Mathematical Structures in Computer Science (2022), **32**, pp. 1099–1116
doi:10.1017/S0960129523000014

CAMBRIDGE UNIVERSITY PRESS

PAPER

On function spaces equipped with Isbell topology and Scott topology†

Xiaoquan Xu[1*], Meng Bao[2] and Xiaoyuan Zhang[3]

[1]Fujian Key Laboratory of Granular Computing and Applications, Minnan Normal University, Zhangzhou 363000, China, [2]College of Mathematics, Sichuan University, Chengdu 610064, China and [3]School of Big Data Science, Hebei Finance University, Baoding 071051, China

*Corresponding author. Email: xiqxu2002@163.com

(Received 26 July 2022; revised 15 November 2022; accepted 6 January 2023; first published online 5 February 2023)

Abstract

In this paper, we mainly study the function spaces related to H-sober spaces. For an irreducible subset system H and T_0 spaces X and Y, it is proved that the following three conditions are equivalent: (1) the Scott space $\Sigma\mathcal{O}(X)$ of the lattice of all open sets of X is H-sober; (2) for every H-sober space Y, the function space $\mathbb{C}(X, Y)$ of all continuous mappings from X to Y equipped with the Isbell topology is H-sober; (3) for every H-sober space Y, the Isbell topology on $\mathbb{C}(X, Y)$ has property S with respect to H. One immediate corollary is that for a T_0 space X, Y is a d-space (resp., well-filtered space) iff the function space $\mathbb{C}(X, Y)$ equipped with the Isbell topology is a d-space (resp., well-filtered space). It is shown that for any T_0 space X for which the Scott space $\Sigma\mathcal{O}(X)$ is non-sober, the function space $\mathbb{C}(X, \Sigma 2)$ equipped with the Isbell topology is not sober. The function spaces $\mathbb{C}(X, Y)$ equipped with the Scott topology, the compact-open topology and the pointwise convergence topology are also discussed. Our study also leads to a number of questions, whose answers will deepen our understanding of the function spaces related to H-sober spaces.

Keywords: Function space; Isbell topology; Scott topology; compact-open topology; pointwise convergence topology; H-sober space

1. Introduction

Function spaces (equipped with certain topologies) are important structures in topology and domain theory (see Engelking 1989; Gierz et al. 2003), which was initially introduced by Dana Scott (Scott 1970). As a special kind of mathematical structures, domains serve as mathematical universes within which people can interpret higher-order functional programming languages, and cartesian closed categories of domains (more generally, certain topological spaces) are appropriate for models of various typed and untyped lambda-calculi and functional programming languages (see Gierz et al. 2003). Since whether certain properties of topological spaces are preserved when passing to function spaces is connected with the cartesian closed category of topological spaces, this question has attracted considerable attention in domain theory and non-Hausdorff topology, especially for domains (which are a special kind of topological spaces when endowed with the Scott topology), sober spaces, d-spaces and well-filtered spaces (see Ershov et al. 2020; Gierz et al. 2003; Liu et al. 2021; Xu 2021).

†This research was supported by the National Natural Science Foundation of China (Nos. 12071199, 11661057), and the Science and Technology Project of Education Bureau of Hebei Province, China (No. QN2023023).

There exists a quite satisfactory theory which deals with the cartesian closedness of domains. Jung (1989, 1990), Plotkin (1976) and Smyth (1983) have made essential contributions to this theory. For topological spaces, it is well-known that if X is a T_0 space and Y a sober space, then the function space $\mathbb{C}(X, Y)$ of all continuous functions $f : X \to Y$ equipped with the topology of pointwise convergence is sober (see, for example, Gierz et al. 2003, Exercise O-5.16). Furthermore, in Ershov et al. (2020), it was shown that for any T_0 space X, a T_0 space Y is a d-space (resp., sober space) iff the function space $\mathbb{C}(X, Y)$ equipped with the topology of pointwise convergence is a d-space (resp., sober space). It is known that for a T_0 space X and a d-space Y, the function space $\mathbb{C}(X, Y)$ equipped with the Isbell topology is a d-space (cf. Gierz et al. 2003, Lemma II-4.3). Conversely, in Liu et al. (2021), the authors showed that if the function space $\mathbb{C}(X, Y)$ equipped with the Isbell topology is a d-space, then Y is a d-space. For the well-filteredness, it was proved in Liu et al. (2021) that for any core-compact space X and well-filtered space Y, the function space $\mathbb{C}(X, Y)$ equipped with the Isbell topology is well-filtered.

In order to provide a uniform approach to d-spaces, sober spaces and well-filtered spaces, Xu (2021) introduced the concepts of irreducible subset system H and H-sober spaces and developed a general framework for dealing with all these spaces. The irreducible subset systems $\mathscr{D}$, $\mathscr{R}$, WD and RD are four important ones, where for each T_0 space X, $\mathscr{D}(X)$ is the set of all directed subsets of X, $\mathscr{R}(X)$ is the set of all irreducible subsets of X, $\mathsf{WD}(X)$ is the set of all well-filtered determined subsets of X, and $\mathsf{RD}(X)$ is the set of all Rudin subsets of X (see Xu 2021; Xu et al. 2020). So the d-spaces, sober spaces and well-filtered spaces are three special types of H-sober spaces. It was proved in Xu (2021) that for a T_0 space X and an H-sober space Y, the function space $\mathbb{C}(X, Y)$ equipped with the topology of pointwise convergence is H-sober. One immediate corollary is that for a T_0 space X and a sober space (resp., d-space, well-filtered space) Y, the function space $\mathbb{C}(X, Y)$ equipped with the topology of pointwise convergence is a sober space (resp., d-space, well-filtered space).

In this paper, we mainly study the function spaces related to H-sober spaces, especially to d-spaces, well-filtered spaces and sober spaces. The paper is organized as follows:

In Section 2, some fundamental concepts and notation are introduced which will be used in the whole paper.

In Section 3, we briefly recall some basic concepts and results about irreducible subset systems and H-sober spaces that will be used in the other sections.

In Section 4, for the convenience of discussions of function spaces equipped with the pointwise convergence topology, compact-open topology, Isbell topology and Scott topology, several basic lemmas are given.

In Section 5, we mainly discuss the function spaces equipped with Isbell topology. For an irreducible subset system H and T_0 spaces X and Y, the interlinks between the H-sobriety of Scott space $\Sigma\mathscr{O}(X)$, the H-sobriety of Y and the H-sobriety of the function space $\mathbb{C}(X, Y)$ equipped with the Isbell topology are discussed. It is proved that the following three conditions are equivalent: (1) $\Sigma\mathscr{O}(X)$ is H-sober; (2) for every H-sober space Y, the function space $\mathbb{C}(X, Y)$ equipped with the Isbell topology is H-sober; (3) for every H-sober space Y, the Isbell topology on $\mathbb{C}(X, Y)$ has property S with respect to H. As an immediate corollary, we get that for a T_0 space X, Y is a d-space (resp., well-filtered space) iff the function space $\mathbb{C}(X, Y)$ equipped with the Isbell topology is a d-space (resp., well-filtered space). It is shown that for any T_0 space X for which the Scott space $\Sigma\mathscr{O}(X)$ is non-sober, the function space $\mathbb{C}(X, \Sigma 2)$ equipped with the Isbell topology is not sober.

In Sections 6 and 7, we investigate the function spaces equipped with the pointwise convergence topology and compact-open topology, and the Scott topology, respectively.

2. Preliminaries

In this section, we briefly recall some fundamental concepts and notation that will be used in this paper; more details can be founded in Engelking (1989), Gierz et al. (2003), Goubault-Larrecq (2013).

For a poset P and $A \subseteq P$, let $\uparrow A = \{x \in P : a \leq x \text{ for some } a \in A\}$ (dually $\downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\}$). For $A = \{x\}$, $\uparrow A$ and $\downarrow A$ are shortly denoted by $\uparrow x$ and $\downarrow x$, respectively. A subset A is called a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). Let $P^{(<\omega)} = \{F \subseteq P : F \text{ is a finite set}\}$ and $\mathbf{Fin}P = \{\uparrow F : F \in P^{(<\omega)} \setminus \{\emptyset\}\}$. A subset D of P is *directed* provided that it is non-empty, and every finite subset of D has an upper bound in D. The set of all directed sets of P is denoted by $\mathcal{D}(P)$. P is said to be a *directed complete poset*, a *dcpo* for short, if every directed subset of P has the least upper bound in P. As in Gierz et al. (2003), the *upper topology* on a poset P, generated by the complements of the principal ideals of P, is denoted by $\upsilon(P)$. The upper sets of P form the (*upper*) *Alexandroff topology* $\gamma(P)$. The space $\Gamma P = (P, \gamma(P))$ is called the *Alexandroff space* of P. A subset U of a poset P is called *Scott open* if $U = \uparrow U$ and $D \cap U \neq \emptyset$ for all directed sets $D \subseteq P$ with $\vee D \in U$ whenever $\vee D$ exists. The topology formed by all Scott open sets of P is called the *Scott topology*, written as $\sigma(P)$. $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P. Clearly, ΣP is a T_0 space.

The following result is well-known (cf. Gierz et al. 2003).

Lemma 1. *Let P, Q be posets and $f : P \to Q$. Then, the following two conditions are equivalent:*

(1) f is Scott continuous, that is, $f : \Sigma P \to \Sigma Q$ is continuous.
(2) For any $D \in \mathcal{D}(P)$ for which $\vee D$ exists, $f(\vee D) = \vee f(D)$.

Given a T_0 space X, we can define a partial order $\leq_X$, called the *specialization order*, which is defined by $x \leq_X y$ iff $x \in \overline{\{y\}}$. Let ΩX denote the poset $(X, \leq_X)$. Clearly, each open set is an upper set, and each closed set is a lower set with respect to the partial order $\leq_X$. Unless otherwise stated, throughout the paper, whenever an order-theoretic concept is mentioned in a T_0 space, it is to be interpreted with respect to the specialization order. We often use X itself to denote the poset ΩX. Let $\mathcal{O}(X)$ (resp., $\mathcal{C}(X)$) be the set of all open subsets (resp., closed subsets) of X and denote $\mathcal{S}(X) = \{\{x\} : x \in X\}$ and $\mathcal{D}(X) = \{D \subseteq X : D \text{ is a directed set of } X\}$. Let $\mathcal{S}_c(X) = \{\overline{\{x\}} : x \in X\}$ and $\mathcal{D}_c(X) = \{\overline{D} : D \in \mathcal{D}(X)\}$. A T_0 topology τ on a poset P is called *order compatible* if the specialization order agrees with the original order on P or, equivalently, $\upsilon(P) \subseteq \tau \subseteq \gamma(P)$.

