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ON A NEW CIRCLE PROBLEM

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Abstract

We attempt to discuss a new circle problem. Let $\zeta(s)$ denote the Riemann zeta-function $\sum_{n=1}^{\infty} n^{-s}$ ($\operatorname{Re} s > 1$) and $L(s,\chi_4)$ the Dirichlet L-function $\sum_{n=1}^{\infty} \chi_4(n)n^{-s}$ ($\operatorname{Re} s > 1$) with the primitive Dirichlet character mod 4. We shall define an arithmetical function $R_{(1,1)}(n)$ by the coefficient of the Dirichlet series $\zeta'(s)L'(s,\chi_4) = \sum_{n=1}^{\infty} R_{(1,1)}(n)n^{-s}$ ($\operatorname{Re} s > 1$). This is an analogue of $r(n)/4 = \sum_{d|n} \chi_4(d)$. In the circle problem, there are many researches of estimations and related topics on the error term in the asymptotic formula for $\sum_{n\le x} r(n)$. As a new problem, we deduce a 'truncated Voronoï formula' for the error term in the asymptotic formula for $\sum_{n\le x} R_{(1,1)}(n)$. As a direct application, we show the mean square for the error term in our new problem.

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1. Introduction

For each positive integer n, let $r(n)$ be the number of the pairs of integers (k, l) satisfying $k^2 + l^2 = n$, and

$$P(x) = \sum_{n\le x} r(n) - \pi x + 1.$$

The study on the estimation of $P(x)$ is called the circle problem. It is one of the important problems in number theory. There are many researches concerned with $P(x)$. For instance, it is known that

$$P(x) = \sqrt{x}\sum_{n=1}^{\infty} \frac{r(n)}{\sqrt{n}} J_1(2\pi\sqrt{nx}),$$

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where $J_1(y)$ is the Bessel function defined by

$$J_1(y) = \frac{y}{2}\sum_{j=0}^{\infty}\frac{(-1)^j(\frac{y}{2})^{2j}}{\Gamma(j+1)\Gamma(j+2)}$$

for $y > 0$ and $\Gamma(y)$ is the Gamma function. For further topics of the circle problem, see, for example, [I] and [K].

Many studies of the circle problem are connected deeply with those of the divisor problem (see, for example, [T] and [I]). In [FMT] and [M], we proposed a new type of the divisor problem and derived several formulas analogous to the classical divisor problem. In these two articles, the authors considered the error term for

$$\Delta_{(k,l)}(x) = \sum_{n\le x} D_{(k,l)}(n) - xP_{(k+l+1)}(\log x),$$

where

$$D_{(k,l)}(n) := (-1)^{k+l}\sum_{d|n}(\log n)^k\Big(\log\frac{n}{d}\Big)^l$$

and $P_{(k+l+1)}(y)$ is a polynomial of degree $k+l+1$ in y.

The above new divisor function $D_{(k,l)}(n)$ (it is an analogue of the divisor function $d(n) = \sum_{d|n} 1$) is the nth coefficient of the Dirichlet series

$$\zeta^{(k)}(s)\zeta^{(l)}(s) = \sum_{n=1}^{\infty}\frac{D_{(k,l)}(n)}{n^s} \quad (\mathrm{Re}\, s > 1), \tag{1.1}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ $(\mathrm{Re}\, s > 1)$ is the Riemann zeta-function and $f^{(\mu)}(s)$ with a (at least) μ times continuously differentiable function $f(s)$ means the μth derivative of the function $f(s)$, especially $f^{(0)}(s) = f(s)$.

In this paper, we shall consider a new problem analogous to (1.1) in the case of the circle problem.

We shall recall that the generating Dirichlet series of $r(n)$ is

$$4\zeta(s)L(s,\chi_4) = \sum_{n=1}^{\infty}\frac{r(n)}{n^s} \quad (\mathrm{Re}\, s > 1),$$

where $L(s,\chi_4)$ is the Dirichlet L-function $\sum_{n=1}^{\infty}\chi_4(n)n^{-s}$ with the Dirichlet character mod 4. Now we write

$$\zeta^{(\mu)}(s)L^{(\nu)}(s,\chi_4) = \sum_{n=1}^{\infty}\frac{R_{(\mu,\nu)}(n)}{n^s} \quad (\mathrm{Re}\, s > 1)$$

for $\mu,\nu = 0,1$ and investigate the error term $P_{(1)}(x)$ defined by

$$\sum_{n\le x} R_{(1,1)}(n) = a_2 x\log x + a_1 x + a_0 + P_{(1)}(x) \tag{1.2}$$

(for the values of a_i $(i = 0,1,2)$, see (3.2) below); especially, we shall show the ‘truncated Voronoï formula’ for $P_{(1)}(x)$. The first theorem of this paper is as follows.

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THEOREM 1.1. *For an arbitrary small positive ε, a large parameter x, and a large natural number $N \ll x^A$ with some positive constant A,*

$$\begin{aligned} P_{(1)}(x) = {} & \frac{\sqrt{x}}{4} \sum_{n \le N} \frac{R_{(0,0)}(n) \log^2(\pi^2 nx)}{\sqrt{n}} J_1(2\pi\sqrt{nx}) \\ & + \frac{\sqrt{x}}{2} \sum_{n \le N} \frac{S(n) \log(\pi^2 nx)}{\sqrt{n}} J_1(2\pi\sqrt{nx}) + \sqrt{x} \sum_{n \le N} \frac{T(n)}{\sqrt{n}} J_1(2\pi\sqrt{nx}) \\ & + O(x^{\varepsilon}) + O(x^{1/2+\varepsilon} N^{-1/2}), \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} S(n) &= -(2\log\pi) R_{(0,0)}(n) + R_{(0,1)}(n) + R_{(1,0)}(n), \\ T(n) &= (2\log\pi)\Big(\log\frac{\pi}{2}\Big) R_{(0,0)}(n) - \log(2\pi) R_{(0,1)}(n) \\ &\qquad - \log\Big(\frac{\pi}{2}\Big) R_{(1,0)}(n) + R_{(1,1)}(n), \end{aligned} \tag{1.4}$$

and in the right-hand side of (1.3) the implied constants in the symbol O depend at most on ε.

Immediately we have the following corollary.

COROLLARY 1.2. *Using the same notation in Theorem 1.1,*

$$\begin{aligned} P_{(1)}(x) = {} & -\frac{x^{1/4}}{4\pi} \sum_{n \le N} \frac{R_{(0,0)}(n) \log^2(\pi^2 nx)}{n^{3/4}} \cos\Big(2\pi\sqrt{nx} + \frac{\pi}{4}\Big) \\ & - \frac{x^{1/4}}{2\pi} \sum_{n \le N} \frac{S(n) \log(\pi^2 nx)}{n^{3/4}} \cos\Big(2\pi\sqrt{nx} + \frac{\pi}{4}\Big) \\ & - \frac{x^{1/4}}{\pi} \sum_{n \le N} \frac{T(n)}{n^{3/4}} \cos\Big(2\pi\sqrt{nx} + \frac{\pi}{4}\Big) + O(x^{\varepsilon}) + O(x^{1/2+\varepsilon} N^{-1/2}). \end{aligned}$$

Moreover, we have the mean square formula for $P_{(1)}(x)$:

THEOREM 1.3.

$$\begin{aligned} \int_1^X P_{(1)}^2(x)\,dx = {} & \frac{1}{768\pi^2} \sum_{n=1}^{\infty} \frac{r^2(n)}{n^{3/2}} X^{3/2} \log^4 X + C_3 X^{3/2} \log^3 X + C_2 X^{3/2} \log^2 X \\ & + C_1 X^{3/2} \log X + C_0 X^{3/2} + O(X^{5/4+\varepsilon}) \end{aligned}$$

for $X \ge 2$, where the coefficients C_j are absolute constants defined in Section 4.

As a behaviour of the error function $P_{(1)}(x)$, we obtain the following corollary.

