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Existence and multiplicity of positive solutions for the nonlinear Schrödinger–Poisson equations

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We study the existence and multiplicity of positive solutions for the following nonlinear Schrödinger–Poisson equations:

$$\begin{aligned} -\Delta u + \lambda u + \phi u &= Q(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ -\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^3, \end{aligned}$$

where $2 < p \leqslant 3$ or $4 \leqslant p < 6$, $\lambda > 0$ and $Q \in C(\mathbb{R}^3)$. We show that the number of positive solutions is dependent on the profile of $Q(x)$.

1. Introduction

In this paper, we are concerned with the coupled system of Schrödinger–Poisson equations of the form:

$$\left.\begin{aligned} -\Delta u + \lambda u + \phi u &= f(x,u) \quad \text{in } \mathbb{R}^3, \\ -\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^3, \end{aligned}\right\} \tag{1.1}$$

with the function $f(x,u)$ being nonlinear in u. In the case when $f(x,u) = 0$, the wave function u satisfies the stationary solution of a quantum system proposed by Benci and Fortunato [4], describing the interaction of a particle with an electromagnetic field. The time-independent ϕ is the electrostatic potential and is dependent on u according to Maxwell's equations. The Schrödinger–Poisson equations are thus also known as the Schröndinger–Maxwell equations. Another linear version including an additional linear term $V(x)u$ describing the effect of an external potential has been treated in [8, 10], with the potential $V(x)$ assumed to be radially symmetric. The existence of a sequence of solutions has been proved for both of the linear systems [4, 8].

More recently, systems of a nonlinear version of the Schrödinger equation coupled with a Poisson equation, of a form similar to (1.1), have been widely studied; see, for example, [1, 2, 12, 19, 21] and the references therein. The nonlinearity of the Schrödinger equation has its origin in the interaction among particles; many-particle systems can be found, for example, in the study of condensed matter or problems in nonlinear optics. A plethora of problems has been investigated under various conditions of f concerning the existence of solutions, ground-state solutions and the multiplicity of results. Azzollini and Pomponio [2] and Zhao and Zhao [21],

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for example, studied ground-state solutions depending on the value of p for $\lambda > 0$ and for cases when λ is dependent on x in systems with power-type nonlinearities. Ruiz [19] investigated the existence of positive solutions for the nonlinearity $f(x,u) = |u|^{p-2}u$, $2 < p < 6$; the results were further improved by Ambrosetti and Ruiz [1] by showing the presence of multiple bound states when certain conditions on the parameters are satisfied. The semiclassical limit of the nonlinear system, where the Planck constant $\hbar \to 0$, has also been investigated [11,18]. The existence and asymptotic behaviours of the solutions describe the particle-like matter in the transition from quantum to classical mechanics.

In this paper, we are particularly interested in the existence and multiplicity of positive solutions for the following system:

$$\left.\begin{aligned} -\Delta u + \lambda u + \phi u &= Q(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ -\Delta \phi &= u^2 \qquad\qquad\quad\; \text{in } \mathbb{R}^3, \end{aligned}\right\} \tag{E_λ}$$

where two ranges $2 < p \leqslant 3$ and $4 \leqslant p < 6$ are considered, with $\lambda > 0$, and $Q \in C(\mathbb{R}^3)$ a non-negative function for both cases. The following theorems are our main results.

Theorem 1.1. *Suppose that $4 \leqslant p < 6$, and the following conditions hold.*

(Q1) $\lim_{|x|\to\infty} Q(x) = Q^\infty > 0$.

(Q2) *There exist some points $x^1, x^2, \dots, x^k$ in $\mathbb{R}^3$ such that $Q(x^i)$ are strict maxima and satisfy*

$$Q(x^i) = Q_{\max} \equiv \sup_{x\in\mathbb{R}^3} Q(x) > 0 \quad \textit{for all } i = 1, 2, \dots, k.$$

Then there exists $\lambda_0 > 0$ such that, for every $\lambda > \lambda_0$, (E_λ) has at least k positive solutions.

Theorem 1.2. *If $Q^\infty < Q_{\max}$, then there exists $\hat{\lambda} \geqslant \lambda_0$ such that for every $\lambda > \hat{\lambda}$ we can find at least one ground-state solution among the solutions of theorem 1.1.*

Furthermore, using a similar argument to that employed by Ruiz [19, theorem 4.1], we have the following non-existence result.

Theorem 1.3.

(i) *Suppose that $p = 3$ and $\sup_{x\in\mathbb{R}^3} Q(x) \leqslant 1$. Then, for any $\lambda > 0$, $u = 0$ is the unique solution of (E_λ).*

(ii) *Suppose that $2 < p < 3$ and $\sup_{x\in\mathbb{R}^3} Q(x) < (p-2)^{2-p}(3-p)^{p-3}$. Then, for any $\lambda \geqslant 2^{(p-2)/(p-3)}$, $u = 0$ is the unique solution of (E_λ).*

From the results above, we are inclined to the possibility that $p = 3$ is the critical value and that (multiple) positive solutions may exist for $3 < p < 4$. Work is currently ongoing to verify such a result; additional conditions on $Q(x)$, however, may be required in order to prove that this is indeed the case. Note that for this particular range of $3 < p < 4$, and with $Q(x) = 1$, Azzollini and Pomponio [2] and

Zhao and Zhao [21] have demonstrated the existence of ground-state solutions; in the problem considered in [21], an external potential is considered and the solution exists subject to various conditions imposed on the potential.

This paper is organized as follows. We first outline the notations and preliminaries in § 2, before proving theorem 1.1 in § 3 and theorems 1.2 and 1.3 in § 4.

2. Notation and preliminaries

By the change of variables $\varepsilon = \lambda^{-1/2}$, $v(x) = \varepsilon^{2/(p-2)}u(\varepsilon x)$, (E_λ) can be rewritten as

$$\left.\begin{aligned} -\Delta v + v + \varepsilon^{4(p-3)/(p-2)}\phi v &= Q_\varepsilon |v|^{p-2}v \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi &= v^2 \qquad\qquad\quad\; \text{in } \mathbb{R}^3, \end{aligned}\right\} \tag{$\bar{E}_\varepsilon$}$$

where $Q_\varepsilon = Q(\varepsilon x)$.

We first recall some well-known results (see, for example, [2, 4, 8–10, 12, 19]). For every $u \in L^{12/5}(\mathbb{R}^3)$, there exists a unique solution $\phi_u \in D^{1,2}(\mathbb{R}^3)$ of

$$-\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3.$$

It follows that $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of $(\bar{E}_\varepsilon)$ if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of the functional $I_{\varepsilon,Q_\varepsilon} \colon H^1(\mathbb{R}^3) \to \mathbb{R}$, defined as

$$I_{\varepsilon,Q_\varepsilon}(u) = \tfrac{1}{2}\|u\|_{H^1}^2 + \tfrac{1}{4}\varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{p}\int_{\mathbb{R}^3} Q_\varepsilon |u|^p \tag{2.1}$$

with

$$\|u\|_{H^1} = \biggl(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2\biggr)^{1/2}$$

being a standard norm in $H^1(\mathbb{R}^3)$, and $\phi = \phi_u$. Moreover, the function ϕ_u possesses certain properties (see [2, 12, 19]) that we shall outline below using a functional defined with a more general non-negative function $b(x)$ in place of Q_ε; the functional given by (2.1) defined using Q_ε above is thus a special case. The properties obtained will automatically be applicable if Q_ε is used instead.

