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BOOTSTRAP PERCOLATION ON RANDOM GEOMETRIC GRAPHS

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Bootstrap percolation (BP) has been used effectively to model phenomena as diverse as emergence of magnetism in materials, spread of infection, diffusion of software viruses in computer networks, adoption of new technologies, and emergence of collective action and cultural fads in human societies. It is defined on an (arbitrary) network of interacting agents whose state is determined by the state of their neighbors according to a threshold rule. In a typical setting, BP starts by random and independent "activation" of nodes with a fixed probability p, followed by a deterministic process for additional activations based on the density of active nodes in each neighborhood (θ activated nodes). Here, we study BP on random geometric graphs (RGGs) in the regime when the latter are (almost surely) connected. Random geometric graphs provide an appropriate model in settings where the neighborhood structure of each node is determined by geographical distance, as in wireless *ad hoc* and sensor networks as well as in contagion. We derive bounds p_c', p_c'' on the critical thresholds such that for all $p > p_c''$ full percolation takes place, whereas for $p < p_c'$ it does not. We conclude with simulations that compare numerical thresholds with those obtained analytically.

1. INTRODUCTION

Some crystals or lattices studied in physics and chemistry can be modeled as consisting of atoms occupying sites with specified probabilities. The lattice as a whole would then exhibit certain macroscopic properties, such as (ferro)magnetism, only when a sufficient number of neighboring sites of each atom are also similarly occupied. In computer memory arrays, each functional memory unit can be considered as an occupied site, and a minimum percentage of functioning units are needed in the vicinity of each memory unit in order to maintain the array with proper functioning. In adoption of new technology or emergence of cultural fads, an individual is positively influenced when a sufficient number of its close friends have also done so.

All the three examples cited above may be modeled via a formal process called "bootstrap percolation" (BP), which is a dynamic process that evolves similarly to a cellular automaton. Unlike traditional cellular automata, however, this process can be defined on

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arbitrary graphs and starts with random initial conditions. Nodes are either active or inactive. Once activated, a node remains active forever. Each node is initially active with a (given) probability p. Subsequently and at each discrete time step, a node becomes active if θ of its nearest neighbors are active, for a fixed value of $\theta = 1, 2, 3, \ldots$. As time evolves, a fraction Φ of all the nodes are activated. The emergence of macroscopic properties of interest typically involve Φ to be at or close to 1.

Gersho and Mitra [10] studied a similar model for adoption of new communication services using a random regular graph and obtained (implicit) critical thresholds for widespread adoption. Chalupa et al. [7] were the first to introduce BP formally to explain ferromagnetism. Their analysis is carried out on regular trees (Bethe lattices) and a fundamental recursion is derived for computation of the critical threshold that has since been used extensively. In the more recent past, results for non-regular (infinite) trees have also been derived by Balogh et al. [4]. Aizenman and Lebowitz [1] studied metastability of BP on the d-dimensional Euclidean lattice $\mathbb{Z}^d$ which has now been thoroughly investigated in two and three dimensions; see [6,14]. The existence of a sharp metastability threshold for BP in two-dimensional lattices was proved by Holroyd [14] and recently generalized to d-dimensional lattices by Balogh et al. [3]. Even more recently, BP has been studied on random graphs $G(n, p)$ by Luczak and co-workers [15]. In [19] Watts proposed a model of formation of opinions in social networks in which the percolation threshold is a certain fraction of the size of each neighborhood rather than a fixed value, a departure from the standard model that is used by Amini in [2] for random graphs with a given degree sequence.

Many diffusion processes of interest have a *physical contact* element. A link in an *ad hoc* wireless network, a sensor network, or an epidemiological graph connotes physical proximity within a certain locality. Study of diffusion of virus spread in *ad hoc* wireless, sensor or epidemiological graphs requires this notion of neighborhood for accurate estimation of likelihood of full percolation. This is in contrast to models with long-range reach where physical proximity plays little, if any, role. The natural random model for such phenomena is the random geometric graph (RGG). In this work, we focus on BP on RGGs, a topic that has not been investigated, to the best of our knowledge, and obtain tight bounds on their critical thresholds for full percolation.

2. RGG MODEL

One of the transitions from the random graph model $G(n, p)$ of Erdős and Rényi [8,9] and Gilbert [11] to models that may describe processes constrained by geometric distances among the nodes is the model of RGGs by Gilbert [12]. This model takes as input nodes placed randomly on the plane and a pair is connected when their distance is within a prescribed value. (We shall make this definition precise in the sequel.) The RGG model has been used in many disciplines: for modeling of wireless sensor networks [18], cluster analysis, statistical physics, hypothesis testing, spread of computer viruses in wired networks, processes involving physical contact among individuals, as well as other related disciplines, see [17] for more details. For example, a wireless sensor network typically contains a large number of randomly deployed nodes with links determined by geometric proximity enabled by (a small) radio range among the nodes that is sufficient to enable successful signal transmission across the network. A further application of RGG is in representing d-attribute data where numerical attributes are used as coordinates in $\mathbb{R}^d$ and two nodes are considered connected if they are within a threshold (Euclidean) distance r of each other. The metric distance imposed on such an RGG captures the similarity between data elements.

