldsymbol{q}-\boldsymbol{q}')\gamma_\eta(\boldsymbol{q}-\boldsymbol{q}')} \langle \hat{f}_m^>(\boldsymbol{q}')\hat{f}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle \\
&- \frac{Ro\varepsilon\epsilon_{kmn}\Gamma_{np}}{\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')\gamma_\eta(\boldsymbol{q}-\boldsymbol{q}')} \langle \hat{f}_m^>(\boldsymbol{q}')\hat{f}_p^>(\boldsymbol{q}-\boldsymbol{q}')\rangle \\
&+ \frac{Ro\epsilon_{kmn}}{2\gamma_\nu(\boldsymbol{q}')\gamma_\eta(\boldsymbol{q}-\boldsymbol{q}')}(k_l-k_l') \left[\frac{\langle \hat{f}_m^>(\boldsymbol{q}')\hat{f}_n((\boldsymbol{k}-\boldsymbol{k}')^> - \varepsilon \boldsymbol{\Gamma}_l, \omega-\omega')\rangle}{\gamma_u((\boldsymbol{k}-\boldsymbol{k}')^> - \varepsilon \boldsymbol{\Gamma}_l, \omega-\omega')} \right. \\
&\left. - \frac{\langle \hat{f}_m^>(\boldsymbol{q}')\hat{f}_n((\boldsymbol{k}-\boldsymbol{k}')^> + \varepsilon \boldsymbol{\Gamma}_l, \omega-\omega')\rangle}{\gamma_u((\boldsymbol{k}-\boldsymbol{k}')^> + \varepsilon \boldsymbol{\Gamma}_l, \omega-\omega')} \right] + o(Ro^3). \qquad \text{(A2}f\text{)}
\end{aligned}
$$

Let us note that the inclusion of the $O(Ro(Ro\beta))$-terms in the fluctuational equations and $O(Ro^3(Ro\beta))$-terms in the equations for the means would lead to inclusion of the effect of the Lorentz force on the turbulent diffusivities. In particular, the effect of the Lorentz force on the turbulent magnetic diffusivity could be important, as it would lead to a complicated dependence $\eta(\langle \boldsymbol{B}\rangle)$. Nevertheless, we will neglect this effect here for simplicity. More precisely, the term (A2*c*) accounts for the Lorentz-force effect of short-wavelength fluctuations on the long-wavelength ones (fluctuational Lorentz force acting on fluctuations), whereas the term (A2*d*) and the first term in (A2*b*) contribute directly to the effective mean Lorentz force, acting on the mean field $\langle \boldsymbol{U}\rangle$ (mean fluctuational Lorentz force acting on the mean velocity); of course, to get the Lorentz force from the first term in (A2*b*) we need to expand the γ_u^{-1} factors in powers of $(Ro\beta)^2$. The former, that is the fluctuational Lorentz force acting on the fluctuational velocity, is of less importance, as it turns out to influence the effective diffusivities, the α-effect and the mean Lorentz force only at higher, negligible orders and this is why it will be neglected. This will still allow us to obtain the leading-order form of the turbulent EMF and the mean Lorentz force in strong turbulence with $Ro \ll Ro\beta \ll 1$, that is with inclusion of the nonlinear evolution of turbulent fluctuations. Substituting once again for $\hat{\boldsymbol{u}}^{>}$ and $\hat{\boldsymbol{b}}^{>}$ from (4.27*a*,*b*) into all the $\mathbb{J}$-terms in (A2*a*) and (A2*e*) and making use of the symmetry $\boldsymbol{q}' \mapsto \boldsymbol{q} - \boldsymbol{q}'$ under the integral $\int \mathrm{d}^4 q'$ one obtains

$$\begin{aligned}
&\theta_{\Lambda_\nu} \int^{\Lambda_\nu} \mathrm{d}^4 q' [\langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{u}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{u}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle] \\
&\quad = -\mathrm{i}Ro\theta_{\Lambda_\nu} \int^{\Lambda_\nu} \mathrm{d}^4 q' \int^{\Lambda_\nu} \mathrm{d}^4 q'' \frac{\hat{u}_q^{<}(\boldsymbol{q}'')P_{nqp}(\boldsymbol{k}-\boldsymbol{k}')\langle \hat{f}_m^{>}(\boldsymbol{q}')\hat{f}_p^{>}(\boldsymbol{q}-\boldsymbol{q}'-\boldsymbol{q}'')\rangle_c}{\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}'-\boldsymbol{q}'')} \\
&\qquad - \mathrm{i}Ro\theta_{\Lambda_\nu} \int^{\Lambda_\nu} \mathrm{d}^4 q' \int^{\Lambda_\nu} \mathrm{d}^4 q'' \frac{\hat{u}_q^{<}(\boldsymbol{q}'')P_{mqp}(\boldsymbol{k}-\boldsymbol{k}')\langle \hat{f}_n^{>}(\boldsymbol{q}')\hat{f}_p^{>}(\boldsymbol{q}-\boldsymbol{q}'-\boldsymbol{q}'')\rangle_c}{\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}'-\boldsymbol{q}'')} \\
&\qquad + O(Ro(Ro\beta)),
\end{aligned} \tag{A3a}$$

$$\begin{aligned}
&\int^{\Lambda} \mathrm{d}^4 q' [\langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle] \\
&\quad = \mathrm{i}Ro\epsilon_{nsr}\epsilon_{rqp} \int^{\Lambda} \mathrm{d}^4 q' \int^{\Lambda} \mathrm{d}^4 q'' \frac{(k_s - k_s')\hat{b}_p^{<}(\boldsymbol{q}-\boldsymbol{q}'-\boldsymbol{q}'')}{\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}'')\gamma_\eta(\boldsymbol{q}-\boldsymbol{q}')} \langle \hat{f}_m^{>}(\boldsymbol{q}')\hat{f}_q^{>}(\boldsymbol{q}'')\rangle_c \\
&\qquad + O(Ro(Ro\beta)),
\end{aligned} \tag{A3b}$$

where now the q'-integrals are taken over an intersection of the domains $\lambda_1 < k' < \Lambda_{max}$ and $\lambda_1 < |\boldsymbol{k} - \boldsymbol{k}'| < \Lambda_{max}$, i.e.

$$\{\boldsymbol{k}' : \lambda_1 < k' < \Lambda_{max}, \lambda_1 < |\boldsymbol{k} - \boldsymbol{k}'| < \Lambda_{max}\}. \tag{A4}$$

Following the approach of Yakhot & Orszag (1986) and Smith & Woodruff (1998) we calculate the q'-integrals to lowest non-trivial order in the distant-interaction limit

$$\frac{k}{k'} \to 0, \quad \frac{\omega}{\omega'} \to 0, \tag{A5a,b}$$

which stems from the assumption of local energy transfer in the Fourier spectrum of a turbulent cascade. The lowest non-trivial order in k/k' and ω/ω' is obtained by simply setting $\boldsymbol{k} = 0$ and $\omega = 0$ in the integrands in the expressions for corrections to the means (A2*b*), (A2*d*) and (A2*f*) (accordingly they will all turn out to be proportional to $\delta(\boldsymbol{k})\delta(\omega)$), but in the formulae for corrections to fluctuations (A3*a*), (A3*b*) we set $\omega = 0$ and we need to expand the integrands in powers of k up to the first order (in the non-magnetic case considered by Yakhot & Orszag (1986) the integrals vanish at the order $(k/k')^0$ and expansion in powers of k/k' of the integrands to the first order is also necessary; this results in a viscous correction of the order $\sim \nu k^2$). The integrals are then calculated by substitution $\boldsymbol{k}' \mapsto \boldsymbol{k}' + \boldsymbol{k}/2$ hence symmetrization of the integration domain which in spherical spectral variables (k, θ, φ) at the first step takes the form

$$\left\{ \boldsymbol{k}' : .\lambda_1 < k' < \frac{\Lambda_{max} + \lambda_1}{2}, -\frac{k'^2 - \lambda_1^2}{kk'} < \cos\theta < \frac{k'^2 - \lambda_1^2}{kk'} \right.$$
$$\left. \text{and } \frac{\Lambda_{max} + \lambda_1}{2} < k' < \Lambda_{max}, -\frac{\Lambda_{max}^2 - k'^2}{kk'} < \cos\theta < \frac{\Lambda_{max}^2 - k'^2}{kk'} \right\}. \tag{A6}$$

In the case when the zeroth-order term $\sim (k/k')^0$ vanishes, no corrections of the order k (and higher) from the integration domain are necessary, hence it simplifies to

$$\{\boldsymbol{k}' : \lambda_1 < k' < \Lambda_{max}\}. \tag{A7}$$

This way the total renormalized corrections from short-wavelength modes in (4.26*a*,*b*) are proportional to k^2, which implies that the lowest non-trivial order in distant interactions produces corrections to diffusivities.

We also utilize the assumption $Ro\beta \ll 1$ to expand the inverse γ_u-factors

$$\frac{1}{\gamma_u(\omega', \boldsymbol{k}')\gamma_u(-\omega', -\boldsymbol{k}')} = \frac{1}{|\gamma_\nu(\omega', k')|^2}$$
$$- H(Ro\beta)^2(\boldsymbol{k}' \cdot \langle \boldsymbol{B}\rangle)^2 \frac{1}{|\gamma_\nu(\omega', k')|^2}\left[\frac{1}{\gamma_\nu(-\omega', k')\gamma_\eta(-\omega', k')} + \frac{1}{\gamma_\nu(\omega', k')\gamma_\eta(\omega', k')}\right]$$
$$+ O((Ro\beta)^4), \tag{A8}$$

where

$$\gamma_u(\omega, k) = -\mathrm{i}\omega + E_\nu k^2 + H(Ro\beta)^2 \frac{(\boldsymbol{k} \cdot \langle \boldsymbol{B}\rangle)^2}{\gamma_\eta}, \tag{A9}$$

$$\gamma_\nu(\omega, k) = -\mathrm{i}\omega + E_\nu k^2, \quad \gamma_\eta(\omega, k) = -\mathrm{i}\omega + E_\eta k^2. \tag{A10a,b}$$

Substituting for the force correlations from (2.12) into (A2*b*), (A2*d*) and (A2*f*), taking the limit of distant wavenumber interactions (A5) and making use of

$$P_{imn}(\boldsymbol{k})\delta_{mn} = 0, \quad P_{imn}(\boldsymbol{k})\epsilon_{mnk} = 0, \quad \text{for all } i, k, \tag{A11a}$$

$$\epsilon_{ijk}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}, \tag{A11b}$$

$$\int \mathrm{d}\mathring{\Omega}\underbrace{k_m \ldots k_n k_k}_{N} = 0, \quad \text{for any odd } N \text{ and all } m, \ldots, n, k, \tag{A11c}$$

$$\int k^2 \, \mathrm{d}\mathring{\Omega}\frac{k_m k_n}{k^4} = \frac{4\pi}{3}\delta_{mn}, \tag{A11d}$$

$$\int k^2 \, \mathrm{d}\mathring{\Omega}\frac{k_m k_n k_p k_q}{k^6} = \frac{4\pi}{15}(\delta_{mn}\delta_{pq} + \delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np}), \tag{A11e}$$

where $\mathring{\Omega}$ denotes a solid angle, we obtain

$$- \theta_{\Lambda_\nu} \int^{\Lambda_\nu} \mathrm{d}^4 q \, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \int^{\Lambda_\nu} \mathrm{d}^4 q' \langle \hat{u}_i^{>}(\boldsymbol{q}')\hat{u}_j^{>}(\boldsymbol{q} - \boldsymbol{q}')\rangle$$
$$= -2(Ro\beta)^2 H D_0 \langle B\rangle_q \langle B\rangle_p \theta_{\Lambda_\nu} \int^{\Lambda_\nu} \frac{\mathrm{d}^4 q'}{k'^{\sigma_0}} \frac{k'_q k'_p P_{ij}(\boldsymbol{k}')(\omega'^2 - E_\nu E_\eta k'^4)}{(\omega'^2 + E_\nu^2 k'^4)^2(\omega'^2 + E_\eta^2 k'^4)}$$

$$
\begin{aligned}
&+ RoE_\nu D_0 \frac{\partial \langle U\rangle_p}{\partial x_k}\theta_{\Lambda_\nu}\int^{\Lambda_\nu}\frac{\mathrm{d}^4 q'}{k'^{\sigma_0-2}}\frac{P_{jp}(\boldsymbol{k}')P_{ik}(\boldsymbol{k}')+P_{ip}(\boldsymbol{k}')P_{jk}(\boldsymbol{k}')}{(\omega'^2+E_\nu^2k'^4)^2}\\
&-\tfrac{1}{2}RoD_0\theta_{\Lambda_\nu}\int^{\Lambda_\nu}\frac{\mathrm{d}^4q'k'_p}{|\gamma_\nu(\boldsymbol{q}')|^2}\left[\frac{D_0}{k'^{\sigma_0}}P_{iq}(\boldsymbol{k}')\right.\\
&\left.+\mathrm{i}\frac{D_1}{k'^{\sigma_1}}\epsilon_{iqt}k'_t\right]\left[\frac{P_{jq}(\boldsymbol{k}'-\varepsilon\boldsymbol{G}_p)(1+\mathrm{i}\varepsilon\boldsymbol{G}_p\boldsymbol{\cdot}\boldsymbol{x})}{\gamma_\nu(\boldsymbol{k}'-\varepsilon\boldsymbol{G}_p,-\omega')}-\frac{P_{jq}(\boldsymbol{k}'+\varepsilon\boldsymbol{G}_p)(1-\mathrm{i}\varepsilon\boldsymbol{G}_p\boldsymbol{\cdot}\boldsymbol{x})}{\gamma_\nu(\boldsymbol{k}'+\varepsilon\boldsymbol{G}_p,-\omega')}\right]\\
&-\tfrac{1}{2}RoD_0\theta_{\Lambda_\nu}\int^{\Lambda_\nu}\frac{\mathrm{d}^4q'k'_p}{|\gamma_\nu(\boldsymbol{q}')|^2}\left[\frac{D_0}{k'^{\sigma_0}}P_{jq}(\boldsymbol{k}')\right.\\
&\left.+\mathrm{i}\frac{D_1}{k'^{\sigma_1}}\epsilon_{jqt}k'_t\right]\left[\frac{P_{iq}(\boldsymbol{k}'-\varepsilon\boldsymbol{G}_p)(1+\mathrm{i}\varepsilon\boldsymbol{G}_p\boldsymbol{\cdot}\boldsymbol{x})}{\gamma_\nu(\boldsymbol{k}'-\varepsilon\boldsymbol{G}_p,-\omega')}-\frac{P_{iq}(\boldsymbol{k}'+\varepsilon\boldsymbol{G}_p)(1-\mathrm{i}\varepsilon\boldsymbol{G}_p\boldsymbol{\cdot}\boldsymbol{x})}{\gamma_\nu(\boldsymbol{k}'+\varepsilon\boldsymbol{G}_p,-\omega')}\right]\\
&+\text{const.}+o(Ro^3)+O((Ro\beta)^4),
\end{aligned}
\tag{A12a}
$$