Lemma 2.1. *For each $u \in H^1(\mathbb{R}^3)$, we have the following.*

(i) $\|\phi_u\|_{D^{1,2}(\mathbb{R}^3)} \leqslant C\|u\|_{H^1}^2$, *where C does not depend on u. As a consequence, there exists $C_0 > 0$ such that*

$$\int_{\mathbb{R}^3} \phi_u u^2 \leqslant C_0 \|u\|_{H^1}^4.$$

(ii) $\phi_u \geqslant 0$ *and*

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\,\mathrm{d}y.$$

(iii) *For any $t > 0$, $\phi_{tu} = t^2\phi_u$.*

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For $u \in H^1(\mathbb{R}^3)$, $\varepsilon \geqslant 0$ and a non-negative bounded function $b \in C(\mathbb{R}^3)$, there exists $x_0 \in \mathbb{R}^3$ such that

$$b(x_0) = b_{\max} := \sup\{b(x) \mid x \in \mathbb{R}^3\} > 0.$$

Define

$$\begin{aligned} I_{\varepsilon,b(x)}(u) &= \tfrac{1}{2}\|u\|_{H^1}^2 + \tfrac{1}{4}\varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{p}\int_{\mathbb{R}^3} b(x)|u|^p, \\ \boldsymbol{M}_{\varepsilon,b(x)} &= \{u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid \langle I'_{\varepsilon,b(x)}(u), u\rangle = 0\}, \\ \alpha_{\varepsilon,b(x)} &= \inf_{u \in \boldsymbol{M}_{\varepsilon,b(x)}} I_{\varepsilon,b(x)}(u), \end{aligned}$$

where $I'_{\varepsilon,b(x)}$ denotes the Fréchet derivative of $I_{\varepsilon,b(x)}$. For brevity, we write $I_{\varepsilon,b(x)}$, $I'_{\varepsilon,b(x)}$, $\boldsymbol{M}_{\varepsilon,b(x)}$ and $\alpha_{\varepsilon,b(x)}$ as $I_{\varepsilon,b}$, $I'_{\varepsilon,b}$, $\boldsymbol{M}_{\varepsilon,b}$ and $\alpha_{\varepsilon,b}$, respectively. The Sobolev inequality,

$$\begin{aligned} \|u\|_{H^1}^2 &\leqslant \|u\|_{H^1}^2 + \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 \\ &= \int_{\mathbb{R}^3} b|u|^p \leqslant c b_{\max}\|u\|_{H^1}^p \quad \text{for all } u \in \boldsymbol{M}_{\varepsilon,b}, \end{aligned}$$

implies that there exists $c_0 > 0$ such that $\|u\|_{H^1} \geqslant c_0$ and

$$I_{\varepsilon,b}(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_{H^1}^2 + \left(\frac{1}{4} - \frac{1}{p}\right)\varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 \geqslant \frac{p-2}{2p} c_0^2,$$

for all $u \in \boldsymbol{M}_{\varepsilon,b}$. Thus, the functional $I_{\varepsilon,b}$ is bounded below on $\boldsymbol{M}_{\varepsilon,b}$.

Define

$$\psi_\varepsilon(u) = \langle I'_{\varepsilon,b}(u), u\rangle = \|u\|_{H^1}^2 + \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} b|u|^p.$$

Then, for $u \in \boldsymbol{M}_{\varepsilon,b}$,

$$\begin{aligned} \langle \psi'_\varepsilon(u), u\rangle &= 2\|u\|_{H^1}^2 + 4\varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 - p\int_{\mathbb{R}^3} b|u|^p \\ &= (2-p)\|u\|_{H^1}^2 + (4-p)\varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 \\ &= -2\|u\|_{H^1}^2 - (p-4)\int_{\mathbb{R}^3} b|u|^p \\ &< -2c_0^2 \\ &< 0, \end{aligned}$$

which implies that $\boldsymbol{M}_{\varepsilon,b}$ is a C^1 manifold, and so the Nehari manifold $\boldsymbol{M}_{\varepsilon,b}$ is a natural constraint for the functional $I_{\varepsilon,b}$.

For each $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, we define

$$t_{0,u} = \left(\frac{\|u\|_{H^1}^2}{\int_{\mathbb{R}^3} |u|^p}\right)^{1/(p-2)} > 0.$$

Then we have the following result.

Lemma 2.2.

(i) *Suppose that* $4 < p < 6$ *and* $\varepsilon > 0$. *Then, for each* $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, *there is a unique* $t_\varepsilon > t_{0,u}$ *such that* $t_\varepsilon u \in \boldsymbol{M}_{\varepsilon,b}$ *and*

$$I_{\varepsilon,b}(t_\varepsilon u) = \sup_{t \geqslant 0} I_{\varepsilon,b}(tu) = \sup_{t \geqslant t_{0,u}} I_{\varepsilon,b}(tu).$$

(ii) *Suppose that* $p = 4$ *and* $\varepsilon > 0$. *Then, for each* $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ *with*

$$\int_{\mathbb{R}^3} b|u|^p - \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 > 0,$$

there is a unique

$$t_\varepsilon = \left(\frac{\|u\|_{H^1}^2}{\int_{\mathbb{R}^3} |u|^p - \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2} \right)^{1/(p-2)} > t_{0,u}$$

such that $t_\varepsilon u \in \boldsymbol{M}_{\varepsilon,b}$ *and*

$$I_{\varepsilon,b}(t_\varepsilon u) = \sup_{t \geqslant 0} I_{\varepsilon,b}(tu) = \sup_{t \geqslant t_{0,u}} I_{\varepsilon,b}(tu).$$

Proof. (i) Fix $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. Let

$$h_u(t) = t^{-2}\|u\|_{H_1}^2 - t^{p-4} \int_{\mathbb{R}^3} b|u|^p \quad \text{for } t > 0.$$

Clearly, $tu \in \boldsymbol{M}_{\varepsilon,b}$ if and only if

$$h_u(t) + \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 = 0.$$

We have $h_u(t_{0,u}) = 0$, $\lim_{t\to 0^+} h_u(t) = \infty$ and $\lim_{t\to\infty} h_u(t) = -\infty$. Since $4 < p < 6$ and

$$\begin{aligned} h_u'(t) &= -2t^{-3}\|u\|_{H_1}^2 - (p-4)t^{p-5} \int_{\mathbb{R}^3} |u|^p \\ &= t^{-3}\left(-2\|u\|_{H_1}^2 - (p-4)t^{p-2} \int_{\mathbb{R}^3} b|u|^p \right) \\ &< 0 \quad \text{for all } t > 0, \end{aligned}$$

$h_u(t)$ is decreasing for $t > 0$. Since $h_u(t_{0,u}) = 0$ and $\lim_{t\to\infty} h_u(t) = -\infty$, there is a unique $t_\varepsilon > t_{0,u}$ such that

$$h_u(t_\varepsilon) + \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 = 0.$$

Thus, $t_\varepsilon u \in \boldsymbol{M}_{\varepsilon,b}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{\varepsilon,b}(tu) = t^3 \left(h_u(t) + \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 \right),$$

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which implies that $I_{\varepsilon,b}(tu)$ is increasing for $t \in [0, t_\varepsilon)$, decreasing for $t \in (t_\varepsilon, \infty)$ and

$$I_{\varepsilon,b}(t_\varepsilon u) = \sup_{t \geqslant 0} I_{\varepsilon,b}(tu) = \sup_{t \geqslant t_{0,u}} I_{\varepsilon,b}(tu).$$

(ii) Fix $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ with

$$\int_{\mathbb{R}^3} |u|^p - \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 > 0.$$

Let

$$m_u(t) = I_{\varepsilon,b}(tu) = \tfrac{1}{2} t^2 \|u\|_{H_1}^2 - \frac{t^p}{p} \left(\int_{\mathbb{R}^3} |u|^p - \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 \right) \quad \text{for } t > 0.$$

Since

$$m_u'(t) = t \|u\|_{H_1}^2 - t^{p-1} \left(\int_{\mathbb{R}^3} b|u|^p - \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2 \right),$$

there is a unique

$$t_\varepsilon = \left(\frac{\|u\|_{H^1}^2}{\int_{\mathbb{R}^3} |u|^p - \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_u u^2} \right)^{1/(p-2)} > t_{0,u}$$

such that $t_\varepsilon u \in \boldsymbol{M}_{\varepsilon,b}$ and

$$I_{\varepsilon,b}(t_\varepsilon u) = \sup_{t \geqslant 0} I_{\varepsilon,b}(tu) = \sup_{t \geqslant t_{0,u}} I_{\varepsilon,b}(tu).$$

This completes the proof. □

Furthermore, we have the following lemma.

Lemma 2.3.

(i) $\alpha_{0,b_{\max}} < \alpha_{\varepsilon,b_{\max}}$ *for all* $\varepsilon > 0$.

(ii) $\alpha_{\varepsilon,b_1} < \alpha_{\varepsilon,b_2}$ *for all* $\varepsilon > 0$ *and for all* $b_1, b_2 > 0$ *with* $b_1 < b_2$.