Lemma 2. *(Keimel et al. 2009, Lemma 6.2) Let $f : X \to Y$ be a continuous mapping of T_0 spaces. If $D \in \mathcal{D}(X)$ has a supremum to which it converges, then $f(D)$ is directed and has a supremum in Y to which it converges, and $f(\vee D) = \vee f(D)$.*

Corollary 3. *Let P be a poset and Y a T_0 space. If $f : \Sigma P \to Y$ is continuous, then $f : \Sigma P \to \Sigma Y$ is continuous.*

A non-empty subset A of a T_0 space X is called *irreducible* if for any $\{F_1, F_2\} \subseteq \mathcal{C}(X)$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. We denote by $\mathsf{Irr}(X)$ (resp., $\mathsf{Irr}_c(X)$) the set of all irreducible (resp., irreducible closed) subsets of X. Clearly, every subset of X that is directed under $\leq_X$ is irreducible. A topological space X is called *sober*, if for any $F \in \mathsf{Irr}_c(X)$, there is a unique point $x \in X$ such that $F = \overline{\{x\}}$.

The category of all sets and mappings is denoted by **Set** and the category of all T_0 spaces with continuous mappings is denoted by $\mathbf{Top}_0$. A T_0 space X is called a *d-space* (or *monotone convergence space*) if X (with the specialization order) is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X)$ (cf. Gierz et al. 2003). Clearly, for a dcpo P, ΣP is a d-space.

One can directly get the following result (cf. Xu et al. 2020, Proposition 3.3).

Proposition 4. *For a T_0 space X, the following conditions are equivalent:*

(1) X is a d-space.

(2) $\mathcal{D}_c(X) = \mathcal{S}_c(X)$, *that is, for each* $D \in \mathcal{D}(X)$, *there exists a (unique) point* $x \in X$ *such that* $\overline{D} = \overline{\{x\}}$.

For a T_0 space X, let 2^X be the set of all subsets of X. A subset A of X is called *saturated* if A equals the intersection of all open sets containing it (equivalently, A is an upper set in the specialization order). We denote by $\mathsf{K}(X)$ the poset of non-empty compact saturated subsets of X with the *Smyth preorder*, i.e., for $K_1, K_2 \in \mathsf{K}(X)$, $K_1 \sqsubseteq K_2$ iff $K_2 \subseteq K_1$. For $G \subseteq X$, let $\Phi(G) = \{U \in \mathcal{O}(X) : G \subseteq U\}$. We denote by $\phi(\mathcal{O}(X))$ the topology on $\mathcal{O}(X)$ generated by $\{\Phi(C) : C$ is a compact subset of $X\}$ (as a base). It is easy to verify that $\upsilon(\mathcal{O}(X)) \subseteq \phi(\mathcal{O}(X)) \subseteq \sigma(\mathcal{O}(X))$ and the family $\{\Phi(K) : K \in \mathsf{K}(X) \cup \{\emptyset\}\}$ is also a base of $\phi(\mathcal{O}(X))$. The space X is called *well-filtered*, if for any open set U and any filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$.

In Xi et al. (2017), the following useful result was given.

Proposition 5. *(Xi et al. 2017, Corollary 3.2) If L is a complete lattice, then ΣL is well-filtered.*

For a dcpo P and $x, y \in P$, we say x is *way below* y, written $x \ll y$, if for each $D \in \mathcal{D}(P)$, $y \leq \vee D$ implies $x \leq d$ for some $d \in D$. Let $\Downarrow x = \{u \in P : u \ll x\}$. P is called a *continuous domain*, if for each $x \in P$, $\Downarrow x$ is directed and $x = \vee \Downarrow x$. When a complete lattice L is continuous, we call L a *continuous lattice*. A topological space X is called *core-compact* if $\mathcal{O}(X)$ is a *continuous lattice*.

3. Irreducible Subset Systems and H-Sober Spaces

In this section, we briefly recall some basic concepts and results about irreducible subset systems and H-sober spaces that will be used in the other sections. For further details, we refer the reader to Shen et al. (2019), Xu (2021), Xu et al. (2020). In what follows, all topological spaces will be supposed to be non-empty spaces.

For a T_0 space X and $\mathcal{K} \subseteq \mathsf{K}(X)$, let $M(\mathcal{K}) = \{A \in \mathcal{C}(X) : K \cap A \neq \emptyset \text{ for all } K \in \mathcal{K}\}$ and $m(\mathcal{K}) = \{A \in \mathcal{C}(X) : A \text{ is a minimal member of } M(\mathcal{K})\}$.

Definition 6. *(Shen et al. 2019; Xu et al. 2020) Let X be a T_0 space and A a non-empty subset of X.*

(1) A is said to have the Rudin property *(which is called* compactly filtered property *in Shen et al. 2019), if there exists a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $\overline{A} \in m(\mathcal{K})$ (that is, $\overline{A}$ is a minimal closed set that intersects all members of $\mathcal{K}$). Let $\mathsf{RD}(X) = \{A \subseteq X : A$ has Rudin property$\}$ and $\mathsf{RD}_c(X) = \{\overline{A} : A \in \mathsf{RD}(X)\}$. The sets in $\mathsf{RD}(X)$ will also be called* Rudin sets.

(2) A is called a well-filtered determined set *(WD-set for short) if for any continuous mapping $f : X \to Y$ into a well-filtered space Y, there exists a unique $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$. Denote by $\mathsf{WD}(X)$ the set of all well-filtered determined subsets of X and $\mathsf{WD}_c(X) = \{\overline{A} : A \in \mathsf{WD}(X)\}$. Clearly, a subset A of a space X is well-filtered determined iff $\overline{A}$ is well-filtered determined.*

Lemma 7. *(Xu et al. 2020, Proposition 6.2) Let X be a T_0 space. Then, $\mathcal{S}(X) \subseteq \mathcal{D}(X) \subseteq \mathsf{RD}(X) \subseteq \mathsf{WD}(X) \subseteq \mathsf{Irr}(X)$.*

Lemma 8. *(Shen et al. 2019, Lemma 2.5; Xu et al. 2020, Lemma 6.23) Let X, Y be two T_0 spaces and $f : X \to Y$ a continuous mapping. If $A \in \mathsf{RD}(X)$ (resp., $A \in \mathsf{WD}(X)$), then $f(A) \in \mathsf{RD}(Y)$ (resp., $f(A) \in \mathsf{WD}(Y)$).*

Using Rudin sets and WD-sets, we have the following characterization of well-filtered spaces.

https://doi.org/10.1017/S0960129523000014 Published online by Cambridge University Press

Proposition 9. *(Xu et al. 2020, Corollary 7.11) Let X be a T_0 space. Then, the following conditions are equivalent:*

(1) X is well-filtered.
(2) $\mathsf{WD}_c(X) = \mathscr{S}_c(X)$, that is, for any $A \in \mathsf{WD}(X)$, there exists a unique $a \in X$ such that $\overline{A} = \overline{\{a\}}$.
(3) $\mathsf{RD}_c(X) = \mathscr{S}_c(X)$, that is, for any $A \in \mathsf{RD}(X)$, there exists a unique $a \in X$ such that $\overline{A} = \overline{\{a\}}$.

In order to provide a uniform approach to d-spaces, sober spaces and well-filtered spaces and develop a general framework for dealing with all these spaces, Xu (2021) introduced the following concepts.

Definition 10. *(Xu 2021)* (1) *A covariant functor* $\mathrm{H} : \mathbf{Top}_0 \to \mathbf{Set}$ *is called a* subset system *on* $\mathbf{Top}_0$ *provided that the following two conditions are satisfied:*

(1) $\mathscr{S}(X) \subseteq \mathrm{H}(X) \subseteq 2^X$ (the set of all subsets of X) for each $X \in \mathrm{ob}(\mathbf{Top}_0)$.
(2) For any continuous mapping $f : X \to Y$ in $\mathbf{Top}_0$, $\mathrm{H}(f)(A) = f(A) \in \mathrm{H}(Y)$ for all $A \in \mathrm{H}(X)$.

(2) A subset system $\mathrm{H} : \mathbf{Top}_0 \to \mathbf{Set}$ *is called an* irreducible subset system*, or an* R-subset system *for short, if* $\mathrm{H}(X) \subseteq \mathsf{Irr}(X)$ *for all* $X \in \mathrm{ob}(\mathbf{Top}_0)$*.*

In what follows, the capital letter H always stands for an R-subset system $\mathrm{H} : \mathbf{Top}_0 \to \mathbf{Set}$. For a T_0 space X, let $\mathrm{H}_c(X) = \{\overline{A} : A \in \mathrm{H}(X)\}$. Define a partial order $\leq$ on the set of all R-subset systems by $\mathrm{H}_1 \leq \mathrm{H}_2$ iff $\mathrm{H}_1(X) \subseteq \mathrm{H}_2(X)$ for all T_0 spaces X.

Here are some important examples of R-subset systems used in this paper:

(1) $\mathscr{S}$ (for $X \in ob(\mathbf{Top}_0)$, $\mathscr{S}(X)$ is the set of all single point subsets of X).
(2) $\mathscr{D}$ (for $X \in ob(\mathbf{Top}_0)$, $\mathscr{D}(X)$ is the set of all directed subsets of X).
(3) $\mathscr{R}$ (for $X \in ob(\mathbf{Top}_0)$, $\mathscr{R}(X)$ is the set of all irreducible subsets of X).
(4) RD (for $X \in ob(\mathbf{Top}_0)$, $\mathsf{RD}(X)$ is the set of all Rudin subsets of X).
(5) WD (for $X \in ob(\mathbf{Top}_0)$, $\mathsf{WD}(X)$ is the set of all well-filtered determined subsets of X).

By Lemmas 7 and 8, WD and RD are R-subset systems, and $\mathscr{S} \leq \mathscr{D} \leq \mathsf{RD} \leq \mathsf{WD} \leq \mathscr{R}$.