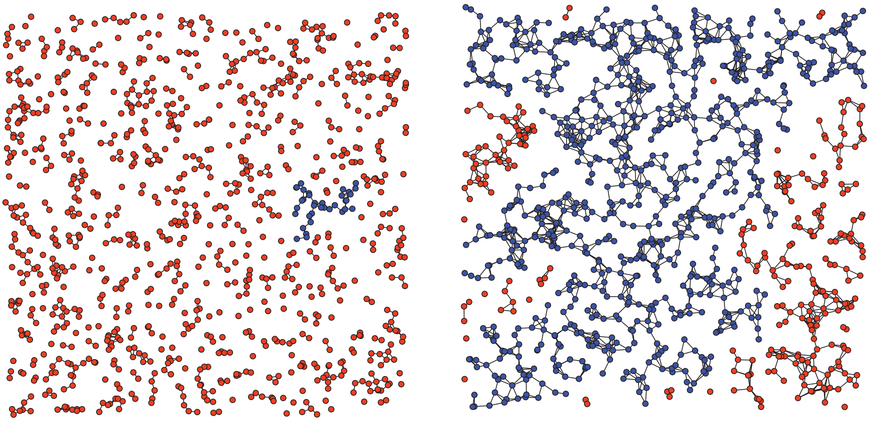

Figure 1. (Color online) (Left) the "very" sparse regime ($r = 0.02$), and (Right) emergence of the giant component regime ($r = 0.035$).

We shall use the following formal definition of RGG. Let $\mathcal{X}$ be a Poisson point process of intensity 1 on $\mathbb{R}^2$. Consider points of $\mathcal{X}$ contained in $[0, \sqrt{n}]^2$ representing the nodes of a graph denoted by $G_{n,r}$. Two nodes of $G_{n,r}$ are connected if their Euclidean distance is at most $r \equiv r(n)$. Before we cite the result on the connectedness of $G_{n,r}$, we introduce the standard asymptotic notation, which will be used in this work. For two functions $f(x)$ and $g(x)$ we write: (i) $f(x) = o(g(x))$ if $\lim_{x\to\infty} f(x)/g(x) = 0$, (ii) $f(x) = \mathcal{O}(g(x))$ if $\lim_{x\to\infty} |f(x)/g(x)| < \infty$, and (iii) $f(x) = \omega(g(x))$ if $\lim_{x\to\infty} f(x)/g(x) = \infty$. When $r_{\mathrm{t}}(n) = \sqrt{(\ln n + \omega(1))/\pi}$, then $G_{n,r}$ is connected whp[1]; see [13,16]. We parametrize $r(n)$ by introducing a parameter a which ensures connectivity at $r(n) = \sqrt{a \ln n/\pi}$ when $a > 1$ and measures how much denser an instance of $G_{n,r}$ is compared to $G_{n,r_{\mathrm{t}}}$ at the connectedness threshold $r_{\mathrm{t}}(n)$. For sufficiently large n the expected degree of a node in RGG is concentrated around its mean value $a \ln n$, a fact that is easily derived from the Chernoff and union bounds.

For example when $n = 1000$, the critical thresholds for (1) the existence of a giant component and (2) connectedness in $G_{n,r}$ satisfy $r_c \approx 0.0316$ and $r_{\mathrm{t}} \approx 0.0469$, respectively. In Figures 1 and 2, we present $G_{n,r}$ for four different regimes, when r takes the values $0.020, 0.035, 0.045$, and 0.050, respectively. The values 0.020 and 0.035 correspond to (1) the very sparse regime and (2) emergence of a giant component, Figure 1 (left and right). The values 0.045 and 0.050 correspond to the (1) "almost" connected and (2) connected regimes, Figure 2 (left and right).

3. BOOTSTRAP PERCOLATION

BP is a cellular automaton defined on an underlying graph $G = (V, E)$ with state space $\{0, 1\}^V$ whose initial configuration is chosen by a Bernoulli product measure. In other words, every node is in one of two different states 0 or 1 (*inactive* or *active* respectively), and a node becomes active with probability p independently of other nodes within the initial configuration.

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[1] We will use the notation "with high probability" and denote it as whp, meaning with probability $1 - o(1)$ as the area n of the square $[0, \sqrt{n}]^2$ tends to infinity.

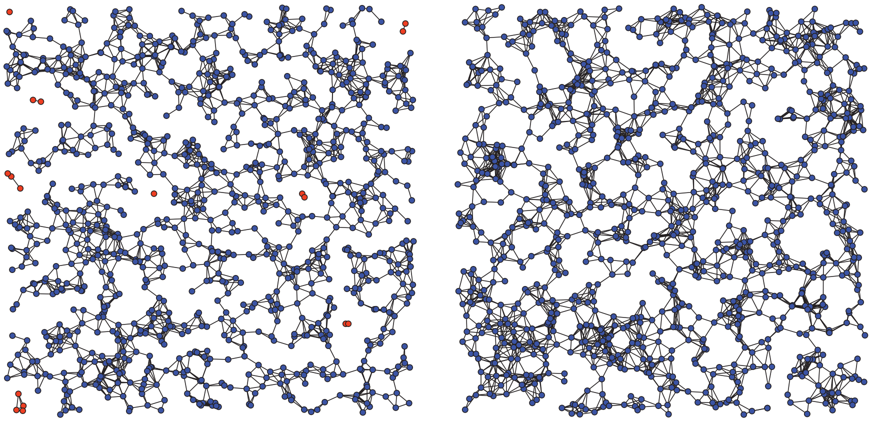

Figure 2. (Color online) (Left) the "almost" connected regime ($r = 0.045$), and (Right) the connected regime ($r = 0.05$).