Proof. The proofs are almost identical to that in Azzollini and Pomponio [2, lemma 2.12]. □

3. Proof of theorem 1.1

We shall first make use of the profile of Q to construct Palais–Smale (PS) sequences which are used later to prove theorem 1.1. For $a > 0$, let $C_a(z)$ denote the hypercube $\prod_{j=1}^{3}(z_j - a, z_j - a)$ centred at $z = (z_1, z_2, z_3)$, and $\overline{C_a(z)}$ and $\partial C_a(z)$ denote the closure and the boundary of $C_a(z)$, respectively. By conditions (Q1) and (Q2), we can choose a number $l > 0$ such that $C_l(x^i)$ is disjoint and $Q(x) < Q(x^i)$ for all $x \in \partial C_l(x^i)$ and for all $i = 1, 2, \dots, k$.

Next, we need a generalized barycentre map. By this we mean a continuous map $\Phi\colon L^p(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}^3$ that is equivalent to the action of the group of Euclidean motions in $\mathbb{R}^3$, that is, for every $\xi \in \mathbb{R}^3$ and $u \in L^p(\mathbb{R}^3) \setminus \{0\}$, we have $\Phi(u) = \Phi(|u|)$,

$$\Phi(u(x - \xi)) = \xi + \Phi(u(x)) \quad \text{and} \quad \Phi(u(\varepsilon x)) = \varepsilon^{-1} \Phi(u(x)). \tag{3.1}$$

Such a map has been constructed in Bartsch and Weth [3, theorem 2.1] and Cerami and Passaseo [7].

Let $C^i_{l/\varepsilon} \equiv C_{l/\varepsilon}(x^i/\varepsilon)$ and

$$N^i_\varepsilon = \{u \in \boldsymbol{M}_{\varepsilon,Q_\varepsilon} \mid u \geqslant 0 \text{ and } \Phi_\varepsilon(u) \in C^i_{l/\varepsilon}\},$$
$$\partial N^i_\varepsilon = \{u \in \boldsymbol{M}_{\varepsilon,Q_\varepsilon} \mid u \geqslant 0 \text{ and } \Phi_\varepsilon(u) \in \partial C^i_{l/\varepsilon}\},$$

for $i = 1, 2, \dots, k$. It can be readily verified that N^i_ε and ∂N^i_ε are non-empty sets for all $i = 1, 2, \dots, k$. Consider the minimization problems in N^i_ε and ∂N^i_ε for $I_{\varepsilon,Q_\varepsilon}$,

$$\gamma^i_\varepsilon = \inf_{u \in N^i_\varepsilon} I_{\varepsilon,Q_\varepsilon}(u) \quad \text{and} \quad \tilde{\gamma}^i_\varepsilon = \inf_{u \in \partial N^i_\varepsilon} I_{\varepsilon,Q_\varepsilon}(u) \quad \text{for } i = 1, 2, \dots, k.$$

Let w be a unique positive radial solution of

$$\left.\begin{aligned} -\Delta u + u = Q_{\max}|u|^{p-2}u \quad \text{in } \mathbb{R}^3,\\ u \in H^1(\mathbb{R}^3), \end{aligned}\right\} \tag{$\bar{E}_{0,Q_{\max}}$}$$

such that $I_{0,Q_{\max}}(w) = \alpha_{0,Q_{\max}}$. For small $\varepsilon > 0$ satisfying $\sqrt{\varepsilon} < 1$, we define a function $\psi_\varepsilon \in C^1(\mathbb{R}^3, [0, 1])$ such that

$$\psi_\varepsilon(x) = \begin{cases} 1, & |x| < \dfrac{1}{\sqrt{\varepsilon}} - 1, \\ 0, & |x| > \dfrac{1}{\sqrt{\varepsilon}}, \end{cases}$$

and $|\nabla\psi_\varepsilon| \leqslant 2$ in $\mathbb{R}^3$. Let

$$v_{\varepsilon,i}(x) = w\left(x - \frac{x^i}{\varepsilon}\right)\psi_\varepsilon\left(x - \frac{x^i}{\varepsilon}\right) \quad \text{for } i = 1, 2, \dots, k.$$

Then we have the following result.

LEMMA 3.1. *Suppose that* $4 \leqslant p < 6$. *Then*

(i)
$$\varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_{v_{\varepsilon,i}(x)} v^2_{\varepsilon,i}(x) \to 0 \quad \textit{as } \varepsilon \to 0,$$

(ii) *there exist positive numbers* ε_1, D_0 *such that*

$$\int_{\mathbb{R}^3} Q_\varepsilon v^p_{\varepsilon,i}(x) - \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_{v_{\varepsilon,i}(x)} v_{\varepsilon,i}(x) \geqslant D_0 \quad \textit{for all } \varepsilon \in (0, \varepsilon_1).$$

Proof. (i) Since

$$0 \leqslant \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_{v_{\varepsilon,i}(x)} v_{\varepsilon,i}(x) \leqslant C_0 \varepsilon^{4(p-3)/(p-2)} \|v_{\varepsilon,i}\|^4_{H^1}$$

and

$$\|v_{\varepsilon,i}\|^2_{H^1} \to \frac{2p}{p-2}\alpha_{0,Q_{\max}} \quad \text{as } \varepsilon \to 0,$$

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we obtain

$$\varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_{v_{\varepsilon,i}(x)} v_{\varepsilon,i}(x) \to 0 \quad \text{as } \varepsilon \to 0.$$

(ii) Since

$$\begin{aligned} \int_{\mathbb{R}^3} Q_\varepsilon v^p_{\varepsilon,i}(x) &= \int_{\mathbb{R}^3} Q_\varepsilon w^p\Big(x - \frac{x^i}{\varepsilon}\Big)\psi^p_\varepsilon\Big(x - \frac{x^i}{\varepsilon}\Big) \\ &= \int_{\mathbb{R}^3} Q(x^i) w^p + o(\varepsilon) \end{aligned}$$

with $o(\varepsilon) \to 0$ as $\varepsilon \to 0$, by

$$\int_{\mathbb{R}^3} Q(x^i) w^p = \frac{2p}{p-2}\alpha_{0,Q_{\max}} > 0$$

and case (i), there exist positive numbers ε_1, D_0 such that

$$\int_{\mathbb{R}^3} Q_\varepsilon v^p_{\varepsilon,i}(x) - \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_{v_{\varepsilon,i}(x)} v_{\varepsilon,i}(x) \geqslant D_0 \quad \text{for all } \varepsilon \in (0, \varepsilon_1).$$

This completes the proof. □

Using lemmas 2.2 and 3.1, for each $p \in [4, 6)$ and $\varepsilon \in (0, \varepsilon_1)$ there exists

$$t_{\varepsilon,i} > \bigg(\frac{\|v_{\varepsilon,i}\|^2_{H^1}}{\int_{\mathbb{R}^3} Q_\varepsilon |v_{\varepsilon,i}|^p}\bigg)^{1/(p-2)} > 0 \tag{3.2}$$

such that $t_{\varepsilon,i} v_{\varepsilon,i} \in \boldsymbol{M}_{\varepsilon,Q_\varepsilon}$. The following result is obtained.

Lemma 3.2. *We have $t_{\varepsilon,i} \to 1$ as $\varepsilon \to 0$.*

Proof. Since $t_{\varepsilon,i} v_{\varepsilon,i} \in \boldsymbol{M}_{\varepsilon,Q_\varepsilon}$, we have

$$\begin{aligned} t^2_{\varepsilon,i}&\Big\|w\Big(x - \frac{x^i}{\varepsilon}\Big)\psi_\varepsilon\Big(x - \frac{x^i}{\varepsilon}\Big)\Big\|^2_{H^1} \\ &= t^p_{\varepsilon,i} \int_{\mathbb{R}^3} Q_\varepsilon w^p\Big(x - \frac{x^i}{\varepsilon}\Big)\psi^p_\varepsilon\Big(x - \frac{x^i}{\varepsilon}\Big) \\ &\quad + \varepsilon^{4(p-3)/(p-2)} t^4_{\varepsilon,i} \int_{\mathbb{R}^3} \phi_{w(x - x^i/\varepsilon)\psi_\varepsilon(x - x^i/\varepsilon)} w^2\Big(x - \frac{x^i}{\varepsilon}\Big)\psi^2_\varepsilon\Big(x - \frac{x^i}{\varepsilon}\Big). \end{aligned}$$

Since

$$\|w\|^2_{H^1} = \int_{\mathbb{R}^3} Q_{\max} w^p,$$

from lemma 3.1,

$$\begin{aligned} t^2_{\varepsilon,i}\|w\|^2_{H^1} &= t^2_{\varepsilon,i}\Big\|w\Big(x - \frac{x^i}{\varepsilon}\Big)\psi_\varepsilon\Big(x - \frac{x^i}{\varepsilon}\Big)\Big\|^2_{H^1} + o(\varepsilon) \\ &= t^p_{\varepsilon,i} \int_{\mathbb{R}^3} Q_\varepsilon w^p\Big(x - \frac{x^i}{\varepsilon}\Big)\psi^p_\varepsilon\Big(x - \frac{x^i}{\varepsilon}\Big) + o(\varepsilon) \\ &= t^p_{\varepsilon,i} \int_{\mathbb{R}^3} Q_\varepsilon(\varepsilon x + x^i) w^p + o(\varepsilon), \end{aligned}$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover,

$$t_{\varepsilon,i} > \left(\frac{\|w(x-x^i/\varepsilon)\psi_\varepsilon(x-x^i/\varepsilon)\|_{H^1}^2}{\int_{\mathbb{R}^3} Q_\varepsilon|w(x-x^i/\varepsilon)\psi_\varepsilon(x-x^i/\varepsilon)|^p}\right)^{1/(p-2)} = 1+o(\varepsilon).$$

Thus, $t_{\varepsilon,i} \to 1$ as $\varepsilon \to 0$. □

Using the ideas in [6, 14], we have the following results.