Definition 11. *(Xu 2021) Let* $\mathrm{H} : \mathbf{Top}_0 \to \mathbf{Set}$ *be an R-subset system and X a T_0 space. X is called* H-sober *if for any $A \in \mathrm{H}(X)$, there is a (unique) point $x \in X$ such that $\overline{A} = \overline{\{x\}}$ or, equivalently, if* $\mathrm{H}_c(X) = \mathscr{S}_c(X)$.

For two topological spaces X and Y, Y is said to be a *retract* of X if there are two continuous mappings $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = id_Y$.

Lemma 12. *(Xu 2021, Proposition 4.26) A retract of an H-sober space is H-sober.*

Corollary 13. *A retract of a sober space (resp., d-space, well-filtered space) is a sober space (resp., d-space, well-filtered space).*

4. Some Basic Lemmas

For T_0 spaces X and Y, there are four important topologies on the set $\mathbb{C}(X, Y)$ of all continuous functions from X to Y, namely the pointwise convergence topology, compact-open topology,

https://doi.org/10.1017/S0960129523000014 Published online by Cambridge University Press

Isbell topology and Scott topology. In order to discuss the function spaces equipped with these four topologies, we shall give some basic lemmas in this section.

First, we recall the definitions of pointwise convergence topology, Isbell topology and compact-open topology. For further details, we refer the reader to Engelking (1989), Gierz et al. (2003) and Goubault-Larrecq (2013). In what follows, $\mathbb{C}(X, Y)$ always means the set of all continuous functions from X to Y.

Definition 14. *Let X and Y be topological spaces.*

(1) For a point $x \in X$ and an open set $V \in \mathcal{O}(Y)$, let $S(x, V) = \{f \in \mathbb{C}(X, Y) : f(x) \in V\}$. The set $\{S(x, V) : x \in X, V \in \mathcal{O}(Y)\}$ is a subbasis for the pointwise convergence topology *(i.e., the relative product topology) on $\mathbb{C}(X, Y)$. Let $[X \to Y]_P$ denote the function space $\mathbb{C}(X, Y)$ endowed with the topology of pointwise convergence.*

(2) The compact-open topology *on the set $\mathbb{C}(X, Y)$ is generated by the subsets of the form $N(K \to V) = \{f \in \mathbb{C}(X, Y) : f(K) \subseteq V\}$, where K is compact in X and V is open in Y. Let $[X \to Y]_K$ denote the function space $\mathbb{C}(X, Y)$ endowed with the compact-open topology.*

(3) The Isbell topology *on the set $\mathbb{C}(X, Y)$ is generated by the subsets of the form $N(\mathcal{H} \leftarrow V) = \{f \in \mathbb{C}(X, Y) : f^{-1}(V) \in \mathcal{H}\}$, where $\mathcal{H}$ is a Scott open subset of the complete lattice $\mathcal{O}(X)$ and V is open in Y. Let $[X \to Y]_I$ denote the function space $\mathbb{C}(X, Y)$ endowed with the Isbell topology.*

For a topological space X and a T_0 space Y, Y (with the specialization order) is a poset, whence $\mathbb{C}(X, Y)$ is a poset with the pointwise order. Denote by $[X \to Y]_\Sigma$ the Scott space $\Sigma\mathbb{C}(X, Y)$. It is easy to see that the pointwise convergence topology, the compact-open topology and the Isbell topology on $\mathbb{C}(X, Y)$ are all order compatible, and for $x \in X$, $K \in \mathsf{K}(X)$ and $V \in \mathcal{O}(Y)$, we clearly have that $S(x, V) = N(\{x\} \to V)$ and $N(K \to V) = N(\mathcal{H}_K \leftarrow V)$, where $\mathcal{H}_K = \{U \in \mathcal{O}(X) : K \subseteq U\} \in \sigma(\mathcal{O}(X))$. So we have the following result.

Lemma 15. *(Gierz et al. 2003, Lemma II-4.2) Let X, Y be T_0 spaces. Then,*

$$\upsilon(\mathbb{C}(X, Y)) \subseteq \mathcal{O}([X \to Y]_P) \subseteq \mathcal{O}([X \to Y]_K) \subseteq \mathcal{O}([X \to Y]_I) \subseteq \gamma(\mathbb{C}(X, Y)).$$

If X is a core-compact sober space, then the Isbell topology and the compact-open topology agree.

Lemma 16. *Let X, Y be T_0 spaces and $x \in X$. Then, the evaluation function at x*

$$E^P_x : [X \to Y]_P \to Y, f \mapsto f(x),$$

is continuous. Therefore, if $\mathcal{T}$ is a topology on $\mathbb{C}(X, Y)$ which is finer than the pointwise convergence topology, then the function $E^{\mathcal{T}}_x : (\mathbb{C}(X, Y), \mathcal{T}) \to Y, f \mapsto f(x)$ is continuous.

Proof. For any $V \in \mathcal{O}(Y)$, we have $(E^P_x)^{-1}(V) = \{f \in \mathbb{C}(X, Y) : f(x) \in V\} = S(x, V)$, which is open in $[X \to Y]_P$. So E^P_x is continuous. □

By Lemmas 15 and 16, we have the following corollary.

Corollary 17. *Let X, Y be T_0 spaces and $x \in X$. Then,*

(1) The function $E^K_x : [X \to Y]_K \to Y, f \mapsto f(x)$ is continuous.

(2) The function $E^I_x : [X \to Y]_I \to Y, f \mapsto f(x)$ is continuous.

In the following, for topological spaces X, Y and $y \in Y$, c_y denotes the constant function from X to Y with value y, i.e., $c_y(x) = y$ for all $x \in X$.

https://doi.org/10.1017/S0960129523000014 Published online by Cambridge University Press

Lemma 18. *(Liu et al. 2021, Lemma 3.2) Let X and Y be T_0 spaces. Then, the function*

$$\psi^I : Y \to [X \to Y]_I, y \mapsto c_y,$$

is continuous.

Corollary 19. *Let X, Y be T_0 spaces and $\mathscr{T}$ a topology on $\mathbb{C}(X, Y)$ which is finer than the upper topology and coarser than the Isbell topology. Then, the function $\psi^{\mathscr{T}} : Y \to (\mathbb{C}(X, Y), \mathscr{T}), y \mapsto c_y$ is continuous.*

In particular, by Lemmas 15 and 18, we get the following corollary.

Corollary 20. *Let X and Y be T_0 spaces. Then*

(1) The function $\psi^K : Y \to [X \to Y]_K, y \mapsto c_y$ is continuous.
(2) The function $\psi^P : Y \to [X \to Y]_P, y \mapsto c_y$ is continuous.

Remark 21. The similar results of Lemma 18 and Corollary 20 for the Scott topology does not hold in general. Indeed, let $X = 1 = \{0\}$ be the topological space with single point (clearly, $\mathscr{O}(X) = \mathscr{O}(1) = \{\emptyset, 1\}$ and Y the space $(P, \upsilon(P))$, where P is a poset for which the Scott topology is strictly finer than the upper topology on P (e.g., P is a countably infinite set with the discrete order). Then, $\mathbb{C}(X, Y) = \{c_y : y \in Y\}$ and $c_y \mapsto y : [X \to Y]_\Sigma \to \Sigma Y = \Sigma P$ is a homeomorphism. Since $\upsilon(P) \subseteq \sigma(P)$, the function $\psi^\Sigma : Y \to [X \to Y]_\Sigma, y \mapsto c_y$ is not continuous.

Lemma 22. *Let X, Y be T_0 spaces and $\mathscr{T}$ a topology on $\mathbb{C}(X, Y)$ which is finer than the upper topology and coarser than the Isbell topology. Then, Y is a retract of $(\mathbb{C}(X, Y), \mathscr{T})$. In particular, Y is a retract of $[X \to Y]_P$ (resp., $[X \to Y]_K$, $[X \to Y]_I$).*

Proof. Select an $x \in X$. Then by Lemma 16 and Corollary 19, $E_x^{\mathscr{T}} : (\mathbb{C}(X, Y), \mathscr{T}) \to Y$ and $\psi^{\mathscr{T}} : Y \to (\mathbb{C}(X, Y), \mathscr{T})$ are continuous. Clearly, $E_x^{\mathscr{T}} \circ \psi^{\mathscr{T}} = id_Y$. Thus, Y is a retract of $(\mathbb{C}(X, Y), \mathscr{T})$. □

Lemma 23. *Let X, Y be T_0 spaces and $\emptyset \neq \mathscr{F} \subseteq \mathbb{C}(X, Y)$. For any mapping $g : X \to Y$, the following two conditions are equivalent:*

(1) For each $x \in X$, $\overline{\{f(x) : f \in \mathscr{F}\}} = \overline{\{g(x)\}}$.
(2) For each $V \in \mathscr{O}(Y)$, $g^{-1}(V) = \bigcup\limits_{f \in \mathscr{F}} f^{-1}(V)$.

Therefore, when condition (1) is satisfied, we have that $g \in \mathbb{C}(X, Y)$, $g(x) = \bigvee_{f \in \mathscr{F}} f(x)$ for each $x \in X$ and $g = \bigvee_{\mathbb{C}(X,Y)} \mathscr{F}$.

Proof. (1) ⇒ (2): Let $V \in \mathscr{O}(Y)$. If $x \in g^{-1}(V)$ or, equivalently, $g(x) \in V$, then by (1), $\{f(x) : f \in \mathscr{F}\} \cap V \neq \emptyset$ and hence $f_V(x) \in V$ for some $f_V \in \mathscr{F}$. So $x \in f_V^{-1}(V) \subseteq \bigcup\limits_{f \in \mathscr{F}} f^{-1}(V)$. Thus, $g^{-1}(V) \subseteq \bigcup\limits_{f \in \mathscr{F}} f^{-1}(V)$. Conversely, if $x \in \bigcup\limits_{f \in \mathscr{F}} f^{-1}(V)$, then $f_x(x) \in V$ for some $f_x \in \mathscr{F}$. By (1), $f_x(x) \in \overline{\{g(x)\}}$; whence $g(x) \in V$, that is, $x \in g^{-1}(V)$. Therefore, $\bigcup\limits_{f \in \mathscr{F}} f^{-1}(V) \subseteq g^{-1}(V)$. Thus, $g^{-1}(V) = \bigcup\limits_{f \in \mathscr{F}} f^{-1}(V)$.

https://doi.org/10.1017/S0960129523000014 Published online by Cambridge University Press

(2) $\Rightarrow$ (1): For each $x \in X$ and $V \in \mathcal{O}(Y)$, by (2) we have that $\overline{\{g(x)\}} \cap V \neq \emptyset$ iff $x \in g^{-1}(V)$ iff $x \in f_x^{-1}(V)$ for some $f_x \in \mathcal{F}$ iff $\{f(x) : f \in \mathcal{F}\} \cap V \neq \emptyset$. So $\overline{\{f(x) : f \in \mathcal{F}\}} = \overline{\{g(x)\}}$.