After drawing an initial configuration at time $t = 0$, a discrete time deterministic process updates the configuration according to a local rule: an inactive node becomes active at time $t + 1$ if the number of its active neighbors at t (not necessarily the nearest ones) is greater than some defined *threshold* θ. Once an inactive node becomes active it remains active forever. A configuration that does not change at the next time step is a *stable* configuration. A configuration is *fully active* if all its nodes of are active.

An interesting phenomenon to study is metastability near a first-order phase transition. Do there exist $0 < p'_c \leq p''_c < 1$ such that:

$$(\forall p < p'_c) \lim_{t\to\infty} \mathbb{P}_p\,(V \text{ becomes fully active}) = 0, \tag{3.1}$$

and

$$(\forall p > p''_c) \lim_{t\to\infty} \mathbb{P}_p\,(V \text{ becomes fully active}) = 1? \tag{3.2}$$

Further, is it necessary for p'_c to be equal to p''_c?

A study of BP on a regular infinite tree first appeared in [7]. Subsequently, the relations between the branching number of an infinite (non-regular) tree, threshold value, and p necessary to fully percolate the tree were studied in [4].

A classical example of BP is a d-dimensional lattice $\mathbb{Z}^d$ equipped with Bernoulli product measure with $\theta = d$ [1]. For $\mathbb{Z}^d$ and $V = [0, L-1]^d$ the existence of a unique threshold p_c was shown in [1]. Concretely for $\mathbb{Z}^2$ and $V = [0, L-1]^2$ the exact threshold value is $p_c = \pi^2/(18 \ln L)$ [14]. Furthermore, the sharp threshold for BP in $\mathbb{Z}^d$ in all dimensions was provided in [3].

Additionally to BP on trees and lattices, there has been recent work of BP on random regular graphs [5], Erdős–Rényi random graphs [15], as well as random graphs with a given degree sequence where the threshold depends upon node degree [2].

3.1. BP on Connected RGGs

The structure of $G_{n,r}$ is dictated by random positions of its nodes and radius $r = r(n)$; so it is more 'irregular' than one a tree or a lattice. In this work, we are interested in BP on $G_{n,r}$ which for brevity we denote by $BP(G_{n,r}, p, \theta)$. In this process, a node becomes active with probability p independently of other nodes in the initial configuration and an inactive

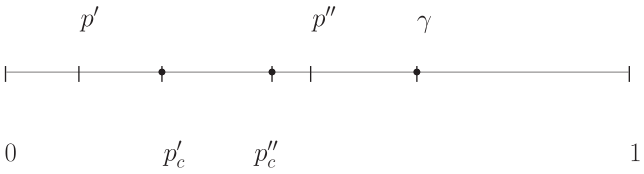

FIGURE 3. Bounds (p', p'') on the critical thresholds (p'_c, p''_c).

node becomes active at each of the following time steps if at least $\theta = \gamma D$ of its neighbors are active at that step, where $\gamma = \gamma(n)$ and $D(n) = \mathbb{E}(\deg) = r^2\pi = a \ln n$ is the expected node degree.

To resolve ambiguity due to the edge effects for the nodes of $G_{n,r}$ within distance r from the boundary of the square $[0, \sqrt{n}]^2$, we allow the activation rule for these nodes to include as neighbors all nodes outside $[0, \sqrt{n}]^2$ but within a distance r from its boundary, from the entire process $\mathcal{X}$. The additional nodes in this "frame" are also activated with probability p at the initial step and participate in the percolation dynamics for the benefit of the boundary nodes within $G_{n,r}$. Note that the ratio of the area of the external r-neighborhood frame around the boundary of the square to the area of the square itself is of order $\Theta\left(\left(r^2\pi + 4r\sqrt{n}\right)/n\right)$ that is $o(1)$ and thus asymptotically inclusion of these extra nodes does not alter our conclusions, but the analysis is now made easier in that we can treat all nodes insides the square similarly.

For the critical thresholds p'_c and p''_c in $BP(G_{n,r}, p, \theta)$, we derive bounds $p' \le p'_c$ and $p'' \ge p''_c$ such that a connected $G_{n,r}$ does not become fully active for $p < p'$ whp, and conversely, becomes fully active for $p > p''$ whp. These bounds are schematically presented in Figure 3.

Before we proceed to the proofs, we need to make an important observation regarding our methodology. Whereas in the case of BP on regular trees and d-dimensional lattices, the entire infinite graph is considered, we study BP on the RGG within $[0, \sqrt{n}]^2$ for a fixed n and let n get larger, as needed, to prove the validity of our results. This is because $r(n) = \sqrt{\pi^{-1} a \ln n}$ goes to infinity as $n \to \infty$ and thus RGG within $[0, \sqrt{n}]^2$ in the actual infinite case is a complete graph and thus uninteresting.

The main ideas of the proofs provided below are the following. We obtain the distribution of the number of active neighbors for each node at the initial configuration. For $p < p'$ we use the Poisson tail bound and the union bound (see (3.3) in the Appendix) to show that an initial configuration is stable whp. For $p > p''$ we use the Stirling approximation to lower bound the number of active neighbors for each node. Then we develop a geometric argument to show that a stable, fully active, configuration is reached within $\mathcal{O}(\sqrt{n}/r)$ steps whp. This geometric argument leverages the following simple observation about BP on $\mathbb{Z}^2$ with $\theta = 1$.

LEMMA 1: *Consider BP in* $\mathbb{Z}^2$ *with the threshold* $\theta = 1$ *and the initial probability* $p > 0$. *For* $p = \omega\left(1/N^2\right)$, *a square grid* $[0, N]^2$ *becomes fully active within* $\mathcal{O}(N)$ *steps with probability* $1 - o(1)$ *as* $N \to \infty$.