COROLLARY 1.4. *We have*

$$P_{(1)}(x) = \begin{cases} O(x^{1/3+\varepsilon}), \\ \Omega(x^{1/4} \log^2 x). \end{cases}$$

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The O-estimate in this corollary can be proved by taking $N = x^{1/3}$ in Corollary 1.2, and the Ω-estimate is a direct consequence from Theorem 1.3.

By the same reasoning as the classical circle problem, we conjecture that a bound

$$P_{(1)}(x) = O(x^{1/4+\varepsilon})$$

holds for $x \geq 2$ from the Ω-estimate in Corollary 1.4.

2. Lemmas

Throughout the paper, ε denotes an arbitrary small positive number which need not be the same at each occurrence, and the implied constants in the symbols $O(\ )$ and $\ll$ depend at most on ε.

To prove Theorem 1.1, we shall prepare several lemmas. First we shall recall the functional equations of $\zeta(s)$ and $L(s)$ (from here we write $L(s)$ instead of $L(s, \chi_4)$ for simplicity):

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) = \pi^{s-1/2}\frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{s}{2})},$$
$$L(s) = \psi(s)L(1-s), \quad \psi(s) = \left(\frac{\pi}{4}\right)^{s-1/2}\frac{\Gamma(\frac{1}{2}(2-s))}{\Gamma(\frac{1}{2}(s+1))},$$
$$\chi(s)\psi(s) = \pi^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}.$$

Since $\zeta'(s) = \chi'(s)\zeta(1-s) - \chi(s)\zeta'(1-s)$ and $L'(s) = \psi'(1-s) - \psi(s)L'(1-s)$,

$$\begin{aligned}\zeta'(s)L'(s) = {} & \chi'(s)\psi'(s)\zeta(1-s)L(1-s) - \chi'(s)\psi(s)\zeta(1-s)L'(1-s) \\ & - \chi(s)\psi'(s)\zeta'(1-s)L(1-s) + \chi(s)\psi(s)\zeta'(1-s)L'(1-s). \end{aligned} \tag{2.1}$$

Let us define

$$\Phi_{(i,j)}(s) = \chi^{(i)}(s)\psi^{(j)}(s), \quad \Phi(s) = \Phi_{(0,0)}(s) = \chi(s)\psi(s).$$

To apply the method used in [M], we recall the following formulas for the Gamma function $\Gamma(s)$ ($s = \sigma + it$, $\sigma, t \in \mathbb{R}$).

Lemma 2.1 (The Stirling formula for $\Gamma(s)$).

$$\Gamma(s) = \sqrt{2\pi}s^{s-1/2}e^{-s}\left(1 + \frac{1}{12s} + O\left(\frac{1}{|s|^2}\right)\right) \quad (|\arg s| < \pi - \varepsilon), \tag{2.2}$$

$$|\Gamma(\sigma + it)| = \sqrt{2\pi}e^{-(\pi|t|)/2}|t|^{\sigma-1/2}\left(1 + O\left(\frac{1}{|t|^2}\right)\right) \quad (\sigma_1 \leq \sigma \leq \sigma_2, |t| \geq 2), \tag{2.3}$$

where σ_1 and σ_2 are fixed real numbers.

Under these preparations, we first observe the following lemma.

LEMMA 2.2. *Let $s = \sigma + it$ ($\sigma,\ t \in \mathbb{R}$), $\sigma_1 \le \sigma \le \sigma_2$ (σ_1, σ_2 are fixed), and $|t| \ge 2$. For the above $\Phi_{(i,j)}(s)$,*

$$|\chi(s)| = \left(\frac{|t|}{2\pi}\right)^{1/2-\sigma}\left(1 + O\left(\frac{1}{|t|^2}\right)\right), \tag{2.4}$$

$$|\psi(s)| = \left(\frac{\pi}{4}\right)^{\sigma-1/2}\left|\frac{t}{2}\right|^{1/2-\sigma}\left(1 + O\left(\frac{1}{|t|^2}\right)\right), \tag{2.5}$$

$$\Phi(s) = O(|t|^{1-2\sigma}), \tag{2.6}$$

$$\chi'(s) = \chi(s)(-\log|t|) + \chi(s)\log(2\pi) + O(|t|^{-1/2-\sigma}), \tag{2.7}$$

$$\psi'(s) = \psi(s)(-\log|t|) + \psi(s)\log\left(\frac{\pi}{2}\right) + O(|t|^{-1/2-\sigma}), \tag{2.8}$$

$$\Phi_{(0,1)}(s) = \Phi(s)(-\log|t|) + \Phi(s)\log\left(\frac{\pi}{2}\right) + O(|t|^{-2\sigma}), \tag{2.9}$$

$$\Phi_{(1,0)}(s) = \Phi(s)(-\log|t|) + \Phi(s)\log(2\pi) + O(|t|^{-2\sigma}), \tag{2.10}$$

$$\begin{aligned}\Phi_{(1,1)}(s) &= \Phi(s)(-\log|t|)^2 + \Phi(s)(-\log|t|)\log(\pi^2)\\ &\quad+\Phi(s)(\log 2\pi)\left(\log\frac{\pi}{2}\right) + O(|t|^{-2\sigma}\log|t|).\end{aligned} \tag{2.11}$$

PROOF. The formulas (2.4) and (2.5) are easily deduced from the Stirling formula (2.3). The estimate (2.6) is a direct result from (2.4) and (2.5).

The formula (2.7) is proved by Gonek [G, page 133, Lemma 6]. By a similar method of Gonek, we get (2.8). Applying the formula

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{|s|}\right) \quad (|t| \ge 2)$$

to the logarithmic derivative of $\psi(s)$,

$$\begin{aligned}\frac{\psi'}{\psi}(s) &= \log\frac{\pi}{4} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}(2-s)\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+\frac{1}{2}\right)\\ &= -\log\frac{|t|}{2} + \log\frac{\pi}{4} + O\left(\frac{1}{|t|}\right).\end{aligned}$$

This and (2.5) imply (2.8).

By (2.4), (2.5), (2.7), and (2.8), we get (2.9), (2.10), and (2.11). □

In a proof of Theorem 1.1, we will consider the integrals of the form

$$\int_{-\varepsilon-iT}^{-\varepsilon+iT} \Phi_{(i,j)}(s)\frac{(nx)^s}{s}\,ds$$

in Section 3. Here we give a lemma on this integral.

LEMMA 2.3. *Let any T, $x > 0$ be large and n a positive integer. For $\Phi_{(i,j)}(s)$ ($i, j = 0, 1$),*

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT}\Phi_{(i,j)}(s)\frac{(nx)^s}{s}\,ds\\ &\quad= \frac{1}{2\pi i}\left(\int_2^T + \int_{-T}^{-2}\right)\Phi_{(i,j)}(-\varepsilon+it)\frac{(nx)^{it}}{t(nx)^{\varepsilon}}\,dt + O\left(\left(\frac{T^2}{nx}\right)^{\varepsilon}(\log T)^{i+j}\right).\end{aligned} \tag{2.12}$$

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Proof. For any positive integer n,

$$\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT}\Phi_{(i,j)}(s)\frac{(nx)^s}{s}\,ds=\frac{1}{2\pi i}\Big(\int_{-\varepsilon+2i}^{-\varepsilon+iT}+\int_{-\varepsilon-iT}^{-\varepsilon-2i}\Big)\Phi_{(i,j)}(s)\frac{(nx)^s}{s}\,ds+O\Big(\frac{1}{(nx)^{\varepsilon}}\Big).$$

By $1/(-\varepsilon+it)=1/(it)+O(1/t^2)$ ($|\varepsilon/t|<1$) and (2.6), (2.9)–(2.11) in Lemma 2.2, we obtain the formula (2.12). □

To express $P_{(1)}(x)$ as certain partial sums involving the Bessel function $J_1(y)$, we apply the formula (2.14) below, which is a consequence of the well-known formula (2.13).

Lemma 2.4 [J, page 20, Lemma 1.5]. *For $x>0$,*

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(1-s)}{\Gamma(s)}\frac{x^s}{s}\,ds=\sqrt{x}J_1(2\sqrt{x}). \tag{2.13}$$

From Lemma 2.4, we have the following result.