Now suppose that condition (1) is satisfied (and hence condition (2) holds). Then, by (2) $g : X \to Y$ is continuous, and for each $x \in X$, as $f(x) \in \overline{\{g(x)\}}$ for each $f \in \mathcal{F}$, $g(x)$ is an upper bound of $\{f(x) : f \in \mathcal{F}\}$. If $y \in Y$ is also an upper bound of $\{f(x) : f \in \mathcal{F}\}$, then $g(x) \in \overline{\{f(x) : f \in \mathcal{F}\}} \subseteq \overline{\{y\}}$, whence $g(x) \leq_Y y$. So $g(x) = \bigvee_{f \in \mathcal{F}} f(x)$. Clearly, g is an upper bound of $\mathcal{F}$. If $h \in \mathbb{C}(X, Y)$ is another upper bound of $\mathcal{F}$, then for each $x \in X$, $\{f(x) : f \in \mathcal{F}\} \subseteq \overline{\{h(x)\}}$; whence $g(x) \in \overline{\{f(x) : f \in \mathcal{F}\}} \subseteq \overline{\{h(x)\}}$, that is, $g(x) \leq h(x)$. Hence, $g \leq h$. So $g = \bigvee_{\mathbb{C}(X,Y)} \mathcal{F}$. $\square$

By Proposition 4 and Lemma 23, we get the following corollary.

Corollary 24. *(Gierz et al. 2003, Lemma II-3.14 and Proposition II-3.15) Let X be a T_0 space and Y a d-space. Then,*

(1) If $(f_d)_{d \in D}$ is a net of continuous functions $f_d : X \to Y$ such that $(f_d(x))_{d \in D}$ is a directed net of Y (with the specialization order) for each $x \in X$, then the pointwise sup $f : X \to Y$ of the net f_d is continuous and f is the least upper bound of $(f_d)_{d \in D}$ in $\mathbb{C}(X, Y)$.
(2) The subset $\mathbb{C}(X, Y)$ of $(\Omega Y)^X$ is closed under directed sups and hence $\mathbb{C}(X, Y)$ is a dcpo.

Corollary 25. *Let X be a T_0 space and Y a d-space. Then, the function $\psi^\sigma : \Sigma Y \to [X \to Y]_\Sigma$, $y \mapsto c_y$ is continuous.*

Proof. For any $D \in \mathcal{D}(Y)$, by Corollary 24, we have $\psi^\sigma(\vee_Y D) = c_{\vee_Y D} = \bigvee_{\mathbb{C}(X,Y)} \psi^\sigma(D)$. Hence by Lemma 1, $\psi^\sigma : \Sigma Y \to [X \to Y]_\Sigma$ is continuous. $\square$

Motivated by Lemma 23, we give the following definition.

Definition 26. *Let $\mathrm{H} : \mathbf{Top}_0 \to \mathbf{Set}$ be an R-subset system and X, Y T_0 spaces. An order compatible topology $\mathcal{T}$ on $\mathbb{C}(X, Y)$ is said to have* property S *with respect to H if it satisfies the following condition:*

(S) For any $\mathcal{F} \in \mathrm{H}((\mathbb{C}(X, Y), \mathcal{T}))$ and $g \in \mathbb{C}(X, Y)$, if $g^{-1}(V) = \bigcup_{f \in \mathcal{F}} f^{-1}(V)$ for each $V \in \mathcal{O}(Y)$, then $g \in \mathrm{cl}_{\mathcal{T}} \mathcal{F}$.

Remark 27. By the order compatibility of $\mathcal{T}$ and Lemma 23, $\mathcal{T}$ has property S with respect to H iff for any $\mathcal{F} \in \mathrm{H}((\mathbb{C}(X, Y), \mathcal{T}))$ and $g \in \mathbb{C}(X, Y)$ with $g^{-1}(V) = \bigcup_{f \in \mathcal{F}} f^{-1}(V)$ for all $V \in \mathcal{O}(Y)$, we have $\mathrm{cl}_{\mathcal{T}} \mathcal{F} = \mathrm{cl}_{\mathcal{T}} \{g\}$.

Lemma 28. *Let X be a T_0 space and Y a d-space and $x \in X$. Then, $\mathbb{C}(X, Y)$ is a dcpo and the following two functions are continuous:*

(1) $E_x^\sigma : [X \to Y]_\Sigma \to \Sigma Y, f \mapsto f(x)$.
(2) $E_x^\Sigma : [X \to Y]_\Sigma \to Y, f \mapsto f(x)$.

Proof. By Corollary 24, $\mathbb{C}(X, Y)$ is a dcpo. Clearly, E_x^σ is monotone. For any $\{f_d : d \in D\} \in \mathcal{D}(\mathbb{C}(X, Y))$, by Corollary 24, we have $E_x^\sigma(\bigvee_{\mathbb{C}(X,Y)} \{f_d : d \in D\}) = \bigvee_{d \in D} f_d(x) = \bigvee_Y \{E_x^\sigma(f_d) : d \in D\}$. Hence by Lemma 1, $E_x^\sigma : [X \to Y]_\Sigma \to \Sigma Y$ is continuous; whence by $\mathcal{O}(Y) \subseteq \sigma(Y)$, $E_x^\Sigma : [X \to Y]_\Sigma \to Y$ is continuous. $\square$

https://doi.org/10.1017/S0960129523000014 Published online by Cambridge University Press

Corollary 29. *Let X be a T_0 space and Y a d-space. Then,*

(1) ΣY is a retract of $[X \to Y]_\Sigma$.
(2) If $\psi^\Sigma : Y \to [X \to Y]_\Sigma$, $y \mapsto c_y$ is continuous, then Y is a retract of $[X \to Y]_\Sigma$.

Proof. (1): Select an $x \in X$. Then by Lemmas 16 and Corollary 25, both the function $E_x^\sigma : [X \to Y]_\Sigma \to \Sigma Y$ and the function $\psi^\sigma : \Sigma Y \to [X \to Y]_\Sigma$ are continuous. Clearly, $E_x^\sigma \circ \psi^\sigma = id_{\Sigma Y}$. Thus, ΣY is a retract of $[X \to Y]_\Sigma$.

(2): Suppose that $\psi^\Sigma : Y \to [X \to Y]_\Sigma$ is continuous. By Lemma 28, $E_x^\Sigma : [X \to Y]_\Sigma \to Y$ is continuous. Clearly, $E_x^\Sigma \circ \psi^\Sigma = id_Y$. So Y is a retract of $[X \to Y]_\Sigma$. □

Lemma 30. *Let X, Y be T_0 spaces and $V \in \mathcal{O}(Y)$. Then, the function*

$$\Theta_V^I : [X \to Y]_I \to \Sigma\mathcal{O}(X), f \mapsto f^{-1}(V),$$

is continuous.

Proof. For each $\mathcal{H} \in \sigma(\mathcal{O}(X))$, we have $(\Theta_V^I)^{-1}(\mathcal{H}) = \{f \in \mathbb{C}(X, Y) : f^{-1}(V) \in \mathcal{H}\} = N(\mathcal{H} \leftarrow V)$, which is open in $[X \to Y]_I$. Thus, $\Theta_V^I : [X \to Y]_I \to \Sigma\mathcal{O}(X)$ is continuous. □

Lemma 31. *Let X be a T_0 space and Y a d-space. Then for any $V \in \mathcal{O}(Y)$, the function*

$$\Theta_V^\Sigma : [X \to Y]_\Sigma \to \Sigma\mathcal{O}(X), f \mapsto f^{-1}(V),$$

is continuous.

Proof. By Corollary 24, $\mathbb{C}(X, Y)$ is a dcpo and the directed sups in $\mathbb{C}(X, Y)$ is the pointwise sups. So for any directed family $\{f_d : d \in D\} \subseteq \mathbb{C}(X, Y)$, by $\mathcal{O}(Y) \subseteq \sigma(Y)$, we have that $\Theta_V^\Sigma(\bigvee_{\mathbb{C}(X,Y)}\{f_d : d \in D\}) = (\bigvee_{\mathbb{C}(X,Y)}\{f_d : d \in D\})^{-1}(V) = \bigcup_{d\in D}(f_d)^{-1}(V) = \bigcup_{d\in D}\Theta_V^\Sigma(f_d)$. Hence by Lemma 1, $\Theta_V^\Sigma : [X \to Y]_\Sigma \to \Sigma\mathcal{O}(X)$ is continuous. □

Lemma 32. *Let X be a T_0 space and Y a d-space having at least two points. Suppose that Y has a least element $\perp_Y$. Select any $y \in Y$ with $y \neq \perp_Y$. Then, the function*

$$\chi^y : \Sigma\mathcal{O}(X) \to [X \to Y]_\Sigma, U \mapsto \chi_U^y,$$

is continuous, where $\chi_U^y : X \to Y$ is defined by

$$\chi_U^y(x) = \begin{cases} y, & x \in U, \\ \perp_Y, & x \notin U. \end{cases}$$

Proof. For each $U \in \mathcal{O}(X)$ and $W \in \mathcal{O}(Y)$, we clearly have

$$(\chi_U^y)^{-1}(W) = \begin{cases} X, & W = Y, \\ \emptyset, & W \neq Y, y \notin U, \\ U, & W \neq Y, y \in U. \end{cases}$$

So $\chi_U^y : X \to Y$ is continuous. For any directed family $\{U_d : d \in D\} \subseteq \mathcal{O}(X)$, let $U = \bigcup_{d\in D} U_d$. It is straightforward to verify that $\chi^y(\bigcup_{d\in D} U_d) = \chi_U^y = \bigvee_{\mathbb{C}(X,Y)}\{\chi_{U_d}^y : d \in D\}$. Hence by Lemma 1, $\chi^y : \Sigma\mathcal{O}(X) \to [X \to Y]_\Sigma$ is continuous. □

A T_0 space X is said to be a *non-trivial* space if it has at least two points.

https://doi.org/10.1017/S0960129523000014 Published online by Cambridge University Press

Definition 33. *A T_0-space Y is called a* pointed d-space *if it is a non-trivial d-space and has a least element $\bot_Y$ (with respect to the specialization order).*

The simplest pointed d-space is the Sierpiński space $\Sigma 2$.