$$
\begin{aligned}
&\theta_{\Lambda_\eta}\int^{\Lambda_\eta}\mathrm{d}^4q\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x}-\omega t)}\int^{\Lambda_\eta}\mathrm{d}^4q'\langle\hat{b}_i^>(\boldsymbol{q}')\hat{b}_j^>(\boldsymbol{q}-\boldsymbol{q}')\rangle\\
&=(Ro\beta)^2D_0\langle B\rangle_q\langle B\rangle_p\theta_{\Lambda_\eta}\int^{\Lambda_\eta}\frac{\mathrm{d}^4q'}{k'^{\sigma_0}}\frac{k'_qk'_pP_{ij}(\boldsymbol{k}')}{(\omega'^2+E_\nu^2k'^4)(\omega'^2+E_\eta^2k'^4)}\\
&+O(Ro^2(Ro\beta)^2,(Ro\beta)^4),
\end{aligned}
\tag{A12b}
$$

$$
\begin{aligned}
&\epsilon_{kmn}\int^{\Lambda}\mathrm{d}^4q\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x}-\omega t)}\int^{\Lambda}\mathrm{d}^4q'\langle\hat{u}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle\\
&=2Ro\beta E_\eta D_1\langle B\rangle_{0s}\int^{\Lambda}\frac{\mathrm{d}^4q'}{k'^{\sigma_1-2}}\frac{k'_sk'_k}{(\omega'^2+E_\nu^2k'^4)(\omega'^2+E_\eta^2k'^4)}\\
&+4H(Ro\beta)^3E_\eta D_1\langle B\rangle_r\langle B\rangle_s\langle B\rangle_t\int^{\Lambda}\frac{\mathrm{d}^4q'}{k'^{\sigma_1-2}}\frac{k'_rk'_sk'_tk'_k(\omega'^2-E_\nu E_\eta k'^4)}{(\omega'^2+E_\nu^2k'^4)^2(\omega'^2+E_\eta^2k'^4)^2}\\
&-RoE_\eta D_0\frac{\partial\langle B\rangle_n}{\partial x_p}\epsilon_{kmn}\int^{\Lambda}\frac{\mathrm{d}^4q'}{k'^{\sigma_0-2}}\frac{P_{mp}(\boldsymbol{k}')}{(\omega'^2+E_\nu^2k'^4)(\omega'^2+E_\eta^2k'^4)}\\
&+RoD_1\int^{\Lambda}\frac{\mathrm{d}^4q'}{k'^{\sigma_1}}\frac{k'_lk'_k}{\omega'^2+E_\nu^2k'^4}\left[\frac{-\mathrm{i}+\varepsilon\boldsymbol{\Gamma}_l\boldsymbol{\cdot}\boldsymbol{x}}{\mathrm{i}\omega'+E_\eta|\boldsymbol{k}'-\varepsilon\boldsymbol{\Gamma}_l|^2}+\frac{\mathrm{i}+\varepsilon\boldsymbol{\Gamma}_l\boldsymbol{\cdot}\boldsymbol{x}}{\mathrm{i}\omega'+E_\eta|\boldsymbol{k}'+\varepsilon\boldsymbol{\Gamma}_l|^2}\right]\\
&+o(Ro^3)+O((Ro\beta)^5),
\end{aligned}
\tag{A12c}
$$

where the corrections $O((Ro\beta)^4)$ and $O((Ro\beta)^5)$ account for the neglected higher-order terms in the expansion (A8). The three last lines of (A12a) correspond exactly to (3.15) times $Ro\theta_{\Lambda_\nu}$, whereas the first, third and fourth lines of (A12c) correspond directly to (3.14) times $Ro\beta$, with ν and η replaced by E_ν and E_η. It follows that, at the first step of the renormalization procedure, the corrections in the equations for the means generated by the averaged couplings of fluctuations take the final form

$$
-\varepsilon Ro\bar{\partial}_j\theta_{\Lambda_\nu}\int^{\Lambda_\nu}\mathrm{d}^4q\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x}-\omega t)}\int^{\Lambda_\nu}\mathrm{d}^4q'\langle\hat{u}_i^>(\boldsymbol{q}')\hat{u}_j^>(\boldsymbol{q}-\boldsymbol{q}')\rangle
$$

$$= \frac{8\pi^2 H D_0}{15E_\nu^2(E_\nu + E_\eta)} \varepsilon Ro(Ro\beta)^2 \left[4\bar{\nabla}\frac{\langle B\rangle^2}{2} - (\langle \boldsymbol{B}\rangle\cdot\bar{\nabla})\langle \boldsymbol{B}\rangle\right]_i \theta_{\Lambda_\nu}\int_{\lambda_1}^{\Lambda_\nu}\frac{\mathrm{d}k'}{k'^{\sigma_0+2}}$$

$$+ \frac{2\pi^2(5-\sigma_0)D_0}{15E_\nu^2}\varepsilon^2 Ro^2\bar{\nabla}^2\langle U\rangle_i\theta_{\Lambda_\nu}\int_{\lambda_1}^{\Lambda_\nu}\frac{\mathrm{d}k'}{k'^{\sigma_0+2}}, \tag{A13a}$$

$$\varepsilon H Ro\bar{\partial}_j\theta_{\Lambda_\eta}\int^{\Lambda_\eta}\mathrm{d}^4 q\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\int^{\Lambda_\eta}\mathrm{d}^4 q'\langle \hat{b}_i^{>}(\boldsymbol{q}')\hat{b}_j^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle$$

$$= \frac{8\pi^2 H D_0}{15E_\nu E_\eta(E_\nu + E_\eta)} \varepsilon Ro(Ro\beta)^2 \left[4\bar{\nabla}\frac{\langle B\rangle^2}{2} - (\langle \boldsymbol{B}\rangle\cdot\bar{\nabla})\langle \boldsymbol{B}\rangle\right]_i \theta_{\Lambda_\eta}\int_{\lambda_1}^{\Lambda_\eta}\frac{\mathrm{d}k'}{k'^{\sigma_0+2}}, \tag{A13b}$$

$$\tfrac{1}{2}\varepsilon H Ro\bar{\partial}_i\theta_{\Lambda_\eta}\int^{\Lambda_\eta}\mathrm{d}^4 q\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\int^{\Lambda_\eta}\mathrm{d}^4 q'\langle \hat{b}_j^{>}(\boldsymbol{q}')\hat{b}_j^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle$$

$$= \frac{8\pi^2 H D_0}{3E_\nu E_\eta(E_\nu + E_\eta)} \varepsilon Ro(Ro\beta)^2 \left[\bar{\nabla}\frac{\langle B\rangle^2}{2}\right]_i \theta_{\Lambda_\eta}\int_{\lambda_1}^{\Lambda_\eta}\frac{\mathrm{d}k'}{k'^{\sigma_0+2}}, \tag{A13c}$$

$$Ro\epsilon_{kmn}\int^{\Lambda}\mathrm{d}^4 q\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\int^{\Lambda}\mathrm{d}^4 q'\langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle$$

$$= \frac{8\pi^2 D_1}{3E_\nu(E_\nu + E_\eta)} Ro^2\beta\langle B\rangle_k\left[\int_{\lambda_1}^{\Lambda}\frac{\mathrm{d}k'}{k'^{\sigma_1}} - \frac{3}{5}\frac{H(Ro\beta)^2}{E_\nu E_\eta}\langle B\rangle^2\int_{\lambda_1}^{\Lambda}\frac{\mathrm{d}k'}{k'^{\sigma_1+2}}\right]$$

$$- \frac{8\pi^2 D_0}{3E_\nu(E_\eta + E_\eta)} Ro^2\varepsilon(\bar{\nabla}\times\langle \boldsymbol{B}\rangle)_k\int_{\lambda_1}^{\Lambda}\frac{\mathrm{d}k'}{k'^{\sigma_0+2}}, \tag{A13d}$$

where we have used

$$\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{E_\nu E_\eta k'^4 - \omega'^2}{(\omega'^2 + E_\eta^2 k'^4)^2(\omega'^2 + E_\nu^2 k'^4)^2} = \frac{\pi}{2E_\nu^2 E_\eta^2(E_\nu + E_\eta)k'^{10}}, \tag{A14}$$

$$\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{E_\nu E_\eta k'^4 - \omega'^2}{(\omega'^2 + E_\eta^2 k'^4)(\omega'^2 + E_\nu^2 k'^4)^2} = \frac{\pi}{2E_\nu^2(E_\nu + E_\eta)k'^6}. \tag{A15}$$

Furthermore, the corrections from short-wavelength modes to the equations for long-wavelength fluctuations in (A3*a*,*b*) can be expressed as follows:

$$\theta_{\Lambda_\nu}\int^{\Lambda_\nu}\mathrm{d}^4 q'[\langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{u}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{u}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle]$$

$$= -\mathrm{i}RoD_0\hat{u}_q^{<}(\boldsymbol{q})\theta_{\Lambda_\nu}\int^{\Lambda_\nu}\frac{\mathrm{d}^4 q'}{k'^{\sigma_0}}\frac{P_{nqp}(\boldsymbol{k}-\boldsymbol{k}')P_{mp}(\boldsymbol{k}')}{\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')\gamma_\nu(-\boldsymbol{q}')}$$

$$- \mathrm{i}RoD_0\hat{u}_q^{<}(\boldsymbol{q})\theta_{\Lambda_\nu}\int^{\Lambda_\nu}\frac{\mathrm{d}^4 q'}{k'^{\sigma_0}}\frac{P_{mqp}(\boldsymbol{k}-\boldsymbol{k}')P_{np}(\boldsymbol{k}')}{\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')\gamma_\nu(-\boldsymbol{q}')}$$

$$+ O(Ro(Ro\beta)), \tag{A16a}$$

$$\int^{\Lambda} \mathrm{d}^4 q' [\langle \hat{u}_m^{>}(\boldsymbol{q}') \hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle \hat{u}_m^{>}(\boldsymbol{q}') \hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle]$$

$$= \mathrm{i} Ro \hat{b}_p^{<}(\boldsymbol{q}) \epsilon_{nsr}\epsilon_{rqp} \int^{\Lambda} \mathrm{d}^4 q' \frac{k_s - k_s'}{\gamma_\nu(\boldsymbol{q}')\gamma_\nu(-\boldsymbol{q}')\gamma_\eta(\boldsymbol{q}-\boldsymbol{q}')} \left[\frac{D_0}{k'^{\sigma_0}} P_{mq}(\boldsymbol{k}') + \mathrm{i}\frac{D_1}{k'^{\sigma_1}}\epsilon_{mql}k_l'\right]$$
$$+ O(Ro(Ro\beta)). \tag{A16b}$$

Making the aforementioned substitution $\boldsymbol{k}' \to \boldsymbol{k}' + \frac{1}{2}\boldsymbol{k}$ to symmetrize the domain of integration one obtains

$$\theta_{\Lambda_\nu} \int^{\Lambda_\nu} \mathrm{d}^4 q' [\langle \hat{u}_m^{>}(\boldsymbol{q}') \hat{u}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle \hat{u}_m^{>}(\boldsymbol{q}') \hat{u}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle]$$

$$= -\mathrm{i} Ro D_0 \hat{u}_q^{<}(\boldsymbol{q})\theta_{\Lambda_\nu} \int^{\Lambda_\nu} \frac{\mathrm{d}^4 q'}{(k'^2 + \boldsymbol{k}'\boldsymbol{\cdot}\boldsymbol{k})^{\sigma_0/2}} \frac{P_{nqp}\left(\frac{1}{2}\boldsymbol{k} - \boldsymbol{k}'\right) P_{mp}(\boldsymbol{k}' + \frac{1}{2}\boldsymbol{k})}{|\gamma_\nu(\boldsymbol{k}' + \frac{1}{2}\boldsymbol{k}, \omega')|^2 \gamma_\nu(\boldsymbol{k}' - \frac{1}{2}\boldsymbol{k}, -\omega')}$$
$$- \mathrm{i} Ro D_0 \hat{u}_q^{<}(\boldsymbol{q})\theta_{\Lambda_\nu} \int^{\Lambda_\nu} \frac{\mathrm{d}^4 q'}{(k'^2 + \boldsymbol{k}'\boldsymbol{\cdot}\boldsymbol{k})^{\sigma_0/2}} \frac{P_{mqp}(\frac{1}{2}\boldsymbol{k} - \boldsymbol{k}') P_{np}(\boldsymbol{k}' + \frac{1}{2}\boldsymbol{k})}{|\gamma_\nu(\boldsymbol{k}' + \frac{1}{2}\boldsymbol{k}, \omega')|^2 \gamma_\nu(\boldsymbol{k}' - \frac{1}{2}\boldsymbol{k}, -\omega')}$$
$$+ O(Ro(Ro\beta)), \tag{A17a}$$

$$\epsilon_{kmn} \int^{\Lambda} \mathrm{d}^4 q' [\langle \hat{u}_m^{>}(\boldsymbol{q}') \hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle \hat{u}_m^{>}(\boldsymbol{q}') \hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle]$$

$$= \mathrm{i} Ro D_0 \hat{b}_p^{<}(\boldsymbol{q}) \epsilon_{kmn}\epsilon_{nsr} \int^{\Lambda} \frac{\mathrm{d}^4 q'}{(k'^2 + k_t' k_t)^{\sigma_0/2}} \frac{\epsilon_{rqp} P_{mq}(\boldsymbol{k}' + \frac{1}{2}\boldsymbol{k})(\frac{1}{2}k_s - k_s')}{|\gamma_\nu(\boldsymbol{k}' + \frac{1}{2}\boldsymbol{k}, \omega')|^2 \gamma_\eta(\boldsymbol{k}' - \frac{1}{2}\boldsymbol{k}, -\omega')}$$
$$+ Ro D_1 \hat{b}_p^{<}(\boldsymbol{q}) \epsilon_{kmn}\epsilon_{nsr} \int^{\Lambda} \frac{\mathrm{d}^4 q'}{(k'^2 + k_t' k_t)^{\sigma_1/2}} \frac{\epsilon_{rqp}\epsilon_{mql}(k_l' + \frac{1}{2}k_l)(k_s' - \frac{1}{2}k_s)}{|\gamma_\nu(\boldsymbol{k}' + \frac{1}{2}\boldsymbol{k}, \omega')|^2 \gamma_\eta(\boldsymbol{k}' - \frac{1}{2}\boldsymbol{k}, -\omega')}$$
$$+ O(Ro(Ro\beta)). \tag{A17b}$$