Lemma 3.3. *Suppose that $4 \leqslant p < 6$. Then, for each positive number $\eta \leqslant \alpha_{0,Q_{\max}}$, there exists $\varepsilon_\eta \in (0, \varepsilon_1]$ such that, for any $\varepsilon \in (0, \varepsilon_\eta)$,*

$$\alpha_{\varepsilon,Q_\varepsilon} \leqslant \gamma_\varepsilon^i < \alpha_{0,Q_{\max}} + \eta \quad \textit{for all } i = 1, 2, \dots, k.$$

In particular, the N_ε^i are non-empty sets.

Proof. First, we show that $\Phi_\varepsilon(t_{\varepsilon,i}v_{\varepsilon,i}) \in C^i_{l/\varepsilon}$. By the definition of ψ_ε and $t_{\varepsilon,i} \to 1$ as $\varepsilon \to 0$,

$$\Phi(t_{\varepsilon,i}v_{\varepsilon,i}) = \frac{x^i}{\varepsilon} + o(\varepsilon),$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$. We conclude that $\Phi_\varepsilon(t_{\varepsilon,i}v_{\varepsilon,i}) \in C^i_{l/\varepsilon}$. Thus, $t_{\varepsilon,i}v_{\varepsilon,i} \in N^i_\varepsilon$. Moreover, by lemmas 3.1 and 2.2,

$$\begin{aligned}
&I_{\varepsilon,Q_\varepsilon}(t_{\varepsilon,i}v_{\varepsilon,i}) \\
&\quad = \frac{t_{\varepsilon,i}^2}{2}\left\|w\left(x-\frac{x^i}{\varepsilon}\right)\psi_\varepsilon\left(x-\frac{x^i}{\varepsilon}\right)\right\|_{H^1}^2 \\
&\qquad + \tfrac{1}{4}\varepsilon^{4(p-3)/(p-2)}t_{\varepsilon,i}^4\int_{\mathbb{R}^3}\phi_{w(x-x^i/\varepsilon)\psi_\varepsilon(x-x^i/\varepsilon)}w^2\left(x-\frac{x^i}{\varepsilon}\right)\psi_\varepsilon^2\left(x-\frac{x^i}{\varepsilon}\right) \\
&\qquad - \frac{t_{\varepsilon,i}^p}{p}\int_{\mathbb{R}^3}Q_\varepsilon w^p\left(x-\frac{x^i}{\varepsilon}\right)\psi_\varepsilon^p\left(x-\frac{x^i}{\varepsilon}\right) \\
&\quad = \tfrac{1}{2}\|w\|_{H^1}^2 - \frac{1}{p}\int_{\mathbb{R}^3}Q(\varepsilon x + x^i)w^p + o(\varepsilon).
\end{aligned} \tag{3.3}$$

From (3.3), we have

$$I_{\varepsilon,Q_\varepsilon}(t_{\varepsilon,i}v_{\varepsilon,i}) = \alpha_{0,Q_{\max}} + o(\varepsilon).$$

Therefore, there exists $\varepsilon_\eta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\eta)$,

$$\gamma_\varepsilon^i < \alpha_{0,Q_{\max}} + \eta \quad \text{for all } i = 1, 2, \dots, k.$$

This completes the proof. □

Lemma 3.4. *Suppose that $4 \leqslant p < 6$. Then there are positive numbers δ and ε_δ such that, for any $i = 1, 2, \dots, k$,*

$$\tilde{\gamma}_\varepsilon^i > \alpha_{0,Q_{\max}} + \delta \quad \textit{for all } \varepsilon \in (0, \varepsilon_\delta).$$

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Proof. Fix $i \in \{1, 2, \ldots, k\}$. Assume to the contrary that there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that $\tilde{\gamma}^i_{\varepsilon_n} \to c \leqslant \alpha_{0,Q_{\max}}$. Then there exists a sequence $\{u_n\} \subset \partial N^i_{\varepsilon_n}$ such that $\Phi(u_n) \in \partial C^i_{l/\varepsilon_n}$,

$$\|u_n\|^2_{H^1} + \varepsilon_n^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 = \int_{\mathbb{R}^3} Q_{\varepsilon_n} |u_n|^p$$

and

$$I_{\varepsilon_n, Q_{\varepsilon_n}}(u_n) \to c \leqslant \alpha_{0,Q_{\max}} \quad \text{as } n \to \infty.$$

It follows that $\{u_n\}$ is uniformly bounded in $H^1(\mathbb{R}^3)$. Moreover, by

$$\int_{\mathbb{R}^3} \phi_u u^2 \leqslant C_0 \|u\|^4_{H^1},$$

we have

$$\varepsilon_n^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to 0 \quad \text{as } n \to \infty,$$

which implies that

$$\|u_n\|^2_{H^1} = \int_{\mathbb{R}^3} Q_{\varepsilon_n} |u_n|^p + o(1). \tag{3.4}$$

Thus, there exists a sequence $\{t_n\} \subset \mathbb{R}^+$ with $t_n \to 0$ such that

$$\|t_n u_n\|^2_{H^1} = \int_{\mathbb{R}^3} Q_{\varepsilon_n} |t_n u_n|^p$$

and

$$I_{\varepsilon_n, Q_{\varepsilon_n}}(t_n u_n) \geqslant \alpha_{0,Q_{\varepsilon_n}} \geqslant \alpha_{0,Q_{\max}},$$

which implies

$$I_{\varepsilon_n, Q_{\varepsilon_n}}(u_n) \to \alpha_{0,Q_{\max}}. \tag{3.5}$$

Next we shall show that

$$\int_{\mathbb{R}^3} [Q_{\max} - Q_{\varepsilon_n}] |u_n|^p = o(1). \tag{3.6}$$

Supposing otherwise, we may assume that there exists a positive constant C_0 such that, for large n,

$$\int_{\mathbb{R}^3} [Q_{\max} - Q_{\varepsilon_n}] |u_n|^p > C_0. \tag{3.7}$$

By (3.4) and (3.7), there exists a sequence $\{s_n\} \subset \mathbb{R}_+$ such that

$$\|s_n u_n\|^2_{H^1} = \int_{\mathbb{R}^3} Q_{\max} |s_n u_n|^p,$$

and, for large n,

$$s_n^{p-2} = \frac{\|u_n\|^2_{H^1}}{\int_{\mathbb{R}^3} Q_{\max} |u_n|^p} < \frac{\|u_n\|^2_{H^1}}{\int_{\mathbb{R}^3} Q_{\varepsilon_n} |u_n|^p + C_0} = \left(1 + \frac{C_0}{\|u_n\|^2_{H^1}}\right)^{-1} + o(1). \tag{3.8}$$

