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A STABLE TRACE FORMULA FOR IGUSA VARIETIES

SUG WOO SHIN

Department of Mathematics, University of Chicago, 5734 South University Avenue, Chicago, IL 60637, USA (swshin@uchicago.edu)

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Abstract Igusa varieties are smooth varieties in positive characteristic p which are closely related to Shimura varieties and Rapoport–Zink spaces. One motivation for studying Igusa varieties is to analyse the representations in the cohomology of Shimura varieties which may be ramified at p. The main purpose of this work is to stabilize the trace formula for the cohomology of Igusa varieties arising from a PEL datum of type (A) or (C). Our proof is unconditional thanks to the recent proof of the fundamental lemma by Ngô, Waldspurger and many others.

An earlier work of Kottwitz, which inspired our work and proves the stable trace formula for the special fibres of PEL Shimura varieties with good reduction, provides an explicit way to stabilize terms at ∞. Stabilization away from p and ∞ is carried out by the usual Langlands–Shelstad transfer as in work of Kottwitz. The key point of our work is to develop an explicit method to handle the orbital integrals at p. Our approach has the technical advantage that we do not need to deal with twisted orbital integrals or the twisted fundamental lemma.

One application of our formula, among others, is the computation of the arithmetic cohomology of some compact PEL-type Shimura varieties of type (A) with non-trivial endoscopy. This is worked out in a preprint of the author's entitled 'Galois representations arising from some compact Shimura varieties'.

Keywords: trace formula; endoscopy; orbital integrals; Shimura varieties; Igusa varieties

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1. Introduction

The l-adic étale cohomology and Hasse–Weil zeta-functions of Shimura varieties have been computed in several cases using the strategy developed by Ihara, Kottwitz, Langlands and others. At the core of the method lies the comparison of the Arthur–Selberg formula (or L^2-Lefschetz formula by Arthur) and the Lefschetz fixed-point formula for the special fibres of Shimura varieties at primes of good reduction ('unramified' primes). In order to compute the cohomology of Shimura varieties at ramified primes, Harris and Taylor introduced a new method making use of the interplay among Shimura varieties, Rapoport–Zink spaces and Igusa varieties. There are two main parts for this method. On one hand, one establishes a formula relating the cohomology spaces of the three geometric objects (see [**11**, Theorem VI.2.9], which is generalized in [**29**, Theorem 22]). On the other hand, one obtains a trace formula for the cohomology of Igusa varieties via counting points (see [**11**, Proposition V.4.8], which is generalized in [**36**]) and compares it to the L^2-Lefschetz formula for Shimura varieties [**1**, Theorem 6.1]. This comparison of the two trace formulae usually requires stabilization. On the geometric side of the trace formula, very roughly speaking, this amounts to rewriting a sum of orbital integrals over the set of conjugacy classes as a sum of stable orbital integrals over the set of stable conjugacy classes on endoscopic groups. In fact, the issue of stabilization was bypassed in the work of Harris and Taylor as they only work with some simple kinds of unitary similitude groups for which endoscopy disappears. However more interesting applications are expected to result from the general case where stabilization is necessary.

The aim of our work is to carry out the stabilization of the trace formula for the cohomology of Igusa varieties, with [**36**, Theorem 13.1] as a starting point. We will use the standard form of the fundamental lemma and the transfer conjecture (Conjectures 2.13 and 2.14), which were recently proved by Ngô and Waldspurger, based on previous work of many others. (See Proposition 2.17 and the subsequent explanation.) Note that there have been results on the stabilization of various trace formulae which are related or analogous to ours. The elliptic part of the Arthur–Selberg trace formula was stabilized by Langlands [**26**] and Kottwitz [**17**]. The characteristic 0 Lefschetz formula for Shimura varieties (as in [**1**] or [**9**]) was stabilized by Kottwitz [**24**] in an unpublished manuscript. The point-counting formula for PEL-type Shimura varieties of type (A) or (C) was stabilized by [**21**], [**31**] and [**30**].* It is worth noting that [**21**], [**31**] and [**30**] use a form of the twisted fundamental lemma while our work does not.

Let us summarize our results more precisely. Let G be the reductive group over $\mathbb{Q}$ attached to a PEL Shimura variety Sh of type (A) or (C), which is a projective system of quasi-projective varieties over a number field. The Newton strata of the special fibre of Sh at a place above p are parametrized by group-theoretic data $b \in B(G_{\mathbb{Q}_p})$, where each b prescribes an isogeny class of Barsotti–Tate groups over $\bar{\mathbb{F}}_p$ with additional structure. Choose Σ_b in that isogeny class. One can define the Igusa variety Ig_{Σ_b}, which is a projective system of smooth varieties over $\bar{\mathbb{F}}_p$ related to the Newton stratum of Sh corresponding to b. From an irreducible finite-dimensional representation ξ of G, we construct an l-adic local system $\mathscr{L}_\xi$ on Ig_{Σ_b} and Sh where $l \neq p$. (We use the same notation $\mathscr{L}_\xi$ for Ig_{Σ_b} and Sh by abuse of language.) We will consider the l-adic cohomology space $H_c(\mathrm{Ig}_{\Sigma_b}, \mathscr{L}_\xi)$ (alternating sum over all cohomological degrees), which is a virtual representation of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$. Here J_b is a certain inner form of a Levi subgroup (of a parabolic subgroup) of $G_{\mathbb{Q}_p}$. (The group action at p is different for the cohomology of Sh. The latter has an action of $G(\mathbb{A}^\infty)$.) When $\phi \in C_c^\infty(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ is acceptable [**36**, Definition 6.2], we have the following formula [**36**, Theorem 13.1] (details are given in § 4.4):

$$\mathrm{tr}(\phi | H_c(\mathrm{Ig}_{\Sigma_b}, \mathscr{L}_\xi)) = \sum_{(\gamma_0;\gamma,\delta)\in \mathrm{KT}_b^{\mathrm{eff}}} \mathrm{vol}(I_\infty(\mathbb{R})^1)^{-1} |A(I_0)| \cdot \mathrm{tr}\,\xi(\gamma_0) \cdot \mathrm{O}_{(\gamma,\delta)}^{G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)}(\phi). \tag{1.1}$$

* Kottwitz stabilized the formula for compactly supported cohomology. This result was extended by Morel to the case of intersection cohomology.

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The upshot of the present paper is Theorem 1.1 (see also Theorem 7.2 later in the paper), which stabilizes (1.1). This is an analogue of Kottwitz's stable trace formula [**19**, Theorem 7.2].*

Theorem 1.1. *Let $\phi \in C_c^\infty(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ be any acceptable function. For each elliptic endoscopic triple (H, s, η) for G, let h^H be the function on $H(\mathbb{A})$ constructed from ϕ as in § 7. Let ST_e^H denote the stable distribution on $H(\mathbb{A})$ given by the sum of stable orbital integrals on elliptic semisimple elements. (See (7.3) for the precise definition.) Then*

$$\mathrm{tr}(\phi | H_c(\mathrm{Ig}_{\Sigma_b}, \mathscr{L}_\xi)) = |\ker^1(\mathbb{Q}, G)| \sum_{(H,s,\eta)} \iota(G, H) \mathrm{ST}_e^H(h^H).$$

Since the orbital integral in (1.1) is essentially a product of local orbital integrals, we can basically stabilize (1.1) at each place. The stabilization away from p is done exactly as in [**19**, § 7]. In fact, it is the usual Langlands–Shelstad transfer of orbital integrals away from p and ∞. At the infinite place of $\mathbb{Q}$, we obtain an *explicit* transfer in exactly the same way as in work of Kottwitz, where the essential inputs are Shelstad's theory of real endoscopy and Clozel–Delorme's result on the existence of pseudo-coefficients for discrete series. (Although the existence of transfer at ∞ can be deduced without exhibiting functions, the mere existence is not enough for applications to the computation of cohomology, at least *a priori*.) So the main issue for us, which takes up § 6, is how to stabilize the term at p in a sensible way. The stabilization at p, which includes an *explicit* process of constructing h_p^H, is considered important because the explicit information about h_p^H would eventually go into the computation of the Galois and automorphic representations (even if they are ramified at p) in the cohomology of Shimura varieties at p. There are two problems for stabilization at p. First, we are not in the usual formalism of the trace formula since the orbital integral at p is computed on a different group, namely on $J_b(\mathbb{Q}_p)$. We resolve this issue by relating the endoscopy of J_b to the endoscopy of G in a systematic way. Second, we need to relate the Kottwitz invariant (§ 4.2) at p to the transfer factors. This is precisely the content of Lemma 6.3, which plays a key role. (An analogous result in the context of Kottwitz's formula is proved by Kottwitz in [**31**, Appendix].)

Our motivation for this work stemmed from two kinds of expected application of Theorem 1.1. (A fair part of that expectation has been realized.) As the first application, given certain PEL-type Shimura varieties arising from unitary groups with *non-trivial endoscopy*, we may compute their l-adic cohomology at *ramified* primes as long as we have some prior knowledge of the cohomology of Rapoport–Zink spaces involved in the computation. Indeed, we studied in [**37**] the cohomology of compact $U(1, n-1)$-type Shimura varieties (which are more general than the ones in [**11**] which have trivial endoscopy) in detail and obtained applications to Galois representations. We would like to make two technical remarks regarding the last result. First, in the special case of $U(1, n-1)$-type, it is actually not necessary to assume that the PEL datum is unramified (§ 4.1) (as we still have nice integral models for Shimura varieties; they also lead to a good notion of

* It is a mere coincidence of numbering that our Theorem 7.2 is an analogue of Theorem 7.2 of [**19**].

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Igusa varieties). Although we wrote this paper only for an unramified PEL datum, the argument and construction carry over to the $U(1, n-1)$-case without the unramified assumption. Next, it is worth noting that a large part of the cohomology of (compact or non-compact) Shimura varieties arising from unitary groups with *arbitrary* signature could also be computed at *ramified* places even if endoscopy is non-trivial, by arguing as in § A.7.3 of [**7**] (even though the latter only deals with the case of trivial endoscopy). The basic strategy is to combine what we know in the case of $U(1, n-1)$ (e.g. [**37**, § 6.2]) with the information about the cohomology of those Shimura varieties at unramified places (e.g. [**31**, § 8.4], which extends the results of [**20**] to the setting of non-compact Shimura varieties with non-trivial endoscopy), and apply the Cebotarev density theorem to obtain the desired information at ramified places.*

The second application of our results is expected in some cases where we have prior knowledge of the cohomology of Shimura varieties. We may compute the cohomology of Rapoport–Zink spaces as an application of Theorem 1.1, by proving a generalization of [**11**, Theorem V.5.4] and using a result of Mantovan [**29**, Theorem 23]. This way we can recover the main results of Fargues [**7**, Chapter 8] and prove some new facts. The second application will appear in our forthcoming work.

Finally, let us sketch the structure of the article. Sections 1–4 are devoted to known facts and background materials from various sources. The reader may try to digest the statement of Theorem 4.4 and then read from § 5, where the stabilization of formula (4.3) in Theorem 4.4 begins. The first four sections may be used as reference along the way. Section 5 is the easier part of stabilization, where local expressions away from p are treated. Here we have not needed any new ideas or insights. The heart of the paper is § 6 and concerned with stabilization at p. After preparatory §§ 6.1 and 6.2, we construct the functions h_p^H whose stable orbital integrals have the desired values. It is fundamentally used in our construction that the Kottwitz invariant at p (denoted by $\tilde{\alpha}_p$) interacts nicely with transfer factors. This relationship is formulated in Lemma 6.3. The proof of Lemma 6.3 is the most technical result of our paper and takes up § 6.4. Section 7 puts together the main results of §§ 5 and 6, culminating in Theorem 7.2 with the fully stabilized formula. Our paper could end here, but we included § 8 for two purposes. By explicitly computing some terms in Lemma 6.3 in simple cases, we wish to help the reader understand the nature of Lemma 6.3. More importantly, the computation of c_{M_H} is a necessary input in the aforementioned application of our main result to the cohomology of Shimura varieties.

1.1. Notation and convention

We will work with various sets of isomorphism classes. By abuse of terminology, we often choose a representative in each isomorphism class and identify the set of isomorphism classes with the set of representatives. When a specific representative is chosen from an isomorphism class, we explain the choice.

* Morel suggested that we include this remark on Shimura varieties attached to unitary groups with arbitrary signature.

Let F be a field of characteristic 0. Let Γ denote $\mathrm{Gal}(\bar{F}/F)$ in §§ 2.1–2.4. Starting from § 3, $\Gamma := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $\Gamma(v) := \mathrm{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v)$ for each place v of $\mathbb{Q}$. For any linear algebraic group G over F, denote by G^0 the neutral component of G.

Let D be a diagonalizable group over F. Let $X^*(D)$ (respectively $X_*(D)$) denote the $\mathbb{Z}$-module $\mathrm{Hom}_F(D, \mathbb{G}_m)$ (respectively $\mathrm{Hom}_F(\mathbb{G}_m, D)$) with $\mathrm{Gal}(\bar{F}/F)$-action. We often write $X^*(D)_{\mathbb{Q}}$ for $X^*(D) \otimes_{\mathbb{Z}} \mathbb{Q}$. The same applies to $X_*(D)_{\mathbb{Q}}$. For a finite abelian group A, let A^D denote the group $\mathrm{Hom}(A, \mathbb{C}^\times)$.

Let G and G' be connected reductive groups over F. For each $g \in G(F)$, use $\mathrm{Int}(g) : G \to G$ to denote the inner automorphism defined by $x \mapsto gxg^{-1}$. Let $\mathrm{Int}_K(G)$ be the group of all inner automorphisms of G defined over a field K (containing F). We say that G and G' are F-*inner forms* with respect to an $\bar{F}$-isomorphism $\psi : G \xrightarrow{\sim} G'$ if $\psi^{-1} \circ \psi^\sigma$ lies in $\mathrm{Int}_{\bar{F}}(G)$ for every $\sigma \in \mathrm{Gal}(\bar{F}/F)$. This notion only depends on the $G(\bar{F})$-conjugacy orbit of ψ. We often omit the reference to ψ when there is no danger of confusion.

Let G be a connected reductive group over F. Let G^{der} denote the derived subgroup of G, and G^* a quasi-split F-inner form of G. Write $Z(G)$ for the centre of G and A_G for the maximal F-split torus in $Z(G)$. If T is a maximal torus of $G \times_F \bar{F}$, we write $R(G,T)$ (respectively $R^\vee(G,T)$) for the set of roots (respectively coroots) of T in G and $\Omega(G,T) := N_G(T)/T$ for the Weyl group. Let $Z_G(\gamma)$ denote the centralizer of $\gamma \in G(F)$ in G. If γ is semisimple and G^{der} is simply connected, then $Z_G(\gamma)$ is connected (see [**12**, § 3]). A semisimple $\gamma \in G(F)$ is called F-*elliptic* if $Z(Z_G(\gamma))^0/Z(G)^0$ is anisotropic over F. An F-*elliptic* torus T in G is one such that $T/Z(G)^0$ is anisotropic over F.

Set $H^1(F,G) := H^1(\mathrm{Gal}(\bar{F}/F), G(\bar{F}))$. When F is a number field, write $\ker^1(F,G)$ for the kernel of $H^1(F,G) \to \prod_v H^1(F_v,G)$ where v runs over all places of F. Similarly define $\ker^1(F,H)$ for any complex Lie group H equipped with an action of $\mathrm{Gal}(\bar{F}/F)$ factoring through a finite quotient.

Suppose that $\gamma, \gamma' \in G(F)$. We say that γ and γ' are *conjugate* in $G(F)$ and write $\gamma \sim \gamma'$ if there exists $g \in G(F)$ such that $\gamma' = g\gamma g^{-1}$. When γ and γ' are conjugate in $G(\bar{F})$ so that $\gamma' = g\gamma g^{-1}$ for some $g \in G(\bar{F})$, the association $\sigma \mapsto g^{-1}g^\sigma$ defines an element of $\ker(H^1(F,I) \to H^1(F,G))$ where $I := Z_G(\gamma)$. If this cohomology class is in the image of $\ker(H^1(F,I^0) \to H^1(F,G))$, we say that γ and γ' are *stably conjugate*, and write $\gamma \sim_{\mathrm{st}} \gamma'$. If I is connected then $G(\bar{F})$-conjugacy and stable conjugacy coincide for γ by definition.

When we say that a field F of characteristic 0 is *global* (respectively *local*), it means that F is a finite extension of $\mathbb{Q}$ (respectively $\mathbb{Q}_v$ for some place v of $\mathbb{Q}$). Suppose that F is global or local. Then the Weil group W_F of F is defined [**38**]. To discuss the L-group ${}^LG$ of a connected reductive F-group G, we fix a $\mathrm{Gal}(\bar{F}/F)$-invariant splitting data $(\mathbb{B}, \mathbb{T}, \{X_\alpha\}_{\alpha\in\Delta})$ once and for all where Δ is the set of $\mathbb{B}$-positive roots for $\mathbb{T}$ in $\hat{G}$. The L-group is defined as a semi-direct product ${}^LG := \hat{G} \rtimes W_F$, where W_F acts on $\hat{G}$ via

$$W_F \to \mathrm{Out}(G) \xrightarrow{\sim} \mathrm{Aut}(\hat{G}, \mathbb{B}, \mathbb{T}, \{X_\alpha\}_{\alpha\in\Delta}).$$

Often a Levi subgroup of a parabolic subgroup of G (respectively ${}^LG$) will be called a Levi subgroup of G (respectively ${}^LG$) by abuse of terminology. (See [**4**, §§ 2–3] for details on L-groups and their Levi subgroups.)

Let F be a global field. When S is a finite set of places of F, we denote by $\mathbb{A}_F^S$ the restricted product of F_v for all $v \notin S$.

Finally, if G is a $\mathbb{Q}$-group and F is any field containing $\mathbb{Q}$, we write G_F for $G \times_{\mathbb{Q}} F$.

1.2. Harmonic analysis on reductive groups

We introduce further notation and convention for harmonic analysis on reductive groups. Let G be a connected reductive group over a field F of characteristic 0.

Suppose that F is local. The Kottwitz sign $e(G) \in \{\pm 1\}$ is defined in [**13**]. If G is quasi-split over F then $e(G) = 1$. If $F = \mathbb{Q}_p$ (respectively $F = \mathbb{R}$) then we often write $e_p(G)$ (respectively $e_\infty(G)$) for this sign. Let $C^\infty(G(F))$ (respectively $C_c^\infty(G(F))$) denote the space of smooth (respectively smooth and compactly supported) functions on $G(F)$ with values in an algebraically closed field Ω. When F is $\mathbb{R}$ or $\mathbb{C}$, take $\Omega = \mathbb{C}$ so that smoothness makes sense. When F is non-archimedean, smooth means locally constant and Ω may be $\mathbb{C}$ or $\bar{\mathbb{Q}}_l$ for some prime l. We will always take $\Omega = \mathbb{C}$ from § 5 until the end. When $F = \mathbb{R}$, let $\chi : A_G(\mathbb{R})^0 \to \mathbb{C}^\times$ be a continuous homomorphism and fix a maximal compact subgroup K_∞ of $G(\mathbb{R})$. Define $C_c^\infty(G(\mathbb{R}), \chi)$ to be the space of smooth functions $G(\mathbb{R}) \to \mathbb{C}$ which are bi-K_∞-finite, compactly supported modulo $A_G(\mathbb{R})^0$, and transform under $A_G(\mathbb{R})^0$ by χ.

Keep assuming that F is local. Let G_1 and G_2 be connected reductive F-groups which are inner forms. Once a Haar measure μ_1 on $G_1(F)$ is chosen, there is a unique Haar measure μ_2 on $G_2(F)$ such that μ_1 and μ_2 are *compatible* in the sense of [**18**, p. 631].

Let us define orbital integrals and stable orbital integrals. Let $\gamma \in G(F)$ be a semisimple element and fix Haar measures on $G(F)$ and $Z_G(\gamma)^0(F)$. For $\phi \in C_c^\infty(G(F))$, define

$$\mathrm{O}_\gamma^{G(F)}(\phi) := \int_{Z_G(\gamma)^0(F)\backslash G(F)} f(x^{-1}\gamma x)\,\mathrm{d}x,$$

where the quotient measure is used for integration. The stable orbital integral is defined as [**18**, p. 638]

$$\mathrm{SO}_\gamma^{G(F)}(\phi) := \sum_{\gamma' \sim_{\mathrm{st}} \gamma} e(Z_G(\gamma')^0) \cdot a(\gamma') \cdot \mathrm{O}_{\gamma'}^{G(F)}(\phi),$$

where γ' runs over a set of representatives for $G(F)$-conjugacy classes in the stable conjugacy class of γ. The number $a(\gamma')$ is defined as the cardinality of the kernel of $H^1(F, Z_G(\gamma')^0) \to H^1(F, Z_G(\gamma'))$. We remark on the choice of Haar measures for $\mathrm{O}_{\gamma'}^{G(F)}(\phi)$ in the definition. If $\gamma' \sim_{\mathrm{st}} \gamma$ then $Z_G(\gamma')^0$ and $Z_G(\gamma)^0$ are F-inner forms. The measure on $Z_G(\gamma')^0$ is chosen to be compatible with that on $Z_G(\gamma)^0$.

Assume that F is a local non-archimedean field. Let $\mathrm{Irr}(G(F))$ denote the set of isomorphism classes of irreducible admissible representations of $G(F)$. The Grothendieck group of admissible representations of $G(F)$ is written as $\mathrm{Groth}(G(F))$. (See [**11**, p. 23] for a precise definition, which also works for representations of any topological group.) Let M be a Levi subgroup of a parabolic subgroup P of G. Write N for the unipotent radical of P. Define a function D_M^G on $M(F)$ and a character $\delta_P : M(F) \to \mathbb{C}^\times$ by

$$D_M^G(m) = \det(1 - \mathrm{ad}(m))|_{\mathrm{Lie}(G)/\mathrm{Lie}(M)}, \qquad \delta_P(m) = |\det(\mathrm{ad}(m))|_{\mathrm{Lie}(P)/\mathrm{Lie}(M)}|_F,$$

where $|\cdot|_F : F^\times \to \mathbb{R}^\times_{>0}$ is the valuation map normalized such that the inverse of the uniformizer maps to the cardinality of the residue field of F. Let π be an admissible representation of $G(F)$ on a Ω-vector space V. Denote by $\mathrm{Jac}^G_P(\pi)$ the admissible representation of $M(F)$ on the quotient of V by the subspace generated by $nv-v$ for $n \in N(F)$ and $v \in V$. Put $J^G_P(\pi) := \mathrm{Jac}^G_P(\pi) \otimes \delta_P^{-1/2}$. Both $\mathrm{Jac}^G_P(\pi)$ and $J^G_P(\pi)$ induce maps from $\mathrm{Groth}(G(F))$ to $\mathrm{Groth}(M(F))$. If π is an admissible representation of $G(F)$ on a vector space V, each $\phi \in C^\infty_c(G(F))$ defines a finite-rank operator $\pi(\phi) := \int_{G(F)} \phi(g)\pi(g)\,\mathrm{d}g$ on V. Thereby $\mathrm{tr}\,\pi(\phi)$, or $\mathrm{tr}(\phi|\pi)$, is defined. This definition extends to $\pi \in \mathrm{Groth}(G(F))$.

Let G be a connected reductive group over $\mathbb{Q}$ and S be a finite set of places of $\mathbb{Q}$. Choose a hyperspecial subgroup K^{hs}_v of $G(\mathbb{Q}_v)$ at every finite place v where $G_{\mathbb{Q}_v}$ is unramified (possibly with finitely many exceptions of v), and define the spaces $C^\infty(G(\mathbb{A}^S))$ and $C^\infty_c(G(\mathbb{A}^S))$ via restricted product over all $v \notin S$ [**8**, § 3]. Let $\mathrm{Groth}(G(\mathbb{A}^S))$ denote the Grothendieck group of admissible representations of $G(\mathbb{A}^S)$ (where we assume $S \supset \{\infty\}$). The definition of orbital integrals, stable orbital integrals and the trace distributions extends to the adelic case in an obvious way. There is a canonical measure on $G(\mathbb{A})/A_G(\mathbb{R})^0$, called the *Tamagawa measure*. The volume of $G(\mathbb{A})/A_G(\mathbb{R})^0$ for this measure is finite and denoted by $\tau(G)$. It is known that [**18**, p. 629]

$$\tau(G) = |\pi_0(Z(\hat{G})^{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})})|/|\ker^1(\mathbb{Q}, G)|. \tag{1.2}$$

Let G be a real reductive group and T be an $\mathbb{R}$-elliptic torus in G. Define $d(G) := |\ker(H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G))|$. This value is finite and independent of the choice of T.

2. Endoscopic groups and the transfer conjecture

Throughout § 2, let G be a connected reductive group over a local or global field F of characteristic 0 and assume that G^{der} is simply connected. More conditions on G or F will be specified as needed. In § 2.1 and § 2.4 various sets such as $\mathcal{E}_F(G)$, $\mathcal{EQ}_F(G)$, $\mathcal{SS}_F(G)$, etc., are defined. In later sections we will write $\mathcal{E}(G)$ (respectively $\mathcal{E}_v(G)$) for $\mathcal{E}_F(G)$ if $F = \mathbb{Q}$ (respectively $F = \mathbb{Q}_v$) and do the same with $\mathcal{EQ}_F(G)$, $\mathcal{SS}_F(G)$, etc.

2.1. Endoscopic triples

We first give the definition of endoscopic triples. Recall that there is an action of $\Gamma := \mathrm{Gal}(\bar{F}/F)$ on $\hat{G}$ given by the choice of splitting data. The definition below is independent of this choice since the Γ-actions for any two splitting data differ by $\hat{G}$-conjugacy.

Definition 2.1 (Kottwitz [15, 7.4]). An *endoscopic triple* for G is a triple (H, s, η) satisfying the following three conditions where H is a quasi-split connected reductive group over F, s is an element of $Z(\hat{H})$, and $\eta : \hat{H} \to \hat{G}$ is an embedding of complex Lie groups.

(i) $\eta(\hat{H}) = Z_{\hat{G}}(\eta(s))^0$.

(ii) The $\hat{G}$-conjugacy class of η is fixed by Γ.

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(iii) The image of s in $Z(\hat{H})/Z(\hat{G})$ is Γ-invariant and its image under the connecting homomorphism $(Z(\hat{H})/Z(\hat{G}))^{\Gamma} \to H^1(F, Z(\hat{G}))$ arising from the Γ-equivariant short exact sequence

$$1 \to Z(\hat{G}) \to Z(\hat{H}) \to Z(\hat{H})/Z(\hat{G}) \to 1$$

is trivial if F is local and locally trivial (i.e. contained in $\ker^1(F, Z(\hat{G}))$) if F is global.

Remark 2.2. The condition (ii) in the above definition implies that the embedding of $Z(\hat{G})$ into $Z(\hat{H})$ via η is Γ-invariant. Thus the condition (iii) makes sense.

Remark 2.3. It is obvious that the endoscopic triples for G are the same as those for G' if G and G' are inner forms over F.

Definition 2.4. An endoscopic triple (H, s, η) for G is called *elliptic* if $(Z(\hat{H})^{\Gamma})^0 \subset Z(\hat{G})$, or equivalently if $Z(\hat{H})^{\Gamma} Z(\hat{G})/Z(\hat{G})$ is a finite group.

Definition 2.5 (Kottwitz [24, Definition 2.5]). An *isomorphism* between endoscopic triples (H, s, η) and (H', s', η') for G is an isomorphism $\alpha : H \xrightarrow{\sim} H'$ such that

(i) $\eta \circ \hat{\alpha}$ and η' are conjugate under an element of $\hat{G}$ (this makes sense as the $\hat{H}$-conjugacy orbit of $\hat{\alpha}$ is well-defined);

(ii) s and $\hat{\alpha}(s')$ are equal in $Z(\hat{H})/Z(\hat{G})$.

The group of automorphisms of (H, s, η) is denoted by $\mathrm{Aut}_F(H, s, \eta)$. Define

$$\mathrm{Out}_F(H, s, \eta) := \mathrm{Aut}_F(H, s, \eta)/\mathrm{Int}_F(H).$$

We write $\mathcal{E}_F(G)$ (respectively $\mathcal{E}_F^{\mathrm{ell}}(G)$) for the set of isomorphism classes of all (respectively elliptic) endoscopic triples for G.