Next, we use the following expansions in k/k' up to the first order:

$$P_{mp}\left(\boldsymbol{k}' + \tfrac{1}{2}\boldsymbol{k}\right) = P_{mp}(\boldsymbol{k}') + \frac{k_m' k_p' k_r'}{k'^4} k_r - \frac{k_m' k_p + k_m k_p'}{2k'^2}, \tag{A18}$$

$$P_{nqp}\left(\tfrac{1}{2}\boldsymbol{k} - \boldsymbol{k}'\right) = -P_{nqp}(\boldsymbol{k}') + 2\frac{k_n' k_p' k_q' k_r'}{k'^4} k_r - \frac{k_n' k_q' k_p + k_n' k_p' k_q + 2k_n k_p' k_q'}{2k'^2}$$
$$+ \tfrac{1}{2}k_q P_{np}(\boldsymbol{k}') + \tfrac{1}{2}k_p P_{nq}(\boldsymbol{k}'), \tag{A19}$$

$$\frac{1}{|\gamma_\nu(\boldsymbol{k}' + \frac{1}{2}\boldsymbol{k}, \omega')|^2 \gamma_\nu(\boldsymbol{k}' - \frac{1}{2}\boldsymbol{k}, -\omega')} = \frac{-\mathrm{i}\omega' + E_\nu k'^2 - E_\nu \boldsymbol{k}' \cdot \boldsymbol{k}}{(\omega'^2 + E_\nu^2 k'^4)^2}, \tag{A20}$$

$$\frac{1}{|\gamma_\nu(\boldsymbol{k}' + \frac{1}{2}\boldsymbol{k}, \omega')|^2 \gamma_\eta(\boldsymbol{k}' - \frac{1}{2}\boldsymbol{k}, -\omega')} = \frac{-\mathrm{i}\omega' + E_\eta k'^2}{(\omega'^2 + E_\nu^2 k'^4)(\omega'^2 + E_\eta^2 k'^4)}$$
$$- \frac{E_\eta \boldsymbol{k}' \cdot \boldsymbol{k}}{(\omega'^2 + E_\nu^2 k'^4)(\omega'^2 + E_\eta^2 k'^4)}$$

$$-\frac{2(E_\nu^2-E_\eta^2)\omega'^2k'^2\boldsymbol{k}'\cdot\boldsymbol{k}(-\mathrm{i}\omega'+E_\eta k'^2)}{(\omega'^2+E_\nu^2k'^4)^2(\omega'^2+E_\eta^2k'^4)^2}, \tag{A21}$$

$$\frac{1}{(k'^2+\boldsymbol{k}'\boldsymbol{\cdot}\boldsymbol{k})^{\sigma_0/2}}=\frac{1}{k'^{\sigma_0}}-\frac{\sigma_0\,\boldsymbol{k}'\boldsymbol{\cdot}\boldsymbol{k}}{2k'^{\sigma_0+2}}, \tag{A22}$$

and symmetry arguments such that $\int_{-\infty}^{\infty}\omega' f_s(\omega')\mathrm{d}\omega'=0$ for any function $f_s(\omega')$ symmetric about $\omega'=0$ which allows us to transform (A17*a*,*b*) into

$$\begin{aligned}&-\tfrac{1}{2}\mathrm{i}RoP_{imn}(\boldsymbol{k})\theta_{\varLambda_\nu}\int^{\varLambda_\nu}\mathrm{d}^4q'[\langle\hat{u}_m^>(\boldsymbol{q}')\hat{u}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle_c-\langle\hat{u}_m^>(\boldsymbol{q}')\hat{u}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle]\\&\quad=-\frac{2\pi^2(5-\sigma_0)}{15E_\nu^2}Ro^2D_0k^2\hat{u}_i^<(\boldsymbol{q})\theta_{\varLambda_\nu}\int_{\lambda_1}^{\varLambda_\nu}\frac{\mathrm{d}k'}{k'^{\sigma_0+2}}\\&\qquad+O(Ro^2(Ro\beta),k^3),\end{aligned} \tag{A23a}$$

$$\begin{aligned}&\mathrm{i}Rok_j\epsilon_{ijk}\epsilon_{kmn}\int^{\varLambda}\mathrm{d}^4q'[\langle\hat{u}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle_c-\langle\hat{u}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle]\\&\quad=-\frac{8\pi^2}{3E_\nu(E_\nu+E_\eta)}Ro^2D_0k^2\hat{b}_i^<(\boldsymbol{q})\int_{\lambda_1}^{\varLambda}\frac{\mathrm{d}k'}{k'^{\sigma_0+2}}\\&\qquad+\frac{8\pi^2Ro^2}{3E_\nu(E_\nu+E_\eta)}D_1(\epsilon_{ijk}\mathrm{i}k_j\hat{b}_k^<(\boldsymbol{q}))\int_{\lambda_1}^{\varLambda}\frac{\mathrm{d}k'}{k'^{\sigma_1}}\\&\qquad+O(Ro^2(Ro\beta),k^3),\end{aligned} \tag{A23b}$$

where we have also used

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{(\omega'^2+E_\nu^2k'^4)^2}=\frac{\pi}{2E_\nu^3k'^6}, \tag{A24}$$

$$\int_{-\infty}^{\infty}\mathrm{d}\omega\frac{1}{(\omega'^2+E_\nu^2k'^4)(\omega'^2+E_\eta^2k'^4)}=\frac{\pi}{E_\nu E_\eta(E_\nu+E_\eta)k'^6}. \tag{A25}$$

We now utilize the assumption of narrowness of the first spectral bite $\varLambda-\lambda_1=\delta\lambda\ll 1$ and define the following coefficients which describe the average effect of the short-wavelength fluctuations with wavenumbers from the narrow band $\lambda_1\leq k\leq\varLambda$ on the means (cf. (A13*a*–*d*))

$$\bar{Q}_p(\lambda_1)=1+\frac{8\pi^2}{15E_\nu^2E_\eta(E_\nu+E_\eta)}(\theta_{\varLambda_\eta}E_\nu-4\theta_{\varLambda_\nu}E_\eta)Ro^2D_0\frac{\delta\lambda}{\lambda_1^{\sigma_0+2}}, \tag{A26a}$$

$$\bar{Q}(\lambda_1)=1-\frac{8\pi^2(\theta_{\varLambda_\eta}E_\nu+\theta_{\varLambda_\nu}E_\eta)}{15E_\nu^2E_\eta(E_\nu+E_\eta)}Ro^2D_0\frac{\delta\lambda}{\lambda_1^{\sigma_0+2}} \tag{A26b}$$

$$\bar{E}_\eta(\lambda_1)=E_\eta+\frac{8\pi^2Ro^2D_0}{3E_\nu(E_\nu+E_\eta)}\frac{\delta\lambda}{\lambda_1^{\sigma_0+2}}, \tag{A26c}$$

$$\bar{E}_\nu(\lambda_1)=E_\nu+\frac{2\pi^2(5-\sigma_0)}{15E_\nu^2}Ro^2D_0\theta_{\varLambda_\nu}\frac{\delta\lambda}{\lambda_1^{\sigma_0+2}}, \tag{A26d}$$

$$\bar{\alpha}_A(\lambda_1) = \frac{8\pi^2}{3}\frac{Ro^2}{E_\nu(E_\nu+E_\eta)}\frac{D_1\delta\lambda}{\lambda_1^{\sigma_1}}, \tag{A26e}$$

$$\bar{\alpha}_S(\lambda_1) = \frac{8\pi^2}{5}\frac{Ro^2(Ro\beta)^2 H}{E_\nu^2 E_\eta(E_\nu+E_\eta)}\frac{D_1\delta\lambda}{\lambda_1^{\sigma_1+2}}, \tag{A26f}$$

(where $\bar{\alpha} = \bar{\alpha}_A - \bar{\alpha}_S\langle B\rangle^2$) and the effect of short-wavelength fluctuations on the long-wavelength fluctuations corresponding to the band $\Lambda_L < k < \lambda_1$ (cf. (A23*a*,*b*)) is contained in

$$\breve{E}_\nu(\lambda_1) = E_\nu + \frac{2\pi^2(5-\sigma_0)}{15E_\nu^2}Ro^2 D_0\theta_{\Lambda_\nu}\frac{\delta\lambda}{\lambda_1^{\sigma_0+2}}, \tag{A27a}$$

$$\breve{E}_\eta(\lambda_1) = E_\eta + \frac{8\pi^2}{3E_\nu(E_\nu+E_\eta)}Ro^2 D_0\frac{\delta\lambda}{\lambda_1^{\sigma_0+2}}, \tag{A27b}$$

$$\breve{\alpha}(\lambda_1) = \frac{8\pi^2 Ro^2}{3E_\nu(E_\nu+E_\eta)}D_1\frac{\delta\lambda}{\lambda_1^{\sigma_1}}. \tag{A27c}$$

With the use of those definitions we can write down the dynamical equations (4.8*a*,*b*) and (4.26*a*,*b*) in the new form, with the effect of the short-wavelength modes $\boldsymbol{u}^>$, $\boldsymbol{b}^>$ expressed through the effective EMF and the Lorentz force

$$\begin{aligned}\frac{\partial\langle\boldsymbol{U}\rangle}{\partial t} &+ \varepsilon Ro(\langle\boldsymbol{U}\rangle\boldsymbol{\cdot}\nabla)\langle\boldsymbol{U}\rangle = -\varepsilon\nabla\left(\frac{\langle p\rangle}{\rho} + Ro\beta^2 H\bar{Q}_p(\lambda_1)\frac{\langle B\rangle^2}{2}\right)\\ &+ \varepsilon Ro\beta^2 H\bar{Q}(\lambda_1)(\langle\boldsymbol{B}\rangle\boldsymbol{\cdot}\nabla)\langle\boldsymbol{B}\rangle\\ &+ \bar{E}_\nu(\lambda_1)\varepsilon^2\nabla^2\langle\boldsymbol{U}\rangle - \varepsilon Ro(\boldsymbol{\nabla}\boldsymbol{\cdot}\langle\boldsymbol{u}^<\boldsymbol{u}^<\rangle - H\boldsymbol{\nabla}\boldsymbol{\cdot}\langle\boldsymbol{b}^<\boldsymbol{b}^<\rangle),\end{aligned} \tag{A28a}$$

$$\begin{aligned}\frac{\partial\langle\boldsymbol{B}\rangle}{\partial t} &= \varepsilon\boldsymbol{\nabla}\times[(\bar{\alpha}_A(\lambda_1) - \bar{\alpha}_S(\lambda_1)\langle B\rangle^2)\langle\boldsymbol{B}\rangle] + \varepsilon Ro\beta^{-1}\boldsymbol{\nabla}\times\langle\boldsymbol{u}^<\times\boldsymbol{b}^<\rangle\\ &+ \varepsilon Ro(\langle\boldsymbol{B}\rangle\boldsymbol{\cdot}\bar{\nabla})\langle\boldsymbol{U}\rangle - \varepsilon\frac{Ro}{\beta}(\langle\boldsymbol{U}\rangle\boldsymbol{\cdot}\bar{\nabla})\langle\boldsymbol{B}\rangle + \frac{\varepsilon^2}{\beta}\bar{E}_\eta(\lambda_1)\nabla^2\langle\boldsymbol{B}\rangle,\end{aligned} \tag{A28b}$$

$$\begin{aligned}(-\mathrm{i}\omega + \breve{E}_\nu k^2)\hat{u}_i^<(\boldsymbol{q}) &- \mathrm{i}Ro\beta H(\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle_0)\hat{b}_i^<(\boldsymbol{q}) = \hat{f}_i^<(\boldsymbol{q})\\ &- \tfrac{1}{2}\mathrm{i}RoP_{imn}(\boldsymbol{k})[\mathbb{I}_{mn}^{(u^<)} - H\mathbb{I}_{mn}^{(b^<)} - \langle\mathbb{I}_{mn}^{(u^<)}\rangle + H\langle\mathbb{I}_{mn}^{(b^<)}\rangle]\\ &- Ro\varepsilon P_{ij}(\boldsymbol{k})G_{jk}\hat{u}_k^<(\boldsymbol{q}) + RoH\varepsilon P_{ij}(\boldsymbol{k})\Gamma_{jk}\hat{b}_k^<(\boldsymbol{q})\\ &- \frac{Ro}{2}k_m P_{ij}(\boldsymbol{k})[\hat{u}_j(\boldsymbol{k}^< - \varepsilon\boldsymbol{G}_m,\omega) - \hat{u}_j(\boldsymbol{k}^< + \varepsilon\boldsymbol{G}_m,\omega)]\\ &+ \frac{Ro}{2}k_m P_{ij}(\boldsymbol{k})[\hat{b}_j(\boldsymbol{k}^< - \varepsilon\boldsymbol{\Gamma}_m,\omega) - \hat{b}_j(\boldsymbol{k}^< + \varepsilon\boldsymbol{\Gamma}_m,\omega)],\end{aligned} \tag{A28c}$$

$$\begin{aligned}(-\mathrm{i}\omega + \breve{E}_\eta k^2)\hat{b}_i^<(\boldsymbol{q}) &- \mathrm{i}\alpha(\lambda_1)\epsilon_{ijk}k_j\hat{b}_k^< = \mathrm{i}Ro\beta(\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle_0)\hat{u}_i^<(\boldsymbol{q})\\ &+ \mathrm{i}Rok_j[\mathbb{I}_{ij}^{(u^<b^<)} - \langle\mathbb{I}_{ij}^{(u^<b^<)}\rangle]\\ &+ Ro\varepsilon G_{ij}\hat{b}_j^<(\boldsymbol{q}) - Ro\varepsilon\Gamma_{ij}\hat{u}_j^<(\boldsymbol{q})\end{aligned}$$

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$$-\frac{Ro}{2}k_m[\hat{b}_i(\boldsymbol{k}^< - \varepsilon\boldsymbol{G}_m, \omega) - \hat{b}_i(\boldsymbol{k}^< + \varepsilon\boldsymbol{G}_m, \omega)]$$
$$+\frac{Ro}{2}k_m[\hat{u}_i(\boldsymbol{k}^< - \varepsilon\boldsymbol{\Gamma}_m, \omega) - \hat{u}_i(\boldsymbol{k}^< + \varepsilon\boldsymbol{\Gamma}_m, \omega)]. \qquad (A28d)$$

In order to proceed to the second step of renormalization we need explicit expressions of the type (4.18*a,b*). To that end, we introduce a short notation

$$\gamma_\nu \hat{\boldsymbol{u}}^<(\boldsymbol{q}) - \mathrm{i}Ro\beta H(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)\hat{\boldsymbol{b}}^<(\boldsymbol{q}) = \boldsymbol{r.h.s.}_u = \hat{\boldsymbol{f}}^<(\boldsymbol{q}) + \boldsymbol{rem}_u, \qquad (A29a)$$
$$\gamma_\eta \hat{\boldsymbol{b}}^<(\boldsymbol{q}) - \mathrm{i}\breve{\alpha}(\lambda_1)\boldsymbol{k}\times\hat{\boldsymbol{b}}^< = \boldsymbol{r.h.s.}_b = \mathrm{i}Ro\beta(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)\hat{\boldsymbol{u}}^<(\boldsymbol{q}) + \boldsymbol{rem}_b, \qquad (A29b)$$

with

$$\gamma_\nu = -\mathrm{i}\omega + \breve{E}_\nu k^2, \quad \gamma_\eta = -\mathrm{i}\omega + \breve{E}_\eta k^2. \qquad (A30a,b)$$

From (A29*b*) we can write

$$\hat{\boldsymbol{b}} = \frac{1}{\gamma_\alpha^2}(\gamma_\eta + \mathrm{i}\breve{\alpha}\boldsymbol{k}\times)(\mathrm{i}Ro\beta(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)\hat{\boldsymbol{u}}(\boldsymbol{q}) + \boldsymbol{rem}_b)$$
$$= \frac{Ro\beta(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)}{\gamma_\alpha^2}(\mathrm{i}\gamma_\eta\hat{\boldsymbol{u}}(\boldsymbol{q}) - \breve{\alpha}\boldsymbol{k}\times\hat{\boldsymbol{u}}(\boldsymbol{q})) + \boldsymbol{K}\cdot\boldsymbol{rem}_b, \qquad (A31)$$

where we have defined the operator

$$K_{ik} = \frac{1}{\gamma_\alpha^2}(\gamma_\eta\delta_{ik} + \mathrm{i}\breve{\alpha}\epsilon_{ijk}k_j), \quad \gamma_\alpha^2 = \gamma_\eta^2 - k^2\breve{\alpha}^2. \qquad (A32)$$

