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PATH TO SURVIVAL FOR THE CRITICAL BRANCHING PROCESSES IN A RANDOM ENVIRONMENT

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Abstract

A critical branching process $\{Z_k, k = 0, 1, 2, \dots\}$ in a random environment is considered. A conditional functional limit theorem for the properly scaled process $\{\log Z_{pu},\ 0 \le u < \infty\}$ is established under the assumptions that $Z_n > 0$ and $p \ll n$. It is shown that the limiting process is a Lévy process conditioned to stay nonnegative. The proof of this result is based on a limit theorem describing the distribution of the initial part of the trajectories of a driftless random walk conditioned to stay nonnegative.

Keywords: Branching process; random environment; random walk conditioned to stay positive; Lévy process conditioned to stay positive; change of measure; functional limit theorem

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1. Introduction

We consider a branching process in a random environment specified by a sequence of independent and identically distributed (i.i.d.) random laws. Denote by Δ the space of probability measures on $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Equipped with the metric of total variation, Δ becomes a Polish space. Let Q be a random variable taking values in Δ. Then, an infinite sequence

$$\Pi = (Q_1, Q_2, \dots) \tag{1}$$

of i.i.d. copies of Q is said to form a *random environment*. A sequence of $\mathbb{N}_0$-valued random variables $Z_0, Z_1, \dots$ is called a *branching process in the random environment* Π, if Z_0 is independent of Π and given Π the process $\mathcal{Z} = (Z_0, Z_1, \dots)$ is a Markov chain with

$$\mathcal{L}(Z_n \mid Z_{n-1} = z,\ \Pi = (q_1, q_2, \dots)) = \mathcal{L}(\xi_{n1} + \cdots + \xi_{nz} \mid q_n)$$

for every $n \ge 1, z \in \mathbb{N}_0$, and $q_1, q_2, \dots \in \Delta$, where $\xi_{n1}, \xi_{n2}, \dots$ are i.i.d. random variables with distribution q_n.

We assume that $Z_0 = 1$ almost surely (a.s.) for convenience and denote the corresponding probability measure on the underlying probability space by $\boldsymbol{P}$ (and expectation by $\boldsymbol{E}$). (If we refer to other probability spaces, then we use notation $\mathbb{P}$ and $\mathbb{E}$ (maybe with some indices) for the respective probability measures and expectations and laws.)

As it turns out, the properties of $\mathcal{Z}$ are first of all determined by its associated random walk $\mathcal{S} := \{S_n,\ n \ge 0\}$. This random walk has initial state $S_0 = 0$ and increments $X_n = S_n - S_{n-1}$,

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$n \geq 1$, defined as

$$X_n := \log\biggl(\sum_{y=0}^{\infty} y Q_n(\{y\})\biggr),$$

which are i.i.d. copies of the logarithmic mean offspring number

$$X := \log\biggl(\sum_{y=0}^{\infty} y Q(\{y\})\biggr).$$

Following [8], we call the process $\mathcal{Z} := \{Z_n,\ n \geq 0\}$ *critical* if and only if the random walk $\mathcal{S}$ is oscillating, that is, $\limsup_{n\to\infty} S_n = \infty$ and $\liminf_{n\to\infty} S_n = -\infty$ with probability 1. It was shown in [8] that the extinction moment of the critical branching process in a random environment is finite with probability 1, and, moreover, if

$$\lim_{n\to\infty} \boldsymbol{P}(S_n > 0) = \rho \in (0, 1), \tag{2}$$

then (under some mild additional assumptions to be specified later on)

$$\boldsymbol{P}(Z_n > 0) \sim \theta \boldsymbol{P}(\min(S_0, S_1, \dots, S_n) \geq 0) = \theta \frac{l(n)}{n^{1-\rho}}, \tag{3}$$

where $l(n)$ is a slowly varying function and θ is a known positive constant whose explicit expression is given by (15) below.