PROOF: The probability that a square grid $[0, N]^2$ becomes fully active is $1 - (1-p)^{N^2}$ and tends to 1 as $N \to \infty$. Once any point is active within $2N$ steps the grid $[0, N]^2$ becomes fully active. ■

We first introduce the following functions upon which our analysis will heavily depend. For the function $H(x) := x \ln x - x + 1$ on $[0, +\infty)$ (see Figure 4 left), define $H_L^{-1} : [0, 1] \to$

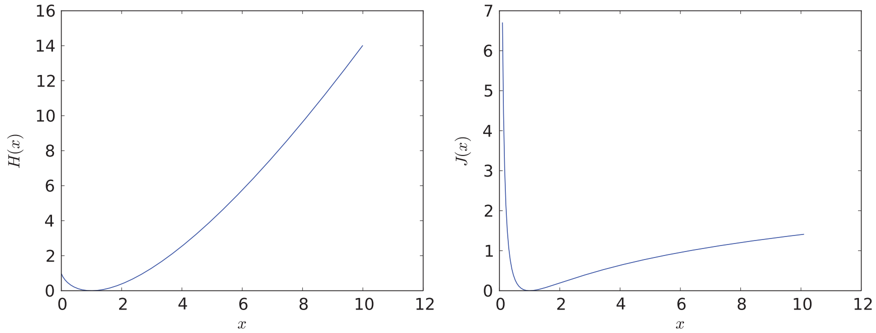

FIGURE 4. (Color online) Functions $H(x)$ and $J(x)$.

$[0,1]$ to be the inverse of $H(x)$ on $[0,1]$, and $H_R^{-1}:[0,+\infty)\to[1,+\infty)$ to be the inverse of $H(x)$ on $[1,+\infty)$. Analogously for the function $J(x):=x^{-1}H(x)=\ln x-1+x^{-1}$ on $(0,+\infty)$ (see Figure 4 right), define $J_L^{-1}:[0,+\infty]\to[0,1]$ to be the inverse of $J(x)$ on $[0,1]$, and $J_R^{-1}:[0,+\infty)\to[1,+\infty)$ to be the inverse of $J(x)$ on $[1,+\infty)$.

We will use the following concentration results on a Poisson random variable, see [17].

LEMMA 2: *A Poisson random variable* Po(λ) *(with* $\lambda>0$*) satisfies:*

$$\mathbb{P}\left(\mathrm{Po}(\lambda)\ge k\right)\le\exp\left(-\lambda H(k/\lambda)\right),\quad k\ge\lambda, \tag{3.3}$$

$$\mathbb{P}\left(\mathrm{Po}(\lambda)\le k\right)\le\exp\left(-\lambda H(k/\lambda)\right),\quad k\le\lambda, \tag{3.4}$$

where

$$H(x)=\begin{cases}1, & x=0\\ x\ln x-x+1, & x>0.\end{cases} \tag{3.5}$$

We now provide bounds on the critical thresholds on p in Theorems 1 and 2.

THEOREM 1: *Consider bootstrap percolation* $BP(G_{n,r},p,\theta)$*, where* $r=\sqrt{\pi^{-1}a\ln n}$ *and* $\theta=\gamma a\ln n$*. For* $a>1$*,* $\gamma\in(0,1)$ *and when*

$$p<p':=\gamma/J_R^{-1}(1/a\gamma), \tag{3.6}$$

$G_{n,r}$ *does not become fully active whp.*

PROOF: We show that for the conditions of the assertion, an initial configuration is stable whp. The number of active nodes in the initial configuration follows Poisson distribution Po(pn). The degree distribution of a node is Po$(r^2\pi)$, and the expected degree $D=r^2\pi=a\ln n$. By the thinning theorem [17] the number of active neighbors in the initial configuration follows Po(pD). We consider the case $p\le\gamma$ and seek a maximal $p'\le\gamma$ for this method (see Figure 3) such that $BP(G_{n,r},p',\gamma)$ does not become fully active whp. Consider the activation rule in $BP(G_{n,r},p,\theta)$. The probability that a node becomes active at the next time step given that it is inactive initially is $\mathbb{P}\left(\mathrm{Po}(pD)\ge\gamma D\right)\le\exp\left(-pDH(\gamma/p)\right)$, see (3.3). From (3.3) and (3.4) it follows that the number of nodes Po(n) within the square $[0,\sqrt{n}]^2$ is concentrated around its mean n whp. Hence, the union bound over all nodes provides

$$\mathbb{P}_p\left(\text{the initial configuration is stable}\right)\ge1-\exp\left((1+\o(1))\ln n-pDH(\gamma/p)\right). \tag{3.7}$$