Next, on defining

$$\tilde{\gamma}_u = \gamma_\nu + \frac{H(Ro\beta)^2(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)^2}{\gamma_\alpha^2}\gamma_\eta, \qquad (A33)$$

and introducing (A31) into (A29*a*) we get

$$\tilde{\gamma}_u\hat{\boldsymbol{u}}(\boldsymbol{q}) + \mathrm{i}\frac{\breve{\alpha}}{\gamma_\alpha^2}H(Ro\beta)^2(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)^2\boldsymbol{k}\times\hat{\boldsymbol{u}}(\boldsymbol{q}) = \boldsymbol{r.h.s.}_u + \mathrm{i}Ro\beta H(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)\boldsymbol{K}\cdot\boldsymbol{rem}_b. \qquad (A34)$$

Inverting the latter equation yields

$$\hat{\boldsymbol{u}}(\boldsymbol{q}) = \tilde{\boldsymbol{K}}\cdot[\boldsymbol{r.h.s.}_u + \mathrm{i}Ro\beta H(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)\boldsymbol{K}\cdot\boldsymbol{rem}_b]$$
$$= \tilde{\boldsymbol{K}}\cdot\hat{\boldsymbol{f}}(\boldsymbol{q}) + \tilde{\boldsymbol{K}}\cdot\boldsymbol{rem}_u + \mathrm{i}Ro\beta H(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)\tilde{\boldsymbol{K}}\cdot\boldsymbol{K}\cdot\boldsymbol{rem}_b, \qquad (A35a)$$
$$\hat{\boldsymbol{b}}(\boldsymbol{q}) = \mathrm{i}Ro\beta(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)\boldsymbol{K}\cdot\tilde{\boldsymbol{K}}\cdot\hat{\boldsymbol{f}}(\boldsymbol{q}) + \mathrm{i}Ro\beta(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)\boldsymbol{K}\cdot\tilde{\boldsymbol{K}}\cdot\boldsymbol{rem}_u$$
$$- H(Ro\beta)^2(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)^2\boldsymbol{K}\cdot\tilde{\boldsymbol{K}}\cdot\boldsymbol{K}\cdot\boldsymbol{rem}_b + \boldsymbol{K}\cdot\boldsymbol{rem}_b, \qquad (A35b)$$

where

$$\tilde{K}_{ik} = \frac{1}{\tilde{\gamma}_\alpha^2}\left(\tilde{\gamma}_u\delta_{ik} - \mathrm{i}\frac{\breve{\alpha}}{\gamma_\alpha^2}H(Ro\beta)^2(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)^2\epsilon_{ijk}k_j\right), \qquad (A36)$$

$$\tilde{\gamma}_\alpha^2 = \tilde{\gamma}_u^2 - k^2\frac{\breve{\alpha}^2}{\gamma_\alpha^4}H^2(Ro\beta)^4(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)^4. \qquad (A37)$$

We can now substitute back for the reminders $\boldsymbol{rem}_u$ and $\boldsymbol{rem}_b$, which leads to explicit expressions for $\hat{u}_i(\boldsymbol{q})$ and $\hat{b}_i(\boldsymbol{q})$

$$\begin{aligned}
\hat{u}_i(\boldsymbol{q}) = {} & \tilde{K}_{ij}\hat{f}_j(\boldsymbol{q}) - \tfrac{1}{2}\mathrm{i}Ro\tilde{K}_{ij}P_{jmn}(\boldsymbol{k})[\mathbb{I}^{(u)}_{mn} - H\mathbb{I}^{(b)}_{mn} - \langle\mathbb{I}^{(u)}_{mn}\rangle + H\langle\mathbb{I}^{(b)}_{mn}\rangle] \\
& - Ro(Ro\beta)H(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)k_m\tilde{K}_{ij}K_{jk}(\mathbb{I}^{(ub)}_{km} - \langle\mathbb{I}^{(ub)}_{km}\rangle) \\
& - Ro\varepsilon\tilde{K}_{ij}P_{jn}(\boldsymbol{k})G_{nk}\hat{u}_k(\boldsymbol{q}) + RoH\varepsilon\tilde{K}_{ij}P_{jn}(\boldsymbol{k})\varGamma_{nk}\hat{b}_k(\boldsymbol{q}) \\
& + \mathrm{i}HRo(Ro\beta)\varepsilon(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)\tilde{K}_{ij}K_{jk}G_{km}\hat{b}_m(\boldsymbol{q}) \\
& - \mathrm{i}HRo(Ro\beta)\varepsilon(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)\tilde{K}_{ij}K_{jk}\varGamma_{km}\hat{u}_m(\boldsymbol{q}) \\
& - \frac{Ro}{2}k_m\tilde{K}_{ij}P_{jk}(\boldsymbol{k})[\hat{u}_k(\boldsymbol{k}-\varepsilon\boldsymbol{G}_m,\omega) - \hat{u}_k(\boldsymbol{k}+\varepsilon\boldsymbol{G}_m,\omega)] \\
& + \frac{Ro}{2}k_m\tilde{K}_{ij}P_{jk}(\boldsymbol{k})[\hat{b}_k(\boldsymbol{k}-\varepsilon\boldsymbol{\varGamma}_m,\omega) - \hat{b}_k(\boldsymbol{k}+\varepsilon\boldsymbol{\varGamma}_m,\omega)] \\
& - \tfrac{1}{2}\mathrm{i}Ro(Ro\beta)H(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)k_m\tilde{K}_{ij}K_{jk}[\hat{b}_k(\boldsymbol{k}-\varepsilon\boldsymbol{G}_m,\omega) - \hat{b}_k(\boldsymbol{k}+\varepsilon\boldsymbol{G}_m,\omega)] \\
& + \tfrac{1}{2}\mathrm{i}Ro(Ro\beta)H(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)k_m\tilde{K}_{ij}K_{jk}[\hat{u}_k(\boldsymbol{k}-\varepsilon\boldsymbol{\varGamma}_m,\omega) - \hat{u}_k(\boldsymbol{k}+\varepsilon\boldsymbol{\varGamma}_m,\omega)],
\end{aligned} \tag{A38a}$$

$$\begin{aligned}
\hat{b}_i(\boldsymbol{q}) = {} & \mathrm{i}Ro\beta(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)K_{ij}\tilde{K}_{jk}\hat{f}_k(\boldsymbol{q}) \\
& + \mathrm{i}Ro[K_{il} - H(Ro\beta)^2(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)^2K_{ij}\tilde{K}_{jk}K_{kl}]k_m(\mathbb{I}^{(ub)}_{lm} - \langle\mathbb{I}^{(ub)}_{lm}\rangle) \\
& + \tfrac{1}{2}Ro(Ro\beta)(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)K_{ij}\tilde{K}_{jk}P_{kmn}(\boldsymbol{k})[\mathbb{I}^{(u)}_{mn} - H\mathbb{I}^{(b)}_{mn} - \langle\mathbb{I}^{(u)}_{mn}\rangle + H\langle\mathbb{I}^{(b)}_{mn}\rangle] \\
& - \mathrm{i}Ro(Ro\beta)\varepsilon(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)K_{ij}\tilde{K}_{jk}P_{km}(\boldsymbol{k})G_{mn}\hat{u}_n(\boldsymbol{q}) \\
& + \mathrm{i}Ro(Ro\beta)H\varepsilon(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)K_{ij}\tilde{K}_{jk}P_{km}(\boldsymbol{k})\varGamma_{mn}\hat{b}_n(\boldsymbol{q}) \\
& + Ro\varepsilon[K_{il} - H(Ro\beta)^2(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)^2K_{ij}\tilde{K}_{jk}K_{kl}]G_{lm}\hat{b}_m(\boldsymbol{q}) \\
& - Ro\varepsilon[K_{il} - H(Ro\beta)^2(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)^2K_{ij}\tilde{K}_{jk}K_{kl}]\varGamma_{lm}\hat{u}_m(\boldsymbol{q}) \\
& - \tfrac{1}{2}\mathrm{i}Ro(Ro\beta)(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)k_mK_{ij}\tilde{K}_{jk}P_{kl}(\boldsymbol{k})[\hat{u}_l(\boldsymbol{k}-\varepsilon\boldsymbol{G}_m,\omega) - \hat{u}_l(\boldsymbol{k}+\varepsilon\boldsymbol{G}_m,\omega)] \\
& + \tfrac{1}{2}\mathrm{i}Ro(Ro\beta)(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)k_mK_{ij}\tilde{K}_{jk}P_{kl}(\boldsymbol{k})[\hat{b}_l(\boldsymbol{k}-\varepsilon\boldsymbol{\varGamma}_m,\omega) - \hat{b}_l(\boldsymbol{k}+\varepsilon\boldsymbol{\varGamma}_m,\omega)] \\
& - \tfrac{1}{2}Ro[K_{il} - H(Ro\beta)^2(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)^2K_{ij}\tilde{K}_{jk}K_{kl}]k_m[\hat{b}_l(\boldsymbol{k}-\varepsilon\boldsymbol{G}_m,\omega) \\
& - \hat{b}_l(\boldsymbol{k}+\varepsilon\boldsymbol{G}_m,\omega)] \\
& + \tfrac{1}{2}Ro[K_{il} - H(Ro\beta)^2(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)^2K_{ij}\tilde{K}_{jk}K_{kl}]k_m[\hat{u}_l(\boldsymbol{k}-\varepsilon\boldsymbol{\varGamma}_m,\omega) \\
& - \hat{u}_l(\boldsymbol{k}+\varepsilon\boldsymbol{\varGamma}_m,\omega)].
\end{aligned} \tag{A38b}$$

We now proceed to the next step of the renormalization procedure which consists of a step-by-step elimination of infinitesimally small wavenumber bands from the Fourier spectrum from the short-wavelength side. We introduce λ_2, which satisfies

$$\delta\lambda = \lambda_1 - \lambda_2 \ll 1, \tag{A39}$$

and, again, split the remaining fluctuational Fourier spectrum $\varLambda_L \leq k \leq \lambda_1$ into two parts by defining new variables (but keeping the same notation)

$$\theta(k-\lambda_2)\hat{u}^{<}_i(\boldsymbol{k},\omega) \mapsto \hat{u}^{>}_i(\boldsymbol{k},\omega), \quad (\text{for } \lambda_2 < k < \lambda_1), \tag{A40}$$

$$\theta(\lambda_2 - k)\hat{u}_i^<(\boldsymbol{k}, \omega) \mapsto \hat{u}_i^<(\boldsymbol{k}, \omega), \quad (\text{for } k < \lambda_2), \tag{A41}$$

and same for $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{f}}$. The equations for the fluctuations are also split, as in the first step (cf. (4.26*a*,*b*) and (4.27*a*,*b*)), i.e. we have

$$\begin{aligned}
(-\mathrm{i}\omega + \breve{E}_\nu k^2)\hat{u}_i^<(\boldsymbol{q}) &= \hat{f}_i^<(\boldsymbol{q}) + \mathrm{i}Ro\beta H(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)\hat{b}_i^<(\boldsymbol{q}) \\
&- \tfrac{1}{2}\mathrm{i}RoP_{imn}(\boldsymbol{k})[\mathbb{I}_{mn}^{(u^<)} - H\mathbb{I}_{mn}^{(b^<)} - \langle\mathbb{I}_{mn}^{(u^<)}\rangle + H\langle\mathbb{I}_{mn}^{(b^<)}\rangle] \\
&- \tfrac{1}{2}\mathrm{i}RoP_{imn}(\boldsymbol{k})\left\{\theta_{\Lambda_\nu}\int^{\Lambda_\nu}\mathrm{d}^4q'[\langle\hat{u}_m^>(\boldsymbol{q}')\hat{u}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle\hat{u}_m^>(\boldsymbol{q}')\hat{u}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle]\right. \\
&\left.-H\theta_{\Lambda_\eta}\int^{\Lambda_\eta}\mathrm{d}^4q'[\langle\hat{b}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle\hat{b}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle]\right\} \\
&- Ro\varepsilon P_{ij}(\boldsymbol{k})G_{jk}\hat{u}_k^<(\boldsymbol{q}) + RoH\varepsilon P_{ij}(\boldsymbol{k})\varGamma_{jk}\hat{b}_k^<(\boldsymbol{q}) \\
&- \frac{Ro}{2}P_{ij}(\boldsymbol{k})k_m[\hat{u}_j(\boldsymbol{k}^< - \varepsilon\boldsymbol{G}_m, \omega) - \hat{u}_j(\boldsymbol{k}^< + \varepsilon\boldsymbol{G}_m, \omega)] \\
&+ \frac{Ro}{2}P_{ij}(\boldsymbol{k})k_m[\hat{b}_j(\boldsymbol{k}^< - \varepsilon\boldsymbol{\varGamma}_m, \omega) - \hat{b}_j(\boldsymbol{k}^< + \varepsilon\boldsymbol{\varGamma}_m, \omega)],
\end{aligned} \tag{A42a}$$

$$\begin{aligned}
(-\mathrm{i}\omega + \breve{E}_\eta k^2)\hat{b}_i^<(\boldsymbol{q}) - \mathrm{i}\breve{\alpha}(\lambda_1)\epsilon_{ijk}k_j\hat{b}_k^<(\boldsymbol{q}) &= \mathrm{i}Ro\beta(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)\hat{u}_i^<(\boldsymbol{q}) \\
&+ \mathrm{i}Rok_j[\mathbb{I}_{ij}^{(u^<b^<)} - \langle\mathbb{I}_{ij}^{(u^<b^<)}\rangle] \\
&+ \mathrm{i}Rok_j\epsilon_{ijk}\epsilon_{kmn}\int^{\Lambda}\mathrm{d}^4q'[\langle\hat{u}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle\hat{u}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle] \\
&+ Ro\varepsilon G_{ij}\hat{b}_j^<(\boldsymbol{q}) - Ro\varepsilon\varGamma_{ij}\hat{u}_j^<(\boldsymbol{q}) \\
&- \frac{Ro}{2}k_m[\hat{b}_i(\boldsymbol{k}^< - \varepsilon\boldsymbol{G}_m, \omega) - \hat{b}_i(\boldsymbol{k}^< + \varepsilon\boldsymbol{G}_m, \omega)] \\
&+ \frac{Ro}{2}k_m[\hat{u}_i(\boldsymbol{k}^< - \varepsilon\boldsymbol{\varGamma}_m, \omega) - \hat{u}_i(\boldsymbol{k}^< + \varepsilon\boldsymbol{\varGamma}_m, \omega)],
\end{aligned} \tag{A42b}$$

for the new long-wavelength modes and for the short-wavelength ones it is enough to write

$$\begin{aligned}
\hat{u}_i^>(\boldsymbol{q}) &= \tilde{K}_{ij}\hat{f}_j^>(\boldsymbol{q}) - \tfrac{1}{2}\mathrm{i}Ro\tilde{K}_{ij}P_{jmn}(\boldsymbol{k})[\mathbb{I}_{mn}^{(u^<)} - H\mathbb{I}_{mn}^{(b^<)} - \langle\mathbb{I}_{mn}^{(u^<)}\rangle + H\langle\mathbb{I}_{mn}^{(b^<)}\rangle] \\
&- Ro\varepsilon\frac{P_{in}(\boldsymbol{k})}{\gamma_\nu^2}G_{nm}\hat{f}_m^>(\boldsymbol{q}) - \frac{Ro}{2\gamma_\nu}k_mP_{ij}(\boldsymbol{k})\left[\frac{\hat{f}_j(\boldsymbol{k}^> - \varepsilon\boldsymbol{G}_m, \omega)}{\gamma_\nu(\boldsymbol{k}^> - \varepsilon\boldsymbol{G}_m, \omega)} - \frac{\hat{f}_j(\boldsymbol{k}^> + \varepsilon\boldsymbol{G}_m, \omega)}{\gamma_\nu(\boldsymbol{k}^> + \varepsilon\boldsymbol{G}_m, \omega)}\right] \\
&- \mathrm{i}\frac{Ro}{\gamma_\nu}P_{imn}(\boldsymbol{k})[\mathbb{J}_{mn}^{(u)} - H\mathbb{J}_{mn}^{(b)}] + \cdots,
\end{aligned} \tag{A43a}$$