Let

$$\mathcal{A} = \{0 < \alpha < 1;\ |\beta| < 1\} \cup \{1 < \alpha < 2;\ |\beta| < 1\} \cup \{\alpha = 1, \beta = 0\} \cup \{\alpha = 2, \beta = 0\}$$

be a subset in $\mathbb{R}^2$. For $(\alpha, \beta) \in \mathcal{A}$ and a random variable X, write $X \in \mathcal{D}(\alpha, \beta)$ if the distribution of X belongs to the domain of attraction of a stable law with characteristic function

$$\mathcal{H}_{\alpha,\beta}(t) := \exp\biggl\{-c|t|^{\alpha}\biggl(1 + \mathrm{i}\beta\frac{t}{|t|}\tan\frac{\pi\alpha}{2}\biggr)\biggr\}, \qquad c > 0, \tag{4}$$

and, in addition, $\boldsymbol{E}X = 0$ if this moment exists. Note that for $X \in \mathcal{D}(\alpha, \beta)$, the quantity ρ in (2) may be calculated explicitly; see, for instance, [29].

Denote $\mathbb{N}_+ := \{1, 2, \dots\}$ and let $\{c_n,\ n \geq 1\}$ be a sequence of positive integers specified by the relation $c_n := \inf\{u \geq 0\colon G(u) \leq n^{-1}\}$, where

$$u^2 G(u) := \boldsymbol{E}X^2 \mathbf{1}\{|X| \leq u\},$$

and $\mathbf{1}\{D\}$ is the indicator of the event D. It is known (see, for instance, [18, Chapter XVII, Section 5]) that, if $X \in \mathcal{D}(\alpha, \beta)$ then there exists a function $l_1(n)$, slowly varying at ∞, such that $c_n = n^{1/\alpha} l_1(n)$. In addition, the scaled sequence $\{S_n/c_n,\ n \geq 1\}$ converges in distribution, as $n \to \infty$, to the stable law given by (4).

Denote

$$M_n := \max(S_1, \dots, S_n), \qquad L_{k,n} := \min_{k\leq j\leq n} S_j, \qquad L_n := L_{0,n} = \min(S_0, S_1, \dots, S_n),$$

and introduce a right-continuous function

$$V(x) := \mathbf{1}\{x \geq 0\} + \sum_{k=1}^{\infty} \boldsymbol{P}(-S_k \leq x, M_k < 0).$$

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The fundamental property of V is the identity

$$\boldsymbol{E}[V(x+X);\ x+X\geq 0]=V(x),\qquad x\geq 0, \tag{5}$$

which holds for any oscillating random walk. The function V gives rise to further probability measures $\boldsymbol{P}_x^+,\ x\geq 0$, specified by corresponding expectations $\boldsymbol{E}_x^+$. The construction procedure of this measure is explained in [10] in detail. This was also carried out in [8] for the branching processes setting. We only recall that if the random walk $\mathcal{S}=(S_n,\ n\geq 0)$ with $S_0=x\geq 0$ is adapted to some filtration $\mathcal{F}=(\mathcal{F}_n)$ and $\zeta_0,\zeta_1,\dots$ is a sequence of random variables, adapted to $\mathcal{F}$, then for each fixed n and a bounded and measurable function $g_n\colon\mathbb{R}^{n+1}\to\mathbb{R}$,

$$\boldsymbol{E}_x^+[g_n(\zeta_0,\dots,\zeta_n)]:=\frac{1}{V(x)}\boldsymbol{E}_x[g_n(\zeta_0,\dots,\zeta_n)V(S_n);\ L_n\geq 0],$$

where $\boldsymbol{E}_x$ is the expectation corresponding to the probability measure $\boldsymbol{P}_x$ which is generated by $\mathcal{S}$. For simplicity we write $\boldsymbol{P}^+$ for $\boldsymbol{P}_0^+$.

We now describe in brief a construction of Lévy processes conditioned to stay positive following basically the definitions given in [15] and [13].

Let $\Omega:=D([0,\infty),\mathbb{R})$ be the space of real-valued càdlàg paths on the real half-line $[0,\infty)$ and let $\mathcal{B}:=\{B_t,\ t\geq 0\}$ be the coordinate process defined as $B_t(\omega)=\omega_t$ for $\omega\in\Omega$. In the sequel we consider also the spaces $\Omega_U:=D([0,U],\mathbb{R}),\ U>0$.

We endow the spaces Ω and Ω_U with Skorokhod topology and denote by $\mathcal{F}=\{\mathcal{F}_t,\ t\geq 0\}$ and by $\mathcal{F}^U=\{\mathcal{F}_t,\ t\in[0,U]\}$ (with some misuse of notation) the natural filtrations of the processes $\mathcal{B}$ and $\mathcal{B}^U=\{B_t,\ t\in[0,U]\}$.