With $\{u_n\}$ being uniformly bounded in $H^1(\mathbb{R}^3)$, there also exists $c_0 > 0$ such that

$$s_n^2 < 1 - c_0 \quad \text{for } n \text{ sufficiently large.}$$

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Thus, by (3.4) and the Sobolev inequality, there exists $d_0 > 0$ such that

$$\begin{aligned} I_{\varepsilon_n,Q_{\varepsilon_n}}(u_n) &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_{H^1}^2 + o(1) \\ &> \left(\frac{1}{2}-\frac{1}{p}\right)\|s_n u_n\|_{H^1}^2 + \left(\frac{1}{2}-\frac{1}{p}\right)c_0\|u_n\|_{H^1}^2 + o(1) \\ &\geqslant \alpha_{0,Q_{\max}} + \left(\frac{1}{2}-\frac{1}{p}\right)c_0 d_0, \end{aligned}$$

for n sufficiently large; this contradicts (3.5). It then follows from (3.4) and (3.6) that

$$\|u_n\|_{H^1}^2 = \int_{\mathbb{R}^3} Q_{\max}|u_n|^p + o(1) \tag{3.9}$$

and

$$I_{\varepsilon_n,Q_{\varepsilon_n}}(u_n) = \tfrac{1}{2}\|u_n\|_{H^1}^2 - \frac{1}{p}\int_{\mathbb{R}^3} Q_{\max}|u_n|^p + o(1) = \alpha_{0,Q_{\max}}. \tag{3.10}$$

Using the results of (3.9), (3.10) and [20, lemma 7], $\{u_n\}$ is thus a $(\mathrm{PS})_{\alpha_{0,Q_{\max}}}$-sequence in $H^1(\mathbb{R}^3)$ for $I_{0,Q_{\max}}$. Since $u_n \in \boldsymbol{M}_{\varepsilon_n,Q_{\varepsilon_n}}$, we deduce from the Sobolev imbedding theorem that $\|u_n\|_{H^1} > \nu > 0$ for some constant ν and for all n. Applying the concentration-compactness principle of Lions [15, 16] to $|u_n|^p$, there exist positive constants R, θ and $\{z_n\} \subset \mathbb{R}^3$ such that

$$\int_{B^N(z_n;R)} |u_n|^p \geqslant \theta \quad \text{for all } n, \tag{3.11}$$

where $B^N(z_n;R) = \{x \in \mathbb{R}^3 \mid |x-z_n| < R\}$. Let $\tilde{u}_n = u_n(z+z_n)$. From the translation invariance of the functional $I_{0,Q_{\max}}$, we conclude that $\{\tilde{u}_n\}$ is also a $(\mathrm{PS})_{\alpha_{0,Q_{\max}}}$-sequence in $H^1(\mathbb{R}^3)$ for $I_{0,Q_{\max}}$. Then, by (3.11), there exist a subsequence $\{\tilde{u}_n\}$ and a non-zero $u_0 \in H^1(\mathbb{R}^3)$ such that

$$\begin{gathered} \tilde{u}_n \rightharpoonup u_0 \quad \text{in } H^1(\mathbb{R}^3), \\ \tilde{u}_n \to u_0 \quad \text{a.e. in } \mathbb{R}^3, \\ \int_{B^N(0;R)} |\tilde{u}_n|^p \to \int_{B^N(0;R)} |u_0|^p \geqslant \theta. \end{gathered}$$

This implies that u_0 is a non-trivial solution of $(\bar{E}_{0,Q_{\max}})$. By the Fatou lemma

$$\alpha_{0,Q_{\max}} \leqslant I_{0,Q_{\max}}(u_0) = \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^3}|u_0|^p \leqslant \liminf \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^3}|\tilde{u}_n|^p = \alpha_{0,Q_{\max}},$$

and so $I_{0,Q_{\max}}(u_0) = \alpha_{0,Q_{\max}}$. Moreover, by the strong maximum principle, u_0 is a positive solution of $(\bar{E}_{0,Q_{\max}})$. Set $w_n = \tilde{u}_n - u_0$. Since $\{\tilde{u}_n\}$ is uniformly bounded, by the Brézis–Lieb lemma [5], we obtain

$$\int_{\mathbb{R}^3} |\tilde{u}_n|^p = \int_{\mathbb{R}^3} |u_0|^p + \int_{\mathbb{R}^3} |w_n|^p + o(1). \tag{3.12}$$

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Moreover, $\tilde{u}_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$; thus,

$$\|\tilde{u}_n\|_{H^1}^2 = \|u_0\|_{H^1}^2 + \|w_n\|_{H^1}^2 + o(1). \tag{3.13}$$

Combining (3.12) and (3.13) gives

$$\|w_n\|_{H^1}^2 = \int_{\mathbb{R}^3} w_n^4 + o(1), \tag{3.14}$$

and so

$$\left(\frac{1}{2} - \frac{1}{p}\right)\|w_n\|_{H^1}^2 = I_{0,Q_{\max}}(w_n) = I_{0,Q_{\max}}(\tilde{u}_n) - I_{0,Q_{\max}}(u_0) + o(1) = o(1).$$

This implies $\tilde{u}_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$. Moreover, $\Phi(u_n) \in \partial C^i_{l/\varepsilon_n}$ and $\tilde{u}_n(z) = u_n(z + z_n)$, we have

$$\varepsilon_n z_n = \varepsilon_n \Phi(u_n) - \varepsilon_n \Phi(\tilde{u}_n) = \varepsilon_n \Phi(u_n) - \varepsilon_n \Phi(u_0),$$

and so $\mathrm{dist}(\varepsilon_n z_n, \partial C_l(x^i)) \to 0$ as $n \to \infty$. Without loss of generality, we may assume that $\varepsilon_n z_n \to z_0 \in \partial C_l(x^i)$. By condition (Q2), $Q(z_0) < Q_{\max}$. Subsequently, using (3.4) and (3.6), we can conclude

$$\|u_0\|_{H^1}^2 = \int_{\mathbb{R}^3} Q(z_0)|u_0|^p < \int_{\mathbb{R}^3} Q_{\max}|u_0|^p;$$

this contradicts the earlier result that u_0 is a positive solution of $(\bar{E}_{0,Q_{\max}})$. This completes the proof. □

Using lemmas 2.3, 3.3 and 3.4 for a positive number $\eta \leqslant \min\{\delta, \alpha_{0,Q_{\max}}\}$ and taking $\varepsilon_0 = \min\{\varepsilon_\eta, \varepsilon_\delta\}$, we obtain, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\alpha_{\varepsilon,Q_\varepsilon} \leqslant \gamma_\varepsilon^i < \min\{2\alpha_{0,Q_{\max}}, \tilde{\gamma}_\varepsilon^i\} \leqslant \min\{2\alpha_{\varepsilon,Q^\infty}, \tilde{\gamma}_\varepsilon^i\} \quad \text{for all } i = 1, 2, \dots, k. \tag{3.15}$$

Adopting the idea of Ni and Takagi [17], we have the following result.

Lemma 3.5. *Suppose that $4 \leqslant p < 6$. Then, for each $\varepsilon \in (0, \varepsilon_0)$ and $u \in N_\varepsilon^i$, there exist $\sigma > 0$ and a differentiable function $t^* : B(0;\sigma) \subset H^1(\mathbb{R}^3) \to \mathbb{R}^+$ such that $t^*(0) = 1$, $t^*(v)(u - v) \in N_\varepsilon^i$ for all $v \in B(0;\sigma)$ and*

$$\langle (t^*)'(0), \varphi\rangle = \frac{2\int_{\mathbb{R}^3}(\nabla u \nabla \varphi + u\varphi) + 4\varepsilon^{4(p-3)/(p-2)}\int_{\mathbb{R}^3}\phi_u u\varphi - p\int_{\mathbb{R}^3} Q_\varepsilon |u|^{p-2}u\varphi}{\|u\|_{H^1}^2 - (p-1)\int_{\mathbb{R}^3} Q_\varepsilon |u|^p}$$

for all $v \in H^1(\mathbb{R}^3)$.