Remark 2.6. The notion of isomorphism in Definition 2.5 is stronger than the one given in [**15**, § 7]. Consider $G = \mathrm{GL}_2$ and $H = \mathrm{GL}_1 \times \mathrm{GL}_1$. Let $s_{a,b} := (a, b) \in \hat{H}$, where $a, b \in \mathbb{C}^{\times}$, and η be such that (a, b) maps to the diagonal matrix with entries a and b. Then $(H, s_{a,b}, \eta)$ belongs to $\mathcal{E}_F(G)$ (but not to $\mathcal{E}_F^{\mathrm{ell}}(G)$). Any two $(H, s_{a,b}, \eta)$ and $(H, s_{c,d}, \eta)$ are isomorphic in the sense of [**15**, § 7], but they are isomorphic in our sense if and only if $a/b = c/d$ or $a/b = d/c$.

Let (H, s, η) be an endoscopic triple for G. Let $T_H \subset H$, $T \subset G$, $\mathbb{T}_H \subset \hat{H}$, $\mathbb{T} \subset \hat{G}$ be maximal tori over $\bar{F}$. Choose Borel subgroups $B_H \subset H$, $B \subset G$, $\mathbb{B}_H \subset \hat{H}$ and $\mathbb{B} \subset \hat{G}$ over $\bar{F}$ such that $T_H \subset B_H$, $T \subset B$, $\mathbb{T}_H \subset \mathbb{B}_H$ and $\mathbb{T} \subset \mathbb{B}$. These determine isomorphisms $\iota_H : \hat{T}_H \simeq \mathbb{T}_H$ and $\iota : \hat{T} \simeq \mathbb{T}$. There exists $\hat{g} \in \hat{G}$ such that $\theta := \mathrm{Int}(\hat{g}) \circ \eta$ sends $\mathbb{T}_H$ to $\mathbb{T}$ and $\mathbb{B}_H$ into $\mathbb{B}$. Thereby we obtain $\hat{T} \xrightarrow{\sim} \hat{T}_H$ given by $\iota_H^{-1} \circ \theta^{-1} \circ \iota$. Get an $\bar{F}$-isomorphism $j : T_H \xrightarrow{\sim} T$ by taking the dual. The $\Omega(G, T)$-orbit of j is independent of the choice of $\hat{g}$ and the Borel subgroups. For a fixed T_H, the $G(\bar{F})$-conjugacy class of embeddings $T_H \hookrightarrow G$ induced by j is independent of the choice of T, $\mathbb{T}_H$, $\mathbb{T}$, $\hat{g}$ and the Borel subgroups. Given an $\bar{F}$-maximal torus T_H of H, there exists a maximal torus defined over F in its $H(\bar{F})$-conjugacy class since H is quasi-split over F. Suppose that G is quasi-split and that T_H is defined over F. Then we may arrange that j is an F-morphism in the above process, replacing (B, T) by a $G(\bar{F})$-conjugate if necessary so that T is defined over F. (Use [**12**, Corollary 2.2] to find an F-embedding $j : T_H \hookrightarrow G$ in the canonical $G(\bar{F})$-conjugacy class and take for T the image of j.)

There is an embedding of $Z(G) \hookrightarrow Z(H)$ given by j^{-1} since $R(H, T_H) \subset R(G, T)$ via $X^*(T_H) \overset{j}{\simeq} X^*(T)$. The embedding $Z(G) \hookrightarrow Z(H)$ is canonical in the sense that it only depends on (H, s, η) and that it is compatible with isomorphisms of endoscopic triples. The embedding $Z(G) \hookrightarrow Z(H)$ is defined over F. Indeed, it is enough to prove this when G is quasi-split over F, and for such a group G we may take j to be defined over F as remarked earlier. Restricting $Z(G) \hookrightarrow Z(H)$ to maximal F-split subtori, we obtain a canonical embedding $A_G \hookrightarrow A_H$ over F. This embedding is an isomorphism if $(H, s, \eta) \in \mathcal{E}^{\mathrm{ell}}_F(G)$.

In practice (from § 5), we will fix a representative in each isomorphism class of endoscopic triples and identify the set of isomorphism classes of endoscopic triples with the set of representatives.

2.2. The groups $A(\cdot)$ and $\mathfrak{K}(I_0/\mathbb{Q})$

Define $A_F(G_0) := \pi_0(Z(\hat{G}_0)^{\Gamma})^D$ for any connected reductive F-group G_0. For the relationship between $A_F(G_0)$ and the Galois cohomology of G_0, see [**17**, §§ 1–2]. If F is local there is a canonical functorial map

$$H^1(F, G_0) \to A_F(G_0). \tag{2.1}$$

(This map is functorial with respect to any F-morphism by [**17**]; cf. the proof of Lemma 2.3 in [**36**].) The map in (2.1) is an isomorphism (of pointed sets) if F is non-archimedean and will form the left vertical arrow of (3.1).

Let F and G be as before. Let γ_0 be a semisimple element and set $I_0 := Z_G(\gamma_0)$. From the canonical Γ-equivariant inclusion $Z(\hat{G}) \hookrightarrow Z(\hat{I}_0)$, obtain an exact sequence

$$1 \to Z(\hat{G}) \to Z(\hat{I}_0) \to Z(\hat{I}_0)/Z(\hat{G}) \to 1$$

and consider the connecting homomorphism $(Z(\hat{I}_0)/Z(\hat{G}))^{\Gamma} \to H^1(F, Z(\hat{G}))$. Define $\mathfrak{K}(I_0/F) = \mathfrak{K}_G(I_0/F)$ to be the subgroup of $(Z(\hat{I}_0)/Z(\hat{G}))^{\Gamma}$ whose image in $H^1(F, Z(\hat{G}))$ is trivial (respectively locally trivial) if F is local (respectively global). Our definition (due to [**24**]) coincides with the one in [**17**, 4.6] when γ_0 is F-elliptic but differs from it in general.

Define $\tilde{\mathfrak{K}}(I_0/F)$ to be $\bigcap_v Z(\hat{I}_0)^{\Gamma(v)} Z(\hat{G})$ if F is global and $Z(\hat{I}_0)^{\Gamma} Z(\hat{G})$ if F is local. By unraveling the definition of $\mathfrak{K}(I_0/F)$ we see that canonically

$$\mathfrak{K}(I_0/F) = \tilde{\mathfrak{K}}(I_0/F)/Z(\hat{G}). \tag{2.2}$$

Now suppose that F is global and that γ_0 is F-elliptic. In particular $\mathfrak{K}(I_0/F)$ is a finite abelian group. Assume that the group homomorphism $\ker^1(F, Z(\hat{G})) \hookrightarrow \ker^1(F, Z(\hat{I}_0))$

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induced by the canonical Γ-equivariant map $Z(\hat{G}) \hookrightarrow Z(\hat{I}_0)$ is injective (cf. Lemma 4.1). Then by dualizing the exact sequence on [**17**, p. 395] we obtain an exact sequence

$$1 \to \mathfrak{K}(I_0/F)^D \to A_F(I_0) \to A_F(G) \to 1. \tag{2.3}$$

2.3. Transfer of conjugacy classes

Let F be local or global and consider $(H, s, \eta) \in \mathcal{E}_F(G)$. We explain how to transfer semisimple stable conjugacy classes from H to G. Let $\gamma_H \in H(\bar{F})$ be a semisimple element of H. Choose a maximal torus T_H of H over $\bar{F}$ containing γ_H. As explained in the paragraph below Remark 2.6, there is a canonical $G(\bar{F})$-conjugacy class of embeddings $j : T_H \to G$. Put $\gamma := j(\gamma_H)$ for one such embedding. The $G(\bar{F})$-conjugacy class of γ is independent of the choice of T_H and j. This $G(\bar{F})$-conjugacy class contains an element $\gamma_0 \in G(F)$ if G is quasi-split over F, but not in general. If such $\gamma_0 \in G(F)$ exists, we say that γ_H *transfers* to γ_0 in $G(F)$, or that γ_H and γ_0 have matching conjugacy classes. The association $\gamma_H \mapsto \gamma_0$ is a partially defined map from the set of semisimple stable conjugacy classes in $H(F)$ to the set of semisimple stable conjugacy classes in $G(F)$. This map is compatible with isomorphisms between endoscopic triples for G. In the above process, we may choose T_H, and also j if G is quasi-split over F, so that T_H and j are defined over F. (Use [**12**, Corollary 2.2].)

Let $T := j(T_H)$. We have an inclusion $R(H, T_H) \hookrightarrow R(G, T) \subset X^*(T)$ via j. The semisimple element γ_H is called (G, H)*-regular* if $\alpha(\gamma_H) \neq 1$ for every α in $R(G, T) \backslash R(H, T_H)$. This notion is independent of the choice of T_H and j.

Define $\mathcal{SS}_F(G)$ (respectively $\mathcal{SS}_F^{\mathrm{ell}}(G)$) to be the set of equivalence classes of (γ_0, κ) where $\gamma_0 \in G(F)$ is semisimple (respectively elliptic) and $\kappa \in \mathfrak{K}_G(I_0/F)$. Two pairs (γ_0, κ) and (γ_0', κ') are considered equivalent if $\gamma_0 \sim_{\mathrm{st}} \gamma_0'$ and $\kappa = \kappa'$ via the canonical isomorphism $Z(\hat{I}_0) \simeq Z(\hat{I}_0')$, where $I_0' := Z_G(\gamma_0')$. At this point, assume temporarily that G is quasi-split over F. Define $\mathcal{EQ}_F(G)$ to be the set of equivalence classes of (endoscopic) quadruples (H, s, η, γ_H) where (H, s, η) is an endoscopic triple for G and γ_H is a (G, H)-regular semisimple element of $H(F)$. As we are assuming that G^{der} is simply connected, we know that $I_H := Z_H(\gamma_H)$ is connected [**17**, Lemma 3.2]. The quadruples (H, s, η, γ_H) and $(H', s', \eta', \gamma_H')$ are *equivalent* if there exists an isomorphism $(H, s, \eta) \xrightarrow{\sim} (H', s', \eta')$ given by $\alpha : H \xrightarrow{\sim} H'$ such that $\alpha(\gamma_H)$ is conjugate to γ_H' in $H'(\bar{F})$ (equivalently, $\alpha(\gamma_H)$ and γ_H' are stably conjugate). Define $\mathcal{EQ}_F^{\mathrm{ell}}(G)$ to be the subset of $\mathcal{EQ}_F(G)$ characterized by the condition that $(H, s, \eta) \in \mathcal{E}_F^{\mathrm{ell}}(G)$. It is worth noting that I_0 and I_H are connected and inner forms of each other [**17**, § 3]. Observe that γ_H transfers to some $\gamma_0 \in G(F)$ since G is quasi-split over F. From $s \in Z(\hat{H})$ we construct $\kappa \in \mathfrak{K}(I_0/F)$ by taking the image of s under $Z(\hat{H}) \hookrightarrow Z(\hat{I}_H) \xrightarrow{\sim} Z(\hat{I}_0)$, which lies in $\tilde{\mathfrak{K}}(I_0/\mathbb{Q})$. (cf. Remark 2.7.) It is easy to check that equivalent endoscopic quadruples give rise to equivalent pairs (γ_0, κ). Thus we have defined a map $(H, s, \eta, \gamma_H) \mapsto (\gamma_0, \kappa)$ from $\mathcal{EQ}_F(G)$ to $\mathcal{SS}_F(G)$.

Now drop the assumption that G is quasi-split over F and let G^* be the quasi-split inner form of G. Define $\mathcal{EQ}_F(G)$ to be the subset of $\mathcal{EQ}_F(G^*)$ consisting of $(H, s, \eta, \gamma_H) \in \mathcal{EQ}_F(G)$ for which γ_H transfers to a stable conjugacy class in $G(F)$.

Remark 2.7. In the situation $(H,s,\eta,\gamma_H)\mapsto(\gamma_0,\kappa)$, we will always use the symbol $\tilde{\kappa}$ to denote the image of s under $Z(\hat{H})\hookrightarrow Z(\hat{I}_H)\xrightarrow{\sim}Z(\hat{I}_0)$. It follows that $\tilde{\kappa}\in\tilde{\mathfrak{K}}(I_0/\mathbb{Q})$. The image of $\tilde{\kappa}$ in $\mathfrak{K}(I_0/\mathbb{Q})$ is κ.

Lemma 2.8 (Kottwitz [24, Lemma 4.1]). *The above map defines a bijection from $\mathcal{EQ}_F(G)$ to $\mathcal{SS}_F(G)$ and restricts to a bijection from $\mathcal{EQ}_F^{\mathrm{ell}}(G)$ to $\mathcal{SS}_F^{\mathrm{ell}}(G)$. An automorphism $\alpha: H\xrightarrow{\sim}H$ inducing a self-equivalence of $(H,s,\eta,\gamma_H)\in\mathcal{EQ}_F(G)$ is unique up to $H(\bar{F})$-conjugacy.*

Proof. It was proved in [**17**, Lemma 9.7] that the map from $\mathcal{EQ}_F^{\mathrm{ell}}(G)$ to $\mathcal{SS}_F^{\mathrm{ell}}(G)$ is a bijection. The general case is proved in the same way. □

As remarked at the end of § 2.1, we will fix a representative for each isomorphism class of $\mathcal{E}_F(G)$ from § 5. When working with $\mathcal{EQ}_F(G)$, it is convenient to consider only those (H,s,η,γ_H) such that (H,s,η) is in the set of fixed representatives. For any given (H,s,η,γ_H), it is easy to see from Lemma 2.8 that there are precisely $|\mathrm{Out}_F(H,s,\eta)|$ stable conjugacy classes of $\gamma'_H\in H(F)$ (including that of γ_H itself) such that (H,s,η,γ_H) and (H,s,η,γ'_H) are equivalent.

2.4. Endoscopic triples for Levi subgroups

Let M be a Levi subgroup of an F-rational parabolic subgroup of G. In § 2.4 we assume for simplicity that G is quasi-split over F. Let J denote an inner form of M over F.

Definition 2.9 (Kottwitz [24, Definition 7.1]). A *G-endoscopic triple* for M is an endoscopic triple (M_H,s_H,η_H) for M such that the condition (iii) of Definition 2.1 holds with M_H and G in place of H and G, respectively. An *isomorphism* between two G-endoscopic triples (M_H,s_H,η_H) and (M'_H,s'_H,η'_H) for M is an isomorphism $\alpha: M_H\xrightarrow{\sim}M'_H$ of endoscopic triples for M such that s_H and $\hat{\alpha}(s'_H)$ are equal (not only in $Z(\hat{M}_H)/Z(\hat{M})$ but also) in $Z(\hat{M}_H)/Z(\hat{G})$. Denote by $\mathcal{E}_F(M,G)$ the set of isomorphism classes of G-endoscopic triples for M. Write $\mathrm{Aut}_F^G(M_H,s_H,\eta_H)$ for the group of automorphisms of (M_H,s_H,η_H) and define $\mathrm{Out}_F^G(M_H,s_H,\eta_H):=\mathrm{Aut}_F^G(M_H,s_H,\eta_H)/\operatorname{Int}_F(M_H)$.

If J is an inner form of M over F, define G-endoscopic triples for J, the notion of isomorphism, and the set $\mathcal{E}_F(J,G)$ in an analogous way, by replacing M with J and using the canonical map $Z(\hat{G})\hookrightarrow Z(\hat{M})\xrightarrow{\sim}Z(\hat{J})$.

Let $\gamma_0\in M(F)$ be a semisimple element. Let T be a maximal torus of M over $\bar{F}$ containing γ_0. We say that γ_0 is (G,M)-*regular* if $\alpha(\gamma_0)\neq 1$ for every root α of T in G which is not a root in M. This notion is independent of the choice of T. Now suppose that γ_0 is (G,M)-regular. Let I_0 denote $Z_M(\gamma_0)$, which is the same as $Z_G(\gamma_0)$. We have a natural map $\mathfrak{K}_G(I_0/F)\to\mathfrak{K}_M(I_0/F)$. (This turns out to be a surjection with kernel $(Z(\hat{M})/Z(\hat{G}))^{\Gamma}$ but we do not need this fact.) Suppose that a semisimple element $\delta\in J(F)$ transfers to a (G,M)-regular $\gamma_0\in M(F)$. It is easy to check that $I_\delta:=Z_J(\delta)$ is an inner form of I_0 over F. Define $\mathfrak{K}_G(I_\delta/F)$, replacing $Z(\hat{I}_0)$ by $Z(\hat{I}_\delta)$ in the definition of $\mathfrak{K}_G(I_0/F)$. (There is a canonical Γ-equivariant embedding $Z(\hat{G})\hookrightarrow Z(\hat{I}_\delta)$.) There are

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canonical isomorphisms $\mathfrak{K}_G(I_\delta/F) \simeq \mathfrak{K}_G(I_0/F)$ and $\mathfrak{K}_J(I_\delta/F) \simeq \mathfrak{K}_M(I_0/F)$ coming from the canonical Γ-equivariant isomorphism $Z(\hat{I}_\delta) \simeq Z(\hat{I}_0)$.

Define $\mathcal{SS}_F(M,G)$ to be the set of equivalence classes of (γ_0, κ) where γ_0 is a (G,M)-regular semisimple element of $M(F)$ and κ belongs to $\mathfrak{K}_G(I_0/F)$. Two pairs (γ_0,κ) and (γ_0',κ') are considered *equivalent* if $\gamma_0 \sim_{\mathrm{st}} \gamma_0'$ in $M(F)$ and $\kappa = \kappa'$ via the canonical isomorphism $Z(\hat{I}_0) \simeq Z(\hat{I}_0')$. The set $\mathcal{SS}_F(J,G)$ is defined analogously, replacing (γ_0,κ) with (δ,κ) where $\kappa \in \mathfrak{K}_G(I_\delta/F)$ and $\delta \in J(F)$ is a semisimple element which transfers to a (G,M)-regular element of $M(F)$. There is a natural injection $\mathcal{SS}_F(J,G) \hookrightarrow \mathcal{SS}_F(M,G)$ given by the transfer of stable conjugacy classes.

Define $\mathcal{EQ}_F(M,G)$ to be the set of equivalence classes of (G-endoscopic) quadruples $(M_H, s_H, \eta_H, \gamma_H)$ where (M_H, s_H, η_H) is a G-endoscopic triple for M and γ_H is an (M, M_H)-regular semisimple element of $M_H(F)$ which transfers to a (G,M)-regular element in $M(F)$. The quadruples $(M_H, s_H, \eta_H, \gamma_H)$ and $(M_H', s_H', \eta_H', \gamma_H')$ are *equivalent* if there is an isomorphism $(M_H, s_H, \eta_H) \xrightarrow{\sim} (M_H', s_H', \eta_H')$ by $\alpha : M_H \xrightarrow{\sim} M_H'$ such that $\alpha(\gamma_H)$ is conjugate to γ_H' in $M_H'(\bar{F})$. For $(M_H, s_H, \eta_H, \gamma_H) \in \mathcal{EQ}_F(M,G)$, put $I_{M_H} := Z_{M_H}(\gamma_H)$ and suppose that γ_H transfers to $\gamma_0 \in M(F)$. As before, I_0 and I_{M_H} are connected and inner forms of each other, and we may construct $\kappa \in \mathfrak{K}_G(I_0/F)$ as the image of $s_H \in Z(\hat{M}_H)$. Thus obtain a map $(M_H, s_H, \eta_H, \gamma_H) \mapsto (\gamma_0, \kappa)$ from $\mathcal{EQ}_F(M,G)$ to $\mathcal{SS}_F(M,G)$. Now define a subset $\mathcal{EQ}_F(J,G)$ of $\mathcal{EQ}_F(M,G)$ by the following condition: the image (γ_0,κ) of $(M_H, s_H, \eta_H, \gamma_H)$ is such that γ_0 transfers to an element $\delta \in J(F)$. Thus we get a map $\mathcal{EQ}_F(J,G) \to \mathcal{SS}_F(J,G)$ given by $(M_H, s_H, \eta_H, \gamma_H) \mapsto (\delta,\kappa)$. There is an analogue of Lemma 2.8.

Lemma 2.10. *The maps that we constructed above are bijections from $\mathcal{EQ}_F(M,G)$ to $\mathcal{SS}_F(M,G)$ and from $\mathcal{EQ}_F(J,G)$ to $\mathcal{SS}_F(J,G)$, respectively. An automorphism $\alpha : M_H \xrightarrow{\sim} M_H$ inducing a self-equivalence of (M_H, s, η, γ_H) in $\mathcal{EQ}_F(M,G)$ is unique up to $M_H(\bar{F})$-conjugacy.*

Proof. In the case of $\mathcal{EQ}_F(M,G)$ and $\mathcal{SS}_F(M,G)$, the proof of [**17**, Lemma 9.7] works without essential change. The analogous assertion for $\mathcal{EQ}_F(J,G)$ and $\mathcal{SS}_F(J,G)$ follows from this. □

2.5. Levi subgroups of *L*-groups

In this subsection we use the notions and facts covered in [**4**, §§ 1–3], omitting proofs most of the time. Choose a Borel subgroup B and a maximal torus T of G over $\bar{F}$. Thus get a based root datum $(X^*(T), \Delta, X_*(T), \Delta^\vee)$. In particular we are given a bijection $\alpha \mapsto \alpha^\vee$ from $R(G,T)$ onto $R^\vee(G,T)$ which restricts to a bijection $\Delta \leftrightarrow \Delta^\vee$. Recall that we choose a $\mathrm{Gal}(\bar{F}/F)$-invariant splitting data $(\mathbb{B}, \mathbb{T}, \{X_\alpha\})$ to define ${}^LG = \hat{G} \rtimes W_F$. In particular ${}^LG$ is equipped with a natural surjection ${}^LG \twoheadrightarrow W_F$.

Definition 2.11. The normalizer $\mathcal{P}$ of a parabolic subgroup of $\hat{G}$ in ${}^LG$ is called a *parabolic subgroup of* ${}^LG$ if $\mathcal{P}$ surjects onto W_F. A *standard* parabolic subgroup of ${}^LG$ is one containing $\mathbb{B} \rtimes W_F$. If $\mathcal{P}$ is a parabolic subgroup of ${}^LG$, the normalizer in $\mathcal{P}$ of a Levi subgroup of $\mathbb{P} := \mathcal{P}^0$ (which is a parabolic subgroup of $\hat{G}$) is called a *Levi subgroup* of $\mathcal{P}$ (or of ${}^LG$ by abuse of terminology).

The bijection $\Delta \leftrightarrow \Delta^\vee$ induces a natural injection from the set of $G(F)$-conjugacy classes of F-rational parabolic subgroups of G to the set of $\hat{G}$-conjugacy classes of parabolic subgroups of ${}^LG$. Similarly there is a natural injection from the set of $G(F)$-conjugacy classes of F-rational Levi subgroups of G to the set of $\hat{G}$-conjugacy classes of Levi subgroups of ${}^LG$ [**4**, 3.3, 3.4]. If G is quasi-split over F then both injections are bijections. So if the image of an F-embedding $i_M : M \hookrightarrow G$ is an F-rational Levi subgroup of G, then i_M determines a $\hat{G}$-conjugacy class of L-embeddings ${}^LM \hookrightarrow {}^LG$ whose images are Levi subgroups of ${}^LG$. Let $\tilde{l}^0_M : {}^LM \hookrightarrow {}^LG$ be one such embedding and put $l^0_M := \tilde{l}^0_M|_{\hat{M}}$.

Lemma 2.12. *If $l_M : \hat{M} \hookrightarrow \hat{G}$ lies in the $\hat{G}$-conjugacy orbit of l^0_M then l_M can be extended to an L-embedding $\tilde{l}_M : {}^LM \hookrightarrow {}^LG$ which is $\hat{G}$-conjugate to $\tilde{l}^0_M$. Moreover, the image of any such extension $\tilde{l}_M$ is the centralizer of $l_M((Z(\hat{M})^\Gamma)^0)$ in ${}^LG$.*

Proof. Let $\hat{g} \in \hat{G}$ be such that $\mathrm{Int}(\hat{g}) \circ l^0_M = l_M$. Then $\tilde{l}_M := \mathrm{Int}(\hat{g}) \circ \tilde{l}^0_M$ is as desired in the first assertion. Let us prove the second assertion. Let $\mathcal{M} := \tilde{l}_M({}^LM)$ and $\mathcal{M}' := Z_{{}^LG}(l_M((Z(\hat{M})^\Gamma)^0))$. Clearly, $\mathcal{M} \subset \mathcal{M}'$ and $(\mathcal{M}')^0 = \mathcal{M}^0$. Moreover, $\mathcal{M}$ and $\mathcal{M}'$ are Levi subgroups of ${}^LG$. This is obvious for $\mathcal{M}$ and follows from [**4**, Lemma 3.5] for $\mathcal{M}'$. From this it is easy to see that $\mathcal{M} = \mathcal{M}'$. □

2.6. Transfer conjecture and the fundamental lemma

In § 2.6 we state the famous transfer conjecture and the fundamental lemma which are at the heart of the stable trace formula formalism. They are now proved in most cases by the work of several mathematicians. (See Proposition 2.17 and the remark below it.)

Assume that F is a local field. For each $(H, s, \eta) \in \mathcal{E}^{\mathrm{ell}}_F(G)$, fix an L-group morphism $\tilde{\eta} : {}^LH \to {}^LG$ extending η. Such an $\tilde{\eta}$ exists since G^{der} is simply connected (see [**25**, Proposition 1], cf. [**15**, 1.8.3]). Consider a (G, H)-regular semisimple element $\gamma_H \in H(F)$ and a semisimple element $\gamma_0 \in G(F)$ with matching stable conjugacy classes. There is a complex-valued function $\Delta(\cdot\,,\cdot)^G_H$, called the *transfer factor* and well-defined up to a constant, defined on any such pair of elements (γ_H, γ). (See [**27**] when γ_H is G-regular and [**28**] in general.) The function $\Delta(\cdot\,,\cdot)^G_H$ depends not only on η but also on the choice of $\tilde{\eta}$. When there is no danger of confusion, we simply write $\Delta(\cdot\,,\cdot)$ for $\Delta(\cdot\,,\cdot)^G_H$. Langlands and Shelstad proposed the following transfer conjecture. Functions ϕ and ϕ^H as in the conjecture are called (Δ-)*matching functions.*

Conjecture 2.13 (Kottwitz [17, Conjecture 5.5], cf. Langlands and Shelstad [28, 2.1]). *For each function $\phi \in C^\infty_c(G(F))$, there exists a function $\phi^H \in C^\infty_c(H(F))$ such that for every (G, H)-regular semisimple element $\gamma_H \in H(F)$, if γ_H transfers to $\gamma_0 \in G(F)$ in the sense of § 2.3, we have*

$$\mathrm{SO}^{H(F)}_{\gamma_H}(\phi^H) = \sum_{\gamma \sim_{\mathrm{st}} \gamma_0} e(Z_G(\gamma)) \cdot \Delta(\gamma_H, \gamma) \cdot \mathrm{O}^{G(F)}_\gamma(\phi),$$

where the sum is taken over a set of representatives for conjugacy classes in the stable conjugacy class of γ_0, and $\mathrm{SO}^{H(F)}_{\gamma_H}(\phi^H) = 0$ if γ_H does not transfer to $G(F)$.

There is freedom in the choice of $\Delta(\gamma_H,\gamma)$. Namely it is fixed only up to a constant. Nevertheless, once the value of $\Delta(\cdot,\cdot)$ is fixed for one pair (γ_H,γ), it is determined for every other pair.

When G is an unramified group over F, there is a more precise conjecture. Suppose that $\tilde{\eta}$ is unramified in the sense that it arises from a map $\hat{H}\rtimes W(F^{\mathrm{ur}}/F)\to\hat{G}\rtimes W(F^{\mathrm{ur}}/F)$ by inflation. (By definition $W(F^{\mathrm{ur}}/F)$ is the free abelian group generated by the Frobenius morphism.) Let K_G and K_H be hyperspecial maximal compact subgroups of G and H, respectively. The following is believed to be true under an appropriate normalization of $\Delta(\gamma_H,\gamma)$.