$$\begin{aligned}
\hat{b}_i^>(\boldsymbol{q}) &= \mathrm{i}Ro\beta(\boldsymbol{k}\cdot\langle\boldsymbol{B}\rangle_0)K_{ij}\tilde{K}_{jk}\hat{f}_k^>(\boldsymbol{q}) + \mathrm{i}RoK_{il}k_m(\mathbb{I}_{lm}^{(u^<b^<)} - \langle\mathbb{I}_{lm}^{(u^<b^<)}\rangle) \\
&- Ro\varepsilon\frac{K_{il}}{\gamma_\nu}\varGamma_{lm}\hat{f}_m^>(\boldsymbol{q}) + \tfrac{1}{2}RoK_{il}k_m\left[\frac{\hat{f}_l(\boldsymbol{k}^> - \varepsilon\boldsymbol{\varGamma}_m, \omega)}{\gamma_\nu(\boldsymbol{k}^> - \varepsilon\boldsymbol{\varGamma}_m, \omega)} - \frac{\hat{f}_l(\boldsymbol{k}^> + \varepsilon\boldsymbol{\varGamma}_m, \omega)}{\gamma_\nu(\boldsymbol{k}^> + \varepsilon\boldsymbol{\varGamma}_m, \omega)}\right] \\
&+ \mathrm{i}RoK_{il}k_m\mathbb{J}_{lm}^{(ub)} + \cdots,
\end{aligned} \tag{A43b}$$

$$K_{ik} = \frac{1}{\gamma_\alpha^2}(\gamma_\eta\delta_{ik} + \mathrm{i}\breve{\alpha}\epsilon_{ijk}k_j), \quad \gamma_\alpha^2 = \gamma_\eta^2 - k^2\breve{\alpha}^2, \tag{A43c}$$

$$\tilde{K}_{ik} = \frac{1}{\tilde{\gamma}_u}\delta_{ik} - \mathrm{i}\frac{\breve{\alpha}}{\tilde{\gamma}_u^2\gamma_\alpha^2}H(Ro\beta)^2(\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle_0)^2\epsilon_{ijk}k_j, \tag{A43d}$$

$$\tilde{\gamma}_u = \gamma_\nu + \frac{H(Ro\beta)^2(\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle_0)^2}{\gamma_\alpha^2}\gamma_\eta, \tag{A43e}$$

as the remaining terms indicated by the dots create negligible corrections within the kept order of accuracy (we neglected terms of order $o(Ro^3)$ and $O(Ro^2(Ro\beta)^2)$; the $\mathbb{J}$-terms will not contribute to the corrections in the equations for the means and in the following calculation of corrections in the fluctuational equations we will neglect terms of order $O(Ro(Ro\beta)^2)$ in the above expressions for $\hat{u}_i^>(\boldsymbol{q})$ and $\hat{b}_i^>(\boldsymbol{q})$; we have also neglected $O((Ro\beta)^4)$ terms in $\tilde{\gamma}_\alpha^2$). Of course now $\langle\boldsymbol{\cdot}\rangle_c$ denotes conditional average over the second shell ($\lambda_2 \leq k \leq \lambda_1$) statistical subensemble.

According to the above remark, for the sake of the fluctuational equations, i.e. calculation of the corrections from short-wavelength couplings of the type $\langle\hat{u}_m^>\hat{b}_n^>\rangle_c - \langle\hat{u}_m^>\hat{b}_n^>\rangle$, all we need from the expressions (A43*a*,*b*) is

$$\hat{u}_i^>(\boldsymbol{q}) = \frac{1}{\gamma_\nu}\hat{f}_i(\boldsymbol{q}) - \mathrm{i}\frac{Ro}{\gamma_\nu}P_{imn}[\mathbb{J}_{mn}^{(u)} - H\mathbb{J}_{mn}^{(b)}] + \cdots, \tag{A44a}$$

$$\hat{b}_i^>(\boldsymbol{q}) = \mathrm{i}Ro\beta\frac{(\boldsymbol{k}\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle_0)}{\gamma_\nu}K_{ij}\hat{f}_j(\boldsymbol{q}) + \mathrm{i}RoK_{im}k_j\mathbb{J}_{mj}^{(ub)} + \cdots. \tag{A44b}$$

Repetition of the sub-steps undertaken in the first step of the renormalization procedure, but with the modified expressions for the short-wavelength modes $\hat{u}_i^>(\boldsymbol{q})$ and $\hat{b}_i^>(\boldsymbol{q})$, leads to modification of the mean and fluctuational EMF (including the turbulent magnetic diffusivities – mean and fluctuational) and the mean Lorentz force. However, it is quite clear from the expressions (A44*a*,*b*) that the fluctuational turbulent viscosity remains uninfluenced by the fluctuational EMF $\breve{\alpha}$ at leading order and is thus unaltered with respect to the previous step of the procedure (recall that we neglect the terms of order $O(Ro(Ro\beta))$ in the fluctuational corrections $\langle\hat{u}_m^>\hat{u}_n^>\rangle_c - \langle\hat{u}_m^>\hat{u}_n^>\rangle$ and $\langle\hat{b}_m^>\hat{b}_n^>\rangle_c - \langle\hat{b}_m^>\hat{b}_n^>\rangle$). The effect of the fluctuational EMF $\breve{\alpha}$ is pronounced in all the remaining short-wavelength couplings as follows:

$$\begin{aligned}&\langle\hat{u}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle\hat{u}_m^>(\boldsymbol{q}')\hat{b}_n^>(\boldsymbol{q}-\boldsymbol{q}')\rangle\\ &\quad= \frac{\mathrm{i}RoK_{nq}(\boldsymbol{q}-\boldsymbol{q}')}{\gamma_\nu(\boldsymbol{q}')}(k_s-k_s')\langle\hat{f}_m^>(\boldsymbol{q}')\mathbb{J}_{qs}^{(ub)}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c + O(Ro(Ro\beta)),\end{aligned} \tag{A45a}$$

$$\begin{aligned}&\langle\hat{u}_i^>(\boldsymbol{q}')\hat{u}_j^>(\boldsymbol{q}-\boldsymbol{q}')\rangle\\ &\quad= \tilde{K}_{ik}(\boldsymbol{q}')\tilde{K}_{jl}(\boldsymbol{q}-\boldsymbol{q}')\langle\hat{f}_k^>(\boldsymbol{q}')\hat{f}_l^>(\boldsymbol{q}-\boldsymbol{q}')\rangle\\ &\qquad- Ro\varepsilon\frac{P_{jp}(\boldsymbol{k}-\boldsymbol{k}')}{\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')^2}G_{pk}\langle\hat{f}_i^>(\boldsymbol{q}')\hat{f}_k^>(\boldsymbol{q}-\boldsymbol{q}')\rangle\\ &\qquad- Ro\varepsilon\frac{P_{ip}(\boldsymbol{k}')}{\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}')^2}G_{pk}\langle\hat{f}_j^>(\boldsymbol{q}-\boldsymbol{q}')\hat{f}_k^>(\boldsymbol{q}')\rangle\\ &\qquad- \frac{Ro(k_p-k_p')P_{jq}(\boldsymbol{k}-\boldsymbol{k}')}{2\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')}\left[\frac{\langle\hat{f}_i^>(\boldsymbol{q}')\hat{f}_q((\boldsymbol{k}-\boldsymbol{k}')^> - \varepsilon\boldsymbol{G}_p, \omega-\omega')\rangle}{\gamma_\nu((\boldsymbol{k}-\boldsymbol{k}')^> - \varepsilon\boldsymbol{G}_p, \omega-\omega')}\right.\end{aligned}$$

$$
- \frac{\langle \hat{f}_i^{>}(\boldsymbol{q}')\hat{f}_q((\boldsymbol{k}-\boldsymbol{k}')^{>}+\varepsilon\boldsymbol{G}_p,\omega-\omega')\rangle}{\gamma_\nu((\boldsymbol{k}-\boldsymbol{k}')^{>}+\varepsilon\boldsymbol{G}_p,\omega-\omega')}\Bigg]
$$

$$
- \frac{Rok'_pP_{iq}(\boldsymbol{k}')}{2\gamma_\nu(\boldsymbol{q}')\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')}\Bigg[\frac{\langle \hat{f}_j^{>}(\boldsymbol{q}-\boldsymbol{q}')\hat{f}_q(\boldsymbol{k}'^{>}-\varepsilon\boldsymbol{G}_p,\omega')\rangle}{\gamma_\nu(\boldsymbol{k}'^{>}-\varepsilon\boldsymbol{G}_p,\omega')}
$$

$$
- \frac{\langle \hat{f}_j^{>}(\boldsymbol{q}-\boldsymbol{q}')\hat{f}_q(\boldsymbol{k}'^{>}+\varepsilon\boldsymbol{G}_p,\omega')\rangle}{\gamma_\nu(\boldsymbol{k}'^{>}+\varepsilon\boldsymbol{G}_p,\omega')}\Bigg] + o(Ro^3), \tag{A45b}
$$

$$
\begin{aligned}
&\langle \hat{b}_i^{>}(\boldsymbol{q}')\hat{b}_j^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle \\
&\quad = -(Ro\beta)^2(\boldsymbol{k}'\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle)((\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle)K_{ik}(\boldsymbol{q}')\tilde{K}_{kl}(\boldsymbol{q}') \\
&\quad \times K_{jm}(\boldsymbol{q}-\boldsymbol{q}')\tilde{K}_{mn}(\boldsymbol{q}-\boldsymbol{q}')\langle \hat{f}_l^{>}(\boldsymbol{q}')\hat{f}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle \\
&\quad + O(Ro^2(Ro\beta)^2),
\end{aligned} \tag{A45c}
$$

$$
\begin{aligned}
&\epsilon_{kmn}\langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle \\
&\quad = \mathrm{i}Ro\beta\epsilon_{kmn}((\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{\cdot}\langle\boldsymbol{B}\rangle_0)\tilde{K}_{mq}(\boldsymbol{q}')K_{np}(\boldsymbol{q}-\boldsymbol{q}')\tilde{K}_{pl}(\boldsymbol{q}-\boldsymbol{q}')\langle \hat{f}_q^{>}(\boldsymbol{q}')\hat{f}_l^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle \\
&\qquad - \frac{Ro\varepsilon\epsilon_{kmn}\tilde{K}_{mq}(\boldsymbol{q}')K_{nl}(\boldsymbol{q}-\boldsymbol{q}')\varGamma_{lp}}{\gamma_\nu(\boldsymbol{q}-\boldsymbol{q}')}\langle \hat{f}_q^{>}(\boldsymbol{q}')\hat{f}_p^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle \\
&\qquad + \tfrac{1}{2}Ro\epsilon_{kmn}\tilde{K}_{mr}(\boldsymbol{q}')K_{nl}(\boldsymbol{q}-\boldsymbol{q}')(k_q-k'_q)\Bigg[\frac{\langle \hat{f}_r^{>}(\boldsymbol{q}')\hat{f}_l((\boldsymbol{k}-\boldsymbol{k}')^{>}-\varepsilon\boldsymbol{\varGamma}_q,\omega-\omega')\rangle}{\gamma_\nu((\boldsymbol{k}-\boldsymbol{k}')^{>}-\varepsilon\boldsymbol{\varGamma}_q,\omega-\omega')} \\
&\qquad - \frac{\langle \hat{f}_r^{>}(\boldsymbol{q}')\hat{f}_l((\boldsymbol{k}-\boldsymbol{k}')^{>}+\varepsilon\boldsymbol{\varGamma}_q,\omega-\omega')\rangle}{\gamma_\nu((\boldsymbol{k}-\boldsymbol{k}')^{>}+\varepsilon\boldsymbol{\varGamma}_q,\omega-\omega')}\Bigg] + o(Ro^3).
\end{aligned} \tag{A45d}
$$

A.1. *Renormalized fluctuational coefficients* $\breve{\nu}$, $\breve{\eta}$ *and* $\breve{\alpha}$

Making use of (4.28*c*) and (2.12) we start by calculating the fluctuational EMF at the second step

$$
\begin{aligned}
&\int^{\varLambda} \mathrm{d}^4q'[\langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle] \\
&\quad = \mathrm{i}RoD_0\hat{b}_w^{<}(\boldsymbol{q})\epsilon_{qst}\epsilon_{trw}\int^{\varLambda}\frac{\mathrm{d}^4q'}{k'^{\sigma_0}}\frac{K_{nq}(\boldsymbol{q}-\boldsymbol{q}')}{|\gamma_\nu(\boldsymbol{q}')|^2}(k_s-k'_s)P_{mr}(\boldsymbol{k}') \\
&\quad - RoD_1\hat{b}_w^{<}(\boldsymbol{q})\epsilon_{qst}\epsilon_{trw}\epsilon_{mrl}\int^{\varLambda}\frac{\mathrm{d}^4q'}{k'^{\sigma_1}}\frac{K_{nq}(\boldsymbol{q}-\boldsymbol{q}')}{|\gamma_\nu(\boldsymbol{q}')|^2}(k_s-k'_s)k'_l \\
&\quad + O(Ro(Ro\beta)),
\end{aligned} \tag{A46}
$$

and, substituting for $K_{nq}(\boldsymbol{q}-\boldsymbol{q}')$, from (A43*c*) one obtains

$$
\begin{aligned}
&\epsilon_{kmn}\int^{\varLambda}\mathrm{d}^4q'[\langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle_c - \langle \hat{u}_m^{>}(\boldsymbol{q}')\hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle] \\
&\quad = \mathrm{i}RoD_0\hat{b}_w^{<}(\boldsymbol{q})\epsilon_{kmn}\epsilon_{nst}\epsilon_{trw}\int^{\varLambda}\frac{\mathrm{d}^4q'}{k'^{\sigma_0}}\frac{\gamma_\eta(\boldsymbol{q}-\boldsymbol{q}')(k_s-k'_s)P_{mr}(\boldsymbol{k}')}{|\gamma_\nu(\boldsymbol{q}')|^2[\gamma_\eta^2(\boldsymbol{q}-\boldsymbol{q}')-(\boldsymbol{k}-\boldsymbol{k}')^2\alpha^2]}
\end{aligned}
$$