Lemma 2.5 (see [M, page 340, Lemma 3.5]). *Let $x>0$ be large and N a large positive integer. Choose a large T satisfying $N+1/2=T^2/(\pi^2x)$. For any positive integer $n\le N$,*

$$\begin{aligned}&\frac{1}{2\pi i}\Big(\int_2^T+\int_{-T}^{-2}\Big)\Phi(-\varepsilon+it)\frac{(nx)^{it}}{t(nx)^{\varepsilon}}\,dt\\&\quad=\sqrt{nx}J_1(2\pi\sqrt{nx})+O\Big(\Big(\frac{N}{n}\Big)^{\varepsilon}\Big)+O\Big(\frac{1}{\log\frac{N+\frac12}{n}}\Big).\end{aligned} \tag{2.14}$$

Proof. We shall choose a large $T>0$ satisfying $N+1/2=T^2/(\pi^2x)$. By (2.13), for any large $x>0$ and any positive integer $n\le N$,

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Phi(s)\frac{(nx)^s}{s}\,ds=\sqrt{nx}J_1(2\pi\sqrt{nx}). \tag{2.15}$$

As in an argument in [T, page 318] by the residue theorem,

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Phi(s)\frac{(nx)^s}{s}\,ds+\frac{1}{2\pi i}\Big(\int_{i\infty}^{iT}+\int_{-iT}^{-i\infty}+\int_{iT}^{-\varepsilon+iT}+\int_{-\varepsilon-iT}^{-iT}\Big)\Phi(s)\frac{(nx)^s}{s}\,ds\\&\quad=\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT}\Phi(s)\frac{(nx)^s}{s}\,ds.\end{aligned} \tag{2.16}$$

Since (2.6),

$$\frac{1}{2\pi i}\Big(\int_{iT}^{-\varepsilon+iT}+\int_{-\varepsilon-iT}^{-iT}\Big)\Phi(s)\frac{(nx)^s}{s}\,ds\ll\int_{-\varepsilon}^{0}\Big(\frac{nx}{T^2}\Big)^{\sigma}\,d\sigma\ll\frac{T^{2\varepsilon}}{n^{\varepsilon}x^{\varepsilon}}. \tag{2.17}$$

Next we shall estimate

$$\int_{iT}^{i\infty}\Phi(s)\frac{(nx)^s}{s}\,ds=\lim_{R\to\infty}\int_T^R\frac{\Gamma(1-it)}{\Gamma(it)}\frac{e^{it\log(\pi^2nx)}}{t}\,dt.$$

By the Stirling formula (2.2),

$$\Gamma(1-it) = \sqrt{2\pi}|t|^{1/2}e^{-\pi/2|t|}e^{i(-t\log|t|+t\mp\pi/4)}\Big(1-\frac{1}{12it}+O\Big(\frac{1}{|t|^2}\Big)\Big),$$
$$\Gamma(it) = \sqrt{2\pi}|t|^{-1/2}e^{-\pi/2|t|}e^{i(t\log|t|-t\mp\pi/4)}\Big(1+\frac{1}{12it}+O\Big(\frac{1}{|t|^2}\Big)\Big),$$
$$\frac{\Gamma(1-it)}{\Gamma(it)} = \Big(1-\frac{1}{6it}+O\Big(\frac{1}{|t|^2}\Big)\Big)|t|e^{i(-2t\log|t|+2t)}.$$

Here we put $G(t) = -2t\log|t| + 2t + t\log(\pi^2 nx)$ for $T \le t \le R$. We observe that

$$G'(t) = -2\log|t| + \log(\pi^2 nx) \le -\log\frac{T^2}{\pi^2 nx} = -\log\frac{N+\frac{1}{2}}{n} < 0 \quad (n \le N).$$

Hence, by the first-derivative test,

$$\int_T^R\Big(1-\frac{1}{6it}+O\Big(\frac{1}{|t|^2}\Big)\Big)e^{iG(t)}\,dt \ll \frac{1}{\log\frac{N+\frac{1}{2}}{n}}+\frac{1}{T}.$$

This upper bound is uniform on R. Therefore, on letting $R\to\infty$,

$$\frac{1}{2\pi i}\Big(\int_{i\infty}^{iT}+\int_{-iT}^{-i\infty}\Big)\Phi(s)\frac{(nx)^s}{s}\,ds = O\Big(\frac{1}{\log\frac{N+\frac{1}{2}}{n}}\Big). \tag{2.18}$$

Collecting (2.15), (2.16), (2.17), and (2.18),

$$\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT}\Phi(s)\frac{(nx)^s}{s}\,ds = \sqrt{nx}J_1(2\pi\sqrt{nx}) + O\Big(\frac{T^{2\varepsilon}}{(nx)^\varepsilon}\Big)+O\Big(\frac{1}{\log\frac{N+\frac{1}{2}}{n}}\Big). \tag{2.19}$$

By applying the relation $T^2 \asymp Nx$, we can see that the first error term on the right-hand side in (2.19) can be replaced by $O((N/n)^\varepsilon)$.

On the other hand, the left-hand side of (2.19) is expressed as (2.12). Then we get (2.14). □

We remark that we shall replace the error term $O(T^{2\varepsilon}/(nx)^\varepsilon\log^j T)$ with $O((N/n)^\varepsilon\log^j T)$ under the assumption that $T^2 \asymp Nx$ in the following process.

From the formula (2.14), we shall deduce important formulas used in a proof of Theorem 1.1.

LEMMA 2.6 (see Gonek [G] and [M, page 340, Lemma 3.6]). *Let $x > 0$ be large and N a large positive integer. Choose a large T satisfying $N + 1/2 = T^2/(\pi^2 x)$. For any positive integer $n \le N$,*

$$\Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}\frac{1}{(nx)^\varepsilon}\frac{1}{2\pi i}\Big(\int_2^T-\int_{-T}^{-2}\Big)|t|^{2\varepsilon}e^{iF(t)}(\pi^2 nx)^{it}\,dt$$
$$= \sqrt{nx}J_1(2\pi\sqrt{nx}) + O\Big(\Big(\frac{N}{n}\Big)^\varepsilon\Big)+O\Big(\frac{1}{\log\frac{N+\frac{1}{2}}{n}}\Big), \tag{2.20}$$

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$$\Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}\frac{1}{(nx)^{\varepsilon}}\frac{1}{2\pi i}\Big(\int_{2}^{T}-\int_{-T}^{-2}\Big)|t|^{2\varepsilon}(-\log|t|)e^{iF(t)}(\pi^2nx)^{it}\,dt$$
$$= -\frac{\sqrt{nx}}{2}J_1(2\pi\sqrt{nx})\log(\pi^2nx) + O\Big(\Big(\frac{N}{n}\Big)^{\varepsilon}\log T\Big) + O\Big(\frac{\log T}{\log\frac{N+\frac12}{n}}\Big), \quad (2.21)$$

$$\Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}\frac{1}{(nx)^{\varepsilon}}\frac{1}{2\pi i}\Big(\int_{2}^{T}-\int_{-T}^{-2}\Big)|t|^{2\varepsilon}(-\log|t|)^2e^{iF(t)}(\pi^2nx)^{it}\,dt$$
$$= \frac{\sqrt{nx}}{4}J_1(2\pi\sqrt{nx})\log^2(\pi^2nx) + O\Big(\Big(\frac{N}{n}\Big)^{\varepsilon}\log^2 T\Big) + O\Big(\frac{\log^2 T}{\log\frac{N+\frac12}{n}}\Big), \quad (2.22)$$

where $F(t) = -2t\log|t| + 2t$.