Conjecture 2.14 (fundamental lemma). *For any (G,H)-regular semisimple element $\gamma_H\in H(F)$, if γ_H transfers to $\gamma_0\in G(F)$ then*

$$\mathrm{SO}^{H(F)}_{\gamma_H}(\mathrm{char}_{K_H})=\sum_{\gamma\sim_{\mathrm{st}}\gamma_0} e(Z_G(\gamma))\cdot\Delta(\gamma_H,\gamma)\cdot\mathrm{O}^{G(F)}_{\gamma}(\mathrm{char}_{K_G})$$

and $\mathrm{SO}^{H(F)}_{\gamma_H}(\mathrm{char}_{K_H})=0$ if γ_H does not transfer to $G(F)$.

Remark 2.15. As proved in [**28**, Lemma 2.4.A], the general case of Conjecture 2.13 and 2.14 follows from the special case where γ_H is G-regular in the sense of [**27**, 1.3].

Remark 2.16. The map $\tilde{\eta}$ induces a map

$$\tilde{\eta}^*: C^\infty_c(K_G\backslash G(\mathbb{Q}_p)/K_G)\to C^\infty_c(K_H\backslash H(\mathbb{Q}_p)/K_H)$$

of unramified Hecke algebras. A more general version of the fundamental lemma says that ϕ and $\tilde{\eta}^*(\phi)$ are Δ-matching functions. (Recall that Δ depends on the choice of $\tilde{\eta}$.) The proof of this general version reduces to the case $\phi=\mathrm{char}_{K_G}$ as proved by Hales [**10**].

Proposition 2.17. *Conjecture 2.13 is true. Conjecture 2.14 is true if the residue characteristic of F is sufficiently large.**

We briefly remark on the proof of the proposition. Waldspurger showed in [**40**] and [**42**] that Conjectures 2.13 and 2.14 follow from a Lie algebra version of the fundamental lemma. The proof of the Lie algebra fundamental lemma over F (with $\mathrm{char}\,F=0$) is reduced by [**41**] to the proof for local fields of positive characteristic if the residue characteristic of F is large enough. The proof of the last case was recently announced by Ngô [**32**]. (In fact it is enough to assume that the residue characteristic of F does not divide the order of the Weyl group.) This implies Conjecture 2.13 for any F and Conjecture 2.14 for any F with large residue characteristic. For more details and related works on the fundamental lemma, we refer to the introduction of [**32**].

2.7. Transfer between GL_n and their inner forms over p-adic fields

The purpose of § 2.7 is to exhibit one of the simplest examples of the Langlands–Shelstad transfer as well as its interaction with representation theory, in the case of

* Michael Harris, as well as Sophie Morel, informed us that the condition on the residue characteristic can be removed by results of [**10**].

general linear groups and their inner forms. The result in this subsection, which is not new, will not be used in later sections but turns out to be useful for applications (as in [**37**]).

Let F be a finite extension of $\mathbb{Q}_p$. In this subsection, let G be an F-inner form of $G^* := \mathrm{GL}_n$. Note that conjugacy classes are the same as stable conjugacy classes in $G(F)$ and $G^*(F)$ by Hilbert 90.

Badulescu defined a morphism $\mathrm{LJ} = \mathrm{LJ}^{G^*(F)}_{G(F)}$ from $\mathrm{Groth}(G^*(F))$ to $\mathrm{Groth}(G(F))$ which is uniquely determined by the character identity [**3**, Proposition 3.3]

$$\mathrm{tr}\,\mathrm{LJ}(\pi)(g) = e(G)\cdot \mathrm{tr}\,\pi(g^*) \tag{2.4}$$

for every $\pi \in \mathrm{Groth}(G^*(F))$ and every pair of regular semisimple elements $g \in G(F)$ and $g^* \in G^*(F)$ with matching conjugacy classes. If $\pi \in \mathrm{Irr}(G^*(F))$ is square-integrable, its image $\mathrm{LJ}(\pi)$ is the inverse image of π under the Jacquet–Langlands correspondence as in [**6**]. In general, an irreducible smooth representation of $G^*(F)$ may not map to an irreducible representation of $G(F)$ under LJ.

Lemma 2.18 (cf. [11, Lemma V.5.1]). *For each $\phi \in C_c^\infty(G(F))$, there exists a function $\phi^* \in C_c^\infty(G^*(F))$ such that*

(i) *for any pair of semisimple elements $\gamma \in G(F)$, $\gamma^* \in G^*(F)$ with matching conjugacy classes,*

$$\mathrm{O}^{G(F)}_{\gamma}(\phi) = e(G)\cdot e(Z_G(\gamma))\cdot \mathrm{O}^{G^*(F)}_{\gamma^*}(\phi^*),$$

where Haar measures are chosen to be compatible between the inner forms $G(F)$ and $G^*(F)$ (respectively $Z_G(\gamma)(F)$ and $Z_{G^*}(\gamma^*)(F)$), and $\mathrm{O}^{G^*(F)}_{\gamma^*}(\phi^*) = 0$ if a semisimple $\gamma^* \in G^*(F)$ does not transfer to $G(F)$;

(ii) *for any $\pi^* \in \mathrm{Groth}(G^*(F))$,*

$$\mathrm{tr}\,\mathrm{LJ}(\pi^*)(\phi) = \mathrm{tr}\,\pi^*(\phi^*).$$

Remark 2.19. Lemma 2.18 admits an obvious generalization to the case where G is an inner form of a product of general linear groups.

Remark 2.20. Note that $(G^*, 1, \mathrm{id})$ is an endoscopic triple for G. Part (i) of the lemma is a basic example of Conjecture 2.13, with the normalization $\Delta(\cdot\,,\cdot)^G_{G^*} \equiv e(G)$.

Proof. By [**6**, Theorem B.2.c], we may choose $\phi^* \in C_c^\infty(G^*(F))$ such that (i) holds, but we need to account for the sign difference. Our sign convention is different from that of [**6**] because we use compatible measures in the sense of [**18**, p. 631]. The ratio of the measures on $G(F)$ and $G^*(F)$ in our case differs by $e(G)$ from that in [**6**], which explains the appearance of $e(G)$. The extra sign factor $e(Z_G(\gamma))$ comes from the fact that we choose compatible measures on $Z_G(\gamma)(F)$ and $Z_{G^*}(\gamma^*)(F)$ in (i).

It remains to verify (ii). Recall the Weyl integration formula in the notation of [**11**, p. 189]

$$\mathrm{tr}\,\pi(\phi) = \sum_T |W_G(T)|^{-1}\int_{T^{\mathrm{reg}}} D_G(t)\mathrm{O}^{G(F)}_t(\phi)\,\mathrm{tr}\,\pi(t)\,\mathrm{d}t,$$

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where the sum runs over $G(F)$-conjugacy classes of maximal tori T in $G(F)$. A similar formula holds for π^* and G^*. Using part (i) and the fact that $\operatorname{tr}\mathrm{LJ}(\pi^*)(t) = e(G)\operatorname{tr}\pi^*(t^*)$ by (2.4), we deduce that $\operatorname{tr}\mathrm{LJ}(\pi^*)(\phi) = \operatorname{tr}\pi^*(\phi^*)$. □

3. More background

3.1. The sets $B(G)$, $N(G)$ and the Newton maps

In §§ 3.1 and 3.2, let G be a connected reductive group over $\mathbb{Q}_p$ which is quasi-split. Choose a maximal torus T of G defined over $\mathbb{Q}_p$. Let $L := \operatorname{Frac} W(\bar{\mathbb{F}}_p)$ and $L_s := \operatorname{Frac} W(\mathbb{F}_{p^s})$ for $s \in \mathbb{Z}_{>0}$. Denote by σ the Frobenius automorphism of L which induces the pth power map on the residue field. In this section $\Gamma(p) := \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Let $\mathbb{D}$ denote the protorus with character group $\mathbb{Q}$. Define

$$B(G) = G(L)/\sim, \quad \text{where } x \sim y \Leftrightarrow \exists g \in G(L),\ x = g^{-1}yg^{\sigma},$$
$$N(G) = (\operatorname{Int} G(L)\backslash \operatorname{Hom}_L(\mathbb{D}, G))^{\langle\sigma\rangle} \simeq (X_*(T)_{\mathbb{Q}}/\Omega)^{\Gamma(p)},$$

where Ω is the Weyl group for T in G over $\bar{F}$. There is a map

$$\nu_G : G(L) \to \operatorname{Hom}_L(\mathbb{D}, G)$$

characterized by various properties (see [**33**, Theorem 1.8] and [**16**, § 4]) which induces the *Newton map* $\bar{\nu}_G : B(G) \to N(G)$. The sets $B(G)$, $N(G)$ and the maps ν_G, $\bar{\nu}_G$ are functorial in G. Moreover, $\bar{\nu}_G$ fits into the commutative diagram (3.1) below, which is functorial in G. The first (respectively second) row of (3.1) is exact in the middle in the sense of pointed sets (respectively abelian groups). See [**33**, Theorem 1.15] about these facts and the maps in the diagram:

$$\begin{array}{ccccc} H^1(\mathbb{Q}_p, G) & \longrightarrow & B(G) & \xrightarrow{\bar{\nu}_G} & N(G) \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\kappa_G} & & \downarrow{\scriptstyle\delta_G} \\ \pi_0(Z(\hat{G})^{\Gamma(p)})^D & \longrightarrow & X^*(Z(\hat{G})^{\Gamma(p)}) & \xrightarrow{\rho_G} & X^*(Z(\hat{G}))^{\Gamma(p)} \otimes_{\mathbb{Z}} \mathbb{Q} \end{array} \tag{3.1}$$

3.2. The groups M_b and J_b

An element $\tilde{b} \in G(L)$ is called *decent* [**34**, Definition 1.8] if for some $s \in \mathbb{Z}_{>0}$, $s\nu_G(\tilde{b})$ arises from a morphism $\mathbb{G}_m \to G$ and

$$\tilde{b}\sigma(\tilde{b}) \cdots \sigma^{s-1}(\tilde{b}) = s\nu_G(\tilde{b})(p). \tag{3.2}$$

In particular this implies that $\tilde{b} \in G(L_s)$. Recall that $\tilde{b} \in G(L)$ is called *basic* [**16**, § 5.1] if $\nu_G(\tilde{b}) : \mathbb{D} \to G$ factors through $Z(G)$. Any $b \in B(G)$ is basic if it has a representative $\tilde{b} \in G(L)$ which is basic.

Fix $b \in B(G)$ for the moment. It is possible to choose a decent representative $\tilde{b}$ of b such that $\nu_G(\tilde{b})$ is defined over $\mathbb{Q}_p$ (see § 4.3 and p. 219 of [**16**]). Write $M_{\tilde{b}}$ for the centralizer of $\nu_G(\tilde{b})$, which is a $\mathbb{Q}_p$-rational Levi subgroup of G. In fact b gives rise to a basic element of $B(M_{\tilde b})$ [**16**, Proposition 6.2]. We may and will arrange that $\tilde b \in M_{\tilde b}(L)$ and that $\tilde b$ is a basic decent element of $M_{\tilde b}(L)$. Fix $s \in \mathbb{Z}_{>0}$ which satisfies (3.2) for $\tilde b$.

Define a $\mathbb{Q}_p$-group $J_{\tilde b}$ by

$$J_{\tilde b}(R) = \{g \in G(L \otimes_{\mathbb{Q}_p} R) \mid g = \tilde b \sigma(g) \tilde b^{-1}\}$$

for any $\mathbb{Q}_p$-algebra R. The representability of $J_{\tilde b}$ is shown in [**34**, 1.12]. We will often use the fact that $J_{\tilde b}$ is an inner form of $M_{\tilde b}$ represented by the cocycle $\sigma \mapsto \mathrm{Int}(\tilde b)$ in $H^1(L_s/\mathbb{Q}_p, \mathrm{Int}(M_{\tilde b}))$ (cf. [**36**, Lemma 4.2]). We may and will fix a choice of an L_s-isomorphism $\psi : J_{\tilde b} \xrightarrow{\sim} M_{\tilde b}$ such that $\psi\psi^{-\sigma} = \mathrm{Int}(\tilde b)$. (The $M_{\tilde b}(\bar{\mathbb{Q}}_p)$-conjugacy class of ψ is canonical.) This allows us to embed $J_{\tilde b}$ into G over $\bar{\mathbb{Q}}_p$ by $J_{\tilde b} \simeq M_{\tilde b} \hookrightarrow G$. If $\tilde b'$ is another choice for $\tilde b$, then there exists $g \in G(\mathbb{Q}_p)$ such that $M_{\tilde b'} = gM_{\tilde b}g^{-1}$ and $\tilde b' = g\tilde b g^{-1}$ [**16**, Proposition 6.3].

From now on we fix the choice of a decent $\tilde b$ for each $b \in B(G)$ and will write J_b, M_b and ν_b for $J_{\tilde b}$, $M_{\tilde b}$ and $\nu_G(\tilde b)$ for simplicity of notation. It is easy to see from the previous discussion that the $G(\mathbb{Q}_p)$-conjugacy class of the $\mathbb{Q}_p$-embedding $M_b \hookrightarrow G$ and the $G(\bar{\mathbb{Q}}_p)$-conjugacy class of the $\bar{\mathbb{Q}}_p$-embedding $J_b \hookrightarrow G$ are canonical in that they are independent of the choice of $\tilde b$.

3.3. Acceptable elements

Consider a triple (G_0, ν, M_0) such that

(i) G_0 is a connected reductive group over $\mathbb{Q}_p$;

(ii) $\nu : \mathbb{D} \to G_0$ is defined over $\mathbb{Q}_p$;

(iii) M_0 is the centralizer of ν in G_0 (thus a $\mathbb{Q}_p$-rational Levi subgroup of G_0).

For any maximal torus T_0 of M_0 over $\bar{\mathbb{Q}}_p$, the map ν may be viewed as an element of $X_*(T_0)_{\mathbb{Q}}$. Choose $s \in \mathbb{Z}_{>0}$ such that $s\nu \in X_*(T_0)$. We assume that

$$\text{for every } \alpha \in R(G_0, T_0)\backslash R(M_0, T_0), \text{ we have } v_p(\alpha(s\nu(p))) \neq 0. \qquad (\star)$$

If condition $(\star)$ is verified for some T_0 and s then it is also true for any other choice of T_0 and s. For $\alpha \in R(G_0, T_0)$, condition (iii) implies that $\langle \alpha, \nu \rangle = 0$ if and only if $\alpha \in R(M_0, T_0)$.

Definition 3.1. A semisimple element $\gamma_0 \in M_0(\mathbb{Q}_p)$ is said to be *ν-acceptable* if the following condition is verified: for every α in $R(G_0, T_0)\backslash R(M_0, T_0)$, we have $\langle \alpha, \nu \rangle > 0$ if and only if $\alpha(\gamma_0) \in \bar{\mathbb{Q}}_p^\times$ has positive (additive) p-adic valuation. An arbitrary element $\gamma_0 \in M_0(\mathbb{Q}_p)$ is said to be *ν-acceptable* if the semisimple part of γ_0 in the Jordan decomposition is ν-acceptable.

Whether γ_0 is ν-acceptable is independent of the choice of T_0. If $\gamma_0, \gamma_0' \in M_0(\mathbb{Q}_p)$ are $M_0(\bar{\mathbb{Q}}_p)$-conjugate, then γ_0 is ν-acceptable if and only if γ_0' is. So it makes sense to ask whether a stable or $M_0(\bar{\mathbb{Q}}_p)$-conjugacy class in $M_0(\mathbb{Q}_p)$ is ν-acceptable. Let J_0 be an inner form of M_0 over F.

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Definition 3.2. For an element $\delta \in J_0(\mathbb{Q}_p)$, let δ_s denote its semisimple part. We say that δ is *ν-acceptable* if the stable conjugacy class of δ_s transfers to a ν-acceptable stable conjugacy class in $M_0(\mathbb{Q}_p)$ (via the transfer between inner forms).

Remark 3.3. This definition coincides with the one given in [**36**, Definition 6.1].

Denote by $P(\nu)$ the unique $\mathbb{Q}_p$-rational parabolic subgroup of G_0 containing M_0 as a Levi subgroup such that $\alpha \in R(G_0, T_0)\backslash R(M_0, T_0)$ satisfies $\langle \alpha, \nu \rangle < 0$ exactly when α is a positive root with respect to $P(\nu)$. The following lemma is obvious.

Lemma 3.4. *If $\gamma_0 \in M_0(\mathbb{Q}_p)$ is ν-acceptable, then γ_0 is (G_0, M_0)-regular (§ 2.4) and $|D^{G_0}_{M_0}(\gamma_0)|_p = \delta_{P(\nu)}(\gamma_0)$. The set of all ν-acceptable elements is open in $M_0(\mathbb{Q}_p)$.*

We record a few other useful lemmas. (We do not assume that the derived subgroups of G_0 and M_0 are *simply connected*.* This does not bother us as we are concerned with elements with connected centralizers when it comes to applications.)

Lemma 3.5. *Let $m \in M_0(\mathbb{Q}_p)$ be a (G_0, M_0)-regular semisimple element. (For instance, m may be any ν-acceptable semisimple element by the preceding lemma.) The inclusion $M_0 \hookrightarrow G_0$ induces a bijection from the set of $M_0(\mathbb{Q}_p)$-conjugacy classes in the stable conjugacy class (in M_0) of m to that of $G_0(\mathbb{Q}_p)$-conjugacy classes in the stable conjugacy class (in G_0) of m.*

Proof. Set $I := Z_{M_0}(m)^0$. We know $I = Z_{G_0}(m)^0$. The first assertion is equivalent to the statement that the natural map

$$\ker(H^1(\mathbb{Q}_p, I) \to H^1(\mathbb{Q}_p, M_0)) \to \ker(H^1(\mathbb{Q}_p, I) \to H^1(\mathbb{Q}_p, G_0))$$

is a bijection. We will prove that the map $H^1(\mathbb{Q}_p, M_0) \to H^1(\mathbb{Q}_p, G_0)$ given by $M_0 \hookrightarrow G_0$ is an injection of sets. Since $H^1(\mathbb{Q}_p, P(\nu)) \to H^1(\mathbb{Q}_p, G_0)$ is an injection [**35**, III.2.1.Exercise 1], it suffices to show that $H^1(\mathbb{Q}_p, M_0) \to H^1(\mathbb{Q}_p, P(\nu))$ is an injection. Let U be the unipotent radical of $P(\nu)$. Since the composition $M_0 \hookrightarrow P(\nu) \twoheadrightarrow P(\nu)/U$ is an isomorphism, the composition

$$H^1(\mathbb{Q}_p, M_0) \to H^1(\mathbb{Q}_p, P(\nu)) \to H^1(\mathbb{Q}_p, P(\nu)/U)$$

is a bijection and the proof is complete. □

Lemma 3.6. *If ν-acceptable semisimple elements $m, m' \in M_0(\mathbb{Q}_p)$ are conjugate in $G_0(\bar{\mathbb{Q}}_p)$ then m and m' are conjugate in $M_0(\bar{\mathbb{Q}}_p)$.*

Remark 3.7. Lemma 3.6 fails if m, m' are assumed not ν-acceptable but only (G_0, M_0)-regular. A counterexample can be given when $M_0 = \mathrm{GL}_1 \times \mathrm{GL}_1$ is the diagonal torus of $G_0 = \mathrm{GL}_2$, by taking $m = (1, -1)$ and $m' = (-1, 1)$.

* In § 6, the role of G_0 is played by H, for instance. For a PEL datum of type (C) (namely when the group G of § 4.1 is a sympletic similitude group), H^{der} is usually not simply connected.

Proof. Let $g_0 \in G_0(\mathbb{Q}_p)$ be such that $m' = g_0 m g_0^{-1}$ and choose maximal tori T_0 and T_0' of M_0 over $\bar{\mathbb{Q}}_p$ containing m and m' respectively. The proof is easily reduced to the case where $T_0 = T_0'$ and $g_0 T_0 g_0^{-1} = T_0$. Then $\mathrm{Int}(g_0)$ acts on $X^*(T_0)$ in the same way as some $w \in \Omega(G_0, T_0)$. As m and m' are ν-acceptable and conjugate under g_0, it follows that w must preserve the parabolic subgroup $P(\nu)$. This proves $w \in \Omega(M_0, T_0)$. Therefore, m and m' are $M_0(\bar{\mathbb{Q}}_p)$-conjugate. □

Corollary 3.8. *Let $m \in M_0(\mathbb{Q}_p)$ be a semisimple element such that $Z_{M_0}(m)$ is connected. In the $G_0(\mathbb{Q}_p)$-conjugacy class of m, there is at most one $M_0(\mathbb{Q}_p)$-conjugacy class which is ν-acceptable.*

Proof. Immediate consequence of Lemma 3.5 and Lemma 3.6. (Recall that if $Z_{M_0}(m)$ is connected, the stable conjugacy class of m is the same as the $M_0(\bar{\mathbb{Q}}_p)$-conjugacy class of m by definition.) □

The discussion so far may be applied to (G, ν_b, M_b) of § 3.2 as the conditions (i)–(iii) and $(\star)$ are clearly satisfied. So we can make sense of ν_b-acceptable elements in $M_b(\mathbb{Q}_p)$ and $J_b(\mathbb{Q}_p)$ as well as the parabolic subgroup $P(\nu_b)$ of G. Another example is given by $(H, i\nu_{M_H}, M_H)$ of § 6.3.

3.4. A lemma on the transfer of functions

Let (G_0, ν, M_0) be a triple as in § 3.3. Fix Haar measures on G_0 and M_0.

Lemma 3.9. *Suppose that $\phi \in C_c^\infty(M_0(\mathbb{Q}_p))$ is supported on ν-acceptable elements and that $\mathrm{O}_m^{M_0(\mathbb{Q}_p)}(\phi) = 0$ whenever m is a semisimple element such that $Z_{M_0}(m)$ is not connected. Then there exists a function $\tilde{\phi} \in C_c^\infty(G_0(\mathbb{Q}_p))$ such that*

(i) *for any semisimple element $g \in G_0(\mathbb{Q}_p)$,*

$$\mathrm{O}_g^{G_0(\mathbb{Q}_p)}(\tilde{\phi}) = \delta_{P(\nu)}(m)^{-1} \cdot \mathrm{O}_m^{M_0(\mathbb{Q}_p)}(\phi)$$

if there exists a ν-acceptable element $m \in M_0(\mathbb{Q}_p)$ which is conjugate to g in $G_0(\mathbb{Q}_p)$ (if so, m is unique up to $M_0(\mathbb{Q}_p)$-conjugacy), and

$$\mathrm{O}_g^{G_0(\mathbb{Q}_p)}(\tilde{\phi}) = 0$$

otherwise; if m is ν-acceptable and $m \sim g$ in $G_0(\mathbb{Q}_p)$ then we choose compatible Haar measures on $Z_{M_0}(m)^0(\mathbb{Q}_p)$ and $Z_{G_0}(g)^0(\mathbb{Q}_p)$, which are isomorphic; and

(ii) *for any $\pi \in \mathrm{Irr}(G_0(\mathbb{Q}_p))$,*

$$\mathrm{tr}\, \pi(\tilde{\phi}) = \mathrm{tr}\, J_{P(\nu)^{\mathrm{op}}}^{G_0}(\pi)(\phi).$$

Proof. In [**11**, Lemma V.5.2] the above lemma is proved when G_0 is a general linear group and M_0 is the Levi subgroup of a maximal parabolic subgroup of G_0. As the same argument works in our case we only sketch the proof indicating the necessary changes that should be made.

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Define a function $\phi^0 \in C_c^\infty(M_0(\mathbb{Q}_p))$ by $\phi^0 := \phi \cdot \delta_{P(\nu)}^{-1}$ and a function W on $G_0(\mathbb{Q}_p)$ by

$$W(g) := \sum_m \mathrm{O}_m^{M_0(\mathbb{Q}_p)}(\phi^0),$$

where m runs over a set of representatives for $M_0(\mathbb{Q}_p)$-conjugacy classes contained in the $G_0(\mathbb{Q}_p)$-conjugacy class of g. The main step for (i), whose proof will be omitted as it is essentially the same as in [**11**, Lemma V.5.2], is to prove that W satisfies the characterizing properties of orbital integrals in [**39**, Theorem B]. It is worth noting that for the proof we need to make use of the fact that $\phi^0 \in C_c^\infty(M_0(\mathbb{Q}_p))$ is supported on ν-acceptable elements. As a result of the main step there exists a function $\tilde{\phi} \in C_c^\infty(G_0(\mathbb{Q}_p))$ such that $W(g) = \mathrm{O}_g^{G_0(\mathbb{Q}_p)}(\tilde{\phi})$. Corollary 3.8 finishes the proof of (i).

Part (ii) follows from part (i) combined with Lemma 3.4, the Weyl integration formula and [**5**, Theorem 5.2]. One may argue exactly as in [**11**, p. 189–190], noting that there is a difference by $\delta_{P(\nu)}^{-1}$ between our normalization and theirs as we replaced ϕ with ϕ^0 in the course of the proof. □

Corollary 3.10. *Let ϕ and $\tilde{\phi}$ be as in Lemma 3.9. Let $g \in G_0(\mathbb{Q}_p)$ be a semisimple element. If there is no ν-acceptable element $m \in M_0(\mathbb{Q}_p)$ such that $m \sim_{\mathrm{st}} g$ in $G_0(\mathbb{Q}_p)$ then $\mathrm{SO}_g^{G_0(\mathbb{Q}_p)}(\tilde{\phi}) = 0$. If there does exist such an element m,*

$$\mathrm{SO}_g^{G_0(\mathbb{Q}_p)}(\tilde{\phi}) = \delta_{P(\nu)}(m)^{-1} \cdot \mathrm{SO}_m^{M_0(\mathbb{Q}_p)}(\phi),$$

where compatible Haar measures are chosen on $Z_{M_0}(m)^0(\mathbb{Q}_p)$ and $Z_{G_0}(g)^0(\mathbb{Q}_p)$.

Proof. Immediate from Lemma 3.5 and Lemma 3.9. □

4. Pre-stabilized counting point formula

In this section we recall the definition of Igusa varieties and related notions. We state the 'counting point' formula for Igusa varieties in § 4.4. We fix a prime p once and for all, until the end of the paper.

4.1. Igusa varieties

We give a brief summary of the material covered in [**36**, § 5] (see also [**29**]). Consider a tuple $(B, *, V, \langle\cdot\,,\cdot\rangle, h)$, called a PEL (Shimura) datum, where

- B is a finite-dimensional simple $\mathbb{Q}$-algebra,
- $*$ is a positive involution on B,
- V is a finite semisimple B-module,
- $\langle\cdot\,,\cdot\rangle : V \times V \to \mathbb{Q}$ is a non-degenerate alternate pairing such that $\langle bv_1, v_2\rangle = \langle v_1, b^* v_2\rangle$ for all $b \in B$, $v_1, v_2 \in V$, and

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- $h : \mathbb{C} \to \mathrm{End}_B(V)_\mathbb{R}$ is an $\mathbb{R}$-algebra homomorphism such that $\forall z \in \mathbb{C},\ h(z^c) = h(z)^*$ and that the bilinear pairing $(v, w) \mapsto \langle v, h(i)w\rangle$ is symmetric and positive definite.

Put $F := Z(B)$ and define a $\mathbb{Q}$-group G by the relation

$$G(R) = \{g \in \mathrm{End}_{B\otimes_\mathbb{Q} R}(V \otimes_\mathbb{Q} R) \mid \exists\varpi(g) \in R^\times,\ \langle gv_1, gv_2\rangle = \varpi(g)\langle v_1, v_2\rangle \text{ for all } v_1, v_2 \in V \otimes_\mathbb{Q} R\}$$

for any $\mathbb{Q}$-algebra R. We define a $\mathbb{C}$-group morphism $\mu = \mu_h : \mathbb{G}_m \to G$ as the composite

$$\mathbb{C}^\times \hookrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \simeq (\mathbb{C} \otimes_\mathbb{R} \mathbb{C})^\times \xrightarrow{(h,\mathrm{id})} (\mathrm{End}_B(V) \otimes_\mathbb{Q} \mathbb{C})^\times,$$

where the first map is $z \mapsto (z, 1)$ and the inverse of the second map is induced by the algebra map given by $z_1 \otimes z_2 \mapsto (z_1z_2, z_1\bar{z}_2)$. Often μ is viewed as a $\bar{\mathbb{Q}}_p$-morphism by making a choice of $\iota_p : \bar{\mathbb{Q}}_p \simeq \mathbb{C}$. The datum $(B, *, V, \langle\cdot\,,\cdot\rangle, h)$ falls into type (A), (C) or (D) [**21**, § 5]. We will consider *only type (A) and (C)* throughout this paper. This has the following consequences.