Corollary 34. *Let X be a T_0 space and Y a pointed d-space. Then, $\Sigma\mathscr{O}(X)$ is a retract of $[X\to Y]_\Sigma$.*

Proof. As Y is a pointed d-space, $V=Y\setminus\{\bot_Y\}=Y\setminus \mathrm{cl}_Y\{\bot_Y\}$ is a non-empty open set of Y, where $\bot_Y$ is the least element of Y. Select a $y\in V$. Then by Lemmas 31 and 32, $\Theta_V^\Sigma:[X\to Y]_\Sigma\to\Sigma\mathscr{O}(X)$ and $\chi^y:\Sigma\mathscr{O}(X)\to[X\to Y]_\Sigma$ are continuous. For each $U\in\mathscr{O}(X)$, we have $\Theta_V^\Sigma(\chi^y(U))=\Theta_V^\Sigma(\chi_U^y)=(\chi_U^y)^{-1}(V)=U$. So $\Theta_V^\Sigma\circ\chi^y=id_{\Sigma\mathscr{O}(X)}$. Thus, $\Sigma\mathscr{O}(X)$ is a retract of $[X\to Y]_\Sigma$. □

5. Function Spaces Equipped with Isbell Topology

Lemma 35. *Let $\mathrm{H}:\mathbf{Top}_0\to\mathbf{Set}$ be an R-subset system for which $\mathscr{S}\le\mathrm{H}\le\mathsf{WD}$ and X, Y two T_0 spaces. Then, the Isbell topology on $\mathbb{C}(X,Y)$ has property S with respect to H.*

Proof. Let $\mathscr{F}\in\mathrm{H}([X\to Y]_I)$ and $g\in\mathbb{C}(X,Y)$ satisfying $g^{-1}(V)=\bigcup_{f\in\mathscr{F}}f^{-1}(V)$ for each $V\in\mathscr{O}(Y)$. For any $V\in\mathscr{O}(Y)$, by Lemma 30, we have $\Theta_V^I(\mathscr{F})=\{f^{-1}(V):f\in\mathscr{F}\}\in\mathrm{H}(\Sigma\mathscr{O}(X))$. By Proposition 5, $\Sigma\mathscr{O}(X)$ is well-filtered (i.e., WD-sober) and hence H-sober since $\mathscr{S}\le\mathrm{H}\le\mathsf{WD}$. So there is a unique $W_V\in\mathscr{O}(X)$ such that $\mathrm{cl}_{\sigma(\mathscr{O}(X))}\{f^{-1}(V):f\in\mathscr{F}\}=\mathrm{cl}_{\sigma(\mathscr{O}(X))}\{W_V\}$, and consequently, $g^{-1}(V)=\bigcup_{f\in\mathscr{F}}f^{-1}(V)=\bigcup\mathrm{cl}_{\sigma(\mathscr{O}(X))}\{f^{-1}(V):f\in\mathscr{F}\}=\bigvee\mathrm{cl}_{\sigma(\mathscr{O}(X))}\{W_V\}=W_V$. Hence, $\mathrm{cl}_{\sigma(\mathscr{O}(X))}\{f^{-1}(V):f\in\mathscr{F}\}=\mathrm{cl}_{\sigma(\mathscr{O}(X))}\{g^{-1}(V)\}$.

Now we verify that $g\in\mathrm{cl}_{[X\to Y]_I}\mathscr{F}$. For any subbasic open set $N(\mathscr{H}\leftarrow W)$ in $[X\to Y]_I$ with $g\in N(\mathscr{H}\leftarrow W)$ (where $\mathscr{H}\in\sigma(\mathscr{O}(X))$ and $W\in\mathscr{O}(Y)$), we have $g^{-1}(W)\in\mathscr{H}$; whence by $g^{-1}(W)\in\mathrm{cl}_{\sigma(\mathscr{O}(X))}\{f^{-1}(W):f\in\mathscr{F}\}$, there is $f_W\in\mathscr{F}$ such that $f_W^{-1}(W)\in\mathscr{H}$. Thus, $\mathscr{F}\bigcap N(\mathscr{H}\leftarrow W)\ne\emptyset$. As $\mathscr{F}\in H([X\to Y]_I)\subseteq\mathsf{Irr}([X\to Y]_I)$, all basic open sets of g in $[X\to Y]_I$ must meet $\mathscr{F}$. So $g\in\mathrm{cl}_{[X\to Y]_I}\mathscr{F}$, proving that the Isbell topology on $\mathbb{C}(X,Y)$ has property S with respect to H. □

However, for T_0 spaces X and Y, the Isbell topology on $\mathbb{C}(X,Y)$ does not have property S with respect to $\mathscr{R}$ in general (see Proposition 39 below).

Theorem 36. *Let $\mathrm{H}:\mathbf{Top}_0\to\mathbf{Set}$ be an R-subset system and X a T_0 space. Suppose that $\mathscr{T}$ is a topology on $\mathbb{C}(X,Y)$ which is finer than the pointwise convergence topology. Consider the following two conditions:*

(1) $(\mathbb{C}(X,Y),\mathscr{T})$ is H-sober.
(2) $\mathscr{T}$ has property S with respect to H.

Then (1) $\Rightarrow$ (2). Moreover, if Y is H-sober, then the two conditions are equivalent.

Proof. (1) $\Rightarrow$ (2): Let $\mathscr{F}\in\mathrm{H}((\mathbb{C}(X,Y),\mathscr{T}))$ and $g\in\mathbb{C}(X,Y)$ satisfying $g^{-1}(V)=\bigcup_{f\in\mathscr{F}}f^{-1}(V)$ for each $V\in\mathscr{O}(Y)$. By the H-sobriety of $(\mathbb{C}(X,Y),\mathscr{T})$, there is a unique $g'\in\mathbb{C}(X,Y)$ such that $\mathrm{cl}_{\mathscr{T}}\mathscr{F}=\mathrm{cl}_{\mathscr{T}}\{g'\}$. Hence, $g'=\bigvee_{\mathbb{C}(X,Y)}\mathscr{F}=g$ by Lemma 23. So $\mathrm{cl}_{\mathscr{T}}\mathscr{F}=\mathrm{cl}_{\mathscr{T}}\{g\}$.

(2) $\Rightarrow$ (1): Suppose that Y is H-sober. For any $\mathscr{F}\in\mathrm{H}((\mathbb{C}(X,Y),\mathscr{T}))$ and $x\in X$, by Lemma 16, we have that $E_x^{\mathscr{T}}(\mathscr{F})=\{f(x):f\in\mathscr{F}\}\in\mathrm{H}(Y)$. As Y is H-sober, there is a unique point $y_x\in Y$ such that $\overline{\{f(x):f\in\mathscr{F}\}}=\overline{\{y_x\}}$. Now we can define a function

$$g:X\to Y\ \ by\ \ g(x)=y_x\text{ for each }x\in X.$$

Then by Lemma 23, $g : X \to Y$ is continuous and $g^{-1}(V) = \bigcup_{f \in \mathcal{F}} f^{-1}(V)$ for each $V \in \mathcal{O}(Y)$. Since $\mathcal{T}$ has property S with respect to H, we have $g \in \mathrm{cl}_{\mathcal{T}} \mathcal{F}$ or, equivalently, $\mathrm{cl}_{\mathcal{T}} \mathcal{F} = \mathrm{cl}_{\mathcal{T}} \{g\}$. Thus, $(\mathbb{C}(X, Y), \mathcal{T})$ is H-sober.

□

Theorem 37. *Let* $\mathrm{H} : \mathbf{Top}_0 \to \mathbf{Set}$ *be an R-subset system and* X *a* T_0 *space. Suppose that* $\mathcal{T}$ *is a topology on* $\mathbb{C}(X, Y)$ *which is finer than the pointwise convergence topology and coarser than the Isbell topology. Consider the following two conditions:*

(1) $(\mathbb{C}(X, Y), \mathcal{T})$ *is H-sober.*
(2) Y *is H-sober.*

Then, (1) $\Rightarrow$ *(2). Moreover, if* $\mathcal{T}$ *has property S with respect to H, then the two conditions are equivalent.*

Proof. (1) $\Rightarrow$ (2): By Lemmas 12 and 22.
(2) $\Rightarrow$ (1): Suppose that $\mathcal{T}$ has property S with respect to H and Y is H-sober. Then by Theorem 36, $(\mathbb{C}(X, Y), \mathcal{T})$ is H-sober. □

Furthermore, we have the following result.

Theorem 38. *Let* $\mathrm{H} : \mathbf{Top}_0 \to \mathbf{Set}$ *be an R-subset system and* X *a* T_0 *space. Then the following three conditions are equivalent:*

(1) $\Sigma\mathcal{O}(X)$ *is H-sober.*
(2) For every H-sober space Y*, the function space* $[X \to Y]_I$ *is H-sober.*
(3) For every H-sober space Y*, the Isbell topology on* $\mathbb{C}(X, Y)$ *has property S with respect to H.*

Proof. (1) $\Rightarrow$ (2): Suppose that $\Sigma\mathcal{O}(X)$ and Y are H-sober. We will show that $[X \to Y]_I$ is H-sober. For any $\mathcal{F} \in \mathrm{H}([X \to Y]_I)$ and $x \in X$, by Corollary 17 (2), $E^I_x(\mathcal{F}) = \{f(x) : f \in \mathcal{F}\} \in \mathrm{H}(Y)$; whence by the H-sobriety of Y, there is a unique $y_x \in Y$ such that $\overline{\{f(x) : f \in \mathcal{F}\}} = \overline{\{y_x\}}$. Now, we can define a function $g : X \to Y$ by $g(x) = y_x$ for each $x \in X$. Then by Lemma 23, $g \in \mathbb{C}(X, Y)$ and $g^{-1}(V) = \bigcup_{f \in \mathcal{F}} f^{-1}(V)$ for each $V \in \mathcal{O}(Y)$.