Proof. To deduce (2.20) from (2.14), by the Stirling formula (2.2), we remark that

$$\Gamma(1+\varepsilon-it) = \sqrt{2\pi}|t|^{1/2+\varepsilon}e^{\mp\pi/2t}e^{i(-t\log|t|\mp\pi/2(\frac12+\varepsilon)+t)}\Big(1+O\Big(\frac{1}{|t|}\Big)\Big),$$
$$\Gamma(-\varepsilon+it) = \sqrt{2\pi}|t|^{-1/2-\varepsilon}e^{\mp\pi/2t}e^{i(-t\log|t|\mp\pi/2(\frac12+\varepsilon)-t)}\Big(1+O\Big(\frac{1}{|t|}\Big)\Big),$$
$$\Phi(-\varepsilon+it)\frac{(nx)^{it}}{t(nx)^{\varepsilon}} = \Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}|t|^{2\varepsilon}e^{iF(t)}\frac{(\pi^2nx)^{it}}{(nx)^{\varepsilon}}\frac{|t|}{t} + O\Big(\frac{|t|^{2\varepsilon}}{|t|(nx)^{\varepsilon}}\Big), \quad (2.23)$$

where $F(t) = -2t\log|t| + 2t$. Then by (2.12) and (2.23) for any n we observe that

$$\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT}\Phi(s)\frac{(nx)^s}{s}\,ds$$
$$= \Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}\frac{1}{(nx)^{\varepsilon}}\frac{1}{2\pi i}\Big(\int_{2}^{T}-\int_{-T}^{-2}\Big)|t|^{2\varepsilon}e^{iF(t)}(\pi^2nx)^{it}\,dt + O\Big(\frac{T^{2\varepsilon}}{(nx)^{\varepsilon}}\Big). \quad (2.24)$$

From this and (2.19), we obtain the assertion (2.20).

We shall deduce (2.21) from (2.20). Since $(e^{iF(t)})' = (-2i\log|t|)e^{iF(t)}$, using integration by parts,

$$\Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}\frac{1}{(nx)^{\varepsilon}}\frac{1}{2\pi i}\Big(\int_{2}^{T}-\int_{-T}^{-2}\Big)|t|^{2\varepsilon}(-\log|t|)e^{iF(t)}(\pi^2nx)^{it}\,dt$$
$$= \Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}\frac{1}{(nx)^{\varepsilon}}\frac{1}{2\pi i}\Big(\int_{2}^{T}-\int_{-T}^{-2}\Big)|t|^{2\varepsilon}\Big(\frac{e^{iF(t)}}{2i}\Big)'(\pi^2nx)^{it}\,dt$$
$$= \Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}\frac{1}{(nx)^{\varepsilon}}\frac{1}{2\pi i}\Big\{\Big[|t|^{2\varepsilon}\frac{e^{iF(t)}}{2i}(\pi^2nx)^{it}\Big]_2^T - \Big[|t|^{2\varepsilon}\frac{e^{iF(t)}}{2i}(\pi^2nx)^{it}\Big]_{-T}^{-2}$$
$$-\Big(\int_{2}^{T}+\int_{-T}^{-2}\Big)2\varepsilon|t|^{2\varepsilon-1}\frac{e^{iF(t)}}{2i}(\pi^2nx)^{it}\,dt$$
$$-\Big(\int_{2}^{T}-\int_{-T}^{-2}\Big)|t|^{2\varepsilon}\frac{e^{iF(t)}}{2i}i\log(\pi^2nx)(\pi^2nx)^{it}\,dt\Big\}$$
$$= -\frac{\log(\pi^2nx)}{2}\Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}\frac{1}{(nx)^{\varepsilon}}\frac{1}{2\pi i}\Big(\int_{2}^{T}-\int_{-T}^{-2}\Big)|t|^{2\varepsilon}e^{iF(t)}(\pi^2nx)^{it}\,dt + O\Big(\frac{T^{2\varepsilon}}{(nx)^{\varepsilon}}\Big).$$

Here we use (2.20); then we get (2.21).

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In a similar way as above, we shall show that

$$\Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}\frac{1}{(nx)^{\varepsilon}}\frac{1}{2\pi i}\Big(\int_2^T-\int_{-T}^{-2}\Big)|t|^{2\varepsilon}(-\log|t|)^2e^{iF(t)}(\pi^2nx)^{it}\,dt$$
$$=\Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}\frac{1}{(nx)^{\varepsilon}}\frac{1}{2\pi i}\Big\{\Big[|t|^{2\varepsilon}(-\log|t|)\frac{e^{iF(t)}}{2i}(\pi^2nx)^{it}\Big]_2^T$$
$$-\Big[|t|^{2\varepsilon}(-\log|t|)\frac{e^{iF(t)}}{2i}(\pi^2nx)^{it}\Big]_{-T}^{-2}$$
$$-\Big(\int_2^T+\int_{-T}^{-2}\Big)2\varepsilon|t|^{2\varepsilon-1}(-\log|t|)\frac{e^{iF(t)}}{2i}(\pi^2nx)^{it}\,dt$$
$$+\Big(\int_2^T-\int_{-T}^{-2}\Big)|t|^{2\varepsilon}\frac{1}{|t|}\frac{e^{iF(t)}}{2i}(\pi^2nx)^{it}\,dt$$
$$-\Big(\int_2^T-\int_{-T}^{-2}\Big)t^{2\varepsilon}(-\log|t|)\frac{e^{iF(t)}}{2i}i\log(\pi^2nx)(\pi^2nx)^{it}\,dt\Big\}$$
$$=-\frac{\log(\pi^2nx)}{2}\Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}\frac{1}{(nx)^{\varepsilon}}\frac{1}{2\pi i}\Big(\int_2^T-\int_{-T}^{-2}\Big)t^{2\varepsilon}(-\log|t|)e^{iF(t)}(\pi^2nx)^{it}\,dt$$
$$+O\Big(\frac{T^{2\varepsilon}\log T}{(nx)^{\varepsilon}}\Big).$$

By (2.21), we obtain the assertion (2.22). □

3. Proof of Theorem 1.1

We shall apply the following Perron formula to the Dirichlet series $\zeta'(s)L'(s) = \sum_{n=1}^{\infty} R_{(1,1)}(n)n^{-s}$ (Re $s > 1$).

Lemma 3.1 (Perron's formula [T, page 60]). *For a Dirichlet series $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ satisfying* (i) *the series $D(s)$ is absolutely convergent for* Re $s = \sigma > 1$, (ii) *the coefficients $\{a_n\}_{n=1}^{\infty}$ are bounded by a positive increasing function $|a_n| \le A(n)$, and* (iii) *there is a positive constant α satisfying*

$$\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma}} = O\Big(\frac{1}{(\sigma-1)^{\alpha}}\Big) \quad (\sigma \to 1^+),$$

$$\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} D(s)\frac{x^s}{s}\,ds + O\Big(\frac{x^b}{T(b-1)^{\alpha}}\Big) + O\Big(\frac{xA(2x)\log x}{T}\Big),$$

where x and T are large parameters also $x = N_0 + 1/2$ (N_0 is a large natural number) and $b > 1$ is a constant.

Since $|R_{(1,1)}(n)| \le d(n)\log^2 n$, and $\zeta'(s)L'(s)$ has a pole at $s = 1$ of order 2, we apply this lemma to $D(s) = \zeta'(s)L'(s)$.

To apply the residue theorem in the above integral, we shall recall estimates on $\zeta(s)$, $L(s)$, $\zeta'(s)$, and $L'(s)$.

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LEMMA 3.2. *Let $\varepsilon > 0$ be any sufficiently small number, $|t| \geq 2$, and k a nonnegative integer. For $\zeta^{(k)}(\sigma + it)$ and $L^{(k)}(\sigma + it)$,*

$$\zeta^{(k)}(\sigma + it),\ L^{(k)}(\sigma + it) \ll \begin{cases} |t|^{1/2-\sigma+\varepsilon}, & \sigma \leq 0, \\ |t|^{1/2(1-\sigma)+\varepsilon}, & 0 \leq \sigma \leq 1, \\ |t|^{\varepsilon}, & \sigma \geq 1. \end{cases}$$

PROOF. The estimates on $\zeta^{(k)}(s)$ are stated in [G, page 127]. We shall check them in the case of $L^{(k)}(s)$. Putting $\sigma = 1 + \varepsilon$, trivially, we have $L(1 + \varepsilon + it) = O(1)$. On the other hand, by the functional equation $L(s) = \psi(s)L(1 - s)$ and (2.3),

$$L(-\varepsilon + it) = O(|t|^{1/2+\varepsilon}) \quad (|t| \geq 2).$$

Then, for $-\varepsilon \leq \sigma \leq 1 + \varepsilon$ and $|t| \geq 2$, by the maximum-modulus principle,

$$L(\sigma + it) \ll |t|^{1/2(1-\sigma)+\varepsilon},$$

which implies the estimates of $L(s)$ for $\sigma \geq 0$. Moreover, using the results and the functional equation for $L(s)$, we see the estimate of $L(s)$ for $\sigma \leq 0$.