Let $\mathbb{P}_x$ be the law on Ω of an α-stable process $\mathcal{B}$, $\alpha\in(0,2]$ started at x and let $\mathbb{P}=\mathbb{P}_0$. Denote by $\rho=\mathbb{P}(B_1\geq 0)$ the positivity parameter of the process $\mathcal{B}$. We now introduce an analogue of the measure $\boldsymbol{P}^+$ for Lévy processes. Namely, following [14], we specify, for all $t>0$, $D\in\mathcal{F}_t$, the law $\mathbb{P}_x^+$ on Ω of the Lévy process starting at point $x>0$ and conditioned to stay positive by the equality

$$\mathbb{P}_x^+(D):=\frac{1}{x^{\alpha(1-\rho)}}\mathbb{E}_x\Big[B_t^{\alpha(1-\rho)}\,\mathbf{1}\{D\}\,\mathbf{1}\Big\{\inf_{0\leq u\leq t}B_u\geq 0\Big\}\Big].$$

This definition has no sense for $x=0$. However, it was shown in [15] that it is possible to construct a law $\mathbb{P}^+:=\mathbb{P}_0^+$ and a càdlàg Markov process with the same semigroup as $(\mathcal{B},\{\mathbb{P}_x^+,\ x>0\})$ and such that $\mathbb{P}^+(B_0=0)=1$.

Let $\mathbb{P}^{(m)}$ be the law on Ω_1 of the meander of length 1 associated with $(\mathcal{B},\mathbb{P})$, i.e.

$$\mathbb{P}^{(m)}(\cdot):=\lim_{x\downarrow 0}\mathbb{P}_x\Big(\cdot\ \Big|\ \inf_{0\leq u\leq 1}B_u\geq 0\Big). \tag{6}$$

Thus, the law $\mathbb{P}^{(m)}$ may be viewed as the law of the Lévy process $(\mathcal{B},\mathbb{P})$ conditioned to stay nonnegative on the time-interval $(0,1)$, while the law $\mathbb{P}^+$ corresponds to the law of the Lévy process conditioned to stay nonnegative on the whole real half-line $(0,\infty)$.

It was proved in [15] that $\mathbb{P}^{(m)}$ and $\mathbb{P}^+$ are absolutely continuous with respect to each other: for every event $D\in\mathcal{F}_1$,

$$\mathbb{P}^+(D)=C_0\mathbb{E}^{(m)}[\mathbf{1}\{D\}B_1^{\alpha(1-\rho)}],$$

where (see, for instance, Equations (3.5), (3.6), and (3.11) in [13])

$$C_0:=\lim_{n\to\infty}V(c_n)\boldsymbol{P}(L_n\geq 0)=\frac{1}{\mathbb{E}^{(m)}[B_1^{\alpha(1-\rho)}]}\in(0,\infty). \tag{7}$$

Set

$$\zeta(b) := \frac{\sum_{y=b}^{\infty} y^2 Q(\{y\})}{(\sum_{y=0}^{\infty} y Q(\{y\}))^2}, \qquad b \in \mathbb{N}_0.$$

In what follows we say that

- condition (A1) is valid if $X \in \mathcal{D}(\alpha, \beta)$;
- condition (A2) is valid if
$$\boldsymbol{E}(\log^+ \zeta(b))^{\alpha+\varepsilon} < \infty$$
for some $\varepsilon > 0$ and $b \in \mathbb{N}_0$;
- condition (A) is valid if conditions (A1) and (A2) hold and, in addition, the parameter $p = p(n)$ tends to ∞ as $n \to \infty$ in such a way that
$$\lim_{n\to\infty} n^{-1} p = \lim_{n\to\infty} n^{-1} p(n) = 0. \tag{8}$$

Introduce two processes

$$\mathcal{H}^p := \left\{ \frac{\log Z_{[pu]}}{c_p},\ 0 \le u < \infty \right\}, \qquad \mathcal{G}^n := \left\{ \frac{\log Z_{[nt]}}{c_n},\ 0 \le t \le 1 \right\}.$$

We are now ready to formulate two main results of the paper.

The first theorem describes the initial evolutionary stage of the critical branching process in a random environment that provides survival of the process for a long time.