Proof. For $u \in N_\varepsilon^i$, define a function $F_u : \mathbb{R} \times H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$\begin{aligned} F_u(t, v) &= \langle I'_{\varepsilon,Q_\varepsilon}(t(u - v)), t(u - v)\rangle \\ &= t^2 \int_{\mathbb{R}^3} [|\nabla(u - v)|^2 + (u - v)^2] + \varepsilon^{4(p-3)/(p-2)} t^4 \int_{\mathbb{R}^3} \phi_u u^2 \\ &\quad - t^p \int_{\mathbb{R}^3} Q_\varepsilon |u - v|^p. \end{aligned}$$

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Then, $F_u(1,0) = \langle I'_{\varepsilon,Q_\varepsilon}(u), u\rangle = 0$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F_u(1,0) &= 2\|u\|_{H^1}^2 + 4\varepsilon^{4(p-3)/(p-2)}\int_{\mathbb{R}^3}\phi_u u^2 - p\int_{\mathbb{R}^3} Q_\varepsilon|u|^p \\ &= -2\|u\|_{H^1}^2 + u^2 - (p-4)\int_{\mathbb{R}^3} Q_\varepsilon|u|^p \\ &< 0.\end{aligned}$$

According to the implicit function theorem, there exist $\sigma > 0$ and a differentiable function $t^*\colon B(0;\sigma) \subset H^1(\mathbb{R}^3) \to \mathbb{R}$ such that $t^*(0) = 1$,

$$\langle (t^*)'(0), \varphi\rangle = \frac{2\int_{\mathbb{R}^3}(\nabla u\nabla\varphi + u\varphi) + 4\varepsilon^{4(p-3)/(p-2)}\int_{\mathbb{R}^3}\phi_u u\varphi - p\int_{\mathbb{R}^3} Q_\varepsilon|u|^{p-2}u\varphi}{\|u\|_{H^1}^2 - (p-1)\int_{\mathbb{R}^3} Q_\varepsilon|u|^p}$$

and

$$F_u(t^*(v), v) = 0 \quad \text{for all } v \in B(0;\sigma),$$

which is equivalent to

$$\langle I'_{\varepsilon,Q_\varepsilon}(t^*(v)(u-v)), t^*(v)(u-v)\rangle = 0 \quad \text{for all } v \in B(0;\sigma).$$

Furthermore, by the continuity of the maps Φ_ε and t^*, we have

$$\begin{aligned}&\langle \psi'_\varepsilon(t^*(v)(u-v)), t^*(v)(u-v)\rangle \\ &\quad = -2\|t^*(v)(u-v)\|_{H^1}^2 - (p-4)\int_{\mathbb{R}^3} Q_\varepsilon|t^*(v)(u-v)|^p \\ &\quad < 0\end{aligned}$$

and

$$\Phi_\varepsilon(t^*(v)(u-v)) \in C^i_{l/\varepsilon}$$

still hold if σ is sufficiently small. Therefore, $t^*(v)(u-v) \in N^i_\varepsilon$ for all $v \in B(0;\sigma)$. This completes the proof. □

Proposition 3.6. *Suppose that $4 \leqslant p < 6$. Then, for each $\varepsilon \in (0, \varepsilon_0)$, there exists a sequence $\{u_n\} \subset N^i_\varepsilon$ such that*

$$I_{\varepsilon,Q_\varepsilon}(u_n) = \gamma^i_\varepsilon + o(1) \quad \textit{and} \quad I'_{\varepsilon,Q_\varepsilon}(u_n) = o(1) \quad \textit{in } H^{-1}(\mathbb{R}^3).$$

Proof. If $\bar{N}^i_\varepsilon$ denotes the closure of N^i_ε, then first we note that $\bar{N}^i_\varepsilon = N^i_\varepsilon \cup \partial N^i_\varepsilon$ for each $i = 1, 2, \dots, k$. Hence,

$$\gamma^i_\varepsilon = \inf\{I_{\varepsilon,Q_\varepsilon}(u) \mid u \in \bar{N}^i_\varepsilon\} \quad \text{for all } i = 1, 2, \dots, k. \tag{3.16}$$

Now we fix $i \in \{1, 2, \dots, k\}$. Applying the Ekeland variational principle [13], there exists a minimizing sequence $\{u_n\} \subset \bar{N}^i_\varepsilon$ such that

$$I_{\varepsilon,Q_\varepsilon}(u_n) < \gamma^i_\varepsilon + \frac{1}{n} \tag{3.17}$$

and

$$I_{\varepsilon,Q_\varepsilon}(u_n) \leqslant I_{\varepsilon,Q_\varepsilon}(w) + \frac{1}{n}\|w - u_n\|_{H^1} \quad \text{for all } w \in \bar{N}^i_\varepsilon. \tag{3.18}$$

Using (3.15), we may assume that $u_n \in N_\varepsilon^i$ for n sufficiently large. Applying lemma 3.5 with $u = u_n$, we obtain the function $t_n^*\colon B(0;\epsilon_n) \to \mathbb{R}$ for some $\epsilon_n > 0$ such that $t_n^*(w)(u_n - w) \in N_\varepsilon^i$. Let $0 < \delta < \epsilon_n$ and $u \in H^1(\mathbb{R}^3)$ with $u \not\equiv 0$. We set $w_\delta = \delta u/\|u\|_{H^1}$ and $z_\delta = t_n^*(w_\delta)(u_n - w_\delta)$. Since $z_\delta \in N_\varepsilon^i$, we deduce from (3.18) that

$$I_{\varepsilon,Q_\varepsilon}(z_\delta) - I_{\varepsilon,Q_\varepsilon}(u_n) \geqslant -\frac{1}{n}\|z_\delta - u_n\|_{H^1}.$$

By the mean value theorem, we obtain

$$\langle I'_{\varepsilon,Q_\varepsilon}(u_n), z_\delta - u_n\rangle + o(\|z_\delta - u_n\|) \geqslant -\frac{1}{n}\|z_\delta - u_n\|_{H^1}.$$

Therefore,

$$\begin{aligned}\langle I'_{\varepsilon,Q_\varepsilon}(u_n), -w_\delta\rangle &+ (t_n^*(w_\delta) - 1)\langle I'_{\varepsilon,Q_\varepsilon}(u_n), (u_n - w_\delta)\rangle \\ &\geqslant -\frac{1}{n}\|z_\delta - u_n\|_{H^1} + o(\|z_\delta - u_n\|). \qquad (3.19)\end{aligned}$$

Now we observe that $t_n^*(w_\delta)(u_n - w_\delta) \in N_\varepsilon^i$ and, consequently, we derive from (3.19) that

$$\begin{aligned}-\delta\left\langle I'_{\varepsilon,Q_\varepsilon}(u_n), \frac{u}{\|u\|_{H^1}}\right\rangle &+ \frac{(t_n^*(w_\delta) - 1)}{t_n^*(w_\delta)}\langle I'_{\varepsilon,Q_\varepsilon}(z_\delta), t_n^*(w_\delta)(u_n - w_\delta)\rangle \\ &+ (t_n^*(w_\delta) - 1)\langle I'_{\varepsilon,Q_\varepsilon}(u_n) - I'_{\varepsilon,Q_\varepsilon}(z_\delta), (u_n - w_\delta)\rangle \\ &\geqslant -\frac{1}{n}\|z_\delta - u_n\|_{H^1} + o(\|z_\delta - u_n\|).\end{aligned}$$

We rewrite the above inequality in the following form:

$$\begin{aligned}\left\langle I'_{\varepsilon,Q_\varepsilon}(u_n), \frac{u}{\|u\|_{H^1}}\right\rangle &\leqslant \frac{\|z_\delta - u_n\|_{H^1}}{\delta n} + \frac{o(\|z_\delta - u_n\|_{H^1})}{\delta} \\ &+ \frac{(t_n^*(w_\delta) - 1)}{\delta}\langle I'_{\varepsilon,Q_\varepsilon}(u_n) - I'_{\varepsilon,Q_\varepsilon}(z_\delta), (u_n - w_\delta)\rangle. \qquad (3.20)\end{aligned}$$

Since we can find a constant $C > 0$ independent of δ such that

$$\|z_\delta - u_n\|_{H^1} \leqslant \delta + C(|t_n^*(w_\delta) - 1|)$$

and

$$\lim_{\delta\to 0}\frac{|t_n^*(w_\delta) - 1|}{\delta} \leqslant \|(t_n^*)'(0)\| \leqslant C$$

for a fixed n, let $\delta \to 0$ in (3.20) and, using the fact that

$$\lim_{\delta\to 0}\|z_\delta - u_n\|_{H^1} = 0,$$

we obtain

$$\left\langle I'_{\varepsilon,Q_\varepsilon}(u_n), \frac{u}{\|u\|_{H^1}}\right\rangle \leqslant \frac{C}{n}.$$

The result implies that

$$I_{\varepsilon,Q_\varepsilon}(u_n) = \gamma_\varepsilon^i + o(1) \quad \text{and} \quad I'_{\varepsilon,Q_\varepsilon}(u_n) = o(1) \quad \text{in } H^{-1}(\mathbb{R}^3).$$

This completes the proof. □

Proof of theorem 1.1. Fix $i \in \{1, 2, \dots, k\}$ and let $\{u_n^i\} \subset N_\varepsilon^i$ be a sequence satisfying

$$I_{\varepsilon,Q_\varepsilon}(u_n) = \gamma_\varepsilon^i + o(1) \quad \text{and} \quad I'_{\varepsilon,Q_\varepsilon}(u_n) = o(1) \quad \text{in } H^{-1}(\mathbb{R}^3).$$