By the Cauchy integral formula,

$$L^{(k)}(s) = \frac{k!}{2\pi i} \int_{C_t} \frac{L(s + w)}{w^{k+1}}\, dw,$$

where C_t is the circle $|w| = 1/\log|t|$. From this and the previous results, we complete the proof of Lemma 3.2. □

Also, we calculate the residues of $\zeta'(s)L'(s)x^s/s$ at $s = 0, 1$. Since $L(s)$ is regular at $s = 1$, we write

$$L(s) = \frac{\pi}{4} + l_1(s - 1) + l_2(s - 1)^2 + \cdots .$$

Using these symbols and the facts (see, for example, [T, page 20] and [AC, page 344], also see [FT])

$$\zeta'(0) = -\tfrac{1}{2}\log 2\pi, \quad L'(0) = \log\Gamma^2(\tfrac{1}{4}) - \log\pi - \tfrac{3}{2}\log 2,$$

we have

$$\begin{aligned} \operatorname*{Res}_{s=1} \zeta'(s)L'(s)\frac{x^s}{s} &= -l_1 x\log x + (l_1 - 2l_2)x, \\ \operatorname*{Res}_{s=0} \zeta'(s)L'(s)\frac{x^s}{s} &= -(\log 2\pi)\Big(\log\Gamma\Big(\frac{1}{4}\Big)\Big) + \frac{1}{2}(\log 2\pi)(\log\pi) + \frac{3}{4}(\log 2\pi)(\log 2). \end{aligned} \tag{3.1}$$

From Lemmas 3.1, 3.2, and (3.1) and the formula (2.1),

$$\begin{aligned}\sum_{n\le x} R_{(1,1)}(n) &= \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \zeta'(s)L'(s)\frac{x^s}{s}\,ds + O\Big(\frac{x^{1+\varepsilon}}{T}\Big)\\
&= \frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT} \Phi_{(1,1)}(s)\zeta(1-s)L(1-s)\frac{x^s}{s}\,ds\\
&\quad - \frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon} \Phi_{(1,0)}(s)\zeta(1-s)L'(1-s)\frac{x^s}{s}\,ds\\
&\quad - \frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT} \Phi_{(0,1)}(s)\zeta'(1-s)L(1-s)\frac{x^s}{s}\,ds\\
&\quad + \frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT} \Phi(s)\zeta'(1-s)L'(1-s)\frac{x^s}{s}\,ds\\
&\quad - l_1 x\log x + (l_1-2l_2)x + O\Big(\frac{x^{1+\varepsilon}}{T}\Big) + O\Big(\frac{T^{2\varepsilon}}{x^{\varepsilon}}\Big)\\
&=: I_1 - I_2 - I_3 + I_4 - l_1 x\log x + (l_1-2l_2)x\\
&\quad - (\log 2\pi)(\log\Gamma(\tfrac14)) + \tfrac12(\log 2\pi)(\log\pi) + \tfrac34(\log 2\pi)(\log 2)\\
&\quad + O\Big(\frac{x^{1+\varepsilon}}{T}\Big) + O\Big(\frac{T^{2\varepsilon}}{x^{\varepsilon}}\Big),\end{aligned}$$

say. Hence, we have exact values of a_0, a_1, and a_2 in (1.2).

$$\begin{aligned}&a_2 = -l_1, \quad a_1 = l_1 - 2l_2,\\
&a_0 = -(\log 2\pi)(\log\Gamma(\tfrac14)) + \tfrac12(\log 2\pi)(\log\pi) + \tfrac34(\log 2\pi)(\log 2).\end{aligned} \tag{3.2}$$

First, using Lemma 2.6, we shall deduce the following lemma.

Lemma 3.3. *Let x be a large real number and N a large positive integer. Choose T satisfying $N + 1/2 = T^2/(\pi^2 x)$. Then*

$$I_4 = \sqrt{x}\sum_{n\le N}\frac{R_{(1,1)}(n)}{\sqrt{n}}J_1(2\pi\sqrt{nx}) + O(N^{\varepsilon}).$$

Proof. By the above N, we shall divide I_4 into two parts as follows.

$$\begin{aligned}I_4 &= \sum_{n\le N}\frac{R_{(1,1)}(n)}{n}\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT}\Phi(s)\frac{(nx)^s}{s}\,ds\\
&\quad + \sum_{n=N+1}^{\infty}\frac{R_{(1,1)}(n)}{n}\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT}\Phi(s)\frac{(nx)^s}{s}\,ds\\
&=: I_{41} + I_{42},\end{aligned} \tag{3.3}$$

say. In the case of I_{42}, by (2.24),

$$I_{42} = \sum_{n=N+1}^{\infty}\Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}\frac{1}{x^{\varepsilon}}\frac{R_{(1,1)}(n)}{n^{1+\varepsilon}}\frac{1}{2\pi i}\Big(\int_2^T - \int_{-T}^{-2}\Big)|t|^{2\varepsilon}e^{iF(t)}(\pi^2 nx)^{it}\,dt + O\Big(\frac{T^{2\varepsilon}}{x^{\varepsilon}}\Big).$$

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Moreover, we remark that

$$\frac{d}{dt}(F(t) + t\log(\pi^2 nx)) = -2\log|t| + \log(\pi^2 nx) \geq \log\frac{n}{N+\frac{1}{2}} > 0.$$

Then, by the first-derivative test,

$$\begin{aligned}
I_{42} &= O\Big(\frac{T^{2\varepsilon}}{x^\varepsilon}\sum_{n=N+1}^{\infty}\frac{|R_{(1,1)}(n)|}{n^{1+\varepsilon}\log\frac{n}{N+\frac{1}{2}}}\Big) + O\Big(\frac{T^{2\varepsilon}}{x^\varepsilon}\Big)\\
&= O\Big(\frac{T^{2\varepsilon}}{x^\varepsilon}\sum_{n=N+1}^{2N-1}\frac{|R_{(1,1)}(n)|}{n^{1+\varepsilon}\log\frac{n}{N+\frac{1}{2}}}\Big) + O\Big(\frac{T^{2\varepsilon}}{x^\varepsilon}\Big)\\
&= O\Big(\frac{T^{2\varepsilon}}{x^\varepsilon}\sum_{m=1}^{N-1}\frac{|R_{(1,1)}(N+m)|}{(N+m)^{1+\varepsilon}\log\frac{N+m}{N+\frac{1}{2}}}\Big) + O\Big(\frac{T^{2\varepsilon}}{x^\varepsilon}\Big)\\
&= O\Big(\frac{T^{2\varepsilon}}{x^\varepsilon}\sum_{m\leq N}\frac{1}{m}\Big) + O\Big(\frac{T^{2\varepsilon}}{x^\varepsilon}\Big)\\
&= O\Big(\frac{T^{2\varepsilon}}{x^\varepsilon}\log N\Big). \qquad (3.4)
\end{aligned}$$

On the other hand, in the case of I_{41}, by (2.24),

$$I_{41} = \sum_{n\leq N}\frac{R_{(1,1)}(n)}{n}\Big(\frac{1}{\pi}\Big)^{1+2\varepsilon}\frac{1}{(nx)^\varepsilon}\frac{1}{2\pi i}\Big(\int_2^T - \int_{-T}^{-2}\Big)|t|^{2\varepsilon}e^{iF(t)}(\pi^2 nx)^{it}\,dt + O\Big(\frac{T^{2\varepsilon}}{x^\varepsilon}\Big).$$

Here we use the formula (2.20) in Lemma 2.6; then

$$I_{41} = \sqrt{x}\sum_{n\leq N}\frac{R_{(1,1)}(n)}{\sqrt{n}}J_1(2\pi\sqrt{nx}) + O(N^\varepsilon). \qquad (3.5)$$

From (3.3), (3.4), and (3.5), we obtain the assertion of Lemma 3.3. □

Next we shall consider I_3 and prove the following.