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$$+ \mathrm{i}RoD_0\hat{b}^<_w(\boldsymbol{q})\epsilon_{kmn}\epsilon_{qst}\epsilon_{trw}\epsilon_{nuq}\int^{\Lambda}\frac{\mathrm{d}^4q'}{k'^{\sigma_0}}\frac{\mathrm{i}\breve{\alpha}(k_u-k'_u)(k_s-k'_s)P_{mr}(\boldsymbol{k}')}{|\gamma_\nu(\boldsymbol{q}')|^2[\gamma^2_\eta(\boldsymbol{q}-\boldsymbol{q}')-(\boldsymbol{k}-\boldsymbol{k}')^2\alpha^2]}$$
$$- RoD_1\hat{b}^<_w(\boldsymbol{q})\epsilon_{kmn}\epsilon_{nst}\epsilon_{trw}\epsilon_{mrl}\int^{\Lambda}\frac{\mathrm{d}^4q'}{k'^{\sigma_1}}\frac{\gamma_\eta(\boldsymbol{q}-\boldsymbol{q}')(k_s-k'_s)k'_l}{|\gamma_\nu(\boldsymbol{q}')|^2[\gamma^2_\eta(\boldsymbol{q}-\boldsymbol{q}')-(\boldsymbol{k}-\boldsymbol{k}')^2\alpha^2]}$$
$$- RoD_1\hat{b}^<_w(\boldsymbol{q})\epsilon_{kmn}\epsilon_{qst}\epsilon_{trw}\epsilon_{mrl}\epsilon_{nuq}\int^{\Lambda}\frac{\mathrm{d}^4q'}{k'^{\sigma_1}}\frac{\mathrm{i}\breve{\alpha}(k_u-k'_u)(k_s-k'_s)k'_l}{|\gamma_\nu(\boldsymbol{q}')|^2[\gamma^2_\eta(\boldsymbol{q}-\boldsymbol{q}')-(\boldsymbol{k}-\boldsymbol{k}')^2\alpha^2]}$$
$$+ O(Ro(Ro\beta)). \tag{A47}$$

Since

$$\gamma^2_\eta(-\boldsymbol{q}')-k'^2\alpha^2=(\mathrm{i}\omega'+E_\eta k'^2)^2-k'^2\alpha^2=-\omega'^2+E^2_\eta k'^4-k'^2\alpha^2+2\mathrm{i}\omega' E_\eta k'^2, \tag{A48}$$

$$|\gamma^2_\eta(-\boldsymbol{q}')-k'^2\alpha^2|^2=(\omega'^2-E^2_\eta k'^4+k'^2\alpha^2)^2+4\omega'^2E^2_\eta k'^4, \tag{A49}$$

we introduce the following notation

$$\mathcal{I}_1(k')=\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{\omega'^2+E^2_\eta k'^4-\alpha^2k'^2}{(\omega'^2+E^2_\nu k'^4)[(\omega'^2-E^2_\eta k'^4+\alpha^2k'^2)^2+4\omega'^2E^2_\eta k'^4]}, \tag{A50a}$$

$$\mathcal{I}_2(k')=\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{\omega'^2-E^2_\eta k'^4+\alpha^2k'^2}{(\omega'^2+E^2_\nu k'^4)[(\omega'^2-E^2_\eta k'^4+\alpha^2k'^2)^2+4\omega'^2E^2_\eta k'^4]}, \tag{A50b}$$

which allows us to express the fluctuational EMF in the form

$$\mathrm{i}Rok_j\epsilon_{ijk}\epsilon_{kmn}\int^{\Lambda}\mathrm{d}^4q'[\langle\hat{u}^>_m(\boldsymbol{q}')\hat{b}^>_n(\boldsymbol{q}-\boldsymbol{q}')\rangle_c-\langle\hat{u}^>_m(\boldsymbol{q}')\hat{b}^>_n(\boldsymbol{q}-\boldsymbol{q}')\rangle]$$
$$=-\frac{8\pi D_0}{3}Ro^2\breve{E}_\eta\frac{\mathcal{I}_1(\lambda_2)\delta\lambda}{\lambda_2^{\sigma_0-4}}k^2\hat{b}^<_i(\boldsymbol{q})$$
$$+\frac{8\pi D_0}{3}Ro^2\breve{\alpha}\frac{\mathcal{I}_2(\lambda_2)\delta\lambda}{\lambda_2^{\sigma_0-4}}(\mathrm{i}\epsilon_{ijk}k_j\hat{b}^<_k(\boldsymbol{q}))$$
$$+\frac{8\pi D_1}{3}Ro^2\breve{E}_\eta\frac{\mathcal{I}_1(\lambda_2)\delta\lambda}{\lambda_2^{\sigma_1-6}}(\mathrm{i}\epsilon_{ijk}k_j\hat{b}^<_k(\boldsymbol{q}))$$
$$-\frac{8\pi D_1}{3}Ro^2\alpha\frac{\mathcal{I}_2(\lambda_2)\delta\lambda}{\lambda_2^{\sigma_1-4}}k^2\hat{b}^<_i(\boldsymbol{q})$$
$$+O(Ro(Ro\beta),k^3). \tag{A51}$$

This leads to the following expressions for the corrections to the fluctuational diffusivity $\breve{\eta}$ and the $\breve{\alpha}$ coefficient

$$\delta\breve{\eta}=\breve{\eta}(\lambda_1)-\breve{\eta}(\lambda_2)=-\frac{8\pi D_0}{3}\breve{\eta}\frac{\mathcal{I}_1(\lambda_2)\delta\lambda}{\lambda_2^{\sigma_0-4}}-\frac{8\pi D_1}{3}\breve{\alpha}\frac{\mathcal{I}_2(\lambda_2)\delta\lambda}{\lambda_2^{\sigma_1-4}}, \tag{A52}$$

$$\delta\breve{\alpha}=\breve{\alpha}(\lambda_1)-\breve{\alpha}(\lambda_2)=-\frac{8\pi D_1}{3}\breve{\eta}\frac{\mathcal{I}_1(\lambda_2)\delta\lambda}{\lambda_2^{\sigma_1-6}}-\frac{8\pi D_0}{3}\breve{\alpha}\frac{\mathcal{I}_2(\lambda_2)\delta\lambda}{\lambda_2^{\sigma_0-4}}, \tag{A53}$$

and we have returned to the dimensional variables (cf. discussion below (4.20)). It is now clear that in all the following steps of the renormalization procedure no terms with a new structure can appear in the fluctuational equations and thus we can now take the continuous limit $\delta\lambda \to 0$ of the obtain recursions, which yields

$$\frac{\mathrm{d}\breve{\eta}}{\mathrm{d}\lambda} = -\frac{8\pi D_0}{3}\breve{\eta}\frac{\mathcal{I}_1(\lambda)}{\lambda^{\sigma_0-4}} - \frac{8\pi D_1}{3}\breve{\alpha}\frac{\mathcal{I}_2(\lambda)}{\lambda^{\sigma_1-4}}, \tag{A54a}$$

$$\frac{\mathrm{d}\breve{\alpha}}{\mathrm{d}\lambda} = -\frac{8\pi D_1}{3}\breve{\eta}\frac{\mathcal{I}_1(\lambda)}{\lambda^{\sigma_1-6}} - \frac{8\pi D_0}{3}\breve{\alpha}\frac{\mathcal{I}_2(\lambda)}{\lambda^{\sigma_0-4}}. \tag{A54b}$$

The latter constitute the general form of renormalized recursion differential equations for the fluctuational magnetic diffusivity $\breve{\eta}$ and the $\breve{\alpha}$ coefficient. On the other hand, the equation for the fluctuational turbulent viscosity can be obtained from (A27*a*)

$$\frac{\mathrm{d}\breve{\nu}}{\mathrm{d}\lambda} = -\frac{2\pi^2(5-\sigma_0)}{15\breve{\nu}^2}\theta_{\Lambda_\nu}\frac{D_0}{\lambda^{\sigma_0+2}}. \tag{A55}$$

Given that we assume ν and η finite (non-zero), the fact that we have written down the differential recursion equations in the above form means that we are allowed to group terms in the $Ro-$, $Ro\beta-$expansions in the following way:

$$\breve{\eta} = Ro^2(f_1 + O(Ro(Ro\beta))) + Ro^4(f_2 + O(Ro(Ro\beta))) + \cdots . \tag{A56}$$

In other words, at each Ro^{2n} order we can take the leading-order form of the coefficient next to Ro^{2n} (neglecting the $O(Ro(Ro\beta))$ terms) and we are left with a series of the form $\sum_{n=1}^{\infty} f_n Ro^{2n}$. In this way we grasp the full leading-order dependence of the turbulent fluctuational diffusivities and $\breve{\alpha}$ on the wavenumber k in the limit $\Lambda_L \le k \ll \Lambda$ by keeping in the corrections from short-wavelength modes of the form $Ro^2 f(\breve{\nu}, \breve{\eta}, \breve{\alpha})$ at each step of the renormalization procedure, where $f(\cdot)$ denotes here a function different for each of the coefficients $\breve{\nu}, \breve{\eta}, \breve{\alpha}$.

We start by considering the viscosity (A55) which is independent of $\breve{\eta}$ and $\breve{\alpha}$

$$\frac{\mathrm{d}\breve{\nu}^3}{\mathrm{d}\lambda} = -\theta_{\Lambda_\nu}\frac{2\pi^2(5-\sigma_0)D_0}{5\lambda^{\sigma_0+2}}, \tag{A57}$$

which can be solved explicitly with a boundary condition $\breve{\nu}(\Lambda) = \nu$,

$$\breve{\nu}(\lambda) = \left[\nu^3 + \theta_{\Lambda_\nu}\frac{2\pi^2 D_0(5-\sigma_0)}{5(\sigma_0+1)}\left(\frac{1}{\lambda^{\sigma_0+1}} - \frac{1}{\Lambda_\nu^{\sigma_0+1}}\right)\right]^{1/3}, \tag{A58}$$

and since, ultimately, we are interested in the limiting case $\Lambda_L \le k \ll \Lambda$ when the θ_{Λ_ν} coefficient becomes irrelevant, it will be dropped in the following. In strong turbulence, when $\Lambda_L \le \lambda \ll \Lambda$, we may expect

$$\breve{\nu}(\lambda) = \left(\frac{\pi^2 D_0}{5}\right)^{1/3}\frac{1}{\lambda^{4/3}}, \tag{A59}$$

where we have substituted $\sigma_0 = 3$ since this value leads to the Kolmogorov-type scaling for the kinetic energy – cf. § 5.1 and Yakhot & Orszag (1986).

A.1.1. *The case of non-helical turbulence $D_1 = 0$*
When the helical component of the driving vanishes, $D_1 = 0$, we have

$$\breve{\alpha} = 0, \quad \mathcal{I}_1(k) = \frac{\pi}{\breve{\nu}\breve{\eta}(\breve{\nu} + \breve{\eta})k^6}, \tag{A60a,b}$$

and (A54*a*) takes the form

$$(\breve{\nu} + \breve{\eta})\frac{\mathrm{d}\breve{\eta}}{\mathrm{d}\lambda} = -\frac{8\pi^2}{3\breve{\nu}}\frac{D_0}{\lambda^{\sigma_0+2}}. \tag{A61}$$

We take $\sigma_0 = 3$ and change variables according to (A59) in the limit $\Lambda_L \le \lambda \ll \Lambda$, which yields

$$(\breve{\nu} + \breve{\eta})\frac{\mathrm{d}\breve{\eta}}{\mathrm{d}\breve{\nu}} = 10\breve{\nu}. \tag{A62}$$

The general solution of the latter equation obtained via the method of characteristics can be written in the parametric form,

$$\breve{\eta} = aC_0 s - \frac{(\breve{\nu}_0 - C_0)\tilde{a}}{s^{a/\tilde{a}}}, \tag{A63a}$$

$$\breve{\nu} = C_0 s + \frac{\breve{\nu}_0 - C_0}{s^{a/\tilde{a}}}, \tag{A63b}$$

where s is the parameter along the curve $\breve{\eta}(\breve{\nu})$,

$$a = \tfrac{1}{2}(\sqrt{41} - 1), \quad \tilde{a} = \tfrac{1}{2}(\sqrt{41} + 1), \quad C_0 = \frac{\breve{\eta}_0 + \tilde{a}\breve{\nu}_0}{\sqrt{41}}, \tag{A64a–c}$$

and $\breve{\nu}_0 = \breve{\nu}(s = 0)$, $\breve{\eta}_0 = \breve{\eta}(s = 0)$. General solutions are not attainable in the explicit form $\breve{\eta}(\breve{\nu})$, however, in the considered limit of small wavenumber $\lambda \ll \Lambda$ the turbulent viscosity becomes large, which implies that the terms $\sim s^{a/\tilde{a}}$ are negligible and the solution of (A62) takes the following simple form at leading order (a simple example with $a/\tilde{a}$ approximated by 0.5 may be helpful to get some bearings. In such a case (A63*b*) can be inverted with the use of the Cardano's formulae for a cubic equation. In the limit of small wavenumber $\lambda \ll \Lambda$ one can neglect $(\breve{\nu}_0 - C_0)^2/C_0^2$ with respect to $2\breve{\nu}^3/27C_0^3$, which implies that the discriminant of the cubic equation vanishes. It follows that there is only one non-trivial root $s = \breve{\nu}/C_0$ and hence $\eta_f = a\nu_f$ at the leading order):

$$\breve{\eta} = a\breve{\nu}. \tag{A65}$$

We gather the strong-turbulence results for renormalized diffusivities with $\sigma_0 = 3$,

$$\breve{\nu}(k) = \left(\frac{\pi^2 D_0}{5}\right)^{1/3}\frac{1}{k^{4/3}}, \quad \breve{\eta}(k) = a\left(\frac{\pi^2 D_0}{5}\right)^{1/3}\frac{1}{k^{4/3}}, \quad \Lambda_L \le k \ll \Lambda, \tag{A66a–c}$$

and also provide the simple solution of the form $\breve{\eta} = a\breve{\nu}$ for the case of general σ_0

$$\breve{\eta} = a\breve{\nu}(\lambda) = \left[\frac{2\pi^2 D_0(5 - \sigma_0)}{5(\sigma_0 + 1)}\right]^{1/3}\frac{a}{\lambda^{(\sigma_0+1)/3}}, \tag{A67}$$

where in general

$$a = \tfrac{1}{2}\left(\sqrt{1 + \frac{240}{3(5 - \sigma_0)}} - 1\right). \tag{A68}$$

A.1.2. *Helical turbulence with $D_0\varLambda_L \gg D_1$*
In the case at hand the effect of D_1 is weaker than that of D_0 and we have $|\breve{\alpha}|k \ll \breve{\nu}k^2, \breve{\eta}k^2$, which allows us to write

$$\mathcal{I}_1(k) \approx \frac{\pi}{\breve{\nu}\breve{\eta}(\breve{\nu}+\breve{\eta})k^6}, \quad \mathcal{I}_2(k) \approx -\frac{\pi}{\breve{\nu}(\breve{\nu}+\breve{\eta})^2k^6}. \tag{A69a,b}$$

The expressions (A54*a*,*b*) simplify to

$$\frac{\mathrm{d}\breve{\eta}}{\mathrm{d}\lambda} = \frac{8\pi^2 D_0}{3\ell}\frac{\breve{\alpha} - (\ell\lambda)(\breve{\nu}+\breve{\eta})\lambda}{\breve{\nu}(\breve{\nu}+\breve{\eta})^2\lambda^7}, \tag{A70a}$$

$$\frac{\mathrm{d}\breve{\alpha}}{\mathrm{d}\lambda} = \frac{8\pi^2 D_1}{3}\frac{\ell\breve{\alpha} - (\breve{\nu}+\breve{\eta})}{\breve{\nu}(\breve{\nu}+\breve{\eta})^2\lambda^5}, \tag{A70b}$$

where we have introduced the length scale

$$\frac{D_0}{D_1} = \ell. \tag{A71}$$

The current limit is defined by

$$\lambda\ell \gg 1, \tag{A72}$$

which in tandem with $|\breve{\alpha}|k \ll \breve{\nu}k^2, \breve{\eta}k^2$ allows us to further simplify the magnetic diffusivity equation

$$\frac{\mathrm{d}\breve{\eta}}{\mathrm{d}\lambda} = -\frac{8\pi^2 D_0}{3}\frac{1}{\breve{\nu}(\breve{\nu}+\breve{\eta})\lambda^5}. \tag{A73}$$