Given $D=a\ln n$, the condition $paH(\gamma/p)>1$ suffices that the initial configuration is stable whp. The function $J(x)=x^{-1}H(x)$ is monotonically decreasing on $(0,1)$, monotonically increasing on $(1, +\infty)$, with the minimum 0 attained at $x = 1$. Hence, for any positive $\gamma < +\infty$ there are two solutions of $J(x) = 1/a\gamma$, denoted $x_1 < 1 < x_2$. This yields $p > \gamma/x_1 > \gamma$ or $p < \gamma/x_2 < \gamma$. The acceptable solution is $p < \gamma/x_2$, since we consider the case $p < \gamma$. For $J(\gamma/p) > 1/a\gamma$ from (3.7) it follows that the probability that the initial configuration is stable tends to one as n tends to infinity. Finally, a bound on p is given by

$$p < p' := \gamma/J_R^{-1}(1/a\gamma), \tag{3.8}$$

which concludes the proof. ■

Theorem 2: *Consider bootstrap percolation* $BP(G_{n,r}, p, \theta)$*, where* $r = \sqrt{\pi^{-1} a \ln n}$*,* $\theta = \gamma a \ln n$*,* $a \geq 5\pi/H(5\pi\gamma)$ *and* $\gamma \in (0, 1/(5\pi))$*. Then, for*

$$p > p'' := \frac{5\pi\gamma}{J_R^{-1}(1/a\gamma)}, \tag{3.9}$$

$G_{n,r}$ *becomes fully active within* $\mathcal{O}(\sqrt{n}/r)$ *steps whp.*

The proof of Theorem 2 consists of two parts. We tile the square $[0, \sqrt{n}]^2$ into cells $r/\sqrt{5} \times r/\sqrt{5}$ and show that in the initial configuration: (i) When $a > 1$, $\gamma \in (0, 1/(5\pi))$, and $a \geq 5\pi/H(5\pi\gamma)$, every cell contains at least γD nodes whp; (ii) When $p > p''$ at least one cell contains γD or more active nodes. By Lemma 1 it follows that for a, γ, p in the specified ranges $G_{n,r}$ becomes fully active within $\mathcal{O}(\sqrt{n}/r)$ steps whp.

Proof: Tile the square $[0, \sqrt{n}]^2$ into cells $r/\sqrt{5} \times r/\sqrt{5}$, see Figure 5. Define the area of a cell $A := r^2/5 = a \ln n/5\pi$. Call two cells neighboring if they share one side. Note that every pair of nodes within the same cell or within two neighboring cells are adjacent by the choice of the size of a cell. Define $\widehat{G}_{n,r}$ on the set of nodes of $G_{n,r}$ as follows. The set of edges of $\widehat{G}_{n,r}$ consists of the subset of edges of $G_{n,r}$ whose terminal nodes belong to the same cell or two neighboring cells. Then the monotonicity of BP yields

$$\mathbb{P}_p\left(\widehat{G}_{n,r} \text{ becomes fully active}\right) \leq \mathbb{P}_p\left(G_{n,r} \text{ becomes fully active}\right). \tag{3.10}$$

Therefore it is sufficient to show that whp $\widehat{G}_{n,r}$ becomes fully active when $p > p''$ (3.19).

Part (i) (To show that every cell contains at least γD nodes whp.) We first bound the probability that an arbitrary cell contains at most γD nodes. The number of nodes in a cell follows $\mathrm{Po}(A)$, that is, $\mathrm{Po}(a \ln n/5\pi)$. Moreover, the numbers of nodes in cells are independent random variables (given the Poisson point process $\mathcal{X}$). For $\gamma \leq 1/(5\pi)$, from (3.4) we obtain

$$\begin{aligned}
\mathbb{P}\left(\text{a cell contains at most } \gamma D \text{ nodes}\right) &= \mathbb{P}\left(\mathrm{Po}\left(\frac{a \ln n}{5\pi}\right) \leq \gamma a \ln n\right) \\
&\leq \exp\left(-\frac{a \ln n}{5\pi} H(5\pi\gamma)\right) \\
&= n^{-\frac{a}{5\pi} H(5\pi\gamma)}.
\end{aligned} \tag{3.11}$$

The total number of cells in $[0, \sqrt{n}]^2$ is $5n/r^2 = 5\pi n/(a \ln n) = o(n)$. The union bound taken over all cells yields

$$\mathbb{P}\left(\text{every cell contains at least } \gamma D \text{ nodes}\right) \geq 1 - o\left(n^{1 - \frac{a}{5\pi} H(5\pi\gamma)}\right). \tag{3.12}$$

Finally, for $a \geq 5\pi/H(5\pi\gamma)$, from (3.12) it follows that every cell contains at least γD nodes whp.

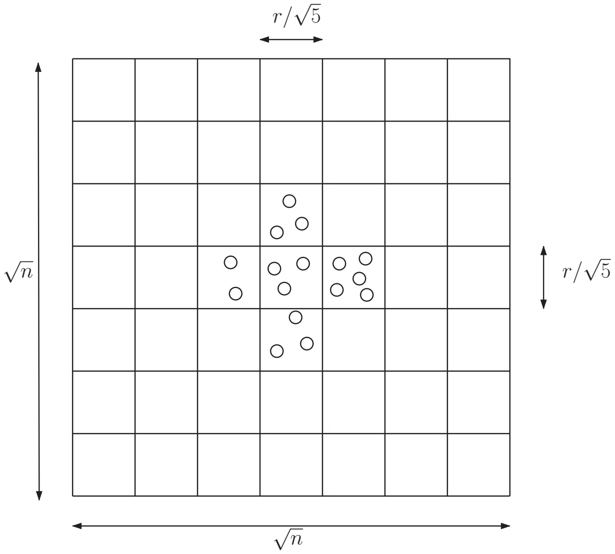

Figure 5. Tiling the square $[0, \sqrt{n}]^2$.