Lemma 3.4. *Keeping the same assumption in Lemma 3.3,*

$$\begin{aligned}
I_3 = &-\frac{\sqrt{x}}{2}\sum_{n\leq N}\frac{R_{(1,0)}(n)\log(\pi^2 nx)}{\sqrt{n}}J_1(2\pi\sqrt{nx})\\
&+\Big(\log\frac{\pi}{2}\Big)\sqrt{x}\sum_{n\leq N}\frac{R_{(1,0)}(n)}{\sqrt{n}}J_1(2\pi\sqrt{nx}) + O(N^\varepsilon).
\end{aligned}$$

Proof. We put I_{31} and I_{32} as follows:

$$\begin{aligned}
I_3 &= \sum_{n\leq N}\frac{R_{(1,0)}(n)}{n}\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT}\Phi_{(0,1)}(s)\frac{(nx)^s}{s}\,ds\\
&\quad + \sum_{n\geq N+1}\frac{R_{(0,1)}(n)}{n}\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT}\Phi_{(0,1)}(s)\frac{(nx)^s}{s}\,ds\\
&=: I_{31} + I_{32}.
\end{aligned}$$

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For $n > N$, by (2.12), (2.6), and (2.9),

$$\begin{aligned}I_{32} &= \sum_{n\geq N+1} \frac{R_{(1,0)}(n)}{n} \frac{1}{2\pi i}\Big(\int_2^T + \int_{-T}^{-2}\Big)\Phi_{(0,1)}(-\varepsilon+it)\frac{(nx)^{it}}{t(nx)^{\varepsilon}}\,dt + O\Big(\frac{T^{2\varepsilon}\log T}{x^{\varepsilon}}\Big)\\
&= \sum_{n\geq N+1} \frac{R_{(1,0)}(n)}{n} \frac{1}{2\pi i}\Big(\int_2^T + \int_{-T}^{-2}\Big)\Phi(-\varepsilon+it)(-\log|t|)\frac{(nx)^{it}}{t(nx)^{\varepsilon}}\,dt\\
&\quad + \Big(\log\frac{\pi}{2}\Big)\sum_{n\geq N+1} \frac{R_{(1,0)}(n)}{n} \frac{1}{2\pi i}\Big(\int_2^T + \int_{-T}^{-2}\Big)\Phi(-\varepsilon+it)\sum_{n\geq N+1}\frac{(nx)^{it}}{t(nx)^{\varepsilon}}\,dt\\
&\quad + O\Big(\frac{T^{2\varepsilon}\log T}{x^{\varepsilon}}\Big)\\
&= \sum_{n\geq N+1} \frac{R_{(1,0)}(n)}{n}\Big(\frac{1}{\pi}\Big)^{1+2\varepsilon} \frac{1}{2\pi i}\Big(\int_2^T - \int_{-T}^{-2}\Big)|t|^{2\varepsilon}(-\log|t|)e^{iF(t)}\frac{(\pi^2 nx)^{it}}{(nx)^{\varepsilon}}\,dt\\
&\quad + \Big(\log\frac{\pi}{2}\Big)\sum_{n\geq N+1} \frac{R_{(1,0)}(n)}{n}\Big(\frac{1}{\pi}\Big)^{1+2\varepsilon} \frac{1}{2\pi i}\Big(\int_2^T - \int_{-T}^{-2}\Big)|t|^{2\varepsilon}e^{iF(t)}\frac{(\pi^2 nx)^{it}}{(nx)^{\varepsilon}}\,dt\\
&\quad + O\Big(\frac{T^{2\varepsilon}\log T}{x^{\varepsilon}}\Big).\end{aligned}$$

Applying a similar method used in the estimation of I_{42},

$$I_{32} = O\Big(\frac{T^{2\varepsilon}\log T}{x^{\varepsilon}}\Big).$$

For $n \leq N$, by (2.12) and (2.9),

$$\begin{aligned}I_{31} &= \sum_{n\leq N} \frac{R_{(1,0)}(n)}{n} \frac{1}{2\pi i}\Big(\int_2^T + \int_{-T}^{-2}\Big)\Phi_{(0,1)}(-\varepsilon+it)\frac{(nx)^{it}}{t(nx)^{\varepsilon}}\,dt + O\Big(\frac{T^{2\varepsilon}\log T}{x^{\varepsilon}}\Big)\\
&= \sum_{n\leq N} \frac{R_{(1,0)}(n)}{n} \frac{1}{2\pi i}\Big(\int_2^T + \int_{-T}^{-2}\Big)\Phi(-\varepsilon+it)(-\log|t|)\frac{(nx)^{it}}{t(nx^{\varepsilon})}\,dt\\
&\quad + \Big(\log\frac{\pi}{2}\Big)\sum_{n\leq N} \frac{R_{(1,0)}(n)}{n} \frac{1}{2\pi i}\Big(\int_2^T + \int_{-T}^{-2}\Big)\Phi(-\varepsilon+it)\frac{(nx)^{it}}{t(nx)^{\varepsilon}}\,dt + O\Big(\frac{T^{2\varepsilon}\log T}{x^{\varepsilon}}\Big).\end{aligned}$$

Here we use (2.23), and (2.20), (2.21) in Lemma 2.6; finally, we get the assertion of Lemma 3.4. □

By using a similar way to (2.10), we obtain the formula for I_2.

Lemma 3.5. *Under the assumption in Lemma 3.3,*

$$\begin{aligned}I_2 = &-\frac{\sqrt{x}}{2}\sum_{n\leq N}\frac{R_{(0,1)}(n)\log(\pi^2 nx)}{\sqrt{n}}J_1(2\pi\sqrt{nx})\\
&+ (\log 2\pi)\sqrt{x}\sum_{n\leq N}\frac{R_{(0,1)}(n)}{\sqrt{n}}J_1(2\pi\sqrt{x}) + O(N^{\varepsilon}).\end{aligned}$$

To complete the proof of Theorem 1.1, we shall show the following formula for I_1.

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Lemma 3.6. *Under the assumption in Lemma 3.3,*

$$I_1 = \frac{\sqrt{x}}{4} \sum_{n\le N} \frac{R_{(0,0)}(n)\log^2(\pi^2 nx)}{\sqrt{n}} J_1(2\pi\sqrt{nx})$$
$$- (\log\pi)\sqrt{x} \sum_{n\le N} \frac{R_{(0,0)}(n)\log(\pi^2 nx)}{\sqrt{n}} J_1(2\pi\sqrt{nx})$$
$$+ (\log\pi^2)\Big(\log\frac{\pi}{2}\Big)\sqrt{x} \sum_{n\le N} \frac{R_{(0,0)}(n)}{\sqrt{n}} J_1(2\pi\sqrt{nx}) + O(N^{\varepsilon}).$$

Proof. Using (2.6) and (2.11),

$$I_1 = \sum_{n\le N} \frac{R_{(0,0)}(n)}{n} \frac{1}{2\pi i}\Big(\int_2^T + \int_{-T}^{-2}\Big)\Phi_{(1,1)}(-\varepsilon+it)\frac{(nx)^{it}}{t(nx)^{\varepsilon}}\,dt + O\Big(\frac{T^{2\varepsilon}\log^2 T}{x^{\varepsilon}}\Big)$$
$$= \sum_{n\le N} \frac{R_{(0,0)}(n)}{n} \frac{1}{2\pi i}\Big(\int_2^T + \int_{-T}^{-2}\Big)\Phi(-\varepsilon+it)(-\log|t|)^2\frac{(nx)^{it}}{t(nx)^{\varepsilon}}\,dt$$
$$+ (2\log\pi)\sum_{n\le N} \frac{R_{(0,0)}(n)}{n} \frac{1}{2\pi i}\Big(\int_2^T + \int_{-T}^{-2}\Big)\Phi(-\varepsilon+it)(-\log|t|)\frac{(nx)^{it}}{t(nx)^{\varepsilon}}\,dt$$
$$+ (\log 2\pi)\Big(\log\frac{\pi}{2}\Big)\sum_{n\le N} \frac{R_{(0,0)}(n)}{n} \frac{1}{2\pi i}\Big(\int_2^T + \int_{-T}^{-2}\Big)\Phi(-\varepsilon+it)\frac{(nx)^{it}}{t(nx)^{\varepsilon}}\,dt$$
$$+ O\Big(\frac{T^{2\varepsilon}\log^2 T}{x^{\varepsilon}}\Big).$$