The latter equation is the same as in the non-helical case (cf. (A61)), thus, in the limit $\varLambda_L \le \lambda \ll \varLambda$, we can write down the solution in the form

$$\breve{\eta} = a\breve{\nu}. \tag{A74}$$

Next we transform (A70*b*) into

$$\frac{\mathrm{d}\left(\frac{\breve{\alpha}}{\breve{\nu}}\right)}{\mathrm{d}\breve{\nu}} + \frac{1}{\breve{\nu}}\left(\frac{\breve{\alpha}}{\breve{\nu}}\right) = -\frac{10}{\ell}\frac{\ell\left(\frac{\breve{\alpha}}{\breve{\nu}}\right) - (1+a)}{(1+a)^2}\frac{1}{\breve{\nu}}, \tag{A75}$$

which in the limit $\varLambda_L \le \lambda \ll \varLambda$ has a solution in the form

$$\breve{\alpha} = \frac{10}{2a+1}\frac{\breve{\nu}}{\ell} = \frac{10D_1}{(2a+1)D_0}\breve{\nu}. \tag{A76}$$

Verification of consistency shows that, indeed,

$$\breve{\alpha} = \frac{10}{2a+1}\frac{\breve{\nu}}{\ell} \ll \breve{\eta}\lambda = a\breve{\nu}\lambda \Longleftrightarrow \lambda\ell \gg 1. \tag{A77}$$

So that, finally, in the limit

$$\frac{D_0\varLambda_L}{D_1} = \frac{2\pi D_0}{\mathcal{L}_L D_1} \gg 1, \quad \varLambda_L \le \lambda \ll \varLambda, \tag{A78}$$

we obtain

$$\breve{\nu}(k) = \frac{A}{k^{4/3}}, \quad \breve{\eta}(k) = \frac{aA}{k^{4/3}}, \quad \breve{\alpha}(k) = \frac{10D_1}{(2a+1)D_0}\frac{A}{k^{4/3}}, \tag{A79a–c}$$

$$A = \left(\frac{\pi^2 D_0}{5}\right)^{1/3}, \quad a = \tfrac{1}{2}(\sqrt{41}-1). \tag{A80a,b}$$

A.2. Renormalized mean coefficients $\bar{\nu}$, $\bar{\eta}$, $\bar{\alpha}$, $\bar{Q}$, $\bar{Q}_p$

Similarly to the case of the fluctuational viscosity, the mean turbulent viscosity at the kept order of accuracy remains the same for both the non-helical and the helical turbulence, and it is obtained from (cf. e.g. (3.19) with $\breve{\nu}$ substituted for ν)

$$\frac{\mathrm{d}\bar{\nu}}{\mathrm{d}\lambda} = -\frac{2\pi^2(5-\sigma_0)D_0}{15\breve{\nu}^2(\lambda)\lambda^{\sigma_0+2}}. \tag{A81}$$

The solution of the latter equation in the case of strong turbulence takes the form

$$\bar{\nu} = \nu + \left(\frac{\pi^2 D_0}{5}\right)^{1/3}\frac{1}{\Lambda_L^{4/3}} \approx \left(\frac{\pi^2 D_0}{5}\right)^{1/3}\frac{1}{\Lambda_L^{4/3}} = \breve{\nu}(\Lambda_L). \tag{A82}$$

Next, we proceed to calculation of the mean EMF

$$\begin{aligned}
&\epsilon_{kmn}\int^{\Lambda}\mathrm{d}^4q\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\langle\hat{u}_m^{>}(\boldsymbol{q}')\hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle \\
&\quad= -\mathrm{i}Ro\beta\epsilon_{kmn}\boldsymbol{k}'\cdot\langle\boldsymbol{B}\rangle_0\tilde{K}_{mq}(\boldsymbol{q}')K_{np}(-\boldsymbol{q}')\tilde{K}_{pl}(-\boldsymbol{q}')\left[\frac{D_0}{k'^{\sigma_0}}P_{ql}(\boldsymbol{k}') + \mathrm{i}\frac{D_1}{k'^{\sigma_1}}\epsilon_{qlk}k_k'\right] \\
&\qquad - \frac{Ro\beta\varepsilon\epsilon_{kmn}K_{nl}(-\boldsymbol{q}')\Gamma_{lp}}{\gamma_\nu(\boldsymbol{q}')\gamma_\nu(-\boldsymbol{q}')}\left[\frac{D_0}{k'^{\sigma_0}}P_{mp}(\boldsymbol{k}') + \mathrm{i}\frac{D_1}{k'^{\sigma_1}}\epsilon_{mpk}k_k'\right] \\
&\qquad - \frac{Ro\epsilon_{kmn}}{2|\gamma_\nu(\boldsymbol{q}')|^2}\left[\frac{D_0}{k'^{\sigma_0}}P_{ml}(\boldsymbol{k}') + \mathrm{i}\frac{D_1}{k'^{\sigma_1}}\epsilon_{mlk}k_k'\right]k_q'[K_{nl}(\varepsilon\boldsymbol{\Gamma}_q-\boldsymbol{k}',-\omega')(1+\mathrm{i}\varepsilon\boldsymbol{\Gamma}_q\cdot\boldsymbol{x}) \\
&\qquad - K_{nl}(-\varepsilon\boldsymbol{\Gamma}_q-\boldsymbol{k}',-\omega')(1-\mathrm{i}\varepsilon\boldsymbol{\Gamma}_q\cdot\boldsymbol{x})] + o(Ro^3).
\end{aligned} \tag{A83}$$

The algebraic manipulations needed to obtain the explicit form of the EMF are cumbersome, but at this stage conceptually straightforward, thus, we now provide the final result

$$\begin{aligned}
\boldsymbol{\mathcal{E}} &= \epsilon_{kmn}\left\langle\int^{\Lambda}\mathrm{d}^4q\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\int^{\Lambda}\mathrm{d}^4q'\langle\hat{u}_m^{>}(\boldsymbol{q}')\hat{b}_n^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle\right\rangle \\
&= \frac{8\pi D_1}{3}Ro\beta\langle B\rangle_k\int_{\lambda_2}^{\lambda_1}\frac{\breve{E}_\eta\,\mathrm{d}k'}{k'^{\sigma_1-6}}\mathcal{I}_1(k') + \frac{8\pi D_0}{3}Ro\beta\langle B\rangle_k\int_{\lambda_2}^{\lambda_1}\frac{\breve{\alpha}\,\mathrm{d}k'}{k'^{\sigma_0-4}}\mathcal{I}_2(k') \\
&\quad + (Ro\beta)^2\frac{8\pi}{5}HRo\beta\langle B\rangle_0^2\langle B\rangle_{0k}\int_{\lambda_2}^{\lambda_1}\mathrm{d}k'\left[\frac{D_0\breve{\alpha}\mathcal{I}_{L2}(k')}{k'^{\sigma_0-6}} - \frac{D_1\mathcal{I}_{L1}(k')}{k'^{\sigma_1-6}}\right] \\
&\quad - \frac{8\pi D_0}{3}Ro\frac{\partial\langle B\rangle_n}{\partial x_m}\epsilon_{kmn}\int_{\lambda_2}^{\lambda_1}\frac{\breve{E}_\eta\,\mathrm{d}k'}{k'^{\sigma_0-4}}\mathcal{I}_1(k') - \frac{8\pi D_1}{3}Ro\frac{\partial\langle B\rangle_n}{\partial x_m}\epsilon_{kmn}\int_{\lambda_2}^{\lambda_1}\frac{\breve{\alpha}\,\mathrm{d}k'}{k'^{\sigma_1-4}}\mathcal{I}_2(k'),
\end{aligned} \tag{A84}$$

where

$$\mathcal{I}_{L1}(k') = \int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{f_{L1}(\omega',k')}{[(\omega'^2-\breve{E}_\eta^2k'^4+\breve{\alpha}^2k'^2)^2+4\breve{E}_\eta^2k'^4\omega'^2]^2(\omega'^2+\breve{E}_\nu^2k'^4)^2}, \tag{A85a}$$

$$f_{L1}(\omega',k') = \breve{E}_\eta k'^2[2\breve{E}_\nu\breve{E}_\eta k'^4(\breve{E}_\eta^2k'^4-\breve{\alpha}^2k'^2+\omega'^2)$$

$$- 2\omega'^2(\breve{E}_\eta^2 k'^4 + \breve{\alpha}^2 k'^2 + \omega'^2)](\breve{E}_\eta^2 k'^4 - \breve{\alpha}^2 k'^2 + \omega'^2)$$
$$+ 2\breve{\alpha}^2 k'^2(\breve{E}_\eta^2 k'^4 - \breve{\alpha}^2 k'^2 - \omega'^2)[\breve{E}_\eta^2 \breve{E}_\nu k'^6 - \breve{\alpha}^2 \breve{E}_\nu k'^4 - (2\breve{E}_\eta + \breve{E}_\nu)\omega'^2 k'^2], \tag{A85b}$$

$$\mathcal{I}_{L2}(k') = \int_{-\infty}^{\infty} \mathrm{d}\omega' \frac{f_{L2}(\omega', k')}{[(\omega'^2 - \breve{E}_\eta^2 k'^4 + \breve{\alpha}^2 k'^2)^2 + 4\breve{E}_\eta^2 k'^4 \omega'^2]^2 (\omega'^2 + \breve{E}_\nu^2 k'^4)^2}, \tag{A85c}$$

$$f_{L2}(\omega', k') = 2\breve{E}_\eta k'^2 (\breve{E}_\eta^2 k'^4 - \breve{\alpha}^2 k'^2 + \omega'^2)(\breve{E}_\eta^2 \breve{E}_\nu k'^6 - \breve{\alpha}^2 \breve{E}_\nu k'^4 - (2\breve{E}_\eta + \breve{E}_\nu)\omega'^2 k'^2)$$
$$+ (\breve{E}_\eta^2 k'^4 - \breve{\alpha}^2 k'^2 - \omega'^2)[2\breve{E}_\nu \breve{E}_\eta k'^4 (\breve{E}_\eta^2 k'^4 - \breve{\alpha}^2 k'^2 + \omega'^2)$$
$$- 2\omega'^2(\breve{E}_\eta^2 k'^4 + \breve{\alpha}^2 k'^2 + \omega'^2)]. \tag{A85d}$$

The curl of the EMF then reads

$$\varepsilon Ro \beta^{-1} \epsilon_{ijk} \frac{\partial}{\partial X_j} \epsilon_{kmn} \left\langle \int^{\Lambda} \mathrm{d}^4 q\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x} - \omega t)} \int^{\Lambda} \mathrm{d}^4 q' \langle \hat{u}_m^>(\boldsymbol{q}') \hat{b}_n^>(\boldsymbol{q} - \boldsymbol{q}') \rangle \right\rangle$$
$$= \frac{8\pi}{3} Ro^2 \left[D_1 \int_{\lambda_2}^{\lambda_1} \frac{\breve{E}_\eta \,\mathrm{d}k'}{k'^{\sigma_1 - 6}} \mathcal{I}_1(k') + D_0 \int_{\lambda_2}^{\lambda_1} \frac{\breve{\alpha}\,\mathrm{d}k'}{k'^{\sigma_0 - 4}} \mathcal{I}_2(k') \right] (\bar{\boldsymbol{\nabla}} \times \langle \boldsymbol{B} \rangle)_i$$
$$+ \frac{8\pi}{5} H Ro^2 (Ro\beta)^2 \int_{\lambda_2}^{\lambda_1} \mathrm{d}k' \left[\frac{D_0 \breve{\alpha} \mathcal{I}_{L2}(k')}{k'^{\sigma_0 - 6}} - \frac{D_1 \mathcal{I}_{L1}(k')}{k'^{\sigma_1 - 6}} \right] [\bar{\boldsymbol{\nabla}} \times (\langle B \rangle^2 \langle \boldsymbol{B} \rangle)]_i$$
$$+ \frac{8\pi}{3} Ro^2 \left[D_0 \int_{\lambda_2}^{\lambda_1} \frac{\breve{E}_\eta \,\mathrm{d}k'}{k'^{\sigma_0 - 4}} \mathcal{I}_1(k') + D_1 \int_{\lambda_2}^{\lambda_1} \frac{\breve{\alpha}\,\mathrm{d}k'}{k'^{\sigma_1 - 4}} \mathcal{I}_2(k') \right] (\bar{\boldsymbol{\nabla}}^2 \langle \boldsymbol{B} \rangle)_i. \tag{A86}$$

Let us point out that, in the limit $D_1/D_0 \ll \Lambda_L$,

$$\mathcal{I}_{L1}(k') \approx 2\breve{E}_\eta k'^2 \int_{-\infty}^{\infty} \mathrm{d}\omega' \frac{\breve{E}_\nu \breve{E}_\eta k'^4 - \omega'^2}{(\omega'^2 + \breve{E}_\eta^2 k'^4)^2 (\omega'^2 + \breve{E}_\nu^2 k'^4)^2} = \frac{\pi}{\breve{E}_\nu^2 \breve{E}_\eta (\breve{E}_\nu + \breve{E}_\eta) k'^8}, \tag{A87a}$$

$$\mathcal{I}_{L2}(k') \approx 2 \int_{-\infty}^{\infty} \mathrm{d}\omega' \frac{2\breve{E}_\nu \breve{E}_\eta^3 k'^8 - (2\breve{E}_\nu + 3\breve{E}_\eta)\breve{E}_\eta k'^4 \omega'^2 + \omega'^4}{(\omega'^2 + \breve{E}_\eta^2 k'^4)^3 (\omega'^2 + \breve{E}_\nu^2 k'^4)^2} = \frac{(\breve{E}_\nu + 2\breve{E}_\eta)\pi}{\breve{E}_\nu^2 \breve{E}_\eta^2 (\breve{E}_\nu + \breve{E}_\eta)^2 k'^{10}}. \tag{A87b}$$