Part (ii). (To show that at least one cell contains γD or more active nodes.) We now derive conditions such that at least one cell contains at least $\theta = \gamma D$ active nodes in the initial configuration. In order to guarantee that whp there is at least one cell among $5n/r^2 = \Theta(n/\ln n)$, which contains at least θ active nodes in the initial configuration, it suffices to find p such that

$$\mathbb{P}\left(\mathrm{Po}(pA) \geq \gamma D\right) = \omega\left(\frac{\ln n}{n}\right), \tag{3.13}$$

since

$$\lim_{n\to\infty} 1 - \left(1 - \omega(\ln n/n)\right)^{\Theta(n/\ln n)} = 1. \tag{3.14}$$

Define $\alpha := 5\pi\gamma/p - 1$, then by rewriting (3.13) we need p such that

$$\mathbb{P}\left(\mathrm{Po}(pA) \geq (1+\alpha)pA\right) = \omega\left(\frac{\ln n}{n}\right). \tag{3.15}$$

Call $\lambda := pA$ and without loss of generality assume that $(1+\alpha)\lambda$ is an integer. By Stirling's approximation we have

$$\begin{aligned}\mathbb{P}\left(\mathrm{Po}(\lambda) \geq (1+\alpha)\lambda\right) \geq \mathbb{P}\left(\mathrm{Po}(\lambda) = (1+\alpha)\lambda\right) &= e^{-\lambda}\frac{\lambda^{\lambda(1+\alpha)}}{((1+\alpha)\lambda)!}\\ &= \exp\left(-\lambda I(\alpha) - \frac{1}{2}\log\left((1+\alpha)\lambda\right) + O(1)\right),\end{aligned} \tag{3.16}$$

where $I(\alpha) = (1+\alpha)\ln(1+\alpha) - \alpha$. From (3.5) note that $I(\alpha) = H(1+\alpha) = H(5\pi\gamma/p)$. Therefore for (3.15) to be satisfied we require

$$\frac{n}{\ln n}\exp\left(-pAI(\alpha) - \frac{1}{2}\log\left((1+\alpha)pA\right) + O(1)\right) = \omega(1). \tag{3.17}$$

The left-hand side of (3.17) equals

$$\exp\left(\left(1-\frac{ap}{5\pi}I(\alpha)\right)\ln n-\frac{3}{2}\ln\ln n-\frac{1}{2}\ln\frac{(1+\alpha)pa}{5\pi}+O(1)\right), \tag{3.18}$$

which is $\omega(1)$ if $1 > apI(\alpha)/5\pi$. Given $\alpha = 5\pi\gamma/p - 1$, the condition $1 > apI(\alpha)/5\pi$ is equivalent to $1/a\gamma > H(5\pi\gamma/p)/(5\pi\gamma/p)$, and moreover to

$$p > p'' := \frac{5\pi\gamma}{J_R^{-1}(1/a\gamma)}. \tag{3.19}$$

To complete the proof notice that once any γD nodes within a (square) cell become active, all nodes within that cell become active at the next time step as would all nodes within its neighboring cells. The resulting process that jointly activates all nodes within one cell is equivalent to activating a site in $\mathbb{Z}^2$. The resulting BP in $\mathbb{Z}^2$ has the threshold $\theta = 1$ by construction, see Figure 5. Thus, $BP(G'_{n,r}, p, \theta)$ becomes fully active when $p > p''$ by Lemma 1. The proof follows from (3.10). ■

The following remark clarifies the feasible region for a and γ required by Theorem 2. It is verified by inspection of the function $H(x)$.

Remark 1: The condition $a \geq 5\pi/H(5\pi\gamma)$ is equivalent to:

$$\gamma \in \left[\frac{1}{5\pi}H_R^{-1}(5\pi/a), \infty\right), \qquad \text{for } a < 5\pi \tag{3.20}$$

and

$$\gamma \in \left[0, \frac{1}{5\pi}H_L^{-1}(5\pi/a)\right] \bigcup \left[\frac{1}{5\pi}H_R^{-1}(5\pi/a), \infty\right), \qquad \text{for } a \geq 5\pi. \tag{3.21}$$

Next, we observe that γ is a trivial lower bound for p'' because if the initial probability of occupancy is greater than γ then full percolation takes place with high probability. The following remark characterizes a necessary condition for $p'' < \gamma$.

Remark 2: From (3.19), for non-trival percolation threshold, that is, $p'' < \gamma$, it is necessary

$$a\gamma \leq \frac{1}{J_R(5\pi)} \approx 0.55. \tag{3.22}$$

3.2. Analysis of Bounds on Critical Thresholds

A bound on the critical threshold $p' = \gamma/J_R^{-1}(1/a\gamma)$ can be rewritten as

$$\ln p' = -\ln a - \ln\left((1/a\gamma)J_R^{-1}(1/a\gamma)\right). \tag{3.23}$$

The function $-\ln\left(xJ_R^{-1}(x)\right)$ is monotonically decreasing in x, hence p' is monotonically increasing in a and monotonically decreasing in γ. As an example we numerically compute and tabulate p' for $\gamma = 1/20$ and different values of a in Table 1. In Figure 6, p' is plotted as a function of a for different values of

$$\gamma \in \{1/70, 1/60, 1/50, 1/40, 1/30, 1/20\}. \tag{3.24}$$

Table 1. Bounds p', p'' on a bound on the critical thresholds for different values of a when $\gamma = 1/20$

a	p'	p''
3	0.0000234198	0.0003678767
4	0.0001242460	0.0019516511
5	0.0003391906	0.0053279940
6	0.0006649716	0.0104453500
7	0.0010794693	0.0169562642
8	0.0015576467	0.0244674579
9	0.0020779022	0.0326396121
10	0.0026234549	0.0412091329
25	0.0101188498	0.1589465210
50	0.0174952121	0.0174952120
100	0.0246619916	0.3873896589