Since $\{u_n^i\}$ is bounded in $H^1(\mathbb{R}^3)$, we can assume that there exists $u_0^i \in H^1(\mathbb{R}^3)$ such that

$$u_n^i \rightharpoonup u_0^i \quad \text{weakly in } H^1(\mathbb{R}^3); \tag{3.21}$$

$$u_n^i \to u_0^i \quad \text{strongly in } L^r_{\text{loc}}(\mathbb{R}^3) \text{ for } 2 \leqslant r < 6; \tag{3.22}$$

$$u_n^i \to u_0^i \quad \text{a.e. in } \mathbb{R}^3. \tag{3.23}$$

First, we claim that $u_0^i \not\equiv 0$. Suppose the contrary, i.e. $u_0^i \equiv 0$. Since $\{u_n^i\} \subset N_\varepsilon^i$ and $\gamma_\varepsilon^i > 0$, we deduce from the Sobolev imbedding theorem that $\|u_n^i\|_{H^1} > \nu > 0$ for some constant ν and for all n. Applying the concentration-compactness principle of Lions [15, 16], there are positive constants R, θ and a sequence $\{z_n\} \subset \mathbb{R}^3$ such that

$$\int_{B^N(0;R)} |u_n^i(x + z_n)|^p \geqslant \theta \quad \text{for } n \text{ sufficiently large.} \tag{3.24}$$

We shall show that $\{z_n\}$ is an unbounded sequence in $\mathbb{R}^3$. Suppose the contrary. Then we can assume that $z_n \to z_0$ for some $z_0 \in \mathbb{R}^3$. By (3.22) and (3.19),

$$\int_{B^N(z_0;R)} |u_0^i|^p \geqslant \theta;$$

this contradicts $u_0^i \equiv 0$. Thus, $\{z_n\}$ is an unbounded sequence in $\mathbb{R}^3$. Set $\tilde{u}_n^i(z) = u_n^i(z + z_n)$. Since $\{\tilde{u}_n^i\}$ is bounded in $H^1(\mathbb{R}^3)$, we may assume that there exists $\tilde{u}_0^i \in H^1(\mathbb{R}^3)$ such that

$$\tilde{u}_n^i \rightharpoonup \tilde{u}_0^i \quad \text{weakly in } H^1(\mathbb{R}^3). \tag{3.25}$$

From (3.24), we have $\tilde{u}_0^i \geqslant 0$ and $\tilde{u}_0^i \not\equiv 0$ in $\mathbb{R}^3$. Set $v_n = \tilde{u}_n^i - \tilde{u}_0^i$. We distinguish the following two cases:

(I) $\|v_n\|_{H^1} \to 0$ as $n \to \infty$;

(II) $\|v_n\|_{H^1} \geqslant \theta$ for large n and for some constant $\theta > 0$.

Assuming case I, we employ the argument in lemma 3.3 to obtain

$$z_n = \Phi(u_n^i) - \Phi(\tilde{u}_n^i) + o(1),$$

and so $|\Phi(u_n^i)| \to \infty$ as $n \to \infty$. This contradicts $\Phi(u_n^i) \in C^i_{l/\varepsilon}$.

In case II, we notice first that $I'_{\varepsilon,Q_\varepsilon}(u_n^i) \to 0$ strongly in $H^{-1}(\mathbb{R}^3)$. Condition (Q1) and $\{u_n^i\} \subset N_\varepsilon^i$ imply

$$\|\tilde{u}_0^i\|^2_{H^1} + \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_{\tilde{u}_0^i}(\tilde{u}_0^i)^2 - \int_{\mathbb{R}^3} Q^\infty |\tilde{u}_0^i|^p = 0 \tag{3.26}$$

and

$$\|\tilde{u}_n^i\|^2_{H^1} + \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi_{\tilde{u}_n^i}(\tilde{u}_n^i)^2 - \int_{\mathbb{R}^3} Q^\infty |\tilde{u}_n^i|^p = o(1). \tag{3.27}$$

By (3.26), (3.27), the Brézis–Lieb lemma [5] and [21, lemma 2.2], we obtain

$$\|v_n\|_{H^1}^2 + \varepsilon^{4(p-3)/(p-2)}\int_{\mathbb{R}^3}\phi_{v_n}v_n^2 - \int_{\mathbb{R}^3}Q^\infty|v_n|^p = o(1).$$

Since $\|v_n\|_{H^1} \geqslant \theta$ for large n, it is straightforward to find a sequence $\{s_n\} \subset \mathbb{R}^+$ with $s_n \to 1$ as $n \to \infty$ such that

$$\|s_nv_n\|_{H^1}^2 + \varepsilon^{4(p-3)/(p-2)}\int_{\mathbb{R}^3}\phi_{s_nv_n}(s_nv_n)^2 = \int_{\mathbb{R}^3}Q^\infty|s_nv_n|^p,$$

and so

$$\tfrac{1}{2}\|v_n\|_{H^1}^2 + \tfrac{1}{4}\varepsilon^{4(p-3)/(p-2)}\int_{\mathbb{R}^3}\phi_{v_n}v_n^2 - \frac{1}{p}\int_{\mathbb{R}^3}Q^\infty|v_n|^p \geqslant \alpha_{\varepsilon,Q^\infty} + o(1).$$

Similarly,

$$\tfrac{1}{2}\|\tilde{u}_0^i\|_{H^1}^2 + \tfrac{1}{4}\varepsilon^{4(p-3)/(p-2)}\int_{\mathbb{R}^3}\phi_{\tilde{u}_0^i}(\tilde{u}_0^i)^2 - \frac{1}{p}\int_{\mathbb{R}^3}Q^\infty|\tilde{u}_0^i|^p \geqslant \alpha_{\varepsilon,Q^\infty}.$$

Thus, by the Brézis–Lieb lemma [5] and [21, lemma 2.2],

$$\begin{aligned}
I_{\varepsilon,Q_\varepsilon}(u_n^i) &= \tfrac{1}{2}\|\tilde{u}_n^i\|_{H^1}^2 + \tfrac{1}{4}\varepsilon^{4(p-3)/(p-2)}\int_{\mathbb{R}^3}\phi_{\tilde{u}_n^i}(\tilde{u}_n^i)^2 - \frac{1}{p}\int_{\mathbb{R}^3}Q^\infty|\tilde{u}_n^i|^p + o(1)\\
&= \tfrac{1}{2}\|v_n\|_{H^1}^2 + \tfrac{1}{4}\varepsilon^{4(p-3)/(p-2)}\int_{\mathbb{R}^3}\phi_{v_n}v_n^2 - \frac{1}{p}\int_{\mathbb{R}^3}Q^\infty|v_n|^p\\
&\quad + \tfrac{1}{2}\|\tilde{u}_0^i\|_{H^1}^2 + \tfrac{1}{4}\varepsilon^{4(p-3)/(p-2)}\int_{\mathbb{R}^3}\phi_{\tilde{u}_0^i}(\tilde{u}_0^i)^2 - \frac{1}{p}\int_{\mathbb{R}^3}Q^\infty|\tilde{u}_0^i|^p + o(1)\\
&\geqslant 2\alpha_{\varepsilon,Q^\infty} + o(1),
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} I_{\varepsilon,Q_\varepsilon}(u_n^i) = \gamma_\varepsilon^i \geqslant 2\alpha_{\varepsilon,Q^\infty}; \tag{3.28}$$

this contradicts (3.15). Next we shall show that $u_n^i \to u_0^i$ strongly in $H^1(\mathbb{R}^3)$. This can be done either by using case II or by adopting a similar argument to that above in order to arrive at the contradiction (3.28). Finally, we shall show that $u_0^i \in N_\varepsilon^i$. Since $\{u_n^i\} \subset N_\varepsilon^i$, we have $u_0^i \in N_\varepsilon^i \cup \partial N_\varepsilon^i$. Moreover, $I_{\varepsilon,Q_\varepsilon}(u_0^i) = \gamma_\varepsilon^i < \tilde{\gamma}_\varepsilon^i$ and so $u_0^i \notin \partial N_\varepsilon^i$. Thus, $u_0^i \in N_\varepsilon^i$. It is clear that u_0^i is non-negative, and, by the maximum principle, u_0^i is therefore positive for $i = 1, 2, \dots, k$. Moreover, the u_0^i are different and the $(u_0^i, \phi_{u_0^i})$ are positive solutions of $(\bar{E}_\varepsilon)$. Taking $\lambda_0 = \varepsilon_0^{-2}$ and $U_i(x) = \lambda^{1/(p-2)}u_0^i(\sqrt{\lambda}x)$, we conclude that the $(U_i, \phi_{U_i})'$ are positive solutions of (E_λ). □

4. Proofs of theorems 1.2 and 1.3

By lemma 3.3, there exists a positive function η_ε with $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that the sublevel set

$$M(\varepsilon,\eta_\varepsilon) = \{u \in \boldsymbol{M}_{\varepsilon,Q_\varepsilon} \mid I_{\varepsilon,Q_\varepsilon}(u_0^i) < \alpha_{0,Q_{\max}} + \eta_\varepsilon\}$$

is non-empty for ε sufficiently small. Then we have the following result.