The Lorentz force can also be calculated using

$$\int^{\Lambda} \mathrm{d}^4 q\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x} - \omega t)} \langle \hat{u}_i^>(\boldsymbol{q}') \hat{u}_j^>(\boldsymbol{q} - \boldsymbol{q}') \rangle$$
$$= \tilde{K}_{ik}(\boldsymbol{q}') \tilde{K}_{jl}(-\boldsymbol{q}') \left[\frac{D_0}{k'^{\sigma_0}} P_{kl}(\boldsymbol{k}') + \mathrm{i} \frac{D_1}{k'^{\sigma_1}} \epsilon_{klu} k'_u \right]$$
$$+ \cdots \tag{A88a}$$

$$\int^{\Lambda} \mathrm{d}^4 q\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x} - \omega t)} \langle \hat{b}_i^>(\boldsymbol{q}') \hat{b}_j^>(\boldsymbol{q} - \boldsymbol{q}') \rangle$$
$$= (Ro\beta)^2 \langle B \rangle_v \langle B \rangle_w k'_v k'_w K_{ik}(\boldsymbol{q}') K_{jm}(-\boldsymbol{q}')$$

$$\times \tilde{K}_{kl}(\boldsymbol{q}')\tilde{K}_{mn}(-\boldsymbol{q}')\left[\frac{D_0}{k'^{\sigma_0}}P_{ln}(\boldsymbol{k}') + \mathrm{i}\frac{D_1}{k'^{\sigma_1}}\epsilon_{lnu}k'_u\right]$$
$$+ O(Ro^2(Ro\beta)^2), \tag{A88b}$$

where the dots indicate the terms proportional to $\bar{\nabla}\langle \boldsymbol{U}\rangle$ which contribute to the mean turbulent viscosity. Finally, we get

$$-\varepsilon Ro\bar{\nabla}\cdot\int^{\Lambda}\mathrm{d}^4q'\int^{\Lambda}\mathrm{d}^4q\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\langle\hat{u}_i^{>}(\boldsymbol{q}')\hat{u}_j^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle$$
$$=\varepsilon\frac{16\pi}{15}HRo^2(Ro\beta)\left[4\bar{\nabla}\frac{\langle B\rangle^2}{2}-(\langle\boldsymbol{B}\rangle\cdot\bar{\nabla})\langle\boldsymbol{B}\rangle\right]\left[D_0\int_{\lambda_1}^{\lambda_2}\frac{\mathrm{d}k'}{k'^{\sigma_0-4}}\mathcal{I}_{Q1}(k')\right.$$
$$\left.-D_1\int_{\lambda_1}^{\lambda_2}\frac{\breve{\alpha}\,\mathrm{d}k'}{k'^{\sigma_1-6}}\mathcal{I}_{Q2}(k')\right]+\mathrm{o}(Ro^4)+\mathrm{const.}, \tag{A89a}$$

$$\varepsilon RoH\bar{\nabla}\cdot\int^{\Lambda}\mathrm{d}^4q'\int^{\Lambda}\mathrm{d}^4q\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\langle\hat{b}_i^{>}(\boldsymbol{q}')\hat{b}_j^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle$$
$$=\varepsilon\frac{16\pi}{15}HRo^2(Ro\beta)\left[4\bar{\nabla}\frac{\langle B\rangle^2}{2}-(\langle\boldsymbol{B}\rangle\cdot\bar{\nabla})\langle\boldsymbol{B}\rangle\right]\left[\frac{D_0}{2}\int_{\lambda_1}^{\lambda_2}\frac{\mathrm{d}k'}{k'^{\sigma_0-4}}\mathcal{I}_{Q3}(k')\right.$$
$$\left.-D_1\int_{\lambda_1}^{\lambda_2}\frac{\breve{\alpha}E_\eta\,\mathrm{d}k'}{k'^{\sigma_1-8}}\mathcal{I}_{Q4}(k')\right]+\mathrm{o}(Ro^4(Ro\beta)^2), \tag{A89b}$$

$$\tfrac{1}{2}HRo\int^{\Lambda}\mathrm{d}^4q'\int^{\Lambda}\mathrm{d}^4q\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\langle\hat{b}_i^{>}(\boldsymbol{q}')\hat{b}_i^{>}(\boldsymbol{q}-\boldsymbol{q}')\rangle$$
$$=\frac{16\pi}{3}HRo^2(Ro\beta)\frac{\langle B\rangle^2}{2}\left[\frac{D_0}{2}\int_{\lambda_1}^{\lambda_2}\frac{\mathrm{d}k'}{k'^{\sigma_0-4}}\mathcal{I}_{Q3}(k')-D_1\int_{\lambda_1}^{\lambda_2}\frac{\breve{\alpha}E_\eta\,\mathrm{d}k'}{k'^{\sigma_1-8}}\mathcal{I}_{Q4}(k')\right]$$
$$+O(Ro^2(Ro\beta)^2), \tag{A89c}$$

where

$$\mathcal{I}_{Q1}(k')=\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{E_\nu E_\eta k'^4(E_\eta^2k'^4-\alpha^2k'^2+\omega'^2)-\omega'^2(E_\eta^2k'^4+\alpha^2k'^2+\omega'^2)}{(\omega'^2+E_\nu^2k'^4)^2[(\omega'^2-E_\eta^2k'^4+\alpha^2k'^2)^2+4\omega'^2E_\eta^2k'^4]}, \tag{A90a}$$

$$\mathcal{I}_{Q2}(k')=\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{E_\eta^2E_\nu k'^6-\alpha^2E_\nu k'^4-(2E_\eta k'^2+E_\nu k'^2)\omega'^2}{(\omega'^2+E_\nu^2k'^4)^2[(\omega'^2-E_\eta^2k'^4+\alpha^2k'^2)^2+4\omega'^2E_\eta^2k'^4]}, \tag{A90b}$$

$$\mathcal{I}_{Q3}(k')=\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{E_\eta^2k'^4(E_\eta^2k'^4-\alpha^2k'^2+\omega'^2)^2+\omega'^2(\omega'^2+k'^2\alpha^2+E_\eta^2k'^4)^2}{(\omega'^2+E_\nu^2k'^4)[(\omega'^2-E_\eta^2k'^4+\alpha^2k'^2)^2+4\omega'^2E_\eta^2k'^4]^2}$$
$$+\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{\breve{\alpha}^2k'^2[(E_\eta^2k'^4-\alpha^2k'^2-\omega'^2)^2+4\omega'^2E_\eta^2k'^4]}{(\omega'^2+E_\nu^2k'^4)[(\omega'^2-E_\eta^2k'^4+\alpha^2k'^2)^2+4\omega'^2E_\eta^2k'^4]^2}, \tag{A90c}$$

$$\mathcal{I}_{Q4}(k')=\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{(E_\eta^2k'^4-\alpha^2k'^2)^2+(E_\eta^2k'^4+\alpha^2k'^2)\omega'^2}{(\omega'^2+E_\nu^2k'^4)[(\omega'^2-E_\eta^2k'^4+\alpha^2k'^2)^2+4\omega'^2E_\eta^2k'^4]^2}. \tag{A90d}$$

It is clear now that following the steps of the renormalization procedure does not introduce any new terms of an order lower than $O(Ro^2(Ro\beta)^2, Ro^4)$ in the expressions for the coefficients $\bar{Q}$, $\bar{Q}_p$, $\bar{\eta}$ and $\bar{\alpha}$. In the limit $\delta\lambda \to 0$, (A84) and (A89*a*–*c*) transform into the following set of recursion differential equations (cf. (A26*a*–*f*)):

$$\frac{\mathrm{d}\bar{Q}}{\mathrm{d}\lambda} = \frac{16\pi}{15} Ro^2 \left\{ D_0 \frac{1}{\lambda^{\sigma_0 - 4}} \left[\mathcal{I}_{Q1}(\lambda) + \tfrac{1}{2}\mathcal{I}_{Q3}(\lambda) \right] - D_1 \frac{\breve{\alpha}}{\lambda^{\sigma_1 - 6}} [\mathcal{I}_{Q2}(\lambda) + \breve{E}_\eta \lambda^2 \mathcal{I}_{Q4}(\lambda)] \right\}, \tag{A91a}$$

$$\frac{\mathrm{d}\bar{Q}_p}{\mathrm{d}\lambda} = \frac{64\pi}{15} Ro^2 \left\{ D_0 \frac{1}{\lambda^{\sigma_0 - 4}} \left[\mathcal{I}_{Q1}(\lambda) - \frac{1}{8}\mathcal{I}_{Q3}(\lambda) \right] - D_1 \frac{\breve{\alpha}}{\lambda^{\sigma_1 - 6}} \left[\mathcal{I}_{Q2}(\lambda) - \frac{1}{4}\breve{E}_\eta \lambda^2 \mathcal{I}_{Q4}(\lambda) \right] \right\}, \tag{A91b}$$

$$\frac{\mathrm{d}\bar{E}_\eta}{\mathrm{d}\lambda} = -\frac{8\pi}{3} Ro^2 \left[D_0 \frac{\breve{E}_\eta(\lambda)}{\lambda^{\sigma_0 - 4}} \mathcal{I}_1(\lambda) + D_1 \frac{\breve{\alpha}(\lambda)}{\lambda^{\sigma_1 - 4}} \mathcal{I}_2(\lambda) \right], \tag{A91c}$$

$$\begin{aligned} \frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}\lambda} = &-\frac{8\pi}{3} Ro^2 \left[D_1 \frac{\breve{E}_\eta(\lambda)}{\lambda^{\sigma_1 - 6}} \mathcal{I}_1(\lambda) + D_0 \frac{\breve{\alpha}(\lambda)}{\lambda^{\sigma_0 - 4}} \mathcal{I}_2(\lambda) \right] \\ &- \frac{8\pi}{5} H Ro^2 (Ro\beta)^2 \left[\frac{D_0 \breve{\alpha}(\lambda) \mathcal{I}_{L2}(\lambda)}{\lambda^{\sigma_0 - 6}} - \frac{D_1 \mathcal{I}_{L1}(\lambda)}{\lambda^{\sigma_1 - 6}} \right] \langle B \rangle^2. \end{aligned} \tag{A91d}$$

Finally, we solve the above set of differential equations, with the obvious 'boundary' conditions

$$\bar{E}_\eta(\varLambda) = 0, \quad \bar{\alpha}(\varLambda) = 0, \quad \bar{Q}(\varLambda) = 1, \quad \bar{Q}_p(\varLambda) = 1, \tag{A92a–d}$$

in the limit $\varLambda_L \gg \ell^{-1} = D_1/D_0$. Returning to dimensional units (cf. (4.21*a*–*e*)), and taking the limit $\lambda \to \varLambda_L$ (with $\varLambda_L \ll \varLambda$), for strong turbulence with $\sigma_0 = 3$ and $\sigma_1 = 5$ one obtains for the EMF

$$\mathcal{I}_1(k) = \frac{\pi}{\breve{\nu}\breve{\eta}(\breve{\nu} + \breve{\eta})k^6}, \quad \mathcal{I}_2(k) = -\frac{\pi}{\breve{\nu}(\breve{\nu} + \breve{\eta})^2 k^6}, \tag{A93a}$$

$$\mathcal{I}_{L1}(k) \approx 2\breve{\eta}k^2 \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\breve{\nu}\breve{\eta}k^4 - \omega^2}{(\omega^2 + \breve{\eta}^2 k^4)^2 (\omega^2 + \breve{\nu}^2 k^4)^2} = \frac{\pi}{\breve{\nu}^2 \breve{\eta}(\breve{\nu} + \breve{\eta})k^8}, \tag{A93b}$$

$$\mathcal{I}_{L2}(k') \approx 2 \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{2\breve{\nu}\breve{\eta}^3 k^8 - (2\breve{\nu} + 3\breve{\eta})\breve{\eta}k^4\omega^2 + \omega^4}{(\omega^2 + \breve{\eta}^2 k^4)^3 (\omega^2 + \breve{\nu}^2 k^4)^2} = \frac{(\breve{\nu} + 2\breve{\eta})\pi}{\breve{\nu}^2 \breve{\eta}^2 (\breve{\nu} + \breve{\eta})^2 k^{10}}, \tag{A93c}$$

$$\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}\lambda} = -\frac{8\pi^2 D_0}{3(1+a)A^2} \left[\frac{1}{\lambda^{7/3}} - \frac{10}{(2a+1)(1+a)\lambda^2 \ell^2} \frac{1}{\lambda^{7/3}} \right] \approx -\frac{8\pi^2 D_0}{3(1+a)A^2} \frac{1}{\lambda^{7/3}}, \tag{A94a}$$

$$\begin{aligned} \frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}\lambda} = &-\frac{40}{3(1+a)\ell} \left(\frac{\pi^2 D_0}{5} \right)^{1/3} \left[1 - \frac{10}{(2a+1)(1+a)} \right] \frac{1}{\lambda^{7/3}} \\ &- \frac{8\pi^2}{5} \left[\frac{D_0 \breve{\alpha}(\breve{\nu} + 2\breve{\eta})}{\breve{\nu}^2 \breve{\eta}^2 (\breve{\nu} + \breve{\eta})^2 \lambda^7} - \frac{D_1}{\breve{\nu}^2 \breve{\eta}(\breve{\nu} + \breve{\eta})\lambda^7} \right] \langle B \rangle^2, \end{aligned} \tag{A94b}$$

and hence with $\breve{\nu}$, $\breve{\eta}$ and $\breve{\alpha}$ given by (A79)

$$\bar{\eta} = \breve{\eta}(\varLambda_L) = \frac{aA}{\varLambda_L^{4/3}}, \quad \bar{\alpha} = \frac{10D_1}{(2a+1)D_0}\frac{A}{\varLambda_L^{4/3}}. \tag{A95a,b}$$

Note that the correction proportional to $\langle B\rangle^2$ in the $\bar{\alpha}$ coefficient vanishes, thus, in strong turbulence, the effect of the Lorentz force on the α-effect is pronounced only at higher orders. Furthermore, in the limit $D_0\varLambda_L \gg D_1$ we have

$$\mathcal{I}_{Q1}(k) \approx \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\breve{\nu}\breve{\eta}k^4 - \omega^2}{(\omega^2+\breve{\nu}^2k^4)^2(\omega^2+\breve{\eta}^2k^4)} = \frac{\pi}{2\breve{\nu}^2(\breve{\nu}+\breve{\eta})k^6}, \tag{A96a}$$

$$\mathcal{I}_{Q2}(k) \approx \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\breve{\eta}^2\breve{\nu}k^6 - (2\breve{\eta}k^2+\breve{\nu}k^2)\omega^2}{(\omega^2+\breve{\nu}^2k^4)^2(\omega^2+\breve{\eta}^2k^4)^2} = \frac{\pi}{2\breve{\nu}^2(\breve{\nu}+\breve{\eta})^2k^8}, \tag{A96b}$$

$$\mathcal{I}_{Q3}(k) \approx \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{1}{(\omega^2+\breve{\nu}^2k^4)(\omega^2+\breve{\eta}^2k^4)} = \frac{\pi}{\breve{\nu}\breve{\eta}(\breve{\nu}+\breve{\eta})k^6}, \tag{A96c}$$

$$\mathcal{I}_{Q4}(k) \approx \breve{\eta}^2k^4\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{1}{(\omega^2+\breve{\nu}^2k^4)(\omega^2+\breve{\eta}^2k^4)^3} = \frac{(3\breve{\nu}^2+9\breve{\nu}\breve{\eta}+8\breve{\eta}^2)\pi}{8\breve{\nu}\breve{\eta}^3(\breve{\nu}+\breve{\eta})^3k^{10}}, \tag{A96d}$$

thus the Lorentz-force coefficients are

$$\bar{Q}(\varLambda_L) \approx -\frac{8}{3a}\ln\frac{\varLambda}{\varLambda_L}, \quad \bar{Q}_p(\varLambda_L) \approx -\frac{4(4a-1)}{15}\ln\frac{\varLambda}{\varLambda_L}. \tag{A97a,b}$$

It is of interest to point out that, in the case of weak turbulence, the mean diffusivities and the $\bar{\alpha}$, $\bar{Q}$ and $\bar{Q}_p$ coefficients are obtained from (A91*a*–*d*) with the integrals given by (A93*a*–*c*) and (A96*a*–*d*) but all the fluctuational diffusivities replaced by the molecular ones $\breve{\nu} = \nu$ and $\breve{\eta} = \eta$ and with $\breve{\alpha} = 0$.

As a final note, it is to be stressed once again that the entire renormalization technique fundamentally relies on two important assumptions regarding the properties of the flow. Firstly, we recall that the statistical correlations between short-wavelength fluctuations of the order higher than second, i.e. terms of the type $\langle \hat{u}_i^> \hat{u}_j^> \hat{f}_k^> \rangle_c$ have been neglected. Secondly, the limit of distant interactions (A5) corresponding to an assumption of ergodicity of the system has greatly simplified the calculations.

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