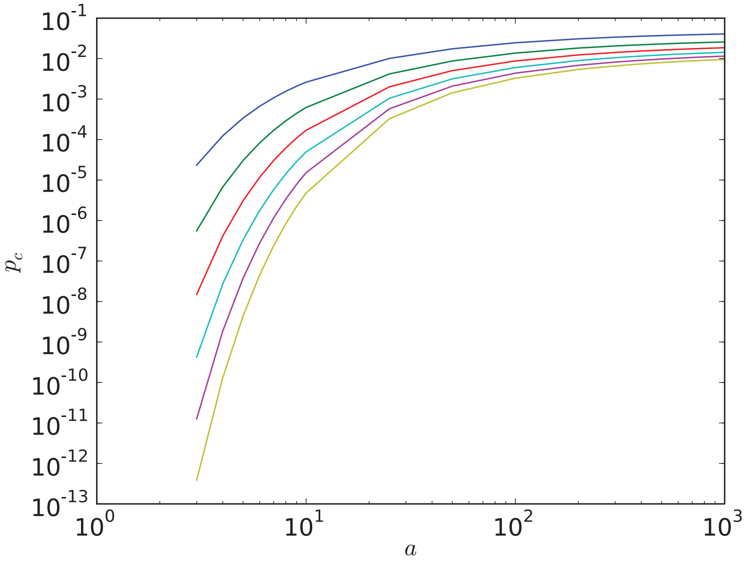

Figure 6. (Color online) The bound p' for $\gamma \in \{1/70, 1/60, 1/50, 1/40, 1/30, 1/20\}$ as a function of a.

The experiments are performed on $G_{n,r}$ with $n = 15000$ and $n = 25000$ nodes, and $r = \sqrt{a \ln n/\pi}$ for the cases: (i) $a = 30$ and $\gamma = 1/100$, and (ii) $a = 35$ and $\gamma = 1/75$. On these instances of graphs, for each chosen value of p in $(0, 1)$ we simulate BP 100 times. More precisely, within each experiment we generate a random initial configuration with the probability p and perform BP with the threshold $\theta = \gamma D$, where the expected degree D is calculated for a given input $G_{n,r}$.

Numerical results are presented with the initial probability p on the horizontal axis, and the percentage of fully active stable configurations on the vertical axis. Four cases when $(a = 30, \gamma = 1/100)$, $(a = 35, \gamma = 1/75)$, for $n = 15000, 25000$, are presented in Figures 7–10, respectively. These charts match the bounds derived theoretically for p' and p''. Furthermore, they appear to suggest that $p'_{\mathrm{c}} \neq p''_{\mathrm{c}}$ even though we do not currently have a proof one way or the other.

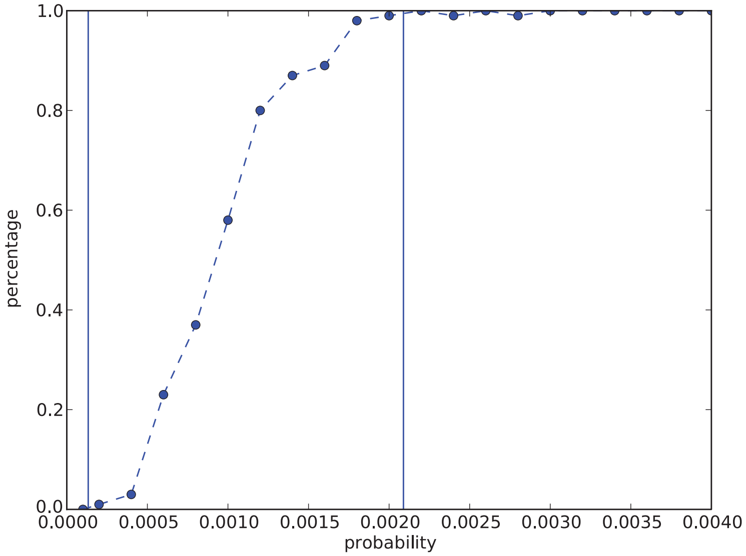

FIGURE 7. (Color online) Percentage of fully percolated configurations in 100 simulations of $BP(G_{n,r},p,\theta)$ when $a=30,\gamma=1/100,\ n=15000,\ r=\sqrt{30\ln n/\pi n}\approx 0.07824,\ D=30\ln n\approx 288.47$ and $\theta=\lceil 100^{-1}\mathbb{E}(\deg)\rceil=\lceil 2.88\rceil=3,$ where $a=30>5\pi$ and $\gamma=1/100\in\left[0,H_L^{-1}(5\pi/a)/5\pi\right]\approx[0,0.0110]$. The bounds are $p'=0.000133$ and $p''=0.002089$.