- G^{der} and $\hat{G}^{\mathrm{der}}$ are simply connected. So are M^{der} and $\hat{M}^{\mathrm{der}}$ for each $\mathbb{Q}_p$-Levi subgroup M of $G_{\mathbb{Q}_p}$.
- $G_\mathbb{R}$ has an elliptic torus and $(A_G)_\mathbb{R} = A_{G_\mathbb{R}}$ canonically.
- For any semisimple $\gamma_0 \in G(\mathbb{Q})$ and $I_0 := Z_G(\gamma_0)$, the canonical map

  $$\ker^1(\mathbb{Q}, Z(\hat{G})) \to \ker^1(\mathbb{Q}, Z(\hat{I}_0))$$

  is injective.

Lemma 4.1. *The last assertion in the list above is true.*

Proof. Write $I_0^{\mathrm{ab}} := I_0/I_0^{\mathrm{der}}$ and $G^{\mathrm{ab}} := G/G^{\mathrm{der}}$. It suffices to prove that $\ker^1(\mathbb{Q}, I_0^{\mathrm{ab}}) \to \ker^1(\mathbb{Q}, G^{\mathrm{ab}})$ is surjective by dualization [**14**, (1.8.3), (4.2.2)]. The following commutative diagram is induced by the obvious commutative diagram of morphisms between groups:

$$\begin{array}{ccccc}
\ker^1(\mathbb{Q}, Z(G)) & \longrightarrow & \ker^1(\mathbb{Q}, G) & \longrightarrow & \ker^1(\mathbb{Q}, G^{\mathrm{ab}}) \\
\downarrow & & \uparrow & & \uparrow \\
\ker^1(\mathbb{Q}, Z(I_0)) & \longrightarrow & \ker^1(\mathbb{Q}, I_0) & \longrightarrow & \ker^1(\mathbb{Q}, I_0^{\mathrm{ab}})
\end{array}$$

The right arrow in the top row is a bijection by [**14**, Lemma 4.3.1]. According to [**21**, p. 393–394], $\ker^1(\mathbb{Q}, G) = 1$ or $\ker^1(\mathbb{Q}, Z(G)) \to \ker^1(\mathbb{Q}, G)$ is a bijection. Therefore, $\ker^1(\mathbb{Q}, I_0^{\mathrm{ab}}) \to \ker^1(\mathbb{Q}, G^{\mathrm{ab}})$ is surjective. □

The PEL datum determines a Shimura variety Sh which is a projective system of quasi-projective varieties Sh_U defined over the reflex field E where U runs over sufficiently small open compact subgroups of $G(\mathbb{A}^\infty)$ [**21**, § 5]. Here E is a number field determined by the PEL datum. Let ξ be a finite-dimensional irreducible representation of G over $\bar{\mathbb{Q}}_l$. We obtain from ξ an l-adic local system $\mathscr{L}_\xi$ on each Sh_U.

We suppose that $(B, *, V, \langle\cdot\,,\cdot\rangle, h)$ can be extended to a p-unramified integral Shimura datum [**36**, Definition 5.2] and fix one such extension. In particular p is unramified in F and $G_{\mathbb{Q}_p}$ is unramified. The p-unramified integral Shimura datum determines a hyperspecial subgroup U_p^{hs} of $G_{\mathbb{Q}_p}$. The Shimura variety $\mathrm{Sh}_{U^p} := \mathrm{Sh}_{U^p\times U_p^{\mathrm{hs}}}$ has an integral model with smooth fibre $\overline{\mathrm{Sh}}_{U^p}$ over $\bar{\mathbb{F}}_p$, which in turn has a Newton polygon stratification

$$\overline{\mathrm{Sh}}_{U^p} = \coprod_b \overline{\mathrm{Sh}}_{U^p}^{(b)}$$

parametrized by $b \in B(G_{\mathbb{Q}_p}, -\mu)$.

From here on, fix b once and for all and also fix a representative $\tilde{b}$ as in § 3.2. Let Σ_b be a Barsotti–Tate group over $\bar{\mathbb{F}}_p$ of isogeny type b, satisfying the additional conditions (i)–(iv) in § 5 of [**36**]. We briefly remark that Σ_b comes equipped with the compatible structure of a $\mathbb{Q}_p$-algebra morphism $B\otimes_{\mathbb{Q}}\mathbb{Q}_p \hookrightarrow \mathrm{End}(\Sigma_b)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and a polarization $\Sigma_b \to \Sigma_b^\vee$, and that $J_b(\mathbb{Q}_p)$ is isomorphic to the group of self-quasi-isogenies of Σ_b preserving these additional structures [**36**, Lemma 4.14]. The Igusa variety Ig_{Σ_b} is a projective system $\{\mathrm{Ig}_{\Sigma_b,U^p,m}\}$ over open compact subgroups U^p (which are small enough) and positive integers m. Each $\mathrm{Ig}_{\Sigma_b,U^p,m}$ is a finite Galois covering of the locus in $\overline{\mathrm{Sh}}_{U^p}^{(b)}$ where the fibres of the universal abelian scheme have their associated Barsotti–Tate groups isomorphic to Σ_b. The representation ξ determines an l-adic local system on each $\mathrm{Ig}_{\Sigma_b,U^p,m}$, to be written as $\mathscr{L}_\xi$ by abuse of notation. Define

$$H_c(\mathrm{Ig}_{\Sigma_b}, \mathscr{L}_\xi) := \sum_k (-1)^k \varinjlim_{U^p,m} H_c^k(\mathrm{Ig}_{\Sigma_b,U^p,m}, \mathscr{L}_\xi),$$

where we use the étale cohomology with compact support. As the summand is an admissible representation of $G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)$ for each k, we may view $H_c(\mathrm{Ig}_b, \mathscr{L}_\xi)$ as a virtual representation in $\mathrm{Groth}(G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p))$.

4.2. Kottwitz triples and Kottwitz invariant

Definition 4.2. By a *Kottwitz triple* (of type b), we mean a triple $(\gamma_0;\gamma,\delta)$ where

- $\gamma_0 \in G(\mathbb{Q})$ is semisimple, and elliptic in $G(\mathbb{R})$,
- $\gamma \in G(\mathbb{A}^{\infty,p})$ and $\gamma \sim \gamma_0$ in $G(\bar{\mathbb{A}}^{\infty,p})$,
- $\delta \in J_b(\mathbb{Q}_p)$ is ν_b-acceptable and $\delta \sim \gamma_0$ in $G(\bar{\mathbb{Q}}_p)$ via any $\bar{\mathbb{Q}}_p$-embedding $J_b \hookrightarrow G$ whose $G(\bar{\mathbb{Q}}_p)$-conjugacy class is canonical (§ 3.2); we will simply write $\delta \sim_{\mathrm{st}} \gamma_0$ for the last condition.

Two triples $(\gamma_0;\gamma,\delta)$ and $(\gamma_0';\gamma',\delta')$ are considered *equivalent* if $\gamma_0 \sim_{\mathrm{st}} \gamma_0'$ in $G(\mathbb{Q})$, $\gamma\sim\gamma'$ in $G(\mathbb{A}^{\infty,p})$, and $\delta\sim\delta'$ in $J_b(\mathbb{Q}_p)$.

Let $(\gamma_0;\gamma,\delta)$ be a Kottwitz triple. We briefly recall the definition of $\alpha(\gamma_0,\gamma,\delta) \in \mathfrak{K}(I_0/\mathbb{Q})^D$, leaving details to [**36**, § 10]. For each place v of $\mathbb{Q}$, we can define $\alpha_v(\gamma_0;\gamma,\delta) \in X^*(Z(\hat{I}_0)^{\Gamma(v)})$, which will be written temporarily as α_v for simplicity. For $v \neq p,\infty$, the invariant α_v equals $\mathrm{inv}_v(\gamma_0,\gamma_v)$ of [**19**, p. 169]. The definition of α_p is reproduced below in § 4.3. See [**36**, § 10] for α_∞.

Recall that α_p (respectively α_∞) restricts to $-\mu_1$ in $X^*(Z(\hat{G})^{\Gamma(p)})$ (respectively μ_1 in $X^*(Z(\hat{G})^{\Gamma(\infty)})$). Also note that $\alpha_v|_{Z(\hat{G})^{\Gamma(v)}}$ is trivial for $v \neq p,\infty$. For each place v we extend α_v to an element $\tilde{\alpha}_v$ of $X^*(Z(\hat{I}_0)^{\Gamma(v)}Z(\hat{G}))$ such that

$$\tilde{\alpha}_v(\gamma_0,\gamma,\delta)|_{Z(\hat{G})} = \begin{cases} 1 & \text{if } v \neq p,\infty, \\ -\mu_1 & \text{if } v = p, \\ \mu_1 & \text{if } v = \infty. \end{cases} \tag{4.1}$$

In view of (2.2), we make the following definition. It makes sense to view $\tilde{\alpha}_v$ $(v \neq p,\infty)$ and $\tilde{\alpha}_p\tilde{\alpha}_\infty$ as characters of $\mathfrak{K}(I_0/\mathbb{Q})$ since each of them is trivial on $Z(\hat{G})$:

$$\alpha(\gamma_0;\gamma,\delta) := \left(\prod_{v\neq p,\infty} \tilde{\alpha}_v|_{\mathfrak{K}(I_0/\mathbb{Q})}\right)\cdot(\tilde{\alpha}_p\tilde{\alpha}_\infty)|_{\mathfrak{K}(I_0/\mathbb{Q})}. \tag{4.2}$$

To clarify what input $\tilde{\alpha}$ depends on, it is helpful to write $\tilde{\alpha}_v$ $(v \neq p,\infty)$, $\tilde{\alpha}_p$, $\tilde{\alpha}_\infty$ as $\tilde{\alpha}_v(\gamma_0,\gamma)$ $(v \neq p,\infty)$, $\tilde{\alpha}_p(\gamma_0,\delta)$, $\tilde{\alpha}_\infty(\gamma_0)$, respectively. Put $\alpha_v(\gamma_0;\gamma) := \tilde{\alpha}_v|_{\mathfrak{K}(I_0/\mathbb{Q})}$.

If $(\gamma_0;\gamma,\delta)$ and $(\gamma_0';\gamma',\delta')$ are equivalent then I_0 and $I_0' := Z_G(\gamma_0')$ are inner forms over $\mathbb{Q}$. In that case $\alpha(\gamma_0;\gamma,\delta)$ and $\alpha(\gamma_0';\gamma',\delta')$ are identified using the canonical Γ-equivariant isomorphism $Z(\hat{I}_0) \xrightarrow{\sim} Z(\hat{I}_0')$.

Denote by KT_b the set of equivalence classes of Kottwitz triples. Let $\mathrm{KT}_b^{\mathrm{eff}}$ denote the subset of KT_b consisting of $(\gamma_0;\gamma,\delta)$ such that $\alpha(\gamma_0;\gamma,\delta)$ is trivial.

4.3. Definition of $\tilde{\alpha}_p(\gamma_0,\delta)$

We will give a definition of the Kottwitz invariant at p which is convenient for our purpose. It is not hard to see that our definition is equivalent to the one given by [**36**, § 10]. Let us freely use the notation of § 3.2. In particular, $\psi : J_b \xrightarrow{\sim} M_b$ is an isomorphism over L_s and satisfies $\psi\psi^{-\sigma} = \mathrm{Int}(\tilde{b})$. Let $(\gamma_0;\gamma,\delta) \in \mathrm{KT}_b$ such that $\gamma_0 \in M_b(\mathbb{Q}_p)$. Then there exists $y \in M_b(L)$ such that $\psi(\delta) = y\gamma_0y^{-1}$. (First find $x \in M_b(\bar{L})$ such that $\psi(\delta) = x\gamma_0x^{-1}$. Since Steinberg's vanishing theorem says $H^1(L, I_0) = 1$, we can replace x by some $y \in M_b(L)$.) In fact, we could find y in $M_b^{\mathrm{der}}(L)$ by the same argument. It is easy to see that $\tilde{b}_\delta := y^{-1}\tilde{b}y^\sigma$ belongs to $I_0(L)$, thus yields an element $b_\delta \in B(I_0)$, which is independent of the choice of y. Define

$$\alpha_p(\gamma_0,\delta) := \kappa_{I_0}(b_\delta).$$

Lemma 4.3. *The above element b_δ is basic in $B(I_0)$.*

Proof. Clearly, b_δ maps to b under the map $B(I_0) \to B(M_b)$ induced by the inclusion $I_0 \hookrightarrow M_b$. Since b is basic in $B(M_b)$ (as noted in § 3.2), b_δ must be basic in $B(I_0)$. The last implication easily follows from [**33**, Proposition 1.12.(i)]. □

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4.4. Point-counting formula for Igusa varieties

Put $K_p^{\mathrm{hs}} := U_p^{\mathrm{hs}}$ (§ 4.1). For each finite place $v \neq p$ where $G_{\mathbb{Q}_v}$ is unramified, choose a hyperspecial subgroup $K_v^{\mathrm{hs}} \subset G(\mathbb{Q}_v)$. These data enable us to define $C_c^\infty(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ via restricted product. An *acceptable* function $\phi \in C_c^\infty(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ is defined in [**36**, Definition 6.2] to be a finite linear combination of functions of the form $\phi^p \times \phi_p$ such that ϕ_p is supported on ν_b-acceptable elements of $J_b(\mathbb{Q}_p)$ and a few other conditions hold. These other conditions ensure that Fujiwara's fixed point formula (also known as Deligne's conjecture) for algebraic correspondences is applicable in the course of the proof of Theorem 4.4, but do not concern us in the stabilization process. In this section ϕ takes values in $\bar{\mathbb{Q}}_l$, but will have values in $\mathbb{C}$ starting from § 5.

We introduce some notation. Let $\gamma_0 \in G(\mathbb{Q})$ be an $\mathbb{R}$-elliptic semisimple element. Write I_0 for $Z_G(\gamma_0)$ as usual and I_∞ for a compact-mod-centre inner form of I_0 over $\mathbb{R}$. Denote by $I_0(\mathbb{A})^1$ the kernel of the map $I_0(\mathbb{A}) \to \mathbb{R}_{>0}^\times$ given by $x \mapsto |\varpi(x)|_{\mathbb{A}^\times}$ where $\varpi : G \to \mathbb{G}_m$ is the multiplier map. Define $G(\mathbb{A})^1$ similarly and set $G(\mathbb{R})^1 := G(\mathbb{A})^1 \cap G(\mathbb{R})$.

Let us explain the choice of Haar measures in Theorem 4.4 below. Fix Haar measures on $G(\mathbb{A}^{\infty,p})$ and $J_b(\mathbb{Q}_p)$ once and for all. Choose the Tamagawa measure on $I_0(\mathbb{A})^1$ and any Haar measure on $I_0(\mathbb{R})^1$, and give $I_0(\mathbb{A}^\infty)$ the quotient measure via the exact sequence

$$1 \to I_0(\mathbb{R})^1 \to I_0(\mathbb{A})^1 \to I_0(\mathbb{A}^\infty) \to 1.$$

Haar measures on $Z_G(\gamma)(\mathbb{Q}_v)$ $(v \neq p, \infty)$, $I_\delta(\mathbb{Q}_p)$ and $I_\infty(\mathbb{R})^1$ are defined compatibly with those on $I_0(\mathbb{Q}_v)$, $I_0(\mathbb{Q}_p)$ and $I_0(\mathbb{R})^1$, respectively. (In fact, our notation $G(\mathbb{A})^1$ coincides with that of [**2**, p. 16], where Arthur gives a natural decomposition $G(\mathbb{A}) = G(\mathbb{A})^1 \times A_G(\mathbb{R})^0$. In our case $A_G(\mathbb{R})^0 \simeq \mathbb{R}_{>0}^\times$. The same applies to I_0 in place of G.)

Theorem 4.4 (Shin [36, Theorem 13.1]). *If $\phi \in C_c^\infty(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ is acceptable, then*

$$\operatorname{tr}(\phi \mid H_c(\mathrm{Ig}_{\Sigma_b}, \mathscr{L}_\xi)) = \sum_{(\gamma_0;\gamma,\delta)\in \mathrm{KT}_b^{\mathrm{eff}}} \mathrm{vol}(I_\infty(\mathbb{R})^1)^{-1} |A_{\mathbb{Q}}(I_0)| \cdot \operatorname{tr}\xi(\gamma_0) \cdot \mathrm{O}_{(\gamma,\delta)}^{G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)}(\phi). \quad (4.3)$$

Even though Theorem 4.4 is valid with any Haar measures on $G(\mathbb{A}^{\infty,p})$ and $J_b(\mathbb{Q}_p)$, we make a particular choice of measures for future convenience. Choose Haar measures μ_v on $G(\mathbb{Q}_v)$ for each v so that whenever $G_{\mathbb{Q}_v}$ is unramified, $\mu_v(K_v^{\mathrm{hs}}) = 1$. For any finite set S of places of $\mathbb{Q}$, take the Haar measure $\prod_{v\notin S} \mu_v$ on $G(\mathbb{A}^S)$. Choose the Tamagawa measure on $G(\mathbb{A})^1$. The measure on $G(\mathbb{R})^1$ is determined by the condition that the quotient measure on $G(\mathbb{A}^\infty)$ via the exact sequence $1 \to G(\mathbb{R})^1 \to G(\mathbb{A})^1 \to G(\mathbb{A}^\infty) \to 1$ is equal to $\prod_{v\neq\infty} \mu_v$. We can arrange that the measure on $G(\mathbb{R})$ induces the usual measure $\mathrm{d}x/x$ on $A_G(\mathbb{R})^0 = \mathbb{R}_{>0}^\times$ via the exact sequence $1 \to G(\mathbb{R})^1 \to G(\mathbb{R}) \to A_G(\mathbb{R})^0 \to 1$. The Haar measure on $M_b(\mathbb{Q}_p)$ is chosen such that $K_p^{\mathrm{hs}} \cap M_b(\mathbb{Q}_p)$ has measure 1. The measure on $J_b(\mathbb{Q}_p)$ is chosen to be compatible with that on $M_b(\mathbb{Q}_p)$.

With Theorem 4.4 as a starting point, our main goal is to obtain a stable trace formula for $\operatorname{tr}(\varphi|H_c(\mathrm{Ig}_{\Sigma_b}, \mathscr{L}_\xi))$. This means that we rewrite the right-hand side of (4.3) in terms of stable orbital integrals on elliptic endoscopic groups for G.

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5. Stabilization away from p

In this section, we assume that the function $\phi \in C_c^\infty(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ is acceptable and has the form

$$\phi = \prod_{v \neq \infty} \phi_v \quad \text{for } \phi_v \in C_c^\infty(G(\mathbb{Q}_v))\ (v \neq p, \infty),\ \phi_p \in C_c^\infty(J_b(\mathbb{Q}_p)). \tag{5.1}$$

Put $\phi^p := \prod_{v \neq p,\infty} \phi_v$. From here on, every test function including ϕ will assume values in $\mathbb{C}$ (rather than $\bar{\mathbb{Q}}_l$). Fix $\iota_l : \bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ once and for all.

5.1. A first step in stabilization

We know from (1.2) and (2.3) (cf. Lemma 4.1) that

$$|A_{\mathbb{Q}}(I_0)| \cdot |\mathfrak{K}(I_0/\mathbb{Q})|^{-1} = \tau(G) \cdot |\ker^1(\mathbb{Q}, G)|.$$

As $\mathfrak{K}(I_0/\mathbb{Q})$ is a finite abelian group, we have

$$|\mathfrak{K}(I_0/\mathbb{Q})|^{-1} \sum_{\kappa \in \mathfrak{K}(I_0/\mathbb{Q})} \langle \alpha(\gamma_0;\gamma,\delta), \kappa\rangle = \begin{cases} 1, & \alpha(\gamma_0;\gamma,\delta) \text{ is trivial,} \\ 0, & \text{otherwise.} \end{cases}$$

Hence (4.3) can be rewritten as

$$\begin{aligned} \operatorname{tr}(\phi|\iota_l H_c(\mathrm{Ig}_{\Sigma_b}, \mathscr{L}_\xi)) = \tau(G)|\ker^1(\mathbb{Q}, G)| &\sum_{(\gamma_0;\gamma,\delta)\in \mathrm{KT}_b} \mathrm{vol}(I_\infty(\mathbb{R})^1)^{-1} \\ &\times \sum_{\kappa \in \mathfrak{K}(I_0/\mathbb{Q})} \langle \alpha(\gamma_0;\gamma,\delta), \kappa\rangle \operatorname{tr}\xi(\gamma_0) \cdot \mathrm{O}^{G(\mathbb{A}^{\infty,p})\times J_b(\mathbb{Q}_p)}_{(\gamma,\delta)}(\phi). \end{aligned} \tag{5.2}$$

As remarked in § 2.1, we fix once and for all a representative (H, s, η) in each isomorphism class of elliptic endoscopic triples for G and view $\mathcal{E}^{\mathrm{ell}}(G)$ as the set of such representatives. For each $(H, s, \eta) \in \mathcal{E}^{\mathrm{ell}}(G)$, we also fix an L-group morphism $\tilde{\eta} : {}^L H \to {}^L G$ extending η once and for all. Fix Haar measures on $H(\mathbb{Q}_v)$ for each v in the same way as we did for $G(\mathbb{Q}_v)$ in the paragraph below Theorem 4.4.

Each pair (γ_0, κ) in the sum of (5.2) can be viewed as an element of $\mathcal{SS}^{\mathrm{ell}}(G)$, which corresponds by Lemma 2.8 to (H, s, η, γ_H) whose isomorphism class in $\mathcal{EQ}^{\mathrm{ell}}(G)$ is uniquely determined. Define $\tilde{\kappa} \in \tilde{\mathfrak{K}}(I_0/\mathbb{Q})$ as in Remark 2.7. By (4.2),

$$\langle \alpha(\gamma_0;\gamma,\delta), \kappa\rangle = \Bigg(\prod_{v\neq p,\infty} \langle \alpha_v(\gamma_0,\gamma), \kappa\rangle\Bigg) \langle \tilde{\alpha}_p(\gamma_0,\delta), \tilde{\kappa}\rangle \langle \tilde{\alpha}_\infty(\gamma_0), \tilde{\kappa}\rangle.$$

If v is a finite place where G is unramified (except finitely many v with small residue characteristics), the transfer factor $\Delta_v(\gamma_H, \gamma_0)$ is pinned down by the formula in Conjecture 2.14 (with $K = \mathbb{Q}_v$) as the relevant Haar measures are fixed. At the other places v,

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the factors $\Delta_v(\gamma_H, \gamma_0)$ are well-defined only up to constant, but will be chosen compatibly so that the following global constraint is satisfied whenever $\gamma_0 \in G(\mathbb{Q})$ [**27**, § 6]:

$$\prod_v \Delta_v(\gamma_H, \gamma_0) = 1. \tag{5.3}$$

Note that $\Delta_v(\gamma_H, \gamma_0) \neq 1$ for only finitely many v. For any $\gamma' \in G(\mathbb{A})$ such that $\gamma' \sim \gamma_0$ in $G(\bar{\mathbb{A}})$, transfer factors satisfy

$$\Delta_v(\gamma_H, \gamma') = \langle \mathrm{inv}_v(\gamma_0, \gamma'), \kappa \rangle \Delta_v(\gamma_H, \gamma_0). \tag{5.4}$$

Put $I_\delta := Z_{J_b}(\delta)$ and $I_v := Z_{G_{\mathbb{Q}_v}}(\gamma)$ for $v \neq p, \infty$.

Lemma 5.1. *Suppose that $(\gamma_0; \gamma, \delta) \in \mathrm{KT}_b$. Then $e_v(I_v) = 1$ for all but finitely many $v \neq p, \infty$. If moreover $\alpha(\gamma_0; \gamma, \delta)$ is trivial, then*

$$\biggl(\prod_{v \neq p,\infty} e_v(I_v)\biggr) e_p(I_\delta) e_\infty(I_\infty) = 1.$$

Proof. As $(I_0)_{\mathbb{Q}_v}$ is isomorphic to I_v for all but finitely many $v \neq p, \infty$, the first assertion is verified. Now assume that $\alpha(\gamma_0; \gamma, \delta)$ is trivial. By Lemma 12.3 of [**36**], there is $((A, \lambda, i), [a]) \in FP_b^{AV}$ corresponding to $(\gamma_0; \gamma, \delta)$ in the notation there. Take the $\mathbb{Q}$-group I' to be the centralizer of a in $\mathrm{End}_B^0(A)$. Then we see that $I'_{\mathbb{Q}_v}$ is isomorphic to I_v, I_δ and I_∞ when $v \neq p, \infty$, $v = p$ and $v = \infty$, respectively. It is a standard fact [**13**, Proposition, p. 297] that $\prod_v e_v(I'_{\mathbb{Q}_v}) = 1$. □

Let $\mathcal{SS}^{\mathrm{KT}}(G)$ denote the subset of $\mathcal{SS}(G)$ consisting of the pairs (γ_0, κ) for which there exist γ and δ such that $(\gamma_0; \gamma, \delta) \in \mathrm{KT}_b$. Observe that $\mathcal{SS}^{\mathrm{KT}}(G) \subset \mathcal{SS}^{\mathrm{ell}}(G)$. Let $\Delta^p(\gamma_H, \gamma_0)$ denote $\prod_{v \neq p,\infty} \Delta_v(\gamma_H, \gamma_0)$. To break up the summand of (5.2) into three parts, we consider pairs $(\gamma_0, \kappa) \in \mathcal{SS}^{\mathrm{KT}}(G)$ and make the following definitions:

$$\left.\begin{aligned} \mathrm{O}^p(\gamma_0, \kappa, \phi^p) &:= \Delta^p(\gamma_H, \gamma_0) \cdot \prod_{v \neq p,\infty} \sum_{\gamma_v \sim_{\mathrm{st}} \gamma_0} \langle \alpha_v(\gamma_0, \gamma), \kappa \rangle^{-1} \cdot e_v(I_v) \cdot \mathrm{O}_{\gamma_v}^{G(\mathbb{Q}_v)}(\phi_v), \\ \mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p) &:= \Delta_p(\gamma_H, \gamma_0) \cdot \sum_{\delta \sim_{\mathrm{st}} \gamma_0} \langle \tilde{\alpha}_p(\gamma_0, \delta), \tilde{\kappa} \rangle^{-1} \cdot e_p(I_\delta) \cdot \mathrm{O}_\delta^{J_b(\mathbb{Q}_p)}(\phi_p), \\ \mathrm{O}_\infty(\gamma_0, \tilde{\kappa}) &:= \Delta_\infty(\gamma_H, \gamma_0) \cdot \mathrm{vol}(I_\infty(\mathbb{R})^1)^{-1} \langle \tilde{\alpha}_\infty(\gamma_0), \tilde{\kappa} \rangle^{-1} \cdot e_\infty(I_\infty) \cdot \mathrm{tr}\, \xi(\gamma_0). \end{aligned}\right\} \tag{5.5}$$

The first (respectively second) sum in (5.5) runs over the set of semisimple conjugacy classes of γ_v in $G(\mathbb{Q}_v)$ (respectively δ in $J_b(\mathbb{Q}_p)$). We generalize the definition in (5.5) to the case where (γ_0, κ) is contained in $\mathcal{SS}(G)$ but not necessarily in $\mathcal{SS}^{\mathrm{KT}}(G)$, as follows. The same definition of $\mathrm{O}^p(\gamma_0, \kappa, \phi^p)$ works in this generality. The expression $\mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p)$ makes sense if we define $\mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p) := 0$ in case there is no δ such that $\delta \sim_{\mathrm{st}} \gamma_0$. Finally, $\mathrm{O}_\infty(\gamma_0, \tilde{\kappa})$ makes sense if γ_0 is elliptic in $G(\mathbb{R})$ and is defined to be zero otherwise.