Theorem 1. *If condition (A) is valid then, as $n \to \infty$,*

$$\mathcal{L}(\mathcal{H}^p \mid Z_n > 0,\ Z_0 = 1) \xrightarrow{\mathrm{w}} \mathbb{P}^+(\mathcal{B}),$$

where the symbol '$\xrightarrow{\mathrm{w}}$' stands for the weak convergence in the space $D([0, \infty), \mathbb{R})$ of càdlàg functions in $[0, \infty)$ endowed with the Skorokhod topology. In particular,

$$\lim_{n\to\infty} \boldsymbol{P}\left(\frac{\log Z_p}{c_p} \le z \mid Z_n > 0,\ Z_0 = 1 \right) = \mathbb{P}^+(B_1 \le z) = C_0 \mathbb{E}^{(m)}[\mathbf{1}\{B_1 \le z\} B_1^{\alpha(1-\rho)}]$$

for any $z > 0$.

Remark 1. This theorem complements Corollary 1.6 in [8], which states that under conditions (A1) and (A2),

$$\mathcal{L}(\mathcal{G}^n \mid Z_n > 0, Z_0 = 1) \xrightarrow{\mathrm{w}} \mathbb{P}^{(m)}(\mathcal{B}^1) \quad \text{as } n \to \infty.$$

In particular,

$$\lim_{n\to\infty} \boldsymbol{P}\left(\frac{\log Z_n}{c_n} \le z \mid Z_n > 0, Z_0 = 1 \right) = \mathbb{P}^{(m)}(B_1 \le z) = \mathbb{E}^{(m)}[\mathbf{1}\{B_1 \le z\}]$$

for any $z > 0$.

Let, for $U > 0$,

$$\mathcal{H}_U^p := \left\{ \frac{\log Z_{[pu]}}{c_p}, 0 \le u \le U \right\}.$$

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Corollary 1. *If condition (A) is valid then, for any $U > 0$,*

$$\mathcal{L}((\mathcal{H}_U^p, \mathcal{G}^n) \mid Z_n > 0, Z_0 = 1) \xrightarrow{\mathrm{w}} \mathbb{P}^+(\mathcal{B}^U) \times \mathbb{P}^{(m)}(\mathcal{B}^1) \quad \text{as } n \to \infty.$$

We have seen by (3) that the asymptotic behavior of the survival probability of the process $\mathcal{Z}$ is primarily determined by the random walk $\mathcal{S}$, since only the constant θ depends on the fine structure of $\mathcal{Z}$; see (15) below. Moreover, the random walk changes its properties drastically, when conditioned on the event $\{Z_n > 0\}$. The next theorem, describing the trajectories of the random walk $\mathcal{S}$ that provide survival of the critical process in a random environment at the initial stage of the development of the population, illustrates this fact.

For $U \in (0, \infty]$, let

$$\mathcal{Q}_U^p := \left\{ \frac{S_{[pu]}}{c_p}, \ 0 \le u \le U \right\}, \qquad \mathcal{Q}^p = \mathcal{Q}_\infty^p,$$

$$\mathcal{S}_U^n := \left\{ \frac{S_{pU+[(n-pU)t]}}{c_n}, \ 0 \le t \le 1 \right\}, \qquad \mathcal{S}^n := \mathcal{S}_0^n.$$

Theorem 2. *If condition (A) is valid then, as $n \to \infty$,*

$$\mathcal{L}(\mathcal{Q}^p \mid Z_n > 0, Z_0 = 1) \xrightarrow{\mathrm{w}} \mathbb{P}^+(\mathcal{B}).$$

Remark 2. This theorem complements Theorem 1.5 in [8], which states that under conditions (A1) and (A2),

$$\mathcal{L}(\mathcal{S}^n \mid Z_n > 0, Z_0 = 1) \xrightarrow{\mathrm{w}} \mathbb{P}^{(m)}(\mathcal{B}^1) \quad \text{as } n \to \infty.$$

Corollary 2. *If condition (A) is valid then, for any $U > 0$,*

$$\mathcal{L}((\mathcal{Q}_U^p, \mathcal{S}^n) \mid Z_n > 0, Z_0 = 1) \xrightarrow{\mathrm{w}} \mathbb{P}^+(\mathcal{B}^U) \times \mathbb{P}^{(m)}(\mathcal{B}^1) \quad \text{as } n \to \infty.$$

The usage of the associated random walks to study branching processes in a random environment has a long history. It seems that Kozlov [20] was the first who observed that to investigate properties of the critical branching processes in a random environment it is convenient to use ladder epochs of the associated random walks. This fact has been used in various situations for the case of the associated random walks with zero or negative drift and finite variance of increments; see [1]–[5] [19], [21], and [23]. The first steps to overcome the assumption of a finite variance random walk in the driftless case were taken in [17] and [25]. In recent years the authors in [6]–[9], [12], [26], and some others provide a systematic approach to the study of branching processes in a random environment under rather general assumptions on the properties of the associated random walk; see [24] and [28] for a detailed exposition.