Now we prove $\mathrm{cl}_{[X\to Y]_I}\{g\} = \mathrm{cl}_{[X\to Y]_I}\mathcal{F}$. Clearly, $f \leq g$ for each $f \in \mathcal{F}$; whence $\mathrm{cl}_{[X\to Y]_I}\mathcal{F} \subseteq \mathrm{cl}_{[X\to Y]_I}\{g\}$. Then, we show $g \in \mathrm{cl}_{[X\to Y]_I}\mathcal{F}$. First, for $V \in \mathcal{O}(Y)$, by Lemma 30, $\Theta^I_V(\mathcal{F}) = \{f^{-1}(V) : f \in \mathcal{F}\} \in \mathrm{H}(\Sigma\mathcal{O}(X))$. As $\Sigma\mathcal{O}(X)$ is H-sober, there is a unique $W_V \in \mathcal{O}(X)$ such that $\mathrm{cl}_{\sigma(\mathcal{O}(X))}\{f^{-1}(V) : f \in \mathcal{F}\} = \mathrm{cl}_{\sigma(\mathcal{O}(X))}\{W_V\}$, and consequently, $g^{-1}(V) = \bigcup_{f \in \mathcal{F}} f^{-1}(V) = \bigcup \mathrm{cl}_{\sigma(\mathcal{O}(X))}\{f^{-1}(V) : f \in \mathcal{F}\} = \bigvee \mathrm{cl}_{\sigma(\mathcal{O}(X))}\{W_V\} = W_V$. So $\mathrm{cl}_{\sigma(\mathcal{O}(X))}\{f^{-1}(V) : f \in \mathcal{F}\} = \mathrm{cl}_{\sigma(\mathcal{O}(X))}\{g^{-1}(V)\}$. Then, we verify that $g \in \mathrm{cl}_{[X\to Y]_I}\mathcal{F}$. For any subbasic open set $N(\mathcal{H} \leftarrow W)$ in $[X \to Y]_I$ with $g \in N(\mathcal{H} \leftarrow W)$ (where $\mathcal{H} \in \sigma(\mathcal{O}(X))$ and $W \in \mathcal{O}(Y)$), we have $g^{-1}(W) \in \mathcal{H}$; whence by $g^{-1}(W) \in \mathrm{cl}_{\sigma(\mathcal{O}(X))}\{f^{-1}(W) : f \in \mathcal{F}\}$, there is $f_W \in \mathcal{F}$ such that $f^{-1}_W(W) \in \mathcal{H}$. Thus, $\mathcal{F} \bigcap N(\mathcal{H} \leftarrow W) \neq \emptyset$. As $\mathcal{F} \in H([X \to Y]_I) \subseteq \mathsf{Irr}([X \to Y]_I)$, all basic open sets of g in $[X \to Y]_I$ must meet $\mathcal{F}$. So $g \in \mathrm{cl}_{[X\to Y]_I}\mathcal{F}$. Thus, $\mathrm{cl}_{[X\to Y]_I}\mathcal{F} = \mathrm{cl}_{[X\to Y]_I}\{g\}$, proving that $[X \to Y]_I$ is H-sober.

(2) $\Rightarrow$ (1): As the Sierpiński space $\Sigma 2$ is sober (and hence H-sober), by (2) $[X \to \Sigma 2]_I$ is H-sober. But $[X \to \Sigma 2]_I$ is homeomorphic to $\Sigma\mathcal{O}(X)$, so $\Sigma\mathcal{O}(X)$ is H-sober.

(2) $\Leftrightarrow$ (3): By Theorem 36.

□

In Xu et al. (2021), it was shown that there exist some T_0 spaces X for which $\Sigma\mathcal{O}(X)$ is non-sober. Using such spaces, we can give the following result.

Proposition 39. *Let X be any T_0 space for which $\Sigma\mathcal{O}(X)$ is non-sober. Then,*

(1) $[X \to \Sigma 2]_I$ *is not sober.*
(2) *The Isbell topology on $\mathbb{C}(X, \Sigma 2)$ does not have property S with respect to $\mathcal{R}$.*

Proof. (1): It is straightforward to verify that $f \mapsto f^{-1}(1) : [X \to \Sigma 2]_I \to \Sigma\mathcal{O}(X)$ is a homeomorphism. As $\Sigma\mathcal{O}(X)$ is non-sober, $[X \to \Sigma 2]_I$ is not sober.

(2): Assume, on the contrary, that the Isbell topology on $\mathbb{C}(X, \Sigma 2)$ has property S with respect to $\mathcal{R}$; then by the sobriety of $\Sigma 2$ and Theorem 38, $[X \to \Sigma 2]_I$ is sober, a contradiction. So the Isbell topology on $\mathbb{C}(X, \Sigma 2)$ does not have property S with respect to $\mathcal{R}$. □

By Lemmas 15, 35 and Theorem 37, we get the following three corollaries.

Corollary 40. *Let X, Y be T_0 spaces.*

(1) *If the function space $[X \to Y]_P$ is H-sober, then Y is H-sober.*
(2) *If the function space $[X \to Y]_K$ is H-sober, then Y is H-sober.*
(3) *If the function space $[X \to Y]_I$ is H-sober, then Y is H-sober.*

Corollary 41. *(Gierz et al. 2003, Lemma II-4.3) Let X, Y be T_0 spaces. Then, the following two conditions are equivalent:*

(1) *Y is a d-space.*
(2) *$[X \to Y]_I$ is a d-space, and hence, the Scott topology on $\mathbb{C}(X, Y)$ is finer than the Isbell topology.*

Corollary 42. *Let X, Y be T_0 spaces. Then, the following two conditions are equivalent:*

(1) *Y is well-filtered.*
(2) *$[X \to Y]_I$ is well-filtered.*

In Liu et al. (2021, Corollary 3.12), it was proved that for a core-compact space X, if Y is well-filtered, then the function space $\mathbb{C}(X, Y)$ equipped with the Isbell topology is a well-filtered space. For the well-filteredness, Corollary 42 improves this conclusion by removing the unnecessary condition that X is core-compact.

Applying Corollary 40 directly to the R-subset systems $\mathcal{D}$, $\mathcal{R}$ and WD (or RD), we get the following corollary.

Corollary 43. *Let X, Y be T_0 spaces.*

(1) *If the function space $[X \to Y]_P$ is a sober space (resp., d-space, well-filtered space), then Y is a sober space (resp., d-space, well-filtered space).*
(2) *If the function space $[X \to Y]_K$ is a sober space (resp., d-space, well-filtered space), then Y is a sober space (resp., d-space, well-filtered space).*
(3) *If the function space $[X \to Y]_I$ is a sober space, then Y is a sober space.*

From Theorem 38 and Corollary 40, we deduce the following result.

Corollary 44. *Let $H : \mathbf{Top}_0 \to \mathbf{Set}$ be an R-subset system and X a T_0 space for which $\Sigma\mathcal{O}(X)$ is H-sober. Then for any T_0 space Y, the following two conditions are equivalent:*

(1) Y is H-sober.
(2) The function space $[X \to Y]_I$ is H-sober.

In particular, we have the following result.

Corollary 45. *For a T_0 space X, the following two conditions are equivalent:*

(1) $\Sigma\mathscr{O}(X)$ is sober.
(2) For every sober space Y, the function space $[X \to Y]_I$ is sober.

6. Function Spaces Equipped with Pointwise Convergence Topology and Compact-Open Topology

In this section, we discuss the function spaces equipped with pointwise convergence topology and compact-open topology.

Proposition 46. *Let $\mathrm{H} : \mathbf{Top}_0 \to \mathbf{Set}$ be an R-subset system and X, Y two T_0 spaces. Then, the pointwise convergent topology on $\mathscr{T}$ has property S with respect to H.*

Proof. Let $\mathscr{F} \in \mathrm{H}([X \to Y]_P)$ and $g \in \mathbb{C}(X, Y)$ satisfying $g^{-1}(V) = \bigcup_{f \in \mathscr{F}} f^{-1}(V)$ for each $V \in \mathscr{O}(Y)$. Then by Lemma 23, $\overline{\{g(x)\}} = \overline{\{f(x) : f \in \mathscr{F}\}}$ for each $x \in X$. For any subbasic open set $S(x, W)$ in $[X \to Y]_P$ with $g \in S(x, W)$ (where $W \in \mathscr{O}(Y)$), we have $g(x) \in W$; whence by $g(x) \in \overline{\{f(x) : f \in \mathscr{F}\}}$, there is $f_W \in \mathscr{F}$ such that $f_W(x) \in W$, that is, $f_W \in S(x, W)$. Thus, $\mathscr{F} \bigcap S(x, W) \neq \emptyset$. As $\mathscr{F} \in H([X \to Y]_P) \subseteq \mathsf{Irr}([X \to Y]_P)$, all basic open sets of g in $[X \to Y]_P$ must meet $\mathscr{F}$. So $g \in \mathrm{cl}_{[X \to Y]_P} \mathscr{F}$, proving that the pointwise convergent topology on $\mathbb{C}(X, Y)$ has property S with respect to H. □

Proposition 47. *Let X, Y be T_0 spaces. Then, the compact-open topology on $\mathbb{C}(X, Y)$ has property S with respect to $\mathscr{D}$.*

Proof. Let $\{f_d : d \in D\} \in \mathscr{D}(\mathbb{C}(X, Y))$ and $g \in \mathbb{C}(X, Y)$ satisfying $g^{-1}(V) = \bigcup_{d \in D} f_d^{-1}(V)$ for each $V \in \mathscr{O}(Y)$. Then by Lemma 23, $\overline{\{g(x)\}} = \overline{\{f_d(x) : d \in D\}}$ for each $x \in X$ and hence $g = \bigvee_{\mathbb{C}(X,Y)} \{f_d : d \in D\}$. Now we show that $g \in \mathrm{cl}_{[X \to Y]_K} \{f_d : d \in D\}$. For each compact subset K of X and $V \in \mathscr{O}(Y)$ with $g \in N(K \to V)$, we have $K \subseteq g^{-1}(V) = \bigcup_{d \in D} f_d^{-1}(V)$. As K is compact and $\{f_d^{-1}(V) : d \in D\}$ is a directed family of open sets of X, there is $d \in D$ such that $K \subseteq f_d^{-1}(V)$ or, equivalently, $f_d \in N(K \to V)$. Hence $\{f_d : d \in D\} \bigcap N(K \to V) \neq \emptyset$. Since $\{f_d : d \in D\} \in \mathscr{D}(\mathbb{C}(X, Y))$, all basic open sets of g in $[X \to Y]_K$ must meet $\{f_d : d \in D\}$. So $g \in \mathrm{cl}_{[X \to Y]_K} \{f_d : d \in D\}$, proving that the compact-open topology on $\mathbb{C}(X, Y)$ has property S with respect to $\mathscr{D}$. □

Question 48. For T_0 spaces X, Y, whether the compact-open topology on $\mathbb{C}(X, Y)$ has property S with respect to WD or RD?