By (5.2), (5.3), (5.5) and Lemma 5.1, we have

$$\begin{aligned} &\mathrm{tr}(\phi | \iota_l H_c(\mathrm{Ig}_{\Sigma_b}, \mathscr{L}_\xi)) \\ &\qquad = \tau(G) |\ker^1(\mathbb{Q}, G)| \sum_{(\gamma_0,\kappa) \in \mathcal{SS}^{\mathrm{KT}}(G)} \mathrm{O}^p(\gamma_0, \kappa, \phi^p) \mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p) \mathrm{O}_\infty(\gamma_0, \tilde{\kappa}). \end{aligned} \tag{5.6}$$

The right-hand side does not get new contributions if the sum is taken over all (γ_0, κ) in $\mathcal{SS}^{\mathrm{ell}}(G)$, or in $\mathcal{SS}(G)$.

5.2. The functions $h^{H,p}$ and h^H_∞

As before, let $(H, s, \eta) \in \mathcal{E}^{\mathrm{ell}}(G)$. The reference for this subsection is [**19**, pp. 178–179, 182–186], where Kottwitz works out stabilization for the terms away from p assuming the validity of Conjecture 2.13 and 2.14 (cf. Proposition 2.17). His method may be adapted to stabilize our terms away from p without change. We state the results of Kottwitz on the functions $h^{H,p}$ and h^H_∞, which are needed to stabilize (5.6).

Since $\alpha_v(\gamma_0, \gamma_v)$ equals $\mathrm{inv}_v(\gamma_0, \gamma_v)$ for $v \neq p, \infty$,

$$\Delta^p(\gamma_H, \gamma_0) \cdot \prod_{v\neq p,\infty} \langle \alpha_v(\gamma_0, \gamma), \kappa\rangle^{-1} = \Delta^p(\gamma_H, \gamma).$$

(Here we use the Langlands–Shelstad definition of transfer factors whereas $\mathrm{inv}_v(\cdot\,,\cdot)$ is as in [**19**, § 2].) We write $e^p(Z_G(\gamma)) := \prod_{v\neq p,\infty} e_v(Z_G(\gamma))$. The usual transfer of κ-orbital integrals yields the following lemma.

Lemma 5.2. *There exists a function $h^{H,p} \in C^\infty_c(H(\mathbb{A}^{\infty,p}))$ such that whenever a (G, H)-regular semisimple $\gamma_H \in H(\mathbb{A}^{\infty,p})$ and a semisimple $\gamma_0 \in G(\mathbb{A}^{\infty,p})$ have matching stable conjugacy classes,*

$$\mathrm{SO}^{H(\mathbb{A}^{\infty,p})}_{\gamma_H}(h^{H,p}) = \sum_{\gamma\sim_{\mathrm{st}}\gamma_0} \Delta^p(\gamma_H, \gamma) \cdot e^p(Z_G(\gamma)) \cdot \mathrm{O}^{G(\mathbb{A}^{\infty,p})}_\gamma(\phi^p) \tag{5.7}$$

and $\mathrm{SO}^{H(\mathbb{A}^{\infty,p})}_{\gamma_H}(h^{H,p}) = 0$ if the (G, H)-regular semisimple element $\gamma_H \in H(\mathbb{A}^{\infty,p})$ does not transfer to $G(\mathbb{A}^{\infty,p})$. The sum in (5.7) is taken over a set of representatives for $G(\mathbb{A}^{\infty,p})$-conjugacy classes which are $G(\bar{\mathbb{A}}^{\infty,p})$-conjugate to γ_0.

Remark 5.3. If $(H, s, \eta, \gamma_H) \mapsto (\gamma_0, \kappa)$ over $\mathbb{Q}$ (from $\mathcal{EQ}^{\mathrm{ell}}(G)$ to $\mathcal{SS}^{\mathrm{ell}}(G)$) then the right-hand side of (5.7) equals $\mathrm{O}^p(\gamma_0, \kappa, \phi^p)$ by (5.4).

We explain the construction of h^H_∞. Assume that the elliptic maximal tori of $G_\mathbb{R}$ come from those of $H_\mathbb{R}$; otherwise simply put $h^H_\infty := 0$. Under this assumption there are canonical isomorphisms among $(A_G)_\mathbb{R}$, $A_{G_\mathbb{R}}$, $(A_H)_\mathbb{R}$ and $A_{H_\mathbb{R}}$. The representation ξ of G yields a (quasi-)character $\chi_\xi : A_G(\mathbb{R})^0 \to \mathbb{C}^\times$ by restricting the central character of ξ. Consider the composition

$$W_\mathbb{R} \to {}^LH_\mathbb{R} \to {}^LG_\mathbb{R} \to {}^L(A_G)_\mathbb{R},$$

where the first map is the standard inclusion, the second is given by $\tilde{\eta}$ and the third by dualizing $A_G \hookrightarrow G$. The above composition map determines a character of $A_G(\mathbb{R})^0$, say χ. Set $\chi_H := \chi\chi_\xi$, viewed as a character of $A_H(\mathbb{R})^0 = A_G(\mathbb{R})^0$. Kottwitz constructed $h^H_\infty \in C^\infty_c(H(\mathbb{R}), \chi_H)$ as a sum of the pseudo-coefficients of certain discrete series representations of $H(\mathbb{R})$ via Shelstad's theory of real endoscopy. (See [**19**, p. 186] for the explicit formula.) In particular h^H_∞ is stable cuspidal in the sense of [**1**, p. 270]. (See [**24**, § 5.5] and [**19**, p. 186] for the fact that h^H_∞ transforms under χ_H.) Observe that if $H = G^*$ and $\tilde{\eta} = \mathrm{id}$ then $\chi_H = \chi_\xi$.

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Lemma 5.4. *There exists a function* $h_\infty^H \in C_c^\infty(H(\mathbb{R}), \chi_H)$ *such that whenever* $(H, s, \eta, \gamma_H) \mapsto (\gamma_0, \kappa)$ *over* $\mathbb{R}$,

$$\mathrm{SO}_{\gamma_H}^{H(\mathbb{R})}(h_\infty^H) = \begin{cases} \mathrm{O}_\infty(\gamma_0, \tilde{\kappa}), & \gamma_H : \textit{elliptic in } H(\mathbb{R}), \\ 0, & \textit{otherwise}, \end{cases} \tag{5.8}$$

and $\mathrm{SO}_{\gamma_H}^{H(\mathbb{R})}(h_\infty^H) = 0$ *if the* (G, H)*-regular semisimple* γ_H *does not transfer to* $G(\mathbb{R})$.

Remark 5.5. We use the convention of Langlands and Shelstad for transfer factors, but Kottwitz [**19**] uses a different normalization from theirs in that he replaces s with s^{-1}, as explained on p. 178 of that paper. (This is why we put exponents -1 in (5.5), which are not seen in Kottwitz's article.) So the formula for h_∞^H on p. 186 of [**19**] needs to be adjusted in our situation, but the validity of Lemma 5.4 remains intact. In case of PEL datum of type (A), we may take s as an order two element so that the distinction between two conventions disappears.

Remark 5.6. In fact, Kottwitz's function h_∞^H has the property that $\mathrm{SO}_{\gamma_H}^{H(\mathbb{R})}(h_\infty^H) = 0$ for every non-(G, H)-regular semisimple γ_H [**31**, Proposition 3.3.4, Remark 3.3.5].

However, Kottwitz's stabilization method does not work for $\mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p)$. (Compare $\mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p)$ with the right-hand side of the formula (7.3) of [**19**].)

6. Stabilization at p

Our goal in this section is to rewrite $\mathrm{O}_p(\tilde{\kappa}, \gamma_0, \phi_p)$ in terms of stable orbital integrals on endoscopic groups of G. This should be more than an abstract statement. For applications of our stable trace formula it is necessary to have a reasonably concrete construction of the test function h_p^H on each endoscopic group H.

6.1. Definition of various sets

Define $\mathcal{SS}_p^{\mathrm{ef}}(M_b, G)$ to be the subset of $\mathcal{SS}_p(M_b, G)$ which contains exactly those (γ_0, κ) such that $\gamma_0 \in M_b(\mathbb{Q}_p)$ is ν_b-acceptable. Similarly define the subset $\mathcal{SS}_p^{\mathrm{eff}}(J_b, G)$ of $\mathcal{SS}_p(J_b, G)$ so that it consists of the pairs (δ, κ) with ν_b-acceptable δ. The transfer of stable conjugacy classes canonically identifies $\mathcal{SS}_p^{\mathrm{eff}}(J_b, G)$ with a subset of $\mathcal{SS}_p^{\mathrm{ef}}(M_b, G)$, which will be denoted by $\mathcal{SS}_p^{\mathrm{eff}}(M_b, G)$. The injection $M_b \hookrightarrow G$ induces a map from $\mathcal{SS}_p^{\mathrm{ef}}(M_b, G)$ (respectively $\mathcal{SS}_p^{\mathrm{eff}}(M_b, G)$) to $\mathcal{SS}_p(G)$, which is an injection by Lemma 3.6. We denote the images of $\mathcal{SS}_p^{\mathrm{ef}}(M_b, G)$ and $\mathcal{SS}_p^{\mathrm{eff}}(M_b, G)$ by $\mathcal{SS}_p^{\mathrm{ef}}(G)$ and $\mathcal{SS}_p^{\mathrm{eff}}(G)$, respectively.

Let $\mathcal{EQ}_p^{\mathrm{ef}}(G)$ (respectively $\mathcal{EQ}_p^{\mathrm{eff}}(G)$) denote the image of $\mathcal{SS}_p^{\mathrm{ef}}(G)$ (respectively $\mathcal{SS}_p^{\mathrm{eff}}(G)$) under the bijection $\mathcal{SS}_p(G) \leftrightarrow \mathcal{EQ}_p(G)$ in Lemma 2.8. Let $\mathcal{EQ}_p^{\mathrm{ef}}(M_b, G)$ (respectively $\mathcal{EQ}_p^{\mathrm{eff}}(J_b, G)$) denote the image of $\mathcal{SS}_p^{\mathrm{ef}}(M_b, G)$ (respectively $\mathcal{SS}_p^{\mathrm{eff}}(J_b, G)$) under the bijection $\mathcal{SS}_p(M_b, G) \leftrightarrow \mathcal{EQ}_p(M_b, G)$ (respectively $\mathcal{SS}_p(J_b, G) \leftrightarrow \mathcal{EQ}_p(J_b, G)$) coming from Lemma 2.10. Similarly let $\mathcal{EQ}_p^{\mathrm{eff}}(M_b, G)$ denote the image of $\mathcal{SS}_p^{\mathrm{eff}}(M_b, G)$. The sets $\mathcal{EQ}_p^{\mathrm{eff}}(M_b, G)$ and $\mathcal{EQ}_p^{\mathrm{eff}}(J_b, G)$ are canonically identified. The discussion so far is put together in the following diagrams:

$$
\begin{array}{ccc}
\mathcal{SS}_p^{\mathrm{ef}}(G) & \xleftrightarrow{1\text{–}1} & \mathcal{EQ}_p^{\mathrm{ef}}(G) \\
\updownarrow{\scriptstyle 1\text{–}1} & & \\
\mathcal{SS}_p^{\mathrm{ef}}(M_b,G) & \xleftrightarrow{1\text{–}1} & \mathcal{EQ}_p^{\mathrm{ef}}(M_b,G)
\end{array}
\qquad
\begin{array}{ccc}
(\gamma_0,\kappa) & \longleftrightarrow & (H,s,\eta,\gamma_H) \\
\updownarrow & & \\
(\gamma_0',\kappa) & \longleftrightarrow & (M_H,s_H,\eta_H,\gamma_{M_H})
\end{array}
\tag{6.1}
$$

$$
\begin{array}{ccc}
\mathcal{SS}_p^{\mathrm{eff}}(G) & \xleftrightarrow{1\text{–}1} & \mathcal{EQ}_p^{\mathrm{eff}}(G) \\
\updownarrow{\scriptstyle 1\text{–}1} & & \\
\mathcal{SS}_p^{\mathrm{eff}}(M_b,G) & \xleftrightarrow{1\text{–}1} & \mathcal{EQ}_p^{\mathrm{eff}}(M_b,G) \\
\updownarrow{\scriptstyle 1\text{–}1} & & \updownarrow{\scriptstyle 1\text{–}1} \\
\mathcal{SS}_p^{\mathrm{eff}}(J_b,G) & \xleftrightarrow{1\text{–}1} & \mathcal{EQ}_p^{\mathrm{eff}}(J_b,G)
\end{array}
\qquad
\begin{array}{ccc}
(\gamma_0,\kappa) & \longleftrightarrow & (H,s,\eta,\gamma_H) \\
\updownarrow & & \\
(\gamma_0',\kappa) & \longleftrightarrow & (M_H,s_H,\eta_H,\gamma_{M_H}) \\
\updownarrow & & \updownarrow \\
(\delta,\kappa) & \longleftrightarrow & (M_H,s_H,\eta_H,\gamma_{M_H})
\end{array}
\tag{6.2}
$$

The top two rows of (6.2) come from the restriction of the diagram (6.1) to subsets.

6.2. Study of the triple (M_H, s_H, η_H)

Throughout § 6.2 we fix $(H,s,\eta) \in \mathcal{E}_p(G)$. Define $\mathcal{E}_p^{\mathrm{ef}}(M_b,G;H)$ to be the set of those isomorphism classes of (M_H,s_H,η_H) in $\mathcal{E}_p(M_b,G)$ for which there exist $\gamma_{M_H} \in M_H(\mathbb{Q}_p)$ and $\gamma_H \in H(\mathbb{Q}_p)$ such that

(i) $(M_H,s_H,\eta_H,\gamma_{M_H}) \in \mathcal{EQ}_p^{\mathrm{ef}}(M_b,G)$ and $(H,s,\eta,\gamma_H) \in \mathcal{EQ}_p^{\mathrm{ef}}(G)$, and

(ii) $(M_H,s_H,\eta_H,\gamma_{M_H})$ and (H,s,η,γ_H) correspond under the bijections in (6.1).

Similarly define the subset $\mathcal{E}_p^{\mathrm{eff}}(J_b,G;H)$ of $\mathcal{E}_p(J_b,G)$. We will explain below how we will fix a representative (M_H,s_H,η_H) for each isomorphism class in $\mathcal{E}_p^{\mathrm{ef}}(M_b,G;H)$. Moreover, we will pin down certain additional data $\tilde{\eta}_H$, $\tilde{l}_{M_b}$, $\tilde{l}_{M_H}$, i_{M_H}, ν_{M_H} for each (M_H,s_H,η_H) and give a more direct way to view the bijection $\mathcal{EQ}_p^{\mathrm{ef}}(G) \leftrightarrow \mathcal{EQ}_p^{\mathrm{ef}}(M_b,G)$ given by (6.1) (cf. Lemma 6.2). The representatives of isomorphism classes for $\mathcal{E}_p^{\mathrm{ef}}(M_b,G;H)$ will also serve as representatives for $\mathcal{E}_p^{\mathrm{eff}}(J_b,G;H)$, in view of the injection $\mathcal{E}_p^{\mathrm{eff}}(J_b,G;H) \hookrightarrow \mathcal{E}_p^{\mathrm{ef}}(M_b,G;H)$.

Consider a triple (T_H,T,j) where

- T_H is a maximal torus of H defined over $\mathbb{Q}_p$,
- T is a maximal torus of M_b defined over $\mathbb{Q}_p$ and
- $j : T_H \xrightarrow{\sim} T$ is a $\mathbb{Q}_p$-isomorphism.

From (T,T_H,j) we would like to construct a $\mathbb{Q}_p$-morphism $\nu : \mathbb{D} \to H$ and $(H_\nu,s_\nu,\eta_\nu) \in \mathcal{E}_p(M_b,G)$ in the next few paragraphs.

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Define $\nu : \mathbb{D} \to H$ by

$$\mathbb{D} \xrightarrow{\nu_b} A_{M_b} \hookrightarrow T \xrightarrow[\sim]{j^{-1}} T_H \hookrightarrow H. \tag{6.3}$$

Put $H_\nu := Z_H(\nu)$ and let $i_\nu : H_\nu \hookrightarrow H$ denote the natural embedding. We are going to complete H_ν into a G-endoscopic triple for M_b but need some preparation first. Use j to identify $X^*(T_H)$ with $X^*(T)$ and $X_*(T_H)$ with $X_*(T)$ as $\mathbb{Z}$-modules. (Here we do not consider Galois actions.) We view ν and ν_b as elements of $X_*(T)_{\mathbb{Q}}$ and $X_*(T_H)_{\mathbb{Q}}$, respectively, which are identified via j. There are the following inclusions.

$$\begin{array}{ccccc} R(M_b,T) & \subset & R(G,T) & \subset & X^*(T) \\ & & \cup & & \| \\ R(H_\nu,T_H) & \subset & R(H,T_H) & \subset & X^*(T_H) \end{array}$$

The set $R(H,T_H)$ (respectively $R(M_b,T)$) consists of the elements $\alpha \in R(G,T)$ satisfying $\alpha^\vee(s) = 1$ (respectively $\alpha \circ \nu_b = 1$). Since $R(H_\nu,T_H)$ is the set of $\alpha \in R(H,T_H)$ such that $\alpha \circ \nu = 1$, we know that $R(H_\nu,T_H) \subset R(M_b,T)$. Similar consideration shows that $R^\vee(H_\nu,T_H)$ is the subset of $R^\vee(M_b,T)$ consisting of those $\alpha^\vee$ satisfying

$$\alpha^\vee(s) = 1. \tag{6.4}$$

Now choose a maximal torus $\mathbb{T}'_H \subset \hat{H}$ and put $\mathbb{T}' := \eta(\mathbb{T}'_H)$. (These are not part of the splitting data used in the definition of ${}^LG$ or ${}^LH$.) Choose a Borel subgroup $\mathbb{B}'$ of $\hat{G}$ containing $\mathbb{T}'$, which determines a Borel subgroup $\mathbb{B}'_H$ of $\hat{H}$ via η. With the choice of Borel subgroups $B_H \subset H$ containing T_H and $B \subset G$ containing T over $\bar{\mathbb{Q}}_p$, we are given isomorphisms $\iota_H : \hat{T}'_H \simeq \mathbb{T}'_H$ and $\iota : \hat{T}' \simeq \mathbb{T}'$ as $\mathbb{C}$-tori. Without loss of generality, we may assume that the previous isomorphism $j : T_H \xrightarrow{\sim} T$ was chosen such that the dual map $\hat{T} \xrightarrow{\sim} \hat{T}_H$ of j is given by $\iota_H^{-1}\eta^{-1}\iota$ (cf. § 2.1). Let us identify $X_*(T_H) = X^*(\mathbb{T}'_H)$ and $X_*(T) = X^*(\mathbb{T}')$ via ι_H and ι. Then the identification $X_*(T_H) = X_*(T)$ via j is transported to the identification $X^*(\mathbb{T}'_H) = X^*(\mathbb{T}')$ via η. These allow us to identify $R^\vee(H_\nu,T_H) = R(\hat{H}_\nu,\mathbb{T}'_H)$, $R^\vee(H,T_H) = R(\hat{H},\mathbb{T}'_H)$, $R^\vee(M_b,T) = R(\hat{M}_b,\mathbb{T}')$ and $R^\vee(G,T) = R(\hat{G},\mathbb{T}')$.

So there is an embedding $l_{M_b} : \hat{M}_b \hookrightarrow \hat{G}$ (respectively $l_{H_\nu} : \hat{H}_\nu \hookrightarrow \hat{H}$) corresponding to the inclusion of the sets of roots of $\mathbb{T}'$ (respectively $\mathbb{T}'_H$). The images of l_{M_b} and l_{H_ν} are Levi subgroups of $\hat{G}$ and $\hat{H}$, respectively. It follows from the construction that the $\hat{G}$-conjugacy orbit of l_{M_b} (respectively $\hat{H}$-conjugacy orbit of l_{H_ν}) is exactly the orbit determined by the given embedding $M_b \hookrightarrow G$ (respectively $H_\nu \hookrightarrow H$) in the sense of the paragraph above Lemma 2.12. In particular the $\hat{G}$-conjugacy class of l_{M_b} and $\hat{H}$-conjugacy class of l_{H_ν} are well-defined regardless of the choice of $\mathbb{T}'$, $\mathbb{T}'_H$, $\mathbb{B}$, $\mathbb{B}'$, B and B_H. (Moreover, the $\hat{G}$-conjugacy class of l_{M_b} is independent of the choice of (T_H, T, j) but the $\hat{H}$-conjugacy class of l_{H_ν} depends on this choice.) Condition (6.4) ensures that the image of $\hat{H}_\nu \hookrightarrow \hat{H} \xrightarrow{\eta} \hat{G}$ is precisely the centralizer of $\eta(s)$ in $l_{M_b}(\hat{M}_b)$. (This centralizer is connected since $\hat{M}_b^{\mathrm{der}}$ is simply connected.) So there is a unique inclusion $\hat{H}_\nu \hookrightarrow \hat{M}_b$, which we call η_ν, making the diagram (6.5) commute. Let s_ν be the inverse image of s under l_{H_ν}. Observe that $\eta_\nu(\hat{H}_\nu) = Z_{\hat{M}_b}(\eta_\nu(s_\nu))$. It is a routine matter to verify that

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(H_ν, s_ν, η_ν) is a G-endoscopic triple for M_b:

$$\begin{array}{ccc} \hat{M}_b & \xrightarrow{l_{M_b}} & \hat{G} \\ {\scriptstyle \eta_\nu}\uparrow & & \uparrow{\scriptstyle \eta} \\ \hat{H}_\nu & \xrightarrow{l_{H_\nu}} & \hat{H} \end{array} \tag{6.5}$$

So far we attached to (T_H, T, j) a morphism $\nu : \mathbb{D} \to H$ and $(H_\nu, s_\nu, \eta_\nu) \in \mathcal{E}_p(M_b, G)$. Actually we are only interested in triples (T_H, T, j) arising from a quadruple $(H, s, \eta, \gamma_H) \in \mathcal{EQ}_p^{\mathrm{ef}}(G)$ in the following way. Let $(\gamma_0, \kappa) \in \mathcal{SS}_p^{\mathrm{ef}}(G)$ be the image of (H, s, η, γ_H). There are maximal tori $T_H \subset H$, $T \subset G$ defined over $\mathbb{Q}_p$ and a $\mathbb{Q}_p$-isomorphism $j : T_H \xrightarrow{\sim} T$, which (after being composed with $T \hookrightarrow G$) belongs to the canonical $G(\bar{\mathbb{Q}}_p)$-conjugacy class of embeddings $T_H \hookrightarrow G$, such that $j(\gamma_H)$ and γ_0 are $G(\bar{\mathbb{Q}}_p)$-conjugate. There also exists a ν_b-acceptable element $\gamma_0' \in M_b(\mathbb{Q}_p)$ such that $\gamma_0' = g\gamma_0 g^{-1}$ for some $g \in G(\bar{\mathbb{Q}}_p)$. We can arrange that $T' := gTg^{-1}$ and $\mathrm{Int}(g) : T \xrightarrow{\sim} T'$ are defined over $\mathbb{Q}_p$. Therefore, it is harmless to assume that T is contained in M_b and that $\gamma_0' = \gamma_0 = j(\gamma_H)$. Now that (T_H, T, j) is among the triples that we considered earlier, we have ν and (H_ν, s_ν, η_ν) attached to (T_H, T, j). Observe that $\gamma_H \in H_\nu(\mathbb{Q}_p)$. We claim that $(H_\nu, s_\nu, \eta_\nu, \gamma_H)$ is equivalent in $\mathcal{EQ}_p(M_b, G)$ to $(M_H, s_H, \eta_H, \gamma_{M_H})$ where the latter corresponds to (H, s, η, γ_H) in (6.1). Indeed, j induces an embedding $T_H \hookrightarrow M_b$ whose $M_b(\bar{\mathbb{Q}}_p)$-conjugacy class coincides with the one determined by (H_ν, s_ν, η_ν) (as in § 2.1), so $(H_\nu, s_\nu, \eta_\nu, \gamma_H)$ maps to $(\gamma_0, \kappa) \in \mathcal{SS}_p^{\mathrm{ef}}(M_b, G)$. Since (γ_0, κ) is also the image of $(M_H, s_H, \eta_H, \gamma_{M_H})$, the claim follows from Lemma 2.10. By the claim, (H_ν, s_ν, η_ν) belongs to $\mathcal{E}_p^{\mathrm{ef}}(M_b, G; H)$.

In the last paragraph, when γ_H is fixed, the choice of (T_H, T, j) is not unique. Let us investigate the dependence of ν on the choice of (T_H, T, j). Suppose that (T_H', T', j') is used to construct $\nu' : \mathbb{D} \to H$. Then $T' = mTm^{-1}$ and $T_H' = hT_Hh^{-1}$ for some $m \in M_b(\bar{\mathbb{Q}}_p)$ and $h \in H(\bar{\mathbb{Q}}_p)$. Let $j'' = \mathrm{Int}(m^{-1}) \circ j' \circ \mathrm{Int}(h)$. We know that j and j'' are in the same $G(\bar{\mathbb{Q}}_p)$-conjugacy class. Since $j(\gamma_H)$ and $j''(\gamma_H)$ are ν_b-acceptable, j and j'' are in fact $M_b(\bar{\mathbb{Q}}_p)$-conjugate. Since the $M_b(\bar{\mathbb{Q}}_p)$-conjugate action is the identity on A_{M_b}, it is easy to see that $\nu' = \mathrm{Int}(h^{-1}) \circ \nu$ in view of (6.3).

On the other hand, for $\alpha \in \mathrm{Aut}_{\mathbb{Q}_p}(H, s, \eta)$ we may replace γ_H by $\alpha(\gamma_H)$ without changing the equivalence class of (H, s, η, γ_H). Changing α by an inner automorphism of H if necessary, we may assume that $\alpha(T_H) = T_H$. Under $j_0 = j \circ \alpha^{-1}$ we see that $\alpha(\gamma_H)$ maps to a ν_b-acceptable element γ_0 in T. The morphism $\mathbb{D} \to H$ constructed from j_0 in (6.3) is given by $\alpha \circ \nu$. To sum up our discussion, the $\mathrm{Aut}_{\mathbb{Q}_p}(H, s, \eta)$-orbit of ν depends only on the equivalence class of (H, s, η, γ_H).

In fact, we can remove the dependence on γ_H in the following sense. Consider $(H, s, \eta, \gamma_H') \in \mathcal{E}_p^{\mathrm{ef}}(G)$. Let $(M_H, s_H, \eta_H, \gamma_{M_H})$ and $(M_H', s_H', \eta_H', \gamma_{M_H}')$ correspond to (H, s, η, γ_H) and (H, s, η, γ_H'), respectively. Construct $\nu : \mathbb{D} \to H$ (respectively $\nu' : \mathbb{D} \to H$) from (H, s, η, γ_H) (respectively (H, s, η, γ_H')) by choosing a triple (T_H, T, j) (respectively (T_H', T', j')).

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Lemma 6.1. *The map ν' is contained in the $\mathrm{Aut}_{\mathbb{Q}_p}(H,s,\eta)$-orbit of ν if (M_H,s_H,η_H) and (M'_H,s'_H,η'_H) are isomorphic in $\mathcal{E}^{\mathrm{ef}}_p(M_b,G;H)$.*

Proof. Arguing as in a few paragraphs above Lemma 6.1, we have that $T'=mTm^{-1}$ and $T'_H=hT_Hh^{-1}$ for some $m\in M_b(\bar{\mathbb{Q}}_p)$ and $h\in H(\bar{\mathbb{Q}}_p)$ and that $j'':=\mathrm{Int}(h^{-1})\circ j'\circ \mathrm{Int}(m)$ belongs to the $\Omega(G,T)$-orbit of j. (Unlike the previous situation we do not know whether j'' is in the $\Omega(M_b,T)$-orbit of j.) Write $j''=wj$ for $w\in\Omega(G,T)$.