2. Auxiliary results

We will use the symbols $K, K_1, K_2, \dots$ to denote different constants. They are not necessarily the same in different formulas.

2.1. Properties of the associated random walk

To prove the main results of the paper we need to know the asymptotic behavior of the function $V(x)$ as $x \to \infty$. The following lemma yields the desired asymptotics.

Lemma 1. (See Lemma 13 in [27] and Corollary 8 in [16].) *If $X \in \mathcal{D}(\alpha, \beta)$ then there exists a slowly varying function $l_0(x)$ such that*

$$V(x) \sim x^{\alpha(1-\rho)} l_0(x) \quad \text{as } x \to \infty. \tag{9}$$

Our next result is a combination (with a slight reformulation) of Lemma 2.1 in [8] and Corollaries 3 and 8 in [16].

Lemma 2. *If $X \in \mathcal{D}(\alpha, \beta)$ then there exist positive constants K, K_1, and K_2 such that, as $n \to \infty$,*

$$\boldsymbol{P}(L_n \geq -w) \sim V(w)\boldsymbol{P}(L_n \geq 0) \sim KV(w)n^{\rho-1}l(n) \tag{10}$$

uniformly for $0 \leq w \ll c_n$, and

$$\boldsymbol{P}(L_n \geq -w) \leq K_1 V(w) n^{\rho-1} l(n) \leq K_2 V(w)\boldsymbol{P}(L_n \geq 0), \qquad w \geq 0,\ n \geq 1. \tag{11}$$

For further references we prove the following simple statement.

Lemma 3. *Let $\mathcal{A}_n \subset \mathbb{R}$, $n \in \mathbb{N}$, be a family of subsets and let $b_n(x)$, $n \in \mathbb{N}$, be a sequence of functions such that, for any fixed sequence $\{a_n,\ n \in \mathbb{N}\}$ such that $a_n \in \mathcal{A}_n$ for all $n \in \mathbb{N}$,*

$$\lim_{n\to\infty} b_n(a_n) = 0. \tag{12}$$

Then $\lim_{n\to\infty} \sup_{a\in\mathcal{A}_n} |b_n(a)| = 0$.

Proof. Assume that the conclusion of the lemma does not hold. Then, there exists $\varepsilon > 0$ such that, for all N, there exist $n(N) \geq N$ and $a_{n(N)} \in \mathcal{A}_{n(N)}$ such that

$$|b_{n(N)}(a_{n(N)})| \geq \varepsilon.$$

This, clearly, contradicts (12). □

In the sequel we agree to consider the expressions of the form $\lim A(p, n)$ or $\limsup A(p, n)$ without lower indices as the lim or lim sup of the triangular array $\{A(p, n),\ p \geq 1, n \geq 1\}$ calculated under the assumption that $pn^{-1} \to 0$ as $p, n \to \infty$. We also write $a_n \ll b_n$if $\lim_{n\to\infty} a_n/b_n = 0$.

Let $\phi_1 \colon \Omega_1 \to \mathbb{R}$ be a bounded uniformly continuous functional and $\{\varepsilon_n,\ n \in \mathbb{N}\}$ be a sequence of positive numbers vanishing as $n \to \infty$.

Lemma 4. *If condition (A1) is valid then*

$$\boldsymbol{E}[\phi_1(\mathcal{S}^n) \mid L_n \geq -x] \to \mathbb{E}^{(m)}[\phi_1(\mathcal{B}^1)] \tag{13}$$

as $n \to \infty$ uniformly in $0 \leq x \leq \varepsilon_n c_n$.

Proof. It was shown in Theorem 1.1 of [13] that, given condition (A1), convergence (13) holds for any sequence $x = x_n$ meeting the restriction $0 \leq x_n \ll c_n$ as $n \to \infty$. This and Lemma 3 with $\mathcal{A}_n := \{0 \leq x \leq \varepsilon_n c_n\}$ imply the desired statement. □

Now we are ready to demonstrate the validity of the following result.