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LEMMA 4.1. *Suppose that the conditions (Q1) and (Q2) hold and that $Q^\infty < Q_{\max}$. Then*

$$\lim_{\varepsilon\to 0} \sup_{u\in M(\varepsilon,\eta_\varepsilon)} \inf_{x\in\bigcup_{i=1}^k C^i_{l/2\varepsilon}} |\Phi(u) - x| = 0. \tag{4.1}$$

Proof. Let $\varepsilon_n \to 0$ as $n\to\infty$; for any n, there exists $u_n \in M(\varepsilon, \eta_{\varepsilon_n})$ such that

$$\inf_{x\in\bigcup_{i=1}^k C^i_{l/2\varepsilon_n}} |\Phi(u_n) - x| = \sup_{u\in M(\varepsilon,\eta_{\varepsilon_n})} \inf_{x\in\bigcup_{i=1}^k C^i_{l/2\varepsilon_n}} |\Phi(u) - x| + o(1).$$

In order to prove (4.1), it suffices to find points $x_n \in \bigcup_{i=1}^k C^i_{l/2\varepsilon_n}$ such that

$$\lim_{n\to\infty} |\Phi(u_n) - x_n| = 0, \tag{4.2}$$

possibly up to a subsequence. Then, similarly to the argument in the proof of lemma 3.4, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} Q_{\varepsilon_n} |u_n|^p = \frac{2p}{p-2}\alpha_{0,Q_{\max}}.$$

Moreover, there exists $\{x_n\} \subset \mathbb{R}^3$ such that $u_n(\cdot + x_n)$ converges strongly in $H^1(\mathbb{R}^N)$ to u_0, a positive ground-state solution of $(E_{0,Q_{\max}})$. We prove that $\{\varepsilon_n x_n\}$ is a bounded sequence in $\mathbb{R}^3$. Arguing by contradiction, we may assume that $|\varepsilon_n x_n| \to \infty$ as $n \to \infty$. Then

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{R}^3} Q_{\varepsilon_n} |u_n|^p &= \lim_{n\to\infty} \int_{\mathbb{R}^3} Q(\varepsilon_n x + \varepsilon_n x_n)|u_n(x + x_n)|^p \\ &= \int_{\mathbb{R}^3} Q^\infty |u_0|^p \\ &< \int_{\mathbb{R}^3} Q_{\max} |u_0|^p \\ &= \frac{2p}{p-2}\alpha_{0,Q_{\max}},\end{aligned}$$

which is a contradiction. In conclusion, the sequence $\{\varepsilon_n x_n\}$ is bounded and it converges to some x_0 (up to a subsequence). We are left to prove $x_0 \in \{x^1, x^2, \dots, x^k\}$. Since

$$\begin{aligned}\frac{2p}{p-2}\alpha_{0,Q_{\max}} &= \lim_{n\to\infty} \int_{\mathbb{R}^3} Q_{\varepsilon_n} |u_n|^p \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^3} Q(\varepsilon_n x + \varepsilon_n x_n)|u_n(x + x_n)|^p \\ &= \int_{\mathbb{R}^3} Q(x_0)|u_0|^p,\end{aligned}$$

which implies that $Q(x_0) = Q_{\max}$, i.e.

$$x_0 \in \{x^1, x^2, \dots, x^k\} \subset \bigcup_{i=1}^k C^i_{l/2}.$$

Then (4.2) holds. □

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Proof of theorem 1.2. We first show that there exists a positive number $\hat{\varepsilon}$ such that, for every $\varepsilon \in (0, \hat{\varepsilon})$, $M(\varepsilon, \eta_\varepsilon) \subset \bigcup_{i=1}^k N_\varepsilon^i$. This is done by using lemma 4.1. We can thus find $\hat{\varepsilon} > 0$ such that, for every $\varepsilon \in (0, \hat{\varepsilon})$,

$$\sup_{u \in M(\varepsilon, \eta_\varepsilon)} \inf_{x \in \bigcup_{i=1}^k C^i_{l/2\varepsilon}} |\Phi(u) - x| < \frac{l}{2\varepsilon}$$

or

$$\operatorname{dist}\biggl(\Phi(u), \bigcup_{i=1}^k C^i_{l/2\varepsilon}\biggr) < \frac{l}{2\varepsilon} \quad \text{for all } u \in M(\varepsilon, \eta_\varepsilon),$$

implying that

$$\Phi(u) \in \bigcup_{i=1}^k C^i_{l/\varepsilon} \quad \text{for all } u \in M(\varepsilon, \eta_\varepsilon).$$

Therefore, $M(\varepsilon, \eta_\varepsilon) \subset \bigcup_{i=1}^k N_\varepsilon^i$ and we can find at least one ground-state solution $(u_0, \phi_{u_0}) \in N_\varepsilon^i$ of $(\bar{E}_\varepsilon)$ for some $i = 1, 2, \dots, k$. By letting $\hat{\lambda} = \hat{\varepsilon}^{-2}$ and $U_0(x) = \lambda^{1/(p-2)} u_0(\sqrt{\lambda} x)$, we obtain (U_0, ϕ_{U_0}) as a ground-state solution of (E_λ). □

Proof of theorem 1.3. Suppose that $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of $(\bar{E}_\varepsilon)$. Then $u \in \boldsymbol{M}_{\varepsilon, Q_\varepsilon}$, i.e.

$$\|u\|^2_{H^1} + \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} \phi u^2 - \int_{\mathbb{R}^3} Q_\varepsilon |u|^p = 0. \tag{4.3}$$

By the definition of ϕ, we have

$$\int_{\mathbb{R}^3} \phi u^2 = \int_{\mathbb{R}^3} \phi(-\Delta\phi) = \int_{\mathbb{R}^3} |\nabla\phi|^2,$$

while, on the other hand,

$$\int_{\mathbb{R}^3} |u|^3 = \int_{\mathbb{R}^3} (-\Delta\phi)|u| = \int_{\mathbb{R}^3} \nabla\phi \cdot \nabla|u|.$$

It follows that

$$\int_{\mathbb{R}^3} |u|^3 \leqslant \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla\phi|^2. \tag{4.4}$$

(i) Since $p = 3$ and $\sup_{x \in \mathbb{R}^3} Q(x) \leqslant 1$, inserting the inequality (4.4) into (4.3), we obtain

$$0 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \int_{\mathbb{R}^3} |\nabla\phi|^2 - \int_{\mathbb{R}^3} Q_\varepsilon |u|^3 \geqslant \int_{\mathbb{R}^3} u^2,$$

implying that u must be equal to zero, and so $u = 0$ is the unique solution of (E_λ).

(ii) Since $2 < p < 3$, $\varepsilon = \lambda^{-1/2}$ and $\lambda \geqslant 2^{(p-2)/(p-3)}$, inserting the inequality (4.4) into (4.3), we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \varepsilon^{4(p-3)/(p-2)} \int_{\mathbb{R}^3} |\nabla \phi|^2 - \int_{\mathbb{R}^3} Q_\varepsilon |u|^p \\ &\geqslant \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 - \sup_{x \in \mathbb{R}^3} Q(x) \int_{\mathbb{R}^3} |u|^p \\ &\geqslant \int_{\mathbb{R}^3} \Big(u^2 + |u|^3 - \sup_{x \in \mathbb{R}^3} Q(x) |u|^p \Big). \end{aligned}$$

However, the function $g(u) = u^2 + |u|^3 - \sup_{x \in \mathbb{R}^3} Q(x)|u|^p$ is non-negative and vanishes only at zero if $\sup_{x \in \mathbb{R}^3} Q(x) < (p-2)^{2-p}(3-p)^{p-3}$. Therefore, u must be equal to zero, and so $u = 0$ is the unique solution of (E_λ). □

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