When u acts on T, we write $\hat{u}$ for its dual action on $\hat{T}$. If $u\in\Omega(G,T)$ then $u\mapsto\hat{u}$ yields an isomorphism $\Omega(G,T)\xrightarrow{\sim}\Omega(\hat{G},\mathbb{T})$ (once an isomorphism $\hat{T}\simeq\mathbb{T}$ is determined by $\mathbb{B}$ and the choice of a Borel $B\supset T$). If the G-endoscopic triples (M_H,s_H,η_H) and (M'_H,s'_H,η'_H) are isomorphic, then $\hat{j}(s)$ and $\hat{j}''(s)$ are $\hat{M}_b$-conjugate in $\hat{T}/Z(\hat{G})$. This happens if and only if $\hat{w}=\hat{w}_0\hat{w}_H$ for some $w_0\in\Omega(M_b,T)$ and $w_H\in\Omega(G,T)$ such that $\hat{w}_H(\eta(s))\cong\eta(s) \mod Z(\hat{G})$.

On the other hand, let us view ν and ν' as maps from $\mathbb{D}$ to T_H. The relation $j''=wj$ implies $\nu'=w\nu$. Since $w\nu=\nu$ if and only if $w\in\Omega(M_b,T)$ (acting on T_H via j), we deduce that ν' is in the $\mathrm{Aut}_{\mathbb{Q}_p}(H,s,\eta)$-orbit of ν if and only if $w=w_Hw_0$ for some $w_0\in\Omega(M_b,T)$ and $w_H\in\Omega(G,T)$ such that w_H acts on T_H (via j) is the same way as some $\alpha\in\mathrm{Aut}_{\mathbb{Q}_p}(H,s,\eta)$ fixing T_H. Such a w_H is precisely characterized by the condition that $\hat{w}_H(\eta(s))\cong\eta(s) \mod Z(\hat{G})$ in view of (ii) of Definition 2.5. So the proof is complete. □

This is a good moment to fix a representative for each isomorphism class, say Ω, in $\mathcal{E}^{\mathrm{ef}}_p(M_b,G;H)$. (This has the effect of fixing a representative for each isomorphism class in $\mathcal{E}^{\mathrm{ef}}_p(J_b,G;H)$ since $\mathcal{E}^{\mathrm{ef}}_p(M_b,G;H)=\mathcal{E}^{\mathrm{ef}}_p(J_b,G;H)$.) Choose any γ_H as in the very beginning of § 6.2 as well as (T_H,T,j), thus obtain $\nu:\mathbb{D}\to H$ and $(H_\nu,s_\nu,\eta_\nu)\in\Omega$. We will fix such ν and (H_ν,s_ν,η_ν), and use the latter as the representative for the isomorphism class Ω. The maps l_{M_b} and l_{H_ν} in (6.5) will also be fixed for (H_ν,s_ν,η_ν).

From now on, the representative (H_ν,s_ν,η_ν) will be denoted by (M_H,s_H,η_H) to save notation, and $\mathcal{E}^{\mathrm{ef}}_p(M_b,G;H)$ will be identified with the set of the representatives we just fixed. Write i_{M_H} for i_ν and $\nu^0_{M_H}$ for ν. Lemma 6.1 tells us that the $\mathrm{Aut}_{\mathbb{Q}_p}(H,s,\eta)$-orbits of i_{M_H} and $\nu^0_{M_H}$ are canonical in that they depend only on the isomorphism class of (M_H,s_H,η_H) in $\mathcal{E}^{\mathrm{ef}}_p(M_b,G;H)$. Define $\nu_{M_H}:\mathbb{D}\to M_H$ by

$$\mathbb{D}\xrightarrow{\nu_b}A_{M_b}\hookrightarrow A_{M_H}\hookrightarrow M_H,$$

where $A_{M_b}\hookrightarrow A_{M_H}$ is the canonical embedding. (See the first paragraph below Remark 2.6.) This embedding is compatible with $j^{-1}:T\xrightarrow{\sim}T_H$, so $\nu^0_{M_H}=i_{M_H}\circ\nu_{M_H}$ by the definition of ν in (6.3).

We claim that (6.5) can be extended to a commutative diagram of L-morphisms ($\hat{H}_\nu$ being rewritten as $\hat{M}_H$). Let us prove the claim. The map η_H induces an injection $(Z(\hat{M}_b)^{\Gamma(p)})^0\hookrightarrow(Z(\hat{M}_H)^{\Gamma(p)})^0$ of groups. Using Lemma 2.12 we can choose an L-morphism $\tilde{l}_{M_b}$ (respectively $\tilde{l}_{M_H}$) extending l_{M_b} (respectively l_{M_H}) such that the $\hat{G}$-conjugacy class of $\tilde{l}_{M_b}$ (respectively $\hat{H}$-conjugacy class of $\tilde{l}_{M_H}$) corresponds to the Levi embedding $M_b\hookrightarrow G$ (respectively $i_{M_H}:M_H\hookrightarrow H$) in the way described in § 2.5. The same lemma tells us that the image of $\tilde{l}_{M_b}$ (respectively $\tilde{l}_{M_H}$) is the centralizer of $(Z(\hat{M}_b)^{\Gamma(p)})^0$ in ${}^LG$ (respectively $(Z(\hat{M}_H)^{\Gamma(p)})^0$ in ${}^LH$). The commutativity of (6.5) shows that $\tilde{\eta} \circ \tilde{l}_{M_H}({}^L M_H) \subset \tilde{l}_{M_b}({}^L M_b)$, hence there is a unique L-morphism $\tilde{\eta}_H : {}^L M_H \to {}^L M_b$ which makes the following diagram commute. We will fix such an $\tilde{\eta}_H$ henceforth:

$$\begin{array}{ccc} {}^L M_b & \xrightarrow{\tilde{l}_{M_b}} & {}^L G \\ \tilde{\eta}_H \uparrow & & \uparrow \tilde{\eta} \\ {}^L M_H & \xrightarrow{\tilde{l}_{M_H}} & {}^L H \end{array} \tag{6.6}$$

There is a natural embedding

$$\iota_{M_H,H} : \mathrm{Out}^G_{\mathbb{Q}_p}(M_H, s_H, \eta_H) \hookrightarrow \mathrm{Out}_{\mathbb{Q}_p}(H, s, \eta)$$

defined as follows. (In the following we often omit the subscript if the field of definition is $\mathbb{Q}_p$.) For each $\bar{\beta} \in \mathrm{Out}^G(M_H, s_H, \eta_H)$, choose a lift $\beta \in \mathrm{Aut}^G(M_H, s_H, \eta_H)$ and also $\hat{m} \in \hat{M}_b$ such that $\mathrm{Int}(\hat{m}) \circ \eta_H = \eta_H \circ \hat{\beta}$ (cf. Definition 2.5 (i)). Note that $\hat{\beta}$ and $\hat{\beta}^\sigma$ are $\hat{M}_H$-conjugate since β is defined over $\mathbb{Q}_p$. There is a unique $\hat{\alpha}_0 \in \mathrm{Aut}_{\mathbb{C}}(\hat{H})$ such that $\eta \circ \hat{\alpha}_0 = \mathrm{Int}(l_{M_b}(\hat{m})) \circ \eta$. Since $\hat{\alpha}_0^\sigma \circ l_{M_H}$ and $l_{M_H} \circ \hat{\beta}^\sigma$ are $\hat{H}$-conjugate, we see that $\hat{\alpha}_0$ and $\hat{\alpha}_0^\sigma$ are $\hat{H}$-conjugate. Thus the $\hat{H}$-conjugacy orbit of $\hat{\alpha}_0^\sigma \hat{\alpha}_0^{-1}$ is defined over $\mathbb{Q}_p$. Choose a $\bar{\mathbb{Q}}_p$-automorphism $\alpha_0 : H \xrightarrow{\sim} H$ such that the outer automorphisms defined by α_0 and $\hat{\alpha}_0$ correspond via the canonical isomorphism $\mathrm{Out}_{\bar{\mathbb{Q}}_p}(H) \simeq \mathrm{Out}_{\mathbb{C}}(\hat{H})$. Then the $H(\bar{\mathbb{Q}}_p)$-conjugacy class of α_0 is defined over $\mathbb{Q}_p$ and we deduce that there is some $\alpha \in \mathrm{Aut}_{\mathbb{Q}_p}(H)$ which is $H(\bar{\mathbb{Q}}_p)$-conjugate to α_0. The properties of $\hat{\alpha}_0$ imply that α actually lies in $\mathrm{Aut}_{\mathbb{Q}_p}(H, s, \eta)$. Finally, we define $\iota_{M_H,H}(\bar{\beta})$ to be the image of α in $\mathrm{Out}_{\mathbb{Q}_p}(H, s, \eta)$. It is not hard to show that $\iota_{M_H,H}$ is well-defined.

Suppose that $\alpha \in \mathrm{Aut}(H, s, \eta)$ and $\beta \in \mathrm{Aut}^G(M_H, s_H, \eta_H)$ are representatives for $\bar{\alpha} \in \mathrm{Out}(H, s, \eta)$ and $\bar{\beta} \in \mathrm{Out}^G(M_H, s_H, \eta_H)$, respectively. If $\iota_{M_H,H}(\bar{\beta}) = \bar{\alpha}$ then we claim that $\alpha \circ i_{M_H}$ and $i_{M_H} \circ \beta$ are $H(\bar{\mathbb{Q}}_p)$-conjugate. Note that α (respectively β) induces $\hat{\alpha} \in \mathrm{Aut}(\hat{H})$ (respectively $\hat{\beta} \in \mathrm{Aut}(\hat{M}_H)$) which is well-defined up to $\mathrm{Int}(\hat{H})$ (respectively $\mathrm{Int}(\hat{M}_H)$) and that there exists $\hat{g} \in \hat{G}$ (respectively $\hat{m} \in \hat{M}_b$) such that $\mathrm{Int}(\hat{g}) \circ \eta = \hat{\alpha} \circ \eta$ (respectively $\mathrm{Int}(\hat{m}) \circ \eta_H = \hat{\beta} \circ \eta_H$) and $\mathrm{Int}(\hat{g})$ (respectively $\mathrm{Int}(\hat{m})$) preserves $\eta(s)$ (respectively $\eta_H(s_H)$) up to $Z(\hat{G})$ (cf. Definitions 2.5 and 2.9). The condition $\iota_{M_H,H}(\bar{\beta}) = \bar{\alpha}$ means that $\mathrm{Int}(l_{M_b}(\hat{m}))$ and $\mathrm{Int}(\hat{g})$ induce the same outer automorphism on $\hat{H}$. In other words, there exists $\hat{h} \in \hat{H}$ such that

$$\mathrm{Int}(\hat{g}) = \mathrm{Int}(l_{M_b}(\hat{m})) \circ \mathrm{Int}(\eta(\hat{h})) \quad \text{on } \eta(\hat{H}). \tag{6.7}$$

On the other hand, the $\hat{H}$-conjugacy class of l_{M_H} corresponds to the $H(\bar{\mathbb{Q}}_p)$-conjugacy class of i_{M_H} in the sense of § 2.5. So the $H(\bar{\mathbb{Q}}_p)$-conjugacy class of $\alpha \circ i_{M_H}$ (respectively $i_{M_H} \circ \beta$) corresponds to the $\hat{H}$-conjugacy class of $\mathrm{Int}(\hat{g}^{-1}) \circ l_{M_H}$ (respectively $l_{M_H} \circ \mathrm{Int}(\hat{m}^{-1})$). (Here $\mathrm{Int}(\hat{g}^{-1})$ and $\mathrm{Int}(\hat{m}^{-1})$ act on $\hat{H}$ and $\hat{M}_H$ via η and η_H, respectively.) Since $\mathrm{Int}(\hat{g}^{-1}) \circ l_{M_H}$ and $l_{M_H} \circ \mathrm{Int}(\hat{m}^{-1})$ are $\hat{H}$-conjugate by (6.7), we see that $\alpha \circ i_{M_H}$ and $i_{M_H} \circ \beta$ are $H(\bar{\mathbb{Q}}_p)$-conjugate.

Choose a finite subset $\{\alpha_r\}_{r \in \mathscr{R}}$ of $\mathrm{Aut}(H, s, \eta)$ such that the natural projection from $\{\alpha_r\}_{r \in \mathscr{R}}$ to $\mathrm{Out}(H, s, \eta) / \mathrm{Out}^G(M_H, s_H, \eta_H)$ is a bijection of sets. We may assume that there exists $r \in \mathscr{R}$ such that α_r is the identity. Define $\mathcal{I}(M_H, H)$ to be the set $\{\alpha_r \circ i_{M_H}\}_{r \in \mathscr{R}}$.

Lemma 6.2.

(i) *For each* $\gamma_{M_H} \in M_H(\mathbb{Q}_p)$ *such that* $(M_H, s_H, \eta_H, \gamma_{M_H}) \in \mathcal{EQ}_p^{\mathrm{ef}}(M_b, G; H)$, *the quadruples* $(H, s, \eta, i(\gamma_{M_H}))$ *are equivalent for all* $i \in \mathcal{I}(M_H, H)$ *and lie in* $\mathcal{EQ}_p^{\mathrm{ef}}(G)$. *If* $(M_H, s_H, \eta_H, \gamma_{M_H}) \in \mathcal{EQ}_p^{\mathrm{eff}}(M_b, G; H)$ *then* $(H, s, \eta, i(\gamma_{M_H})) \in \mathcal{EQ}_p^{\mathrm{eff}}(G)$ *for every* $i \in \mathcal{I}(M_H, H)$.

(ii) *Suppose that* $\gamma_{M_H} \in M_H(\mathbb{Q}_p)$ *and* $\gamma_H \in H(\mathbb{Q}_p)$. *If* $(M_H, s_H, \eta_H, \gamma_{M_H})$ *and* (H, s, η, γ_H) *belong to* $\mathcal{EQ}_p^{\mathrm{ef}}(M_b, G; H)$ *and* $\mathcal{EQ}_p^{\mathrm{ef}}(G)$, *respectively, and correspond to each other as in (6.1), then there exists a unique* $i \in \mathcal{I}(M_H, H)$ *such that the elements* $i(\beta(\gamma_{M_H}))$ *and* γ_H *are* $H(\bar{\mathbb{Q}}_p)$*-conjugate for some* $\beta \in \mathrm{Aut}^G(M_H, s_H, \eta_H)$.

(iii) *If* $\gamma_{M_H} \in M_H(\mathbb{Q}_p)$ *is such that* $(M_H, s_H, \eta_H, \gamma_{M_H}) \in \mathcal{EQ}_p^{\mathrm{ef}}(M_b, G; H)$ *then* γ_{M_H} *is* $i\nu_{M_H}$*-acceptable with respect to any* $i : M_H \hookrightarrow H$ *in* $\mathcal{I}(M_H, H)$.

Proof. Let us prove (i). Suppose that $(M_H, s_H, \eta_H, \gamma_{M_H})$ maps to $(\gamma_0, \kappa) \in \mathcal{SS}_p^{\mathrm{ef}}(M_b, G)$ which may also be viewed as an element of $\mathcal{SS}_p^{\mathrm{ef}}(G)$. We can choose $T_M \subset M_b$, $T_{M_H} \subset M_H$ and $j_M : T_{M_H} \xrightarrow{\sim} T_M$ defined over $\mathbb{Q}_p$ such that $\gamma_0 \sim j_M(\gamma_{M_H})$ in $M_b(\bar{\mathbb{Q}}_p)$. Let $j_{M,G} : T_{M_H} \hookrightarrow G$ be the composition of j_M with the natural injection $T_M \hookrightarrow G$. The $\Omega(G,T)$-orbit of $j_{M,G}$ and the $G(\bar{\mathbb{Q}}_p)$-conjugacy class of $j_{M,G}(\gamma_{M_H})$ are independent of $i \in \mathcal{I}(M_H, H)$. The commutativity of (6.5) ensures that $j_{M,G}$ belongs to the $G(\bar{\mathbb{Q}}_p)$-conjugacy class of the embeddings determined by (H, s, η) (§ 2.1). Therefore, $(\gamma_0, \kappa) \in \mathcal{SS}_p^{\mathrm{ef}}(G)$ is the image of $(H, s, \eta, i(\gamma_{M_H}))$ for every $i \in \mathcal{I}(M_H, H)$. From this the first two assertions of (i) follow. The last assertion is proved similarly.

To prove (ii), observe that (H, s, η, γ_H) and $(H, s, \eta, i_{M_H}(\gamma_{M_H}))$ are equivalent as they have the same image in $\mathcal{SS}^{\mathrm{ef}}(G)$ by the proof of (i). So there exists $\alpha \in \mathrm{Aut}(H, s, \eta)$ such that $\gamma_H = \alpha(i_{M_H}(\gamma_{M_H}))$. It is possible to find $\alpha_r \in \mathrm{Aut}(H, s, \eta)$ which has the same image in $\mathrm{Out}(H, s, \eta)/\mathrm{Out}^G(M_H, s_H, \eta_H)$ as α. We can also find $\beta \in \mathrm{Aut}^G(M_H, s_H, \eta_H)$ such that α is $H(\bar{\mathbb{Q}}_p)$-conjugate to $\alpha_r \circ \iota_{M_H,H}(\beta)$. Here $\iota_{M_H,H}(\beta) \in \mathrm{Aut}(H, s, \eta)$ denotes any automorphism whose image in $\mathrm{Out}(H, s, \eta)$ is $\iota_{M_H,H}(\bar{\beta})$ where $\bar{\beta}$ is the image of β in $\mathrm{Out}^G(M_H, s_H, \eta_H)$. On the other hand, $\iota_{M_H,H}(\beta) \circ i_{M_H}$ is $H(\bar{\mathbb{Q}}_p)$-conjugate to $i_{M_H} \circ \beta$ by the discussion above Lemma 6.2. Therefore, γ_H and $\alpha_r \circ i_{M_H} \circ \beta(\gamma_{M_H})$ are $H(\bar{\mathbb{Q}}_p)$-conjugate.

It remains to prove (iii). Write $i = \alpha \circ i_{M_H}$ for $\alpha \in \mathrm{Aut}(H, s, \eta)$. Let $(\gamma_0, \kappa) \in \mathcal{SS}_p^{\mathrm{ef}}(M_b, G; H)$ be the image of $(M_H, s_H, \eta_H, \gamma_{M_H})$. We can choose maximal tori $T_H \subset H$, $T \subset G$ and an isomorphism $j : T_H \xrightarrow{\sim} T$ over $\mathbb{Q}_p$ such that $\gamma_0 = j(\gamma_H)$, where $M_b(\bar{\mathbb{Q}}_p)$-conjugacy of j is determined by η_H. Set $M_H^\alpha := \alpha(M_H)$, $T_H^\alpha := \alpha(T_H)$ and $j^\alpha := j\alpha^{-1}$. What must be shown is that $\alpha(\gamma_{M_H})$ is $i\nu_{M_H}$-acceptable with respect to the inclusion $M_H^\alpha \subset H$. If we identify $X^*(T) = X^*(T_H^\alpha)$ via j^α then $R(M_H^\alpha, T_H^\alpha) \subset R(M_b, T)$ and $R(H, T_H^\alpha) \subset R(G, T)$. So the ν_b-acceptability of $\gamma_0 = j^\alpha(\alpha(\gamma_{M_H}))$ implies the $i\nu_{M_H}$-acceptability of $\alpha(\gamma_{M_H})$. □

6.3. Stabilization at p

Consider $(H, s, \eta) \in \mathcal{E}_p(G)$ such that $\mathcal{E}_p^{\mathrm{eff}}(J_b, G; H)$ is non-empty. We then fix $(M_H, s_H, \eta_H) \in \mathcal{E}_p^{\mathrm{eff}}(J_b, G; H)$ until we get to (6.10). Suppose that (γ_0, κ), (γ_0', κ), (δ, κ),

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(H, s, η, γ_H) and $(M_H, s_H, \eta_H, \gamma_{M_H})$ are as in diagram (6.2) and correspond to each other. Without loss of generality we can take $\gamma_0' = \gamma_0$ and $\gamma_{M_H} = \gamma_H$. We would like to express $\mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p)$ defined in (5.5) in terms of stable orbital integrals on $H(\mathbb{Q}_p)$.

The first step is to write $\mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p)$ in terms of stable orbital integrals on $M_H(\mathbb{Q}_p)$ using the endoscopic transfer between J_b and M_H with respect to the L-morphism ${}^LM_H \xrightarrow{\tilde{\eta}_H} {}^LM_b = {}^LJ_b$. We will need to relate the transfer factor $\Delta_p(\gamma_H, \delta)^{J_b}_{M_H}$ to $\Delta_p(\gamma_H, \gamma_0)^G_H$. Fix a Haar measure on $M_H(\mathbb{Q}_p)$. In view of diagram (6.6), according to the definition of transfer factors by Langlands–Shelstad, we may and will normalize $\Delta_p(\cdot\,,\cdot)^{M_b}_{M_H}$ so that

$$\Delta_p(\gamma_H, \gamma_0)^{M_b}_{M_H} = |D^G_{M_b}(\gamma_0)|_p^{-1/2} |D^H_{M_H}(\gamma_H)|_p^{1/2} \Delta_p(\gamma_H, \gamma_0)^G_H \tag{6.8}$$

for every γ_H and γ_0 such that $(M_H, s_H, \eta_H, \gamma_H) \in \mathcal{EQ}^{\mathrm{ef}}_p(M_b, G)$ maps to $(\gamma_0, \kappa) \in \mathcal{SS}^{\mathrm{ef}}_p(M_b, G)$. (This remains true if the superscript 'ef' in the last sentence is dropped.) The factor $|D^G_{M_b}(\gamma_0)|_p^{-1/2} |D^H_{M_H}(\gamma_H)|_p^{1/2}$ comes from Δ_{IV} of [**27**, § 3.6]. Here $D^H_{M_H}$ is taken with respect to $i_{M_H} : M_H \hookrightarrow H$.

Lemma 6.3. *There is a non-zero constant c_{M_H}, depending on the normalization of the transfer factor $\Delta_p(\cdot\,,\cdot)^{J_b}_{M_H}$, such that*

$$c_{M_H} \cdot \Delta_p(\gamma_H, \delta)^{J_b}_{M_H} = \langle \tilde{\alpha}_p(\gamma_0, \delta), \tilde{\kappa} \rangle^{-1} \Delta_p(\gamma_H, \gamma_0)^{M_b}_{M_H} \tag{6.9}$$

for every γ_H, γ_0 and δ related to each other as in (6.2) (with $\gamma_H = \gamma_{M_H}$).

We postpone the proof of Lemma 6.3 to § 6.4. In § 8 we will give another proof of Lemma 6.3 in some special case and compute the constant c_{M_H} under a certain normalization of $\Delta_p(\cdot\,,\cdot)^{J_b}_{M_H}$.

Remark 6.4. Lemma 6.3 is easily verified when (γ_0, κ) comes from (H, s, η, γ_H) for $(H, s, \eta) = (G^*, 1, \mathrm{id})$. Note that there is a unique isomorphism class in $\mathcal{E}^{\mathrm{eff}}_p(J_b, G; G^*)$ which is represented by $(M_H, s_H, \eta_H) = (M_b, 1, \mathrm{id})$. Take $\tilde{\eta} = \mathrm{id}$ and $\tilde{\eta}_H = \mathrm{id}$. In (6.9), $\langle \tilde{\alpha}_p(\gamma_0, \delta), \tilde{\kappa} \rangle = 1$ and we may naturally take

$$\Delta_p(\cdot\,,\cdot)^{M_b}_{M_H} \equiv 1 \quad \text{and} \quad \Delta_p(\cdot\,,\cdot)^{J_b}_{M_H} \equiv e_p(J_b).$$

Then Lemma 6.3 is satisfied with $c_{M_H} = e_p(J_b)$.

Let us go back to the task of rewriting $\mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p)$. The identities (6.8) and (6.9) tell us that

$$\mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p) = \sum_{\delta \sim_{\mathrm{st}} \gamma_0} c_{M_H} \cdot \Delta_p(\gamma_H, \delta)^{J_b}_{M_H} \cdot e(I_\delta) \cdot |D^G_{M_b}(\gamma_0)|_p^{1/2} |D^H_{M_H}(\gamma_H)|_p^{-1/2} \cdot \mathrm{O}^{J_b(\mathbb{Q}_p)}_\delta(\phi_p).$$

Since we assume γ_0 is ν_b-acceptable, $|D^G_{M_b}(\gamma_0)|_p = \delta_{P(\nu_b)}(\gamma_0)$ by Lemma 3.4. Define a character $\bar{\delta}_{P(\nu_b)} : J_b(\mathbb{Q}_p) \to \mathbb{C}^\times$ by the formula $\bar{\delta}_{P(\nu_b)}(\delta) = \delta_{P(\nu_b)}(\gamma_0)$ where $\gamma_0 \in M_b(\mathbb{Q}_p)$ is the element whose stable conjugacy class matches that of δ. The function $\phi^0_p := \phi_p \cdot \bar{\delta}^{1/2}_{P(\nu_b)}$ belongs to $C^\infty_c(J_b(\mathbb{Q}_p))$. Let $\phi^{M_H}_p \in C^\infty_c(M_H(\mathbb{Q}_p))$ be a matching function for ϕ^0_p via Conjecture 2.13 (and Proposition 2.17). Then

$$\mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p) = c_{M_H} \cdot |D^H_{M_H}(\gamma_H)|_p^{-1/2} \cdot \mathrm{SO}^{M_H(\mathbb{Q}_p)}_{\gamma_H}(\phi^{M_H}_p)$$

and $\mathrm{SO}^{M_H(\mathbb{Q}_p)}_{\gamma'_H}(\phi_p^{M_H}) = 0$ if the (J_b, M_H)-regular semisimple element γ'_H transfers to $\delta \in J_b(\mathbb{Q}_p)$ which is not ν_b-acceptable or if γ'_H does not transfer to $J_b(\mathbb{Q}_p)$. The last fact and Lemma 6.2 (iii) imply that

$$\mathrm{SO}^{M_H(\mathbb{Q}_p)}_{\gamma'_H}(\phi_p^{M_H}) = 0$$

unless the (J_b, M_H)-regular semisimple γ'_H is $i\nu_{M_H}$-acceptable for every $i : M_H \hookrightarrow H$ in $\mathcal{I}(M_H, H)$. Since the set of $i\nu_{M_H}$-acceptable elements is open in $M_H(\mathbb{Q}_p)$ for every $i \in \mathcal{I}(M_H, H)$ (Lemma 3.4), it is possible to choose $\phi_p^{M_H}$ so that $\phi_p^{M_H}(\gamma'_H) = 0$ unless γ'_H (not necessarily semisimple) is $i\nu_{M_H}$-acceptable for every $i \in \mathcal{I}(M_H, H)$. Define $\phi_p^{M_H,i} \in C_c^\infty(M_H(\mathbb{Q}_p))$ by $\phi_p^{M_H,i} := \phi_p^{M_H} \cdot \delta^{-1/2}_{P(i\nu_{M_H})}$. For every $i \in \mathcal{I}(M_H, H)$, γ_H is $i\nu_{M_H}$-acceptable by Lemma 6.2 (iii) and

$$|D^H_{M_H}(\gamma_H)|_p = \delta_{P(\nu^0_{M_H})}(\gamma_H) = \delta_{P(i\nu_{M_H})}(\gamma_H)$$

where the first equality follows from Lemma 3.4 and the second is obvious. Applying Lemma 3.9 to $\phi_p^{M_H,i}$ with respect to each $i : M_H \hookrightarrow H$, we find a function $\tilde{\phi}_p^{M_H,i} \in C_c^\infty(H(\mathbb{Q}_p))$ and get

$$\mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p) = c_{M_H} \cdot \mathrm{SO}^{H(\mathbb{Q}_p)}_{i(\gamma_H)}(\tilde{\phi}_p^{M_H,i}). \tag{6.10}$$

Note that $\tilde{\phi}_p^{M_H,i}$ depends on ϕ_p, (M_H, s_H, η_H) and i but not on γ_0, δ and γ_H. (As long as (δ, κ) and (γ_0, κ) give rise to the same (M_H, s_H, η_H) in diagram (6.2).)

Define a function $h_p^H \in C_c^\infty(H(\mathbb{Q}_p))$ by

$$h_p^H := \sum_{(M_H, s_H, \eta_H)} \sum_i c_{M_H} \cdot \tilde{\phi}_p^{M_H,i},$$

where the first sum runs over $\mathcal{E}_p^{\mathrm{eff}}(J_b, G; H)$ and the second over $\mathcal{I}(M_H, H)$. The upshot of § 6 is the following lemma.