Question 49. For T_0 spaces X, Y, whether the compact-open topology on $\mathbb{C}(X, Y)$ has property S with respect to $\mathscr{R}$?

From Theorem 37 and Proposition 46 (or Corollary 40) we deduce the following result.

https://doi.org/10.1017/S0960129523000014 Published online by Cambridge University Press

Proposition 50. *(Xu 2021, Theorem 4.28) Let $H : \mathbf{Top}_0 \to \mathbf{Set}$ be an R-subset system and X, Y two T_0 spaces. Then the following two conditions are equivalent:*

(1) Y is H-sober.
(2) The function space $[X \to Y]_P$ is H-sober.

Applying Corollary 40 and Proposition 50 directly to the R-subset systems $\mathcal{D}$, $\mathcal{R}$ and WD (or RD), we get the following corollary.

Corollary 51. *For T_0 spaces X and Y, the following two conditions are equivalent:*

(1) Y is a sober space (resp., d-space, well-filtered space).
(2) The function space $[X \to Y]_P$ is a sober space (resp., d-space, well-filtered space).

The results for d-spaces and sober spaces in Corollary 51 were first shown in Ershov et al. (2020, Theorems 3 and 6) by a different method.

By Lemma 15, Theorem 37 and Proposition 47, we have the following result.

Corollary 52. *For T_0 spaces X and Y, the following two conditions are equivalent:*

(1) Y is a d-space.
(2) The function space $[X \to Y]_K$ is a d-space.

Remark 53. We can give a direct proof of Corollary 52.

Proof. Suppose that X is a T_0 space and Y is a d-space. We show that the function space $\mathbb{C}(X, Y)$ equipped with the compact-open topology is a d-space.

Let $\mathcal{F} \in \mathcal{D}([X \to Y]_K)$. Since the specialization order on $[X \to Y]_K$ is the usually pointwise order on $\mathbb{C}(X, Y)$, we have that for each $x \in X$, $\{f(x) : f \in \mathcal{F}\} \in \mathcal{D}(Y)$. As Y is a d-space, there exists a unique point $a_x \in Y$ such that $\overline{\{f(x) : f \in \mathcal{F}\}} = \overline{\{a_x\}}$. Then, we can define a function $g : X \to Y$ by $g(x) = a_x$ for each $x \in X$. By Lemma 23, g is continuous. Clearly, $\mathrm{cl}_{[X\to Y]_K}\mathcal{F} \subseteq \mathrm{cl}_{[X\to Y]_K}\{g\}$.

Now we show that $g \in \mathrm{cl}_{[X\to Y]_K}\mathcal{F}$. Let $N(K \to V)$ be a subbasic open set in $[X \to Y]_K$ with $g \in N(K \to V)$, where K is a compact subset of X and V is an open subset of Y. Then for any $x \in K$, $g(x) \in V$. As $\overline{\{f(x) : f \in \mathcal{F}\}} = \overline{\{g(x)\}}$, there exists $f \in \mathcal{F}$ with $f(x) \in V$, whence $x \in f^{-1}(V)$. Thus, $K \subseteq \bigcup_{f\in\mathcal{F}} f^{-1}(V)$. By the compactness of K, there exists $\{f_1, f_2, \cdots, f_n\} \subseteq \mathcal{F}$ such that $K \subseteq \bigcup_{i=1}^n f_i^{-1}(V)$. Since $\mathcal{F}$ is directed in $[X \to Y]_K$, there is $h \in \mathcal{F}$ such that $f_i \leq h$ for all $i \in \{1, 2, \cdots, n\}$. Therefore, $K \subseteq \bigcup_{i=1}^n f_i^{-1}(V) \subseteq h^{-1}(V)$ (note that open sets in Y are upper sets with the specialization order of Y). It follows that $h \in N(K \to V)$. So $\mathcal{F} \cap N(K \to V) \neq \emptyset$. Since $\mathcal{F} \in \mathcal{D}([X \to Y]_K) \subseteq \mathsf{Irr}([X \to Y]_K)$ and all basic open sets of g in $[X \to Y]_K$ must meet $\mathcal{F}$, we get that $g \in \mathrm{cl}_{[X\to Y]_K}\mathcal{F}$, and hence, $\mathrm{cl}_{[X\to Y]_K}\mathcal{F} = \mathrm{cl}_{[X\to Y]_K}\{g\}$. Thus, the function space $[X \to Y]_K$ is a d-space. □

But we do not know whether the converse of Proposition 40 (2) holds for a general R-subset system H. So we pose the following question.

Question 54. For an R-subset system H, a T_0 space X and an H-sober space Y, is the function space $[X \to Y]_K$ H-sober?

Especially, we have the following two questions.

https://doi.org/10.1017/S0960129523000014 Published online by Cambridge University Press

Question 55. For a T_0 space X and a sober space Y, is the function space $[X \to Y]_K$ sober?

Question 56. For a T_0 space X and a well-filtered space Y, is the function space $[X \to Y]_K$ well-filtered?

Note that when Y is the Sierpiński space $\Sigma 2$, the compact-open topology on $[X \to Y]$ takes $\{N(K \to 2) = \mathscr{O}(X)\} \bigcup \{N(K \to \{1\}) : K \in \mathsf{K}(X)\}$ as a base. So $[X \to \Sigma 2]_K$ is homeomorphic to the space $(\mathscr{O}(X), \phi(\mathscr{O}(X)))$. Then as a special case of the Question 48 (resp., Question 49, Question 55), Question 56, we have the following ones.

Question 57. For a T_0 space X, whether the compact-open topology on $\mathbb{C}(X, \Sigma 2)$ has property S with respect to WD or RD?

Question 58. For a T_0 space X, whether the compact-open topology on $\mathbb{C}(X, \Sigma 2)$ has property S with respect to $\mathscr{R}$?

Question 59. For a T_0 space X, is the function space $[X \to \Sigma 2]_K$ well-filtered? Or equivalently, is $(\mathscr{O}(X), \phi(\mathscr{O}(X)))$ well-filtered?

Question 60. For a T_0 space X, is the function space $[X \to \Sigma 2]_K$ sober? Or equivalently, is the space $(\mathscr{O}(X), \phi(\mathscr{O}(X)))$ sober?

7. Function Spaces Equipped with Scott Topology

Finally, we investigate the Scott topology on $\mathbb{C}(X, Y)$. First, by Lemma 15, Theorem 36 and Corollary 41, we have the following result.

Theorem 61. *Let* $\mathrm{H} : \mathbf{Top}_0 \to \mathbf{Set}$ *be an R-subset system, X a* T_0 *space and Y a d-space. Consider the following two conditions:*

(1) $[X \to Y]_\Sigma$ *is H-sober.*
(2) The Scott topology on $\mathbb{C}(X, Y)$ *has property S with respect to H.*

Then (1) $\Rightarrow$ *(2). Moreover, if Y is H-sober, then the two conditions are equivalent.*

Proposition 62. *Let X, Y be* T_0 *spaces. Then, the Scott topology on* $\mathbb{C}(X, Y)$ *has property S with respect to* $\mathscr{D}$.

Proof. Let $\{f_d : d \in D\} \in \mathscr{D}(\mathbb{C}(X, Y))$ and $g \in \mathbb{C}(X, Y)$ satisfying $g^{-1}(V) = \bigcup_{d \in D} f_d^{-1}(V)$ for each $V \in \mathscr{O}(Y)$. Then by Lemma 23, $\overline{\{g(x)\}} = \overline{\{f_d(x) : d \in D\}}$ for each $x \in X$ and $g = \bigvee_{\mathbb{C}(X,Y)} \{f_d : d \in D\}$. Now we show that $g \in \mathrm{cl}_{\sigma(\mathbb{C}(X,Y))} \{f_d : d \in D\}$. For each $\mathscr{U} \in \sigma(\mathbb{C}(X, Y))$ with $g \in \mathscr{U}$, we have $\bigvee_{\mathbb{C}(X,Y)} \{f_d : d \in D\} \in \mathscr{U}$; whence $f_d \in \mathscr{U}$ for some $d \in D$. So $g \in \mathrm{cl}_{\sigma(\mathbb{C}(X,Y))} \{f_d : d \in D\}$. Thus, the Scott topology on $\mathbb{C}(X, Y)$ has property S with respect to $\mathscr{D}$. □

For T_0 spaces X and Y, the Scott topology on $\mathbb{C}(X, Y)$ does not have property S with respect to $\mathscr{R}$ in general. Indeed, for a T_0 space X, it is easy to see the Scott topology agrees with the Isbell topology on $\mathbb{C}(X, \Sigma 2)$, namely $[X \to \Sigma 2]_\Sigma = [X \to \Sigma 2]_I$. Hence by Proposition 39, we have the following result.

https://doi.org/10.1017/S0960129523000014 Published online by Cambridge University Press

Proposition 63. *Let X be any T_0 space for which $\Sigma\mathcal{O}(X)$ is non-sober. Then,*

(1) $[X \to \Sigma 2]_\Sigma$ is not sober.
(2) the Scott topology on $\mathbb{C}(X, \Sigma 2)$ does not have property S with respect to $\mathscr{R}$.

We can give a more direct proof of Proposition 63 (not using Proposition 39). Clearly, $f \mapsto f^{-1}(1) : [X \to \Sigma 2]_\Sigma \to \Sigma\mathcal{O}(X)$ is a homeomorphism. As $\Sigma\mathcal{O}(X)$ is non-sober, $[X \to \Sigma 2]_\Sigma$ is not sober. If the Scott topology on $\mathbb{C}(X, \Sigma 2)$ has property S with respect to $\mathscr{R}$, then by Lemma 15, Theorem 36 and Corollary 41, $[X \to \Sigma 2]_\Sigma$ is sober, a contradiction. So the Scott topology on $\mathbb{C}(X, \Sigma 2)$ does not have property S with respect to $\mathscr{R}$.