By (2.23), (2.20) (2.21), and (2.22), we get the assertion of Lemma 3.6. □

Finally, by Lemmas 3.3–3.6, we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.3

In this section we shall investigate the mean square of $P_{(1)}(x)$ and prove Theorem 1.3. In Theorem 1.1, we apply the well-known formula

$$J_1(y) = -\sqrt{\frac{2}{\pi y}}\cos\Big(y+\frac{\pi}{4}\Big) + O\Big(\frac{1}{y^{3/2}}\Big) \quad (y>1).$$

We easily get the assertion of Corollary 1.2. Using the corollary, we shall show Theorem 1.3. Moreover, by noting that $\log\pi^2 nx = \log x + \log\pi^2 n$,

$$P_{(1)}(x) = -\frac{x^{1/4}\log^2 x}{4\pi}\sum_{n\le N}\frac{R_{(0,0)}(n)}{n^{3/4}}\cos\Big(2\pi\sqrt{nx}+\frac{\pi}{4}\Big)$$
$$- \frac{x^{1/4}\log x}{2\pi}\sum_{n\le N}\frac{\alpha(n)}{n^{3/4}}\cos\Big(2\pi\sqrt{nx}+\frac{\pi}{4}\Big)$$
$$- \frac{x^{1/4}}{\pi}\sum_{n\le N}\frac{\beta(n)}{n^{3/4}}\cos\Big(2\pi\sqrt{nx}+\frac{\pi}{4}\Big) + O(x^{\varepsilon}) + O(x^{1/2+\varepsilon}N^{-1/2}), \quad (4.1)$$

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where

$$\alpha(n) = R_{(0,0)}(n)\log(\pi^2 n) + S(n),$$
$$\beta(n) = \frac{R_{(0,0)}(n)\log^2(\pi^2 n)}{4} + \frac{S(n)\log(\pi^2 n)}{2} + T(n).$$

Here $S(n)$ and $T(n)$ are the functions defined by (1.4).

In order to deduce Theorem 1.3 from (4.1), we prepare the following lemma.

Lemma 4.1. *Let $a(n)$ and $b(n)$ be arithmetical functions satisfying $a(n), b(n) \ll n^{\varepsilon}$ for any $\varepsilon > 0$ and X a large parameter. For a fixed nonnegative integer k,*

$$\int_X^{2X} x^{1/2}\log^k x \sum_{m,n\le X} \frac{a(m)b(n)}{(mn)^{3/4}} \cos\Big(2\pi\sqrt{mx} + \frac{\pi}{4}\Big)\cos\Big(2\pi\sqrt{nx} + \frac{\pi}{4}\Big)\,dx$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{a(n)b(n)}{n^{3/2}} \sum_{i=0}^{k}\Big[x^{3/2}\Big(\frac{2}{3}\Big)^{i+1}(-1)^i\frac{k!}{(k-i)!}\log^{k-i} x\Big]_X^{2X} + O(X^{1+\varepsilon}). \quad (4.2)$$

Proof. Since $2\cos\theta\cos\theta' = \cos(\theta-\theta') + \cos(\theta+\theta')$,

$$(\text{LHS of (4.2)}) = \frac{1}{2}\sum_{n\le X}\frac{a(n)b(n)}{n^{3/2}}\int_X^{2X} x^{1/2}\log^k x\,dx$$
$$+ O\bigg(\bigg|\sum_{\substack{m,n\le X\\ m\ne n}}\frac{a(m)b(n)}{(mn)^{3/4}}\int_X^{2X}(x^{1/2}\log^k x)\cos(2\pi\sqrt{mx} - 2\pi\sqrt{nx})\,dx\bigg|\bigg)$$
$$+ O\bigg(\bigg|\sum_{m,n\le X}\frac{a(m)b(n)}{(mn)^{3/4}}\int_X^{2X}(x^{1/2}\log^k x)\sin(2\pi\sqrt{mx} + 2\pi\sqrt{nx}dx)\,dx\bigg|\bigg)$$
$$=: W_1 + W_2 + W_3,$$

say. For W_1, by integration by parts,

$$\int_X^{2X} x^{1/2}\log^k x\,dx = \sum_{i=0}^{k}\Big[x^{3/2}\Big(\frac{2}{3}\Big)^{i+1}(-1)^i\frac{k!}{(k-i)!}\log^{k-i} x\Big]_X^{2X}.$$

Moreover, we easily see that

$$\sum_{n>X}\frac{a(n)b(n)}{n^{3/2}} \ll \int_X^{\infty} t^{-3/2+\varepsilon}\,dt \ll X^{-1/2+\varepsilon}.$$

Therefore,

$$W_1 = \frac{1}{2}\Big(\sum_{n=1}^{\infty}\frac{a(n)b(n)}{n^{3/2}} - \sum_{n>X}\frac{a(n)b(n)}{n^{3/2}}\Big)\sum_{i=0}^{k}\Big[x^{3/2}\Big(\frac{2}{3}\Big)^{i+1}(-1)^i\frac{k!}{(k-i)!}\log^{k-i} x\Big]_X^{2X}$$
$$= \Big(\frac{1}{2}\sum_{n=1}^{\infty}\frac{a(n)b(n)}{n^{3/2}}\Big)\sum_{i=0}^{k}\Big[x^{3/2}\Big(\frac{2}{3}\Big)^{i+1}(-1)^i\frac{k!}{(k-i)!}\log^{k-i} x\Big]_X^{2X} + O(X^{1+\varepsilon}).$$

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For W_2, since $(2\pi\sqrt{mx} - 2\pi\sqrt{nx})' = \pi(\sqrt{m} - \sqrt{n})/\sqrt{x}$,

$$W_2 \ll X \log^k X \sum_{\substack{m,n\le X\\ m\neq n}} \frac{(mn)^{\varepsilon}}{(mn)^{3/4}} \frac{1}{|\sqrt{m} - \sqrt{n}|}$$
$$\ll X^{1+\varepsilon} \sum_{\substack{m,n\le X\\ m\neq n}} \frac{1}{(mn)^{3/4}} \frac{1}{|\sqrt{m} - \sqrt{n}|},$$

by the first-derivative test. In the last sum, we shall divide the range of m and n into two parts. One is $S_1 = \{(m,n) \mid 1 \le m,n \le X, |\sqrt{m} - \sqrt{n}| > (mn)^{1/4}/10\}$ and the other is $S_2 = \{(m,n) \mid 1 \le m \neq n \le X, |\sqrt{m} - \sqrt{n}| \le (mn)^{1/4}/10\}$. We observe that

$$\sum_{S_1} \frac{1}{(mn)^{3/4}} \frac{1}{|\sqrt{m} - \sqrt{n}|} \ll \sum_{S_1} \frac{1}{(mn)^{3/4}} \frac{1}{(mn)^{1/4}}$$
$$\ll \Big(\sum_{n\le X} \frac{1}{n}\Big)^2 \ll \log^2 X.$$

On the other hand, we remark that $n \asymp m$ for $(n,m) \in S_2$. To see this, we can assume that $m < n$. By the condition, we note that $0 < n - m \le \frac{1}{10}(\sqrt{m} + \sqrt{n})(nm)^{1/4} \le \frac{1}{5}n^{3/4}m^{1/4}$, hence $\frac{4}{5}n \le n(1 - \frac{1}{5}(m/n)^{1/4}) \le m$, hence $n \asymp m$. Moreover, by the mean-value theorem of differentiation, there exists t_0 satisfying

$$\left|\frac{\sqrt{m} - \sqrt{n}}{m - n}\right| = \frac{1}{2\sqrt{t_0}} \quad (\min(m,n) < t_0 < \max(m,n)).$$

Since $m \asymp n$,

$$|\sqrt{m} - \sqrt{n}| \asymp \frac{|m - n|}{\sqrt{t_0}} \asymp \frac{|m - n|}{(nm)^{1/4}}.$$

From this observation,

$$\sum_{S_2} \frac{1}{(mn)^{3/4}} \frac{1}{|\sqrt{m} - \sqrt{n}|} \ll \sum_{S_2} \frac{1}{(mn)^{1/2}|m - n|} \ll \sum_{S_2} \Big(\frac{1}{n} + \frac{1}{m}\Big)\frac{1}{|m - n|}$$
$$\ll \log X \sum_{n\le X} \frac{1}{n} \ll \log^2 X.$$

Therefore, we have $W_2 \ll X^{1+\varepsilon}$. Similarly (or more easily), we have $W_3 \ll X^{1+\varepsilon}$.