Lemma 6.5. *Suppose that $(H, s, \eta) \in \mathcal{E}_p(G)$ is such that $\mathcal{E}_p^{\mathrm{eff}}(J_b, G; H)$ is non-empty. For every (G, H)-regular semisimple $\gamma_H \in H(\mathbb{Q}_p)$,*

$$\mathrm{SO}^{H(\mathbb{Q}_p)}_{\gamma_H}(h_p^H) = \mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p) \tag{6.11}$$

if $(H, s, \eta, \gamma_H) \in \mathcal{EQ}_p^{\mathrm{eff}}(G)$ and

$$\mathrm{SO}^{H(\mathbb{Q}_p)}_{\gamma_H}(h_p^H) = 0 \tag{6.12}$$

otherwise.

Proof of (6.12) when $(H, s, \eta, \gamma_H) \notin \mathcal{EQ}_p^{\mathrm{eff}}(G)$. We prove $\mathrm{SO}^{H(\mathbb{Q}_p)}_{\gamma_H}(\tilde{\phi}_p^{M_H,i}) = 0$ for each $(M_H, s_H, \eta_H) \in \mathcal{E}_p^{\mathrm{eff}}(J_b, G; H)$ and $i \in \mathcal{I}(M_H, H)$. We assume that there exists an $i\nu_{M_H}$-acceptable element $\gamma_{M_H} \in M_H(\mathbb{Q}_p)$ such that $i(\gamma_{M_H}) \sim \gamma_H$ in $H(\mathbb{Q}_p)$ as otherwise $\mathrm{SO}^{H(\mathbb{Q}_p)}_{\gamma_H}(\tilde{\phi}_p^{M_H,i}) = 0$ by Lemma 3.9. Let (γ_0, κ) be the image of $(M_H, s_H, \eta_H, \gamma_{M_H})$ under $\mathcal{EQ}_p(M_b) \to \mathcal{SS}_p(M_b)$. (Note that we do not know whether (γ_0, κ) defines an element of $\mathcal{SS}_p(M_b, G)$ as we do not know whether γ_0 is (G, M_b)-regular.) Consider the injection $\mathcal{SS}_p(J_b) \hookrightarrow \mathcal{SS}_p(M_b)$. Since $(M_H, s_H, \eta_H, \gamma_{M_H}) \notin \mathcal{EQ}_p^{\mathrm{eff}}(M_b, G)$ by Lemma 6.2 (i), there are two cases that can occur:

(i) there exist no $(\delta, \kappa) \in \mathcal{SS}_p(J_b)$ mapping to (γ_0, κ), or

(ii) there exists $(\delta, \kappa) \in \mathcal{SS}_p(J_b)$ mapping to (γ_0, κ) but δ is not ν_b-acceptable.

By the construction of $\tilde{\phi}_p^{M_H,i}$,

$$\mathrm{SO}_{\gamma_H}^{H(\mathbb{Q}_p)}(\tilde{\phi}_p^{M_H,i}) = \sum_{\delta'} \Delta_p(\gamma_H, \delta')_{M_H}^{J_b} \cdot e(Z_{J_b}(\delta')) \cdot \mathrm{O}_{\delta'}^{J_b(\mathbb{Q}_p)}(\phi_p^0),$$

where δ' runs over the set of conjugacy classes of $J_b(\mathbb{Q}_p)$ such that $\delta' \sim_{\mathrm{st}} \delta$. The right side is viewed as zero if there is no (δ, κ) mapping to (γ_0, κ). It is now clear that $\mathrm{SO}_{\gamma_H}^{H(\mathbb{Q}_p)}(\tilde{\phi}_p^{M_H,i})$ vanishes in the cases (i) and (ii) alike, noting that the orbital integral of ϕ_p^0 is non-zero only on ν_b-acceptable elements. □

Proof of (6.11) when $(H, s, \eta, \gamma_H) \in \mathcal{EQ}_p^{\mathrm{eff}}(G)$. Suppose that (M_H, s_H, η_H) and (M'_H, s'_H, η'_H) belong to $\mathcal{E}_p(J_b, G; H)$. Let $i \in \mathcal{I}(M_H, H)$ and $i' \in \mathcal{I}(M'_H, H)$. Assuming that $\mathrm{SO}_{\gamma_H}^{H(\mathbb{Q}_p)}(\tilde{\phi}_p^{M_H,i})$ and $\mathrm{SO}_{\gamma_H}^{H(\mathbb{Q}_p)}(\tilde{\phi}_p^{M'_H,i'})$ are both non-zero, we will prove that $(M_H, s_H, \eta_H) \simeq (M'_H, s'_H, \eta'_H)$ and $i = i'$. Once we have done it, (6.11) follows from (6.10).

According to Lemma 3.9, there exist $\gamma_{M_H} \in M_H(\mathbb{Q}_p)$ and $\gamma_{M'_H} \in M'_H(\mathbb{Q}_p)$ such that $i(\gamma_{M_H}), i'(\gamma_{M'_H})$ and γ_H are stably conjugate in $H(\mathbb{Q}_p)$. Let (γ_0, κ_0) (respectively (γ'_0, κ'_0)) be the image of $(M_H, s_H, \eta_H, \gamma_{M_H})$ (respectively $(M'_H, s'_H, \eta'_H, \gamma_{M'_H})$) in $\mathcal{SS}_p(M_b, G)$. The images of (γ_0, κ_0) and (γ'_0, κ'_0) in $\mathcal{SS}_p(G)$ are equivalent since both correspond to (H, s, η, γ_H) via the bijection $\mathcal{SS}_p(G) \leftrightarrow \mathcal{EQ}_p(G)$. In particular, $\gamma_0 \sim_{\mathrm{st}} \gamma'_0$ in $G(\mathbb{Q}_p)$.

We know that $(M_H, s_H, \eta_H, \gamma_{M_H}) \in \mathcal{E}_p^{\mathrm{eff}}(M_b, G; H)$. Indeed, if this were not true, the argument in the previous part of the current proof shows that $\mathrm{SO}_{\gamma_H}^{H(\mathbb{Q}_p)}(\tilde{\phi}_p^{M_H,i})$ vanishes. Similarly $(M'_H, s'_H, \eta'_H, \gamma'_{M_H}) \in \mathcal{E}_p^{\mathrm{eff}}(M_b, G; H)$. So both γ_0 and γ'_0 are ν_b-acceptable in $M_b(\mathbb{Q}_p)$. Lemma 3.5 shows that $\gamma_0 \sim_{\mathrm{st}} \gamma'_0$ in $M_b(\mathbb{Q}_p)$, which implies that (γ_0, κ_0) and (γ'_0, κ'_0) are equivalent in $\mathcal{SS}_p(M_b, G)$. Therefore, $(M_H, s_H, \eta_H, \gamma_{M_H})$ and $(M'_H, s'_H, \eta'_H, \gamma'_{M_H})$ are equivalent in $\mathcal{E}_p^{\mathrm{eff}}(M_b, G; H)$. Finally, we deduce $i = i'$ from Lemma 6.2 (ii). □

So far we assumed that $\mathcal{E}_p^{\mathrm{eff}}(J_b, G; H)$ is non-empty for (H, s, η). Now for arbitrary $(H, s, \eta) \in \mathcal{E}_p(G)$ such that $\mathcal{E}_p^{\mathrm{eff}}(J_b, G; H)$ is empty, we define $h_p^H := 0$. The conclusion of Lemma 6.5 holds in this case since (H, s, η, γ_H) never lies in $\mathcal{EQ}_p^{\mathrm{eff}}(G)$.

6.4. Proof of Lemma 6.3

Subsection 6.4 is devoted to the proof of Lemma 6.3. We recall the setting. Fix

$$(H, s, \eta) \in \mathcal{E}_p(G), \qquad (M_H, s_H, \eta_H) \in \mathcal{E}_p^{\mathrm{ef}}(M_b, G; H).$$

Suppose that $(\gamma_0, \kappa), (\bar{\gamma}_0, \bar{\kappa}) \in \mathcal{SS}_p^{\mathrm{eff}}(M_b, G)$ and $(\delta, \kappa), (\bar{\delta}, \bar{\kappa}) \in \mathcal{SS}_p^{\mathrm{eff}}(J_b, G)$ correspond via the bijection $\mathcal{SS}_p^{\mathrm{eff}}(M_b, G) \leftrightarrow \mathcal{SS}_p^{\mathrm{eff}}(J_b, G)$, respectively. Also suppose that (γ_0, κ) and $(\bar{\gamma}_0, \bar{\kappa})$ correspond to $(M_H, s_H, \eta_H, \gamma_H)$ and $(M_H, s_H, \eta_H, \bar{\gamma}_H)$ via (6.2), respectively (by setting $\gamma'_0 = \gamma_0$, $\gamma_{M_H} = \gamma_H$, etc.).

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Regular case

First we consider the case of regular elements and later extend the proof to the general case. Suppose that γ_0 and $\bar{\gamma}_0$ are regular in M_b. Set $T_0 := Z_{M_b}(\gamma_0)$, $\bar{T}_0 := Z_{M_b}(\bar{\gamma}_0)$, $T := Z_{J_b}(\delta)$, $\bar{T} := Z_{J_b}(\bar{\delta})$, $T_H := Z_{M_H}(\gamma_H)$, $\bar{T}_H := Z_{M_H}(\bar{\gamma}_H)$.* Also set $T_0^{\mathrm{der}} := T_0 \cap M_b^{\mathrm{der}}$, $\bar{T}_0^{\mathrm{der}} := \bar{T}_0 \cap M_b^{\mathrm{der}}$ and $Z^{\mathrm{der}} := Z(M_b^{\mathrm{der}})$. Recall from § 3.2 that the L_s-isomorphism $\psi : J_b \xrightarrow{\sim} M_b$ satisfies $\psi\psi^{-\sigma} = \mathrm{Int}(\tilde{b})$ for the arithmetic Frobenius σ. We can choose $x, \bar{x} \in M_b^{\mathrm{der}}(\bar{\mathbb{Q}}_p)$ and $y, \bar{y} \in M_b^{\mathrm{der}}(L)$ such that

$$\psi(\delta) = x\gamma_0 x^{-1} = y\gamma_0 y^{-1}, \qquad \psi(\bar{\delta}) = \bar{x}\bar{\gamma}_0\bar{x}^{-1} = \bar{y}\bar{\gamma}_0\bar{y}^{-1}. \tag{6.13}$$

(To find such y and $\bar{y}$, use the argument of § 4.3.) Set

$$c = x^{-1}y, \qquad \bar{c} = \bar{x}^{-1}\bar{y}. \tag{6.14}$$

It can be seen from (6.13) that $c \in T_0^{\mathrm{der}}(\bar{L})$ and $\bar{c} \in \bar{T}_0^{\mathrm{der}}(\bar{L})$.

Recall that $\tilde{\kappa}$ and $\bar{\tilde{\kappa}}$ were defined as the images of s under the canonical Γ_p-equivariant isomorphisms $\hat{T}_H \xrightarrow{\sim} \hat{T}$ and $\hat{\bar{T}}_H \xrightarrow{\sim} \hat{\bar{T}}$. Without danger of confusion, we write s for $\tilde{\kappa}$ and $\bar{\tilde{\kappa}}$. (Of course s is viewed as elements of $\hat{T}_H$ and $\hat{\bar{T}}_H$ via $Z(\hat{H}) \hookrightarrow \hat{T}_H$ and $Z(\hat{H}) \hookrightarrow \hat{\bar{T}}_H$.) For the proof of Lemma 6.3 in the regular case, by [**27**, Corollary 4.2.B], it suffices to show that

$$\frac{\langle \tilde{\alpha}_p(\bar{\gamma}_0, \bar{\delta}), s\rangle}{\langle \tilde{\alpha}_p(\gamma_0, \delta), s\rangle} = \left\langle \mathrm{inv}\left(\frac{\gamma_H, \delta}{\bar{\gamma}_H, \bar{\delta}}\right), s_U \right\rangle, \tag{6.15}$$

where we use the notation of [**27**, § 3.4] on the right-hand side. We recall the definitions after setting up more notation.

Let $a := (\tau \mapsto a_\tau)$ be a cocycle in $Z^1(L_s/\mathbb{Q}_p, M_b^{\mathrm{ad}})$ such that a_σ has the same image in $M_b^{\mathrm{ad}}(\bar{\mathbb{Q}}_p)$ as $\tilde{b}$, so that a represents the cohomology class attached to J_b. By inflation a cocycle $\tau \mapsto b_\tau$ in $Z^1(\Gamma_p, M_b^{\mathrm{ad}})$ is obtained from a. For each $\tau \in \Gamma_p$, let $b_\tau^{\mathrm{der}} \in M_b^{\mathrm{der}}(\bar{\mathbb{Q}}_p)$ be any element whose image in $M_b^{\mathrm{ad}}(\bar{\mathbb{Q}}_p)$ is the same as b_τ. (We warn the reader that $\tau \mapsto b_\tau^{\mathrm{der}}$ is not a cocycle in $Z^1(\Gamma_p, M_b^{\mathrm{der}})$ in general.) Similarly, let $\tilde{b}^{\mathrm{der}} \in M_b(L)$ be any element which has the same image in $M_b^{\mathrm{ad}}(L)$ as $\tilde{b}$. Obviously,

$$\tilde{b}^{\mathrm{der}} = z\tilde{b} \tag{6.16}$$

for some $z \in Z(M_b)(L)$. Let $\tilde{b}^{\mathrm{ad}}$ denote the image of $\tilde{b}$ in $M_b^{\mathrm{ad}}(L_s)$. For each $m \in \mathbb{Z}_{>0}$, define

$$(\tilde{b}^{\mathrm{der}})^{(m)} := \tilde{b}^{\mathrm{der}}(\tilde{b}^{\mathrm{der}})^{\sigma} \cdots (\tilde{b}^{\mathrm{der}})^{\sigma^{m-1}}$$

and similarly define $(\tilde{b}^{\mathrm{ad}})^{(m)}$. For each $\tau \in W(\bar{L}/\mathbb{Q}_p)$ (or $\tau \in \Gamma(p)$), define $|\tau| \in \mathbb{Z}$ such that the image of τ in $W(L/\mathbb{Q}_p)$ is $\sigma^{|\tau|}$. Now define $\boldsymbol{b}_\tau^{\mathrm{der}} := b_{\mathrm{pr}(\tau)}^{\mathrm{der}}$ for each $\tau \in W(\bar{L}/\mathbb{Q}_p)$ where $\mathrm{pr} : W(\bar{L}/\mathbb{Q}_p) \to \Gamma(p)$ is the natural projection. For $\tau \in W(\bar{L}/\mathbb{Q}_p)$ with $|\tau| > 0$,

* As G^{der} is simply connected, we know $Z_G(\gamma_0)$ is connected. From this (and [**17**, Lemma 3.2]) it is not hard to see that T_0, $\bar{T}_0$, T, $\bar{T}$, T_H and $\bar{T}_H$ are connected. In other words, γ_0, $\bar{\gamma}_0$, δ, $\bar{\delta}$, γ_H and $\bar{\gamma}_H$ are automatically *strongly* regular.

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it is elementary to check that both $(\tilde{b}^{\mathrm{der}})^{|\tau|}$ and $\boldsymbol{b}_\tau^{\mathrm{der}}$ have image $(\tilde{b}^{\mathrm{ad}})^{|\tau|}$ in $M_b^{\mathrm{ad}}(\bar{L})$. In other words, we can find $z_\tau \in Z(M_b)(\bar{L})$ such that

$$(\tilde{b}^{\mathrm{der}})^{|\tau|} = z_\tau \boldsymbol{b}_\tau^{\mathrm{der}}. \tag{6.17}$$

We need to define the terms on the right-hand side of (6.15). Define a torus

$$U := (T_0^{\mathrm{der}} \times \bar{T}_0^{\mathrm{der}})/\{(z^{-1}, z) \mid z \in Z^{\mathrm{der}}\}. \tag{6.18}$$

By definition,

$$\mathrm{inv}\left(\frac{\gamma_H, \delta}{\bar{\gamma}_H, \bar{\delta}}\right)$$

is the element of $H^1(\mathbb{Q}_p, U)$ given by

$$\tau \mapsto ((x^{-1} b_\tau^{\mathrm{der}} x^\tau)^{-1}, \bar{x}^{-1} b_\tau^{\mathrm{der}} \bar{x}^\tau).$$

Now we recall the definition of s_U. Consider the following commutative diagram where every arrow is $\Gamma(p)$-equivariant. By definition, $Z(\hat{M}_b)^{\mathrm{der}} := Z(\hat{M}_b) \cap (\hat{M}_b)^{\mathrm{der}}$, $\hat{T}^{\mathrm{der}} := \hat{T} \cap (\hat{M}_b)^{\mathrm{der}}$, and $\hat{T}^{\mathrm{ad}}$ is the image of $\hat{T}$ in $\hat{M}_b^{\mathrm{ad}}$. Similarly, define $\hat{\bar{T}}^{\mathrm{der}}$ and $(\hat{\bar{T}})^{\mathrm{ad}}$. We have $Z(\hat{M}_b)^{\mathrm{der}} = Z(\hat{M}_b) \cap \hat{T}^{\mathrm{der}} = Z(\hat{M}_b) \cap \hat{\bar{T}}^{\mathrm{der}}$:

$$\begin{array}{ccc} Z(\hat{M}_H) & \xhookrightarrow{\text{diag.}} & \hat{T}_H \times \hat{\bar{T}}_H \\ \uparrow & & \downarrow \wr \\ Z(\hat{M}_b) & \xhookrightarrow{\text{diag.}} & \hat{T} \times \hat{\bar{T}} \\ \uparrow & & \uparrow \\ Z(\hat{M}_b)^{\mathrm{der}} & \xhookrightarrow{\text{diag.}} & \hat{T}^{\mathrm{der}} \times \hat{\bar{T}}^{\mathrm{der}} \end{array} \tag{6.19}$$

We can choose $z \in Z(\hat{M}_b)$ such that the image $(sz, sz) \in \hat{T} \times \hat{\bar{T}}$ of $sz \in Z(\hat{M}_H)$ belongs to $\hat{T}^{\mathrm{der}} \times \hat{\bar{T}}^{\mathrm{der}}$. (Find one such z so that $sz \in \hat{T}^{\mathrm{der}}$, by using the fact that $\hat{T}^{\mathrm{der}} \cdot Z(\hat{M}_b) = \hat{T}$. Then $sz \in \hat{\bar{T}}^{\mathrm{der}}$ is automatic.) Note that (sz, sz) and (s, s) have the same image, say $(s^{\mathrm{ad}}, s^{\mathrm{ad}})$, in $\hat{T}^{\mathrm{ad}} \times (\hat{\bar{T}})^{\mathrm{ad}}$. Then $s_U \in \hat{U}$ is defined as the image of (sz, sz) in $\hat{U}$. It turns out that s_U is $\Gamma(p)$-invariant and independent of the choice of z [**27**, p. 246]. By abuse of notation, the image of s_U in $\pi_0(\hat{U})$ will be again denoted by s_U. Then the right-hand side of (6.15) is given by the Tate–Nakayama pairing $H^1(\mathbb{Q}_p, U) \times \pi_0(\hat{U}) \to \mathbb{C}^\times$.

Consider the following diagram, which is commutative by the functoriality of the map $\kappa_{(\cdot)}$ (see [**33**, Theorem 1.15.(i)] or [**22**, 4.9.1]):

$$\begin{array}{ccccc} B(T \times \bar{T}) & \longleftarrow & B(T^{\mathrm{der}} \times \bar{T}^{\mathrm{der}}) & \longrightarrow & B(U) \\ \downarrow {\scriptstyle (\kappa_T, \kappa_{\bar{T}})} & & \downarrow {\scriptstyle (\kappa_{T^{\mathrm{der}}}, \kappa_{\bar{T}^{\mathrm{der}}})} & & \downarrow {\scriptstyle \kappa_U} \\ X^*(\hat{T}^{\Gamma(p)} \times \hat{\bar{T}}^{\Gamma(p)}) & \longleftarrow & X^*((\hat{T}^{\mathrm{ad}})^{\Gamma(p)} \times ((\hat{\bar{T}})^{\mathrm{ad}})^{\Gamma(p)}) & \longrightarrow & X^*(\hat{U}^{\Gamma(p)}) \end{array} \tag{6.20}$$

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Recall from § 4.3 that

$$\tilde{\alpha}_p(\gamma_0, \delta) = \kappa_T(y^{-1}\tilde{b}y^{\sigma}), \qquad \tilde{\alpha}_p(\bar{\gamma}_0, \bar{\delta}) = \kappa_{\bar{T}}(\bar{y}^{-1}\tilde{b}\bar{y}^{\sigma}).$$

So the left-hand side of (6.15) can be computed as

$$\frac{\langle \kappa_{\bar{T}}(\bar{y}^{-1}\tilde{b}\bar{y}^{\sigma}), s\rangle}{\langle \kappa_T(y^{-1}\tilde{b}y^{\sigma}), s\rangle} = \frac{\langle \kappa_{\bar{T}^{\mathrm{der}}}(\bar{y}^{-1}\tilde{b}^{\mathrm{der}}\bar{y}^{\sigma}), s^{\mathrm{ad}}\rangle}{\langle \kappa_{T^{\mathrm{der}}}(y^{-1}\tilde{b}^{\mathrm{der}}y^{\sigma}), s^{\mathrm{ad}}\rangle} = \langle Y_{\sigma}, s_U\rangle, \tag{6.21}$$

where Y_{σ} is the image of $((y^{-1}\tilde{b}^{\mathrm{der}}y^{\sigma})^{-1}, \bar{y}^{-1}\tilde{b}^{\mathrm{der}}\bar{y}^{\sigma})$ in $U(L)$. (The notation Y_{σ} also denotes its image in $B(U)$.) The second identity in (6.21) follows from the commutativity of the right rectangle in (6.20). To check the first identity in (6.21), use (6.16) and the functoriality of $\kappa_{(\cdot)}$ with respect to the diagonal embedding $Z(M_b) \hookrightarrow T \times \bar{T}$.

The proof of (6.15) boils down to showing that

$$\left\langle \mathrm{inv}\left(\frac{\gamma_H, \delta}{\bar{\gamma}_H, \bar{\delta}}\right), s_U \right\rangle = \langle Y_{\sigma}, s_U\rangle.$$

In light of the left rectangle of (3.1) for U, the above identify follows if we show that

$$\mathrm{inv}\left(\frac{\gamma_H, \delta}{\bar{\gamma}_H, \bar{\delta}}\right) \mapsto Y_{\sigma}$$

under the map $H^1(\mathbb{Q}_p, U) \to B(U)$ of (3.1).

The last map is defined as the composition of the following:

$$H^1(\mathbb{Q}_p, U) \xrightarrow{\sim} H^1(W(\bar{L}/\mathbb{Q}_p), U(\bar{L})) \xleftarrow{\sim} H^1(W(L/\mathbb{Q}_p), U(L)) = B(U),$$

where the first two arrows are inflation maps. The image of Y_{σ} in $H^1(W(\bar{L}/\mathbb{Q}_p), U(\bar{L}))$ is represented by the cocycle for which

$$\tau \mapsto ((y^{-1}(\tilde{b}^{\mathrm{der}})^{(|\tau|)}y^{\tau})^{-1}, \bar{y}^{-1}(\tilde{b}^{\mathrm{der}})^{(|\tau|)}\bar{y}^{\tau}) \tag{6.22}$$

whenever $|\tau| > 0$. (The images of τ with $|\tau| > 0$ uniquely determine the cocycle.) On the other hand,

$$\tau \mapsto ((x^{-1}\boldsymbol{b}_{\tau}^{\mathrm{der}}x^{\tau})^{-1}, \bar{x}^{-1}\boldsymbol{b}_{\tau}^{\mathrm{der}}\bar{x}^{\tau}) \tag{6.23}$$

represents the image of

$$\mathrm{inv}\left(\frac{\gamma_H, \delta}{\bar{\gamma}_H, \bar{\delta}}\right)$$

in $H^1(W(\bar{L}/\mathbb{Q}_p), U(\bar{L}))$. By (6.14) and (6.17), the cocycle in (6.22) can be rewritten as

$$\tau \mapsto ((c^{-1}x^{-1}\boldsymbol{b}_{\tau}^{\mathrm{der}}x^{\tau}c^{\tau})^{-1}, \bar{c}^{-1}\bar{x}^{-1}\boldsymbol{b}_{\tau}^{\mathrm{der}}\bar{x}^{\tau}\bar{c}^{\tau}) \cdot (z_{\tau}^{-1}, z_{\tau}).$$

Noting that $(z_{\tau}^{-1}, z_{\tau}) = 1$ in $U(\bar{L})$ (see (6.18)), it is now obvious that (6.22) and (6.23) define the same cohomology class in $H^1(W(\bar{L}/\mathbb{Q}_p), U(\bar{L}))$. Hence the proof of Lemma 6.3 is complete in case γ_0 is regular semisimple.

General case

It remains to prove that the identity (6.9) of Lemma 6.3 continues to hold for the same constant c_{M_H} when γ_0 is not regular. We imitate the argument of [**23**, A.3.8].

Changing notation, set

$$I_0 := Z_{M_b}(\gamma_0), \qquad I_\delta := Z_{J_b}(\delta), \qquad I_H := Z_{M_H}(\gamma_H).$$

Note that I_0, I_δ and I_H are connected (cf. the footnote in the current proof for the regular case). Find $y \in M_b(L)$ such that $\psi(\delta) = y\gamma_0 y^{-1}$. The L-isomorphism $\psi_0 := \mathrm{Int}(y^{-1})\psi$ from J_b to M_b restricts to $I_\delta \xrightarrow{\sim} I_0$. Since $\psi\psi^{-\sigma} = \mathrm{Int}(\tilde{b})$, we have

$$\psi_0\psi_0^{-\sigma} = \mathrm{Int}(\tilde{b}_\delta) = \mathrm{Int}(y^{-1}\tilde{b}y^\sigma). \tag{6.24}$$

Choose an elliptic torus T of I_0 over $\mathbb{Q}_p$. Since $b_\delta \in B(I_0)$ is basic by Lemma 4.3, it is in the image of the natural map $B(T) \to B(I_0)$ [**16**, Proposition 5.3]. This means that there exists $i \in I_0(L)$ such that $i^{-1}\tilde{b}_\delta i^\sigma \in T(L)$. It is easy to verify that $k := \psi_0^{-1} \circ \mathrm{Int}(i) = \psi^{-1} \circ \mathrm{Int}(yi)$ gives a $\mathbb{Q}_p$-embedding from T to I_δ. (Namely k and k^σ give the same map from T to I_δ. This is checked using (6.24).) For each $t \in T(\mathbb{Q}_p)$, define

$$\gamma_t := t\gamma_0, \qquad \delta_t := k(t)\delta.$$

We assume that γ_t is regular in M_b so that $T = Z_{M_b}(\gamma_t)$.

The natural inclusion $T \hookrightarrow I_0$ yields the following commutative diagram:

$$\begin{array}{ccc} B(T) & \longrightarrow & B(I_0) \\ \big\downarrow{\scriptstyle \kappa_T} & & \big\downarrow{\scriptstyle \kappa_{I_0}} \\ X^*(\hat{T}^{\Gamma(p)}) & \longrightarrow & X^*(Z(\hat{I}_0)^{\Gamma(p)}) \end{array} \tag{6.25}$$

We claim that

$$\tilde{\alpha}_p(\gamma_t, \delta_t) \mapsto \tilde{\alpha}_p(\gamma_0, \delta) \tag{6.26}$$

via the bottom horizontal map of (6.25). To show this, it is enough to show that $\tilde{b}_\delta := y^{-1}\tilde{b}y^\sigma$ and $\tilde{b}_{\delta_t} := y_t^{-1}\tilde{b}y_t^\sigma$ define the same element in $B(I_0)$. Here y_t is any element of $T(L)$ such that

$$\psi(\delta_t) = y_t^{-1}\gamma_t y_t^\sigma.$$

(A different choice of y_t does not change the image of $\tilde{b}_{\delta_t}$ in $B(T)$.)