Question 64. For T_0 spaces X and Y, whether the Scott topology on $\mathbb{C}(X, Y)$ has property S with respect to WD or RD? Especially, for a T_0 space X and a well-filtered space Y, whether the Scott topology on $\mathbb{C}(X, Y)$ has property S with respect to WD or RD?

By Corollary 24, we directly get the following result.

Proposition 65. *For a T_0 space X and a d-space Y, the function space $[X \to Y]_\Sigma$ is a d-space.*

Proposition 63 shows that for sober spaces, the similar result to Proposition 65 does not hold in general. The following is another counterexample.

Example 66. Let $X = 1$ be the topological space with single point and L the complete lattice constructed by Isbell in Isbell (1982). It is well-known that ΣL is non-sober. Let $Y = (L, \upsilon(L))$. Then by Zhao et al. (2015, Corollary 4.10) or Xu et al. (2020, Proposition 2.9), Y is sober. Clearly, $\mathbb{C}(X, Y) = \{c_y : y \in Y\}$ and $c_y \mapsto y : [X \to Y]_\Sigma \to \Sigma\Omega Y = \Sigma L$ is a homeomorphism. So Y is sober but the function space $[X \to Y]_\Sigma$ is non-sober.

It is still not known whether a similar result to Proposition 65 holds for well-filtered spaces. That is, we have the following question.

Question 67. For a T_0 space X and a well-filtered space Y, is the function space $[X \to Y]_\Sigma$ well-filtered? Or equivalently, whether the Scott topology on $\mathbb{C}(X, Y)$ has property S with respect to WD or RD?

Let $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\omega\})$ with ordering defined by $(j, k) \leq (m, n)$ iff $j = m$ and $k \leq n$, or $n = \omega$ and $k \leq m$. $\mathbb{J}$ is a well-known dcpo constructed by Johnstone in (1981) which is not sober in its Scott topology.

Proposition 68. *There is no well-filtered topology on $\mathbb{J}$ which has the given order as its specialization order, namely, for any topology $\upsilon(\mathbb{J}) \subseteq \tau \subseteq \sigma(\mathbb{J})$, $(\mathbb{J}, \tau)$ is not well-filtered.*

Proof. The set $\mathbb{J}_{max} = \{(n, \infty) : n \in \mathbb{N}\}$ is the set of all maximal elements of $\mathbb{J}$ and $\mathsf{K}(\Sigma\mathbb{J}) = (2^{\mathbb{J}_{max}} \setminus \{\emptyset\}) \bigcup \mathbf{Fin}\, \mathbb{J}$ (see Lu et al. 2017, Example 3.1). Since $\upsilon(\mathbb{J}) \subseteq \tau \subseteq \sigma(\mathbb{J})$, we have that $2^{\mathbb{J}_{max}} \setminus \{\emptyset\} \subseteq \mathsf{K}((\mathbb{J}, \tau))$. Let $\mathscr{K} = \{\mathbb{J}_{max} \setminus F : F \in (\mathbb{J}_{max})^{(<\omega)}\}$. Then, $\mathscr{K} \subseteq \mathsf{K}((\mathbb{J}, \tau))$ is a filtered family and $\bigcap \mathscr{K} = \bigcap_{F \in (\mathbb{J}_{max})^{(<\omega)}} (\mathbb{J}_{max} \setminus F) = \mathbb{J}_{max} \setminus \bigcup (\mathbb{J}_{max})^{(<\omega)}) = \emptyset$, but $\mathbb{J}_{max} \setminus F \neq \emptyset$ for all $F \in (\mathbb{J}_{max})^{(<\omega)}$. Therefore, $(\mathbb{J}, \tau)$ is not well-filtered. □

Corollary 69. *(Gierz et al. 2003, Exercise II-3.16 (V)) There is no sober topology on $\mathbb{J}$ which has the given order as its specialization order.*

Question 70. Is there a well-filtered space Y such that ΣY (i.e., $\Sigma\Omega Y$) is not well-filtered? Or equivalently, is there a dcpo P and a topology $\upsilon(P) \subseteq \tau \subseteq \sigma(P)$ such that (P, τ) is well-filtered but $(P, \sigma(P))$ is not well-filtered? In particular, is there a dcpo P such that $(P, \upsilon(P))$ is well-filtered but $(P, \sigma(P))$ is not well-filtered?

If the answer of Question 70 is "Yes," then the answer of Question 67 is "No"! Indeed, suppose that Y is a well-filtered space for which the Scott space ΣY is not well-filtered and $X = 1$ (the topological space with single point). Then, the function space $\mathbb{C}(X, Y)$ equipped with the Scott topology is not well-filtered since $[X \to Y]_\Sigma$ and ΣY are homeomorphic (see Example 66).

Conversely, for $\mathscr{D} \leq \mathrm{H} \leq \mathscr{R}$ (in particular, $\mathrm{H} = \mathscr{D}, \mathscr{R}$, WD or RD), the following example shows that the H-sobriety of $[X \to Y]_\Sigma$ does not imply the H-sobriety of Y in general.

Example 71. Let X be the topological space with single point and $[0, 1]$ the unit closed interval with the usual order of reals. Then, $\sigma([0, 1]) \neq \gamma([0, 1])$ (note that $\{1\} \in \gamma([0, 1])$ but $\{1\} \notin \sigma([0, 1])$). Clearly, the Alexandroff space $\Gamma[0, 1]$ is not a d-space (since $\gamma([0, 1]) \nsubseteq \sigma([0, 1])$) and the Scott space $\Sigma[0, 1]$ is sober; whence $\Sigma[0, 1]$ is H-sober and $\Gamma[0, 1]$ is not H-sober by $\mathscr{D} \leq \mathrm{H} \leq \mathscr{R}$. As $[X \to \Gamma[0, 1]]_\Sigma$ is homeomorphic to $\Sigma\Omega(\Gamma[0, 1]) = \Sigma[0, 1]$ (see Example 66), the function space $[X \to \Gamma[0, 1]]_\Sigma$ is H-sober but $\Gamma[0, 1]$ is not H-sober.

Finally, by Lemma 12, Corollaries 29 and 34, we get the following two results.

Proposition 72. *Let* $\mathrm{H} : \mathbf{Top}_0 \to \mathbf{Set}$ *be an R-subset system, X a T_0 space and Y a d-space. If $[X \to Y]_\Sigma$ is H-sober, then ΣY is H-sober. In particular, for a dcpo P, if $[X \to \Sigma P]_\Sigma$ is H-sober, then ΣP is H-sober.*

Proposition 73. *Let* $\mathrm{H} : \mathbf{Top}_0 \to \mathbf{Set}$ *be an R-subset system and X a T_0 space and Y a pointed d-space. If $[X \to Y]_\Sigma$ is H-sober, then $\Sigma\mathscr{O}(X)$ is H-sober.*

Corollary 74. *Let X be a T_0 space for which $\Sigma\mathscr{O}(X)$ is non-sober. Then for any pointed d-space Y, $[X \to Y]_\Sigma$ is not sober.*

Acknowledgements. The authors would like to thank the referee for the numerous and very helpful suggestions that have improved this paper substantially.

References

Engelking, R. (1989). *General Topology*, Warzawa, Polish Scientific Publishers.

Ershov, Y. and Schwidefsky, M. (2020). On function spaces. *Siberian Electronic Mathematical Reports* **17** 999–1008.

Gierz, G., Hofmann, K., Keimel, K., Lawson, J., Mislove, M. and Scott, D. (2003). *Continuous Lattices and Domains*, Cambridge, Cambridge University Press.

Goubault-Larrecq, J. (2003). *Non-Hausdorff Topology and Domain Theory*, New Mathematical Monographs, vol. 22, Cambridge, Cambridge University Press.

Isbell, J. (1982). Completion of a construction of Johnstone. *Proceedings of the American Mathematical Society* **85** 333–334.

Johnstone, P. (1981). Scott is not always sober. In *Continuous Lattices*, Lecture Notes in Mathematics, Berlin, Springer-Verlag, 282–283.

Jung, A. (1989). *Cartesian Closed Categories of Domains*, CWI Tract, vol. 66, Amsterdam.

Jung, A. (1990). The classification of continuous domains. In: *Proceedings, Fifth Annual IEEE Symposium on Logic in Computer Science*, IEEE Computer Society Press, 35–40.

Keimel, K. and Lawson, J. (2009). *D*-completion and *d*-topology. *Annals of Pure and Applied Logic* **159** 292–306.

Liu, B., Li, Q. and Ho, W. (2021). On function spaces related to *d*-spaces. *Topology and Its Applications* **300** 107757.

Lu, C. and Li, Q. (2017). Weak well-filtered spaces and coherence. *Topology and Its Applications* **230** 373–380.

Plotkin, G. (1976). A powerdomain construction. *SIAM Journal on Computing* **5** 452–487.

Scott, D. (1970). Outline of a mathematical theory of computation. In: *Proceedings of the 4th Annual Princeton Conference on Information Sciences and Systems*, Princeton University Press, 169–176.

Shen, C., Xi, X., Xu, X. and Zhao, D. (2019). On well-filtered reflections of T_0 spaces. *Topology and Its Applications* **267** 106869.

Smyth, M. (1983). The largest cartesian closed category of domains. *Theoretical Computer Science* **27** 109–119.

Xi, X. and Lawson, J. (2017). On well-filtered spaces and ordered sets. *Topology and Its Applications* **228** 139–144.

Xu, X. (2021). On H-sober spaces and H-sobrifications of T_0 spaces. *Topology and Its Applications* **289** 107548.

Xu, X., Shen, C., Xi, X. and Zhao, D. (2020). On T_0 spaces determined by well-filtered spaces. *Topology and Its Applications* **282** 107323.

Xu, X., Xi, X. and Zhao, D. (2021). A complete Heyting algebra whose Scott topology is not sober. *Fundamenta Mathematicae* **252** 315–323.

Zhao, D. and Ho, W. (2015). On topologies defined by irreducible sets. *Journal of Logical and Algebraic Methods in Programming* **84**(1) 185–195.

Cite this article: Xu X, Bao M and Zhang X (2022). On function spaces equipped with Isbell topology and Scott topology. *Mathematical Structures in Computer Science* **32**, 1099–1116. https://doi.org/10.1017/S0960129523000014