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SUBDIRECTLY IRREDUCIBLE DQC RINGS

BY
W. BURGESS[1] AND M. CHACRON[2]

ABSTRACT Twenty-five years ago McCoy published a characterization of commutative subdirectly irreducible rings. This result was generalized by Thierrin to duo rings with the word "field" which appeared in McCoy's theorem replaced by "division ring". The purpose of this note is to give another generalization in which the words "division ring" will be replaced by "simple ring with 1". The techniques resemble those of McCoy and Thierrin.

The class of rings considered here is the class of rings with dense quasi-centre which was defined by Walt [7]. The *quasi-centre* Q of a ring R is $\{r \in R \mid$ for all $s \in R$ there exist s' and s'' such that $rs = s'r$ and $sr = rs''\}$. The quasi-central elements are those elements $r \in R$ such that $|r) = (r| = (r)$ (the left, right, and two-sided ideals generated by r coincide). R is said to be a DQC *ring* (dense quasi-centre ring) if each ideal I of R is generated by $Q \cap I$. Duo rings are the rings where $Q = R$. Examples of DQC rings include complete matrix rings with entries from a commutative ring with 1.

If S is a subset of R, $\ell(S)$ and $\imath(S)$ are the left and right annihilators of S respectively. A DQC ring R is called *faithful* if $\ell(R) = 0$ (for DQC rings $\ell(R) = \imath(R)$).

THEOREM 1. *Let R be a faithful DQC ring then R is subdirectly irreducible iff there exists $0 \neq j \in Q$ (the quasi-centre) so that*

(I) *$\ell(D) = jR$ where $D = \imath(jR)$*

(II) *R/D is a simple ring with* 1.

(III) *for all $a \in Q$ such that $aR \subseteq D$ and $a \notin \ell(D) = jR$ there exists $b \in D \cap Q$ such that $j = ab$.*

Proof. Since $j \in Q$, $\imath(jR) = D$ is an ideal.

The "if" part. We need to show that jR is the heart of R. That is, to show that for all $0 \neq a \in Q$, (a)$\supseteq jR$. Note that this implies $aR \supseteq jR$ since aR is a sum of ideals of the form (b) where $b \in Q$ and (b)$\supseteq jR$.

There are two cases.

Case 1. $a \notin jR$. If $aR \subseteq D$, by (III) $aR \supseteq jR$ and if $aR \nsubseteq D$ by (II), $aR + D = R$ and $jaR + jD = jaR = jR$; but $jaR \subseteq aR$ since $a \in Q$.

Case 2. $a \in jR$. By assumption R is faithful so $R \neq D$ and it follows that there exists $c \in Q$ such that $(c) \nsubseteq D$. For such c, $(c) + D = R$. Since $a \in jR$, $a = jr$ for some

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$r \in R$. Then, $ac=jrc\neq 0$ since c has an inverse modulo $D=r(jR)$. It follows that $j(rc)\neq 0$ and $rc\notin D$. Thus $(rc)+D=R$ by (II). Hence, $c\equiv nrc+\sum a_\alpha rcb_\alpha+\sum rc_\beta+\sum e_\gamma rc$ mod D for some $n\in Z$, a_α, b_α, d_β, $e_\gamma\in R$ and

$$jc = njrc+\sum a'_\alpha jrb'_\alpha c+\sum jrd'_\beta c+\sum e'_\gamma jrc$$

where $a'_\alpha j=ja_\alpha$, $b'_\alpha c=cb_\alpha$, $d'_\alpha c=cd_\beta$ and $e'_\gamma j=je_\beta$; using the fact that j, $c\in Q$. Hence

$$jc = nac+\sum a'_\alpha ab'_\alpha c+\sum ad'_\beta c+\sum e'_\gamma ac \in (a).$$

But, since $(c)+D=R$, $j(c)=jR=(jc)$, so $jR\subseteq(a)$.

"Only if" part. Let $J\neq 0$ be the heart of R. Suppose $j\in J\cap Q$, $j\neq 0$. It follows that $J=jR$. Put $D=r(J)=r(j)$. If $a\in Q$ is a right zero divisor, $a\in r(j)$ since $0\neq \ell(a)$ is an ideal. Thus D is generated by all the right zero divisors in Q.

To prove (II), note that $jR\neq 0$ implies that $R\neq D$ and for some $c\in Q$, $c\notin D$. For such an element c, $jcR=jR$ and $j=jcr$ for some $r\in R$. To show that cr is the identity modulo D, observe that for any $x\in R$, $xcr-x$ is such that

$$jxcr-jx = x'jcr-jx = x'j-jx = jx-jx = 0$$

where $x'j=jx$; so $xcr-x\in D$. Similarly $jcrx-jx=jx-jx=0$. Thus any ideal properly containing D contains an identity element mod D and R/D is simple. This is (II).