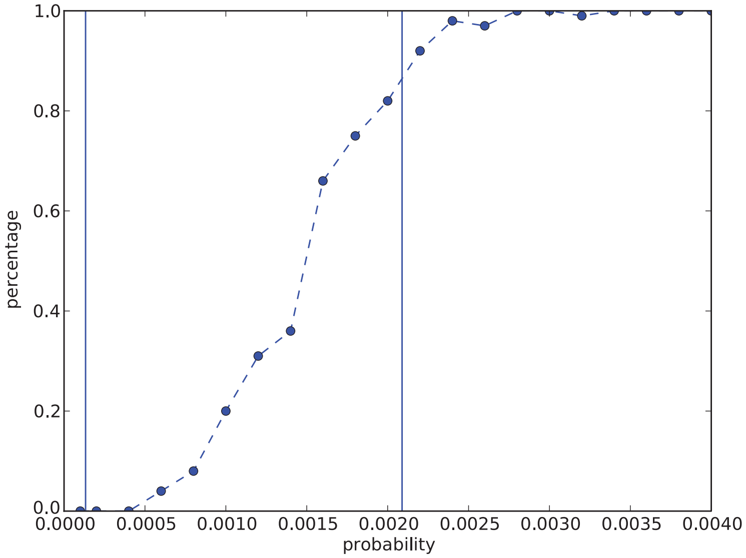

FIGURE 8. (Color online) Percentage of fully percolated configurations in 100 simulations of $BP(G_{n,r},p,\theta)$ when $a=30,\gamma=1/100,\ n=25000,\ r=\sqrt{30\ln n/\pi n}\approx 0.06219,\ D=30\ln n\approx 303.80$ and $\theta=\lceil 100^{-1}\mathbb{E}(\deg)\rceil=\lceil 3.04\rceil=4,$ where $a=30>5\pi$ and $\gamma=1/100\in\left[0,H_L^{-1}(5\pi/a)/5\pi\right]\approx[0,0.0110]$. The bounds are $p'=0.000133$ and $p''=0.002089$.

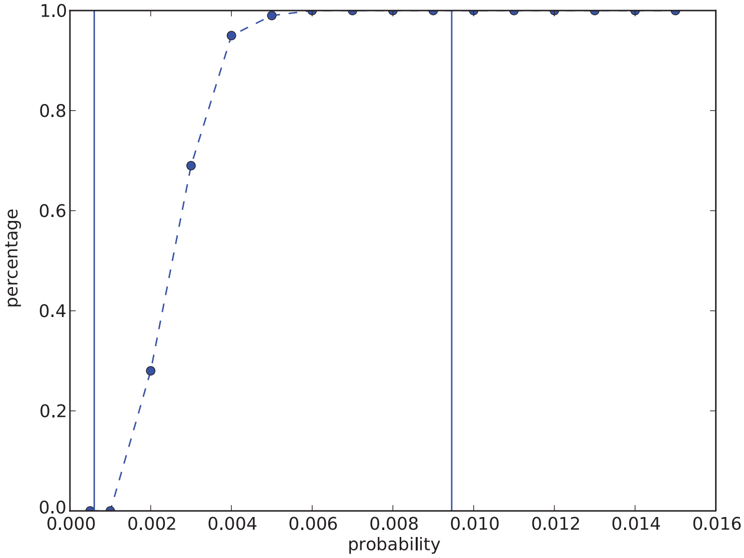

FIGURE 9. (Color online) Percentage of fully percolated configurations in 100 simulations of $BP(G_{n,r}, p, \theta)$ when $a = 35, \gamma = 1/75,\ n = 15000,\ r = \sqrt{35\ln n/\pi n} \approx 0.08451,\ D = 35 \ln n \approx 336.55$ and $\theta = \lceil 75^{-1}\mathbb{E}(\deg)\rceil = \lceil 4.49\rceil = 5,$ where $a = 35 > 5\pi$ and $\gamma = 1/75 \in \left[0, H_L^{-1}(5\pi/a)/5\pi\right] \approx [0, 0.0139]$. The bounds are $p' = 0.000602$ and $p'' = 0.009457$.

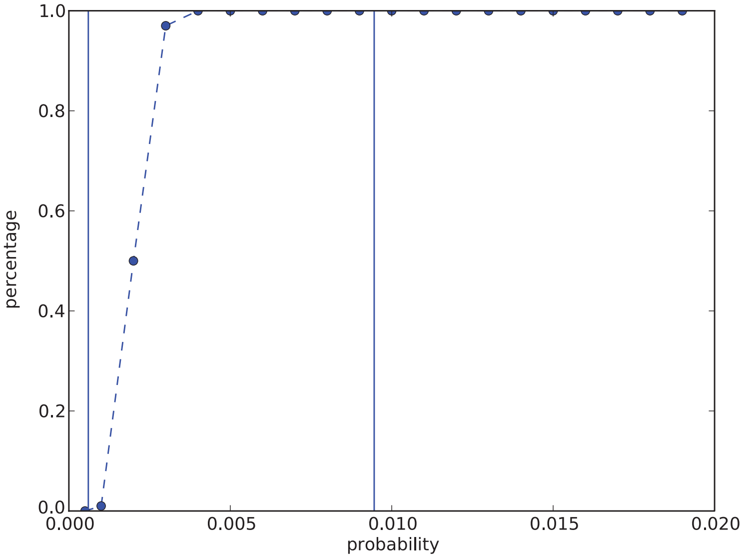

FIGURE 10. (Color online) Percentage of fully percolated configurations in 100 simulations of $BP(G_{n,r}, p, \theta)$ when $a = 35, \gamma = 1/75,\ n = 25000,\ r = \sqrt{35\ln n/\pi n} \approx 0.06718,\ D = 35 \ln n \approx 354.43$ and $\theta = \lceil 75^{-1}\mathbb{E}(\deg)\rceil = \lceil 4.73\rceil = 5,$ where $a = 35 > 5\pi$ and $\gamma = 1/75 \in \left[0, H_L^{-1}(5\pi/a)/5\pi\right] \approx [0, 0.0139]$. The bounds are $p' = 0.000602$ and $p'' = 0.009457$.

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