Let us prove the claim. Observe that

$$\psi(\delta_t) = \psi(k(t))\psi(\delta) = (\mathrm{Int}(yi)t)y\gamma_0 y^{-1} = yiti^{-1}\gamma_0 y^{-1} = yit\gamma_0 i^{-1}y^{-1},$$

where the last identity holds as $i \in I_0(L)$. Hence we can take $y_t = yi$. Then it is obvious that $\tilde{b}_{\delta_t}$ is σ-conjugate to $\tilde{b}_\delta$ in $I_0(L)$. The claim is proved.

We are ready to see that (6.9) holds in general. We deduce from (6.26) that

$$\langle \tilde{\alpha}_p(\gamma_t, \delta_t), \tilde{\kappa} \rangle = \langle \tilde{\alpha}_p(\gamma_0, \delta), \tilde{\kappa} \rangle$$

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for any $t \in T(\mathbb{Q}_p)$ such that γ_t is regular in M_b. On the other hand, $\Delta_p(\gamma_H, \delta)^{J_b}_{M_H}$ (respectively $\Delta_p(\gamma_H, \gamma_0)^{M_b}_{M_H}$) is defined as the value of $\Delta_p(\gamma_{H,t}, \delta_t)^{J_b}_{M_H}$ (respectively $\Delta_p(\gamma_{H,t}, \gamma_t)^{M_b}_{M_H}$) when t is close enough to 1 and γ_t is regular, where $\gamma_{H,t}$ is the transfer of γ_t up to stable conjugacy (see [**28**, § 2.4]). Since we already proved

$$c_{M_H} \cdot \Delta_p(\gamma_{H,t}, \delta_t)^{J_b}_{M_H} = \langle \tilde{\alpha}_p(\gamma_t, \delta_t), \tilde{\kappa} \rangle \Delta_p(\gamma_{H,t}, \gamma_t)^{M_b}_{M_H},$$

for any such t, we conclude that (6.9) is true with the same constant c_{M_H} as in the regular case.

7. End of stabilization

We are ready to obtain a fully stabilized expression for $\mathrm{tr}(\phi | \iota_l H_c(\mathrm{Ig}_{\Sigma_b}, \mathscr{L}_\xi))$ when $\phi \in C_c^\infty(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ is acceptable. For each $(H, s, \eta) \in \mathcal{E}^{\mathrm{ell}}(G)$, put $h^H := h^{H,p} h^H_p h^H_\infty$.

The stable orbital integrals defined by h^H depend on the choice of $\tilde{\eta}$, but are independent of the choice of local transfer factors $\Delta_v(\cdot\,,\cdot)^G_H$, once $\tilde{\eta}$ is fixed. (Despite the fact that each of $h^{H,p}$, h^H_p and h^H_∞ depends on the choice.) The stable orbital integrals defined by h^H remain unchanged if (H, s, η) is replaced by an isomorphic endoscopic triple (H, sz, η) for any $z \in Z(\hat{G})$. (Note that if s is replaced by sz then $\tilde{\kappa}$ changes by z in the process, in view of Remark 2.7.) Both of the above assertions are easy to verify.

Putting these together, we can show that h^H is well-defined in the following sense: keeping the previous notation, suppose that $(H, {}^L H, s, \tilde{\eta})$ and $(H', {}^L H', s', \tilde{\eta}')$ are equivalent as endoscopic data for G via a $\mathbb{Q}$-isomorphism $\alpha : H \xrightarrow{\sim} H'$, in the terminology of [**27**, 1.2]. Then the stable orbital integrals defined by h^H and $h^{H'}$ are the same via α.

Lemma 7.1. *Suppose that $(H, s, \eta) \in \mathcal{E}^{\mathrm{ell}}(G)$. For every (G, H)-regular semisimple $\gamma_H \in H(\mathbb{Q})$,*

(i) *if (H, s, η, γ_H) belongs to $\mathcal{EQ}^{\mathrm{ell}}(G)$, let $(\gamma_0, \kappa) \in \mathcal{SS}^{\mathrm{ell}}(G)$ be its image; then*

$$\mathrm{SO}^{H(\mathbb{A})}_{\gamma_H}(h^H) = \mathrm{O}^p(\gamma_0, \kappa, \phi^p) \cdot \mathrm{O}_p(\gamma_0, \tilde{\kappa}, \phi_p) \cdot \mathrm{O}_\infty(\gamma_0, \tilde{\kappa})$$

if $(\gamma_0, \kappa) \in \mathcal{SS}^{\mathrm{KT}}(G)$ and $\mathrm{SO}^{H(\mathbb{A})}_{\gamma_H}(h^H) = 0$ otherwise (see Remark 2.7 for $\tilde{\kappa}$); and

(ii) *if $(H, s, \eta, \gamma_H) \notin \mathcal{EQ}^{\mathrm{ell}}(G)$ then $\mathrm{SO}^{H(\mathbb{A})}_{\gamma_H}(h^H) = 0$.*

Proof. Let us prove the assertion (i). If $(\gamma_0, \kappa) \in \mathcal{SS}^{\mathrm{KT}}(G)$ then (γ_0, κ) defines an element of $\mathcal{SS}^{\mathrm{eff}}_p(G)$ and $(H, s, \eta, \gamma_H) \in \mathcal{EQ}^{\mathrm{ell}}(G)$ defines an element of $\mathcal{EQ}^{\mathrm{eff}}_p(G)$. The first assertion follows from (5.7), (5.8) and Lemma 6.5.

Now assume $(\gamma_0, \kappa) \notin \mathcal{SS}^{\mathrm{KT}}(G)$. One of the following occurs.

- There is no ν_b-acceptable $\delta \in J_b(\mathbb{Q}_p)$ such that $\delta \sim_{\mathrm{st}} \gamma_0$.
- $\gamma_0 \in G(\mathbb{Q})$ is not $\mathbb{R}$-elliptic.

In the first case, $(\gamma_0, \kappa) \notin \mathcal{SS}^{\mathrm{eff}}_p(G)$ and $(H, s, \eta, \gamma_H) \notin \mathcal{EQ}^{\mathrm{eff}}_p(G)$. By Lemma 6.5 (and the remark below its proof) we conclude that $\mathrm{SO}^{H(\mathbb{A})}_{\gamma_H}(h^H) = 0$. The same equality holds in the second case by (5.8).

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Let us begin the proof of (ii). We may assume that elliptic maximal tori of $G_{\mathbb{R}}$ come from those of $H_{\mathbb{R}}$ as $h^H_\infty = 0$ otherwise. The condition of (ii) means that $\gamma_H \in H(\mathbb{Q})$ does not transfer to $G(\mathbb{Q})$. Consider the case where γ_H as an element of $H(\mathbb{A})$ transfers to some $\gamma_0 \in G(\mathbb{A})$ (up to $G(\bar{\mathbb{A}})$-conjugacy). If γ_0 is not $\mathbb{R}$-elliptic then $\mathrm{SO}^{H(\mathbb{A})}_{\gamma_H}(h^H) = 0$ by (5.8). If γ_0 is $\mathbb{R}$-elliptic then we show that a contradiction occurs. Indeed, the argument of the second paragraph of [**19**, p. 188] shows that $\gamma_0 \in G(\mathbb{A})$ is $G(\bar{\mathbb{A}})$-conjugate to an element of $G(\mathbb{Q})$, which contradicts that γ_H does not transfer to $G(\mathbb{Q})$.

It remains to deal with the case where γ_H does not transfer to $G(\mathbb{A})$. We may assume γ_H is $\mathbb{R}$-elliptic as otherwise $\mathrm{SO}^{H(\mathbb{A})}_{\gamma_H}(h^H) = 0$ by (5.8). Then γ_H transfers to $G(\mathbb{R})$. Since G is quasi-split over $\mathbb{Q}_p$, γ_H transfers to $G(\mathbb{Q}_p)$ as well. So our situation is that γ_H does not transfer to $G(\mathbb{A}^{\infty,p})$, which implies $\mathrm{SO}^{H(\mathbb{A})}_{\gamma_H}(h^H) = 0$ by Lemma 5.2. □

By Lemma 7.1 and Lemma 2.8, the identity (5.6) may be rewritten as

$$\mathrm{tr}(\phi | \iota_l H_c(\mathrm{Ig}_{\Sigma_b}, \mathscr{L}_\xi)) = \tau(G)|\ker^1(\mathbb{Q}, G)| \sum_{(H,s,\eta,\gamma_H)} \mathrm{SO}^{H(\mathbb{A})}_{\gamma_H}(h^H), \tag{7.1}$$

where the sum runs over $\mathcal{EQ}^{\mathrm{ell}}(G)$. By the remark below Lemma 2.8,

$$\sum_{(H,s,\eta,\gamma_H)} \mathrm{SO}^{H(\mathbb{A})}_{\gamma_H}(h^H) = \sum_{(H,s,\eta)} |\mathrm{Out}_{\mathbb{Q}}(H,s,\eta)|^{-1} \sum_{\gamma_H} \mathrm{SO}^{H(\mathbb{A})}_{\gamma_H}(h^H), \tag{7.2}$$

where in the last sum γ_H runs over a set of representatives for $\mathbb{R}$-elliptic semisimple stable conjugacy classes in $H(\mathbb{Q})$ which are (G,H)-regular.

So far we have constructed h^H when $\phi \in C^\infty_c(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ satisfies (5.1). The construction of h^H linearly extends to the general case where ϕ is an arbitrary acceptable function. Define

$$\mathrm{ST}^H_e(h^H) := \sum_{\gamma_H} \tau(H) \cdot |Z_H(\gamma_H)/Z_H(\gamma_H)^0|^{-1} \cdot \mathrm{SO}^{H(\mathbb{A})}_{\gamma_H}(h^H), \tag{7.3}$$

where γ_H runs over a set of representatives for $\mathbb{Q}$-elliptic semisimple stable conjugacy classes in $H(\mathbb{Q})$. (In fact there is no new contribution if we include non-$\mathbb{Q}$-elliptic stable conjugacy classes in the sum since h^H_∞ has trivial stable orbital integrals over them.) Define

$$\iota(G,H) := \tau(G)\tau(H)^{-1}|\mathrm{Out}_{\mathbb{Q}}(H,s,\eta)|^{-1}.$$

Theorem 7.2. *Let $\phi \in C^\infty_c(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ be any acceptable function. For each $(H,s,\eta) \in \mathcal{E}^{\mathrm{ell}}(G)$, let h^H be the function constructed from ϕ as above. Then*

$$\mathrm{tr}(\phi | \iota_l H_c(\mathrm{Ig}_{\Sigma_b}, \mathscr{L}_\xi)) = |\ker^1(\mathbb{Q}, G)| \sum_{(H,s,\eta)} \iota(G,H)\mathrm{ST}^H_e(h^H).$$

Proof. We may assume that ϕ is as in (5.1). Note that only (G,H)-regular γ_H contribute to $\mathrm{ST}^H_e(h^H)$ by Remark 5.6 and that $Z_H(\gamma_H)$ is connected for any such γ_H. The theorem follows from (7.1), (7.2) and (7.3). □

Remark 7.3. Theorem 7.2 is an analogue of [**19**, Theorem 7.2].

8. The constant c_{M_H} of Lemma 6.3 in special cases

The main purpose of this last section is to determine the constant c_{M_H} which shows up in Lemma 6.3, under a particular normalization of transfer factors as in (8.6) and (8.9). (The computation of c_{M_H} would be useful for applications. It is used in [**37**, § 5.5].) When $J_b \simeq M_b$ (§ 8.2) a simpler proof of Lemma 6.3 will be given. We will work in the setting of Lemma 6.3 without mentioning it again. As we are only concerned with $\mathbb{Q}_p$-groups in § 8, we often write G for $G_{\mathbb{Q}_p}$ and similarly for other groups in order to save notation.

8.1. The case of general linear groups

For convenience we introduce a non-standard terminology for a reductive group G_0 over $\mathbb{Q}_p$.

Definition 8.1. We say that G_0 *satisfies* $\mathcal{GL}_p$ if G_0 is $\mathbb{Q}_p$-isomorphic to

$$\prod_{i\in I} R_{K_i/\mathbb{Q}_p}\mathrm{GL}_{n_i}$$

for a finite index set I, positive integers n_i and finite extension fields K_i of $\mathbb{Q}_p$. Here, $R_{K_i/\mathbb{Q}_p}$ means the Weil restriction of scalars.

In this subsection we prove Lemma 6.3 under the simplifying assumption that $G_{\mathbb{Q}_p}$ satisfies $\mathcal{GL}_p$. This assumption is often satisfied for a PEL datum $(B,*,V,\langle\cdot,\cdot\rangle,h)$ of type (A). In the case of type (A) datum, $F = Z(B)$ is a CM field. Let F^+ be the fixed field of F under the complex conjugation. If every place of F^+ above p splits in F, then $G_{\mathbb{Q}_p}$ satisfies $\mathcal{GL}_p$.

Suppose that $G_{\mathbb{Q}_p}$ satisfies $\mathcal{GL}_p$ throughout § 8.1. Then the groups H, M_b, M_H, I_0 and I_H also satisfy $\mathcal{GL}_p$. All these groups and their dual groups have simply connected derived subgroups. In particular, $Z(M_H)$ and $Z(M_b)$ are tori.

One important task for us is to give an explicit formula (8.5) for $\langle\tilde{\alpha}_p(\gamma_0,\delta),\tilde{\kappa}\rangle$. Consequently, its value will be easily seen to be independent of γ_H, δ and γ_0. To this end, we examine the character $\tilde{\alpha}_p(\gamma_0,\delta) \in X^*(Z(\hat{I}_0)^{\Gamma(p)}Z(\hat{G}))$. Consider the following commutative diagram of $\Gamma(p)$-equivariant group homomorphisms:

$$\begin{array}{ccccccc}
Z(\hat{H}) & \hookrightarrow & Z(\hat{M}_H) & \hookrightarrow & Z(\hat{I}_H) & \longrightarrow & \widehat{Z(M_H)} \\
\uparrow & & \uparrow & & \downarrow\wr & & \downarrow \\
Z(\hat{G}) & \hookrightarrow & Z(\hat{M}_b) & \hookrightarrow & Z(\hat{I}_0) & \longrightarrow & \widehat{Z(M_b)}
\end{array} \tag{8.1}$$

The two rows are given by the inclusions $Z(M_H) \subset I_H \subset M_H \subset H$ and $Z(M_b) \subset I_0 \subset M_b \subset G$, respectively, using [**15**, 1.8] and [**17**, 4.2]. The vertical injections $Z(\hat{G}) \hookrightarrow Z(\hat{H})$ and $Z(\hat{M}_b) \hookrightarrow Z(\hat{M}_H)$ are given by η and η_H, respectively. The left-most rectangle is compatible with the diagram (6.6) and thus commutative. The commutativity of other rectangles are straightforward. The element $s \in Z(\hat{H})$ maps to $s_H \in Z(\hat{M}_H)$ and $\tilde{\kappa} \in$

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$Z(\hat{I}_0)$. We may write $s = s_1 s_2$ for $s_1 \in Z(\hat{H})^{\Gamma(p)}$ and $s_2 \in Z(\hat{G})$. Let $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ denote the images of s_1 and s_2 in $Z(\hat{I}_0)$, respectively. In view of (4.1),

$$\langle \tilde{\alpha}_p(\gamma_0, \delta), \tilde{\kappa}_2 \rangle = \mu_1(s_2)^{-1}. \tag{8.2}$$

Let us evaluate $\langle \tilde{\alpha}_p(\gamma_0, \delta), \tilde{\kappa}_1 \rangle$. Let T be a maximal torus of M_b defined over $\mathbb{Q}_p$ containing γ_0. Recall from § 3.2 that $\tilde{b}$ satisfies (3.2) and belongs to $M_b(L_s)$. Let us also recall from § 4.3 that $\tilde{\alpha}_p(\gamma_0, \delta) = \kappa_{I_0}(b_\delta)$ and that $\tilde{b}_\delta$ is σ-conjugate to $\tilde{b}$ in $M_b(L)$. We claim that $\bar{\nu}_{I_0}(b_\delta) = \nu_b$ as elements of $X_*(T)_{\mathbb{Q}}/\Omega(I_0, T)$. Indeed, $\bar{\nu}_{M_b}(b_\delta) = \bar{\nu}_{M_b}(b) = \nu_b$ shows that $\bar{\nu}_{I_0}(b_\delta)$ and ν_b are in the same $\Omega(M_b, T)$-orbit, but since the $\Omega(M_b, T)$-orbit of ν_b consists of ν_b only, the claim is verified.

The commutativity of (3.1) shows that $\rho_{I_0}(\tilde{\alpha}_p(\gamma_0, \delta)) = \delta_{I_0}(\nu_b)$ where ν_b is viewed as an element of $X_*(T)_{\mathbb{Q}} = X^*(\hat{T})_{\mathbb{Q}}$. On the other hand, δ_{I_0} is the pullback map via $Z(\hat{I}_0) \hookrightarrow \hat{T}$ in this case. Since $\nu_b : \mathbb{D} \to T$ factors through $Z(M_b)$, we may view ν_b as an element of $X^*(\widehat{Z(M_b)})_{\mathbb{Q}}$, which will be denoted by $\hat{\nu}_b$. Then $\delta_{I_0}(\nu_b)$ is nothing but the pullback of $\hat{\nu}_b$ via $Z(\hat{I}_0) \to \widehat{Z(M_b)}$.

Let $\tilde{\alpha}'_p \in X^*(Z(\hat{M}_H)^{\Gamma(p)})$ be the pullback of $\tilde{\alpha}_p(\gamma_0, \delta)|_{Z(\hat{I}_0)^{\Gamma(p)}}$ via (8.1). Then

$$\langle \tilde{\alpha}_p(\gamma_0, \delta), \tilde{\kappa}_1 \rangle = \langle \tilde{\alpha}'_p, s_1 \rangle. \tag{8.3}$$

Since $\rho_{(\cdot)}$ is functorial on the category of connected reductive groups over $\mathbb{Q}_p$ (for any $\mathbb{Q}_p$-group morphisms), we see that $\rho_{M_H}(\tilde{\alpha}'_p)$ coincides with the pullback of $\delta_{I_0}(\nu_b) \in X^*(Z(\hat{I}_0))^{\Gamma(p)}_{\mathbb{Q}}$ to $X^*(Z(\hat{M}_H))^{\Gamma(p)}_{\mathbb{Q}}$ via (8.1). Again in view of (8.1), $\rho_{M_H}(\tilde{\alpha}'_p)$ may also be obtained as the pullback of $\hat{\nu}_b$ via

$$Z(\hat{M}_H) \to \widehat{Z(M_H)} \to \widehat{Z(M_b)}.$$

The last map comes from $Z(M_b) \hookrightarrow Z(M_H) \hookrightarrow M_H$, which only depends on the endoscopic datum (M_H, s_H, η_H) and not on γ_H, δ and γ_0. Since M_H satisfies $\mathcal{GL}_p$, ρ_{M_H} is injective. (By (3.1), $\ker(\rho_{M_H})$ is isomorphic to $H^1(\mathbb{Q}_p, M_H)$, which is trivial by Hilbert 90.) Let $\hat{\nu}_b^{M_H}$ denote the pullback of $\hat{\nu}_b$ via

$$Z(\hat{M}_H)^{\Gamma(p)} \to \widehat{Z(M_H)} \to \widehat{Z(M_b)}.$$

We see from the description of ρ_{M_H} [**33**, p. 162] that $\tilde{\alpha}'_p$ is obtained as the pullback of $\rho_{M_H}(\tilde{\alpha}'_p)$ via $Z(\hat{M}_H)^{\Gamma(p)} \hookrightarrow Z(\hat{M}_H)$, hence $\tilde{\alpha}'_p$ coincides with $\hat{\nu}_b^{M_H}$. *A priori* $\hat{\nu}_b^{M_H}$ is just an element of $X^*(Z(\hat{M}_H)^{\Gamma(p)})_{\mathbb{Q}}$, but it belongs to $X^*(Z(\hat{M}_H)^{\Gamma(p)})$ as $\tilde{\alpha}'_p$ does.

To sum up, $\tilde{\alpha}'_p = \hat{\nu}_b^{M_H}$ and

$$\langle \tilde{\alpha}'_p, s_1 \rangle = \langle \hat{\nu}_b^{M_H}, s_1 \rangle. \tag{8.4}$$

By (8.2), (8.3) and (8.4),

$$\langle \tilde{\alpha}_p(\gamma_0, \delta), \tilde{\kappa} \rangle = \mu_1(s_2)^{-1} \cdot \langle \hat{\nu}_b^{M_H}, s_1 \rangle. \tag{8.5}$$

Clearly, the value on the right side is independent of γ_0, δ and the choice of decomposition $s = s_1 s_2$. Therefore, Lemma 6.3 tells us that we may normalize $\Delta_p(\cdot\,,\cdot)^{J_b}_{M_H}$ so that the ratio of $\Delta_p(\gamma_H, \delta)^{J_b}_{M_H}$ to $\Delta_p(\gamma_H, \gamma_0)^{J_b}_{M_H}$ is a non-zero constant. Our normalization is that

$$\frac{\Delta_p(\gamma_H, \delta)^{J_b}_{M_H}}{\Delta_p(\gamma_H, \gamma_0)^{M_b}_{M_H}} = e_p(J_b). \tag{8.6}$$

This choice is to be consistent with our convention that $\Delta_p(\cdot\,,\cdot)^{J_b}_{M_b} \equiv e_p(J_b)$ (cf. Remark 2.20).

It follows from (8.5) and (8.6) that the constant c_{M_H} in Lemma 6.3 is given by

$$c_{M_H} = e_p(J_b) \cdot \mu_1(s_2) \cdot \langle \hat{\nu}_b^{M_H}, s_1 \rangle^{-1}. \tag{8.7}$$

8.2. The case $J_b \simeq M_b$

Recall from § 3.2 that J_b is the $\mathbb{Q}_p$-inner form of M_b given by the cocycle $\sigma \to \tilde{b}$ in $H^1(L_s/\mathbb{Q}_p, \mathrm{Int}(M_b))$. Since G is unramified over $\mathbb{Q}_p$, we may assume that $G \times_{\mathbb{Q}_p} L_s$ is split, by enlarging s if necessary. (Before, $s \in \mathbb{Z}_{>0}$ was chosen in § 3.2.) The aim of § 8.2 is to give an explicit alternative proof for Lemma 6.3, under the assumptions that

- $J_b \simeq M_b$ as $\mathbb{Q}_p$-groups,
- M_H^{der} is simply connected, and
- $Z(M_b)$ and $Z(M_H)$ are connected.

The second and third assumptions are always satisfied for a PEL datum of type (A).

The first assumption implies that the trivial element in $H^1(L_s/\mathbb{Q}_p, \mathrm{Int}(M_b))$ is defined by $\sigma \mapsto \tilde{b}$. So there exists some $b_0 \in M_b(L_s)$ such that $\mathrm{Int}(\tilde{b}) = \mathrm{Int}(b_0^{-1} b_0^{\sigma})$. (This is possible since the natural map $M_b(L_s) \to \mathrm{Int}_{L_s}(M_b)$ is surjective. The last fact follows from the triviality of $H^1(L_s, Z(M_b))$ implied by Hilbert 90.) In other words,

$$\tilde{b} = b_0^{-1} b_0^{\sigma} z \tag{8.8}$$

for some $z \in Z(M_b)(L_s)$.

We claim that $J_b(\mathbb{Q}_p) \xrightarrow{\sim} M_b(\mathbb{Q}_p)$ as subgroups of $M_b(L_s)$ via $\delta \mapsto b_0 \delta b_0^{-1}$. Let us prove the claim. Using (3.2) it is easy to see that $J_b(\mathbb{Q}_p)$ is contained in $M_b(L_s)$. (See also the proof of [**34**, Corollary 1.14].) As subgroups of $M_b(L_s)$, the two groups $J_b(\mathbb{Q}_p)$ and $M_b(\mathbb{Q}_p)$ are cut out by the conditions $g\tilde{b}\sigma = \tilde{b}\sigma g$ and $g\sigma = \sigma g$, respectively. The claim follows from this and (8.8).

We will certainly choose the transfer factor $\Delta(\cdot\,,\cdot)^{J_b}_{M_H}$ so that

$$\Delta(\cdot\,,\cdot)^{J_b}_{M_H} = \Delta(\cdot\,,\cdot)^{M_b}_{M_H} \tag{8.9}$$

via the isomorphism $J_b(\mathbb{Q}_p) \simeq M_b(\mathbb{Q}_p)$ in the last paragraph. Suppose that $\delta \in J_b(\mathbb{Q}_p)$ and $\gamma_0 \in M_b(\mathbb{Q}_p)$ are semisimple elements with matching stable conjugacy classes. This

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amounts to assuming that $b_0\delta b_0^{-1}$ is stably conjugate to γ_0 in $M_b(\mathbb{Q}_p)$. Observe that the identity (6.9) now simplifies as

$$c_{M_H} = \langle \tilde{\alpha}_p(\gamma_0, \delta), \tilde{\kappa}\rangle^{-1} \cdot \langle \mathrm{inv}_p(b_0\delta b_0^{-1}, \gamma_0), \tilde{\kappa}\rangle. \tag{8.10}$$

The proof of Lemma 6.3 comes down to showing that the right side of (8.10) is a constant independent of γ_0 and δ. As in the paragraph preceding (8.2), write $s = s_1 s_2$ for $s_1 \in Z(\hat{H})^{\Gamma(p)}$ and $s_2 \in Z(\hat{G})$ and let $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ denote the images of s_1 and s_2 in $Z(\hat{I}_0)$, respectively. To show Lemma 6.3, it suffices to prove that $\langle \alpha_p(\gamma_0, \delta), \tilde{\kappa}_1\rangle^{-1} \cdot \langle \mathrm{inv}_p(b_0\delta b_0^{-1}, \gamma_0), \tilde{\kappa}_1\rangle$ is constant. In view of [**36**, Lemma 10.9], it is enough to prove that if $\gamma_0 = b_0\delta b_0^{-1}$ then $\langle \alpha_p(\gamma_0, \delta), \tilde{\kappa}\rangle$ is independent of δ.

In the definition $\alpha_p(\gamma_0, \delta) = \kappa_{I_0}(b_\delta)$ (§ 4.3), we may take $y = b_0^{-1}$ so that

$$\tilde{b}_\delta = b_0 \tilde{b} b_0^{-\sigma} = z \in Z(M_b)(L_s)$$

using (8.8). (Its image in $B(I_0)$ is b_δ.) We see that $\alpha_p(\gamma_0, \delta)$ equals the image of $\kappa_{Z(M_b)}(z)$ under the bottom horizontal arrow below:

$$\begin{array}{ccc} B(Z(M_b)) & \longrightarrow & B(I_0) \\ \downarrow{\scriptstyle \kappa_{Z(M_b)}} & & \downarrow{\scriptstyle \kappa_{I_0}} \\ X^*((\widehat{Z(M_b)})^{\Gamma(p)}) & \longrightarrow & X^*(Z(\hat{I}_0)^{\Gamma(p)}) \end{array}$$

Note that the diagram (8.1) makes sense in our case as well. We obtain a character on $Z(\hat{M}_H)^{\Gamma(p)}$ by pulling back $\kappa_{Z(M_b)}(z)$ via (8.1). Since $\tilde{\kappa}_1$ is the image of $s_1 \in Z(\hat{H})^{\Gamma(p)}$, we see that

$$\langle \alpha_p(\gamma_0, \delta), \tilde{\kappa}_1\rangle = \langle \kappa_{Z(M_b)}(z), s_1\rangle,$$

where the pairing on the right is taken between $X^*(Z(\hat{M}_H)^{\Gamma(p)})$ and $Z(\hat{M}_H)^{\Gamma(p)}$. The value on the right-hand side is visibly independent of δ. This finishes the proof of Lemma 6.3 in the setting of the current subsection. Our discussion shows that (with the identification $\Delta_p(\cdot\,,\cdot)^{J_b}_{M_H} = \Delta_p(\cdot\,,\cdot)^{M_b}_{M_H}$)

$$c_{M_H} = \mu_1(s_2)\langle \kappa_{Z(M_b)}(z), s_1\rangle^{-1}. \tag{8.11}$$

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