Collecting these results, we obtain the assertion of (4.2). We complete the proof of Lemma 4.1. □

To prove Theorem 1.3, we first consider the integral $\int_X^{2X} P_{(1)}^2(x)\,dx$ for $X \ge 1$. Taking $N = X$ in section 4, since $x \asymp X$, we can write

$$P_{(1)}(x) = -\frac{1}{4\pi}K_1(x) - \frac{1}{2\pi}K_2(x) - \frac{1}{\pi}K_3(x) + O(x^{\varepsilon}), \tag{4.3}$$

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where

$$\begin{aligned} K_1(x) &= x^{1/4}\log^2 x \sum_{n\le X} \frac{R_{(0,0)}(n)}{n^{3/4}} \cos\Big(2\pi\sqrt{nx}+\frac{\pi}{4}\Big),\\ K_2(x) &= x^{1/4}\log x \sum_{n\le X} \frac{\alpha(n)}{n^{3/4}} \cos\Big(2\pi\sqrt{nx}+\frac{\pi}{4}\Big),\\ K_3(x) &= x^{1/4} \sum_{n\le X} \frac{\beta(n)}{n^{3/4}} \cos\Big(2\pi\sqrt{nx}+\frac{\pi}{4}\Big). \end{aligned} \tag{4.4}$$

Squaring (4.3) and integrating from X to $2X$,

$$\begin{aligned} &\int_X^{2X} P_{(1)}^2(x)\,dx\\ &= \frac{1}{16\pi^2}\int_X^{2X} K_1^2(x)\,dx + \frac{1}{4\pi^2}\int_X^{2X} K_1(x)K_2(x)\,dx\\ &\quad + \frac{1}{4\pi^2}\int_X^{2X} (K_2^2(x)+2K_1(x)K_3(x))\,dx + \frac{1}{\pi^2}\int_X^{2X} K_2(x)K_3(x)\,dx\\ &\quad + \frac{1}{\pi^2}\int_X^{2X} K_3^2(x)\,dx + O\Big(X^{\varepsilon}\int_X^{2X} (|K_1(x)|+|K_2(x)|+|K_3(x)|)\,dx\Big) + O(X^{1+\varepsilon})\\ &=: \sum_{j=1}^{6} J_j + O(X^{1+\varepsilon}), \end{aligned}$$

say. By Lemma 4.1 and (4.4), we have the following explicit representations for J_j $(j = 1, \dots, 5)$:

$$\begin{aligned} J_1 &= \Big(\frac{1}{32\pi^2}\sum_{n=1}^{\infty}\frac{R_{(0,0)}^2(n)}{n^{3/2}}\Big)\sum_{i=0}^{4}\Big[x^{3/2}\Big(\frac{2}{3}\Big)^{i+1}\frac{(-1)^i 4!}{(4-i)!}\log^{4-i}x\Big]_X^{2X} + O(X^{1+\varepsilon}),\\ J_2 &= \Big(\frac{1}{8\pi^2}\sum_{n=1}^{\infty}\frac{R_{(0,0)}(n)\alpha(n)}{n^{3/2}}\Big)\sum_{i=0}^{3}\Big[x^{3/2}\Big(\frac{2}{3}\Big)^{i+1}\frac{(-1)^i 3!}{(3-i)!}\log^{3-i}x\Big]_X^{2X} + O(X^{1+\varepsilon}),\\ J_3 &= \Big(\frac{1}{8\pi^2}\sum_{n=1}^{\infty}\frac{\alpha^2(n)+2R_{(0,0)}(n)\beta(n)}{n^{3/2}}\Big)\sum_{i=0}^{2}\Big[x^{3/2}\Big(\frac{2}{3}\Big)^{i+1}\frac{(-1)^i 2!}{(2-i)!}\log^{2-i}x\Big]_X^{2X} + O(X^{1+\varepsilon}),\\ J_4 &= \Big(\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\alpha(n)\beta(n)}{n^{3/2}}\Big)\sum_{i=0}^{1}\Big[x^{3/2}\Big(\frac{2}{3}\Big)^{i+1}\frac{(-1)^i 1!}{(1-i)!}\log^{1-i}x\Big]_X^{2X} + O(X^{1+\varepsilon}),\\ J_5 &= \Big(\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\beta^2(n)}{n^{3/2}}\Big)\Big[\frac{2}{3}x^{3/2}\Big]_X^{2X} + O(X^{1+\varepsilon}). \end{aligned}$$

As for J_6, we notice the upper bound

$$\int_X^{2X} K_j^2(x)\,dx \ll X^{3/2+\varepsilon} \quad (j = 1, 2, 3)$$

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obtained by Lemma 4.1; therefore, by the Cauchy–Schwarz inequality,

$$J_6 \ll X^{5/4+\varepsilon}.$$

Summing up these results,

$$\begin{aligned}\int_X^{2X} P_{(1)}^2(x)\,dx &= \sum_{n=1}^{\infty} \frac{A_4(n)}{n^{3/2}}((2X)^{3/2}\log^4(2X) - X^{3/2}\log^4 X)\\ &\quad + C_3((2X)^{3/2}\log^3(2X) - X^{3/2}\log^3 X) + C_2((2X)^{3/2}\log^2(2X) - X^{3/2}\log^2 X)\\ &\quad + C_1((2X)^{3/2}\log(2X) - X^{3/2}\log X) + C_0((2X)^{3/2} - X^{3/2}) + O(X^{5/4+\varepsilon}) \qquad (4.5)\end{aligned}$$

with $C_j = \sum_{n=1}^{\infty} A_j(n)n^{-3/2}$, where

$$\begin{aligned}A_4(n) &= \frac{R_{(0,0)}^2(n)}{48\pi^2} = \frac{r^2(n)}{768\pi^2},\\ A_3(n) &= -\frac{R_{(0,0)}^2(n)}{18\pi^2} + \frac{R_{(0,0)}(n)\alpha(n)}{12\pi^2},\\ A_2(n) &= \frac{R_{(0,0)}^2(n)}{9\pi^2} - \frac{R_{(0,0)}(n)\alpha(n)}{6\pi^2} + \frac{2R_{(0,0)}(n)\beta(n) + \alpha^2(n)}{12\pi^2},\\ A_1(n) &= -\frac{4R_{(0,0)}^2(n)}{27\pi^2} + \frac{2R_{(0,0)}(n)\alpha(n)}{9\pi^2} - \frac{2R_{(0,0)}(n)\beta(n) + \alpha^2(n)}{9\pi^2} + \frac{\alpha(n)\beta(n)}{3\pi^2},\\ A_0(n) &= \frac{8R_{(0,0)}^2(n)}{81\pi^2} - \frac{4R_{(0,0)}(n)\alpha(n)}{27\pi^2} + \frac{4R_{(0,0)}(n)\beta(n) + 2\alpha^2(n)}{27\pi^2} - \frac{2\alpha(n)\beta(n)}{9\pi^2} + \frac{\beta^2(n)}{3\pi^2}.\end{aligned}$$

Using (4.5) and calculating $(\int_{X/2}^{X} + \int_{X/4}^{X/2} + \cdots)P_{(1)}^2(x)\,dx$, we obtain the assertion of Theorem 1.3.

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