In (III) it is clear that there exists $b\in R$ with $j=ab$ and it remains to show that we may suppose $b\in D\cap Q$. Certainly $b\in D$ for we have $ab=j$ so if $ab=b'a$, $b'aD=0$; but $aD\neq 0$ is an ideal so $b'jR=0$. It follows that $b'j=ab^2=jb=0$ so $b\in D$. Now $\{b \mid ab\in jR\}=I$ is an ideal generated by quasi-central elements.

It remains to prove that $\ell(D)=J$. Suppose $aD=0$ for some $a\in Q$. For some s, $j=as$ and $jc=asc\neq 0$ for any $c\in Q$, $c\notin D$. Then $sc\notin D$ and by (II)

$$c \equiv \sum a_\alpha scb_\alpha+\sum scd_\beta+\sum e_\gamma sc+nsc \bmod D.$$

With the notation as before,

$$ac = \sum a'_\alpha asb'_\alpha c+\sum asd'_\beta c+\sum e'_\gamma asc+nasc \in jR.$$

Thus $ac = jrc$, but c is not a right zero divisor so $a\in jR$.

Since a simple duo ring with 1 is a division ring and a commutative simple ring with 1 is a field, the theorems of Thierrin and McCoy are special cases of this one.

The statements (I), (II), and (III) have symmetric counterparts starting with $\ell(jR)$.

Corollary 1. *A DQC ring R with 1 is a subdirect product of rings of the type described in the theorem.*

Corollary 2. *A subdirectly irreducible DQC ring which is semi-prime is a simple ring with* 1.

A ring is called *fully idempotent* if each ideal is idempotent (see [1] for charac-

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terizations). Every homomorphic image of a fully idempotent ring is semi-prime. Hence a fully idempotent DQC ring is a subdirect product of simple rings with 1.

Nonfaithful subdirectly irreducible DQC rings can be characterized. The proof is similar to that of the analogous theorem in [4] and is omitted.

THEOREM 2. *Let R be a DQC ring which is not faithful, then R is subdirectly irreducible iff there is some $0 \neq j \in Q$ and a prime integer p so that*

(I) $(j)=\{r \in R \mid pr=0\}$

(II) *if $aR=0$ for $a \in Q$ then $p^k a=0$ for some k*

(III) *if $aR \neq 0$ for $a \in Q$ there exists b so that $ab=j$.*

Other theorems may be generalized from the commutative to the DQC case. Here are two examples, the originals are in [2] and [4].

THEOREM 3. *If R is a faithful subdirectly irreducible DQC ring with either chain condition on (two-sided) ideals then D (as in Theorem 1) is nil and, hence, is the Jacobson radical.*

Proof. Note that an ideal generated by quasi-central nilpotent elements is nil for a typical element of such an ideal is $s=\sum_{i=1}^{r} a_i d_i$ where each d_i is a quasi-central nilpotent element. If $d_i^n=0$ for each i then $s^{nr}=0$.

Now assume the descending chain condition for ideals. For $d \in D \cap Q$, either d is nilpotent or for some n, $(d^n)=(d^{n+1}) \neq 0$. That is, $d^n=d^{n+1}x+md^{n+1}$, $x \in R$, $m \in Z$. For $c \in Q$, $c \notin D$, $d^n c=d^{n+1}xc+md^{n+1}c$ so $d^n[d(xc+mc)-c]=0$ and the expression in brackets is in $\imath(d^n) \subseteq \imath(jR)=D$. Hence $j[d(xc+mc)-c]=0$ and, consequently, $jc=0$. This is a contradiction. Therefore $d^n=0$.

The proof for the ascending chain condition is nearly identical to that for the commutative case (see [2]). Note that either chain condition implies that $D=\imath(jR)$ $=\ell(jR)=$ Jacobson radical of R.

In the commutative (or duo) case when the conditions of Theorem 1 are satisfied, if the identity of R/D can be lifted to an idempotent of R, this idempotent is an identity for R. This does not seem to be the case for DQC rings in general. However, if R is a DQC ring satisfying the conditions of Theorem 1 such that the left socle S of R is nonzero then $S=jR$. If, in addition, S is a large left ideal and the identity of R/D can be lifted to an idempotent of R then R has an identity. Of course, the descending chain condition on left ideals ensures that these additional conditions are satisfied.

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UNIVERSITY OF OTTAWA,
OTTAWA, ONTARIO

CARLETON UNIVERSITY,
OTTAWA, ONTARIO