Lemma 5. *If conditions (A1) and (8) are valid then, for $U > 0$ and any $r \geq 0$,*

$$\mathcal{L}((\mathcal{Q}_U^p, \mathcal{S}^n) \mid L_n \geq -r) \xrightarrow{\text{w}} \mathbb{P}^+(\mathcal{B}^U) \times \mathbb{P}^{(m)}(\mathcal{B}^1) \quad \text{as } n \to \infty.$$

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Proof. Consider the processes $\mathcal{S}^{k,n}$ and $\tilde{\mathcal{S}}^{k,n}$, $0 \le k \le n$, given by

$$S_t^{k,n} := \frac{S_{[nt]\wedge k}}{c_n}, \qquad \tilde{S}^{k,n} := \frac{1}{c_n}(S_{[nt]} - S_{[nt]\wedge k}), \quad 0 \le t \le 1.$$

Clearly,

$$\mathcal{S}^n = \mathcal{S}^{k,n} + \tilde{\mathcal{S}}^{k,n}.$$

Let $\mathcal{S}^* := \{S_n^*,\ n \ge 0\}$ be a probabilistic and independent copy of the random walk $\mathcal{S} = \{S_n,\ n \ge 0\}$ and

$$L_n^* := \min(S_0^*, S_1^*, \dots, S_n^*), \qquad (\mathcal{S}^*)_U^n := \biggl\{\frac{S^*_{[(n-pU)t]}}{c_n},\ 0 \le t \le 1\biggr\}.$$

For a fixed $N > 0$ set

$$I_N(x) := \begin{cases} 0 & \text{if } x \le N^{-1}, \\ Nx - 1 & \text{if } x \in (N^{-1}, 2N^{-1}), \\ 1 & \text{if } 2N^{-1} \le x \le N, \\ N + 1 - x & \text{if } N < x \le N+1, \\ 0 & \text{if } x > N+1, \end{cases}$$

and let

$$\phi\colon \Omega_U \to \mathbb{R} \quad \text{and} \quad \phi_1\colon \Omega_1 \to \mathbb{R}$$

be two continuous and bounded functionals.

Then, for fixed positive U and N and $pU = n\varepsilon_n$, where $\varepsilon \ge \varepsilon_n \downarrow 0$ as $n \to \infty$, we have (with a slight abuse of notation)

$$\begin{aligned} &\boldsymbol{E}\biggl[\phi(\mathcal{Q}_U^p) I_N\biggl(\frac{S_{pU}}{c_p}\biggr)\phi_1(\mathcal{S}^n);\ L_n \ge -r\biggr] \\ &\quad = \boldsymbol{E}\biggl[\phi(\mathcal{Q}_U^p) I_N\biggl(\frac{S_{pU}}{c_p}\biggr)\mathbf{1}\{L_{pU} \ge -r\} \\ &\qquad \times \boldsymbol{E}[\phi_1((\mathcal{S}^*)_U^n + \mathcal{S}^{pU,n})\,\mathbf{1}\{L^*_{n-pU} \ge -S_{pU} - r\}]\biggr]. \end{aligned}$$

Here and in what follows we agree to consider pU and $n - pU$ as $[pU]$ and $[n - pU]$, respectively. Since $c_p/c_n \to 0$ as $n \to \infty$, it follows that, given $L_{pU} \ge -r$,

$$\frac{S_{pU}}{c_n} I_N\biggl(\frac{S_{pU}}{c_p}\biggr) \to 0 \quad \text{a.s.}$$

and $\mathcal{S}^{pU,n}$ vanishes as $n \to \infty$. This observation, Lemma 4, and the continuity of ϕ_1 imply that

$$\boldsymbol{E}[\phi_1((\mathcal{S}^*)_U^n + \mathcal{S}^{pU,n}) \mid L^*_{n(1-\varepsilon_n)} \ge -S_{pU} - r] \to \mathbb{E}^{(m)}[\phi_1(\mathcal{B}^1)]$$

as $n \to \infty$ uniformly for $0 \le S_{pU} \le Nc_p \ll c_n$. On the other hand, by (7), (9), (10), and properties of regularly varying functions (see, for instance, [22]), we deduce that, as $p, n \to \infty$,

$$\begin{aligned} \boldsymbol{P}(L^*_{n-pU} \ge -S_{pU} - r) I_N\bigg(\frac{S_{pU}}{c_p}\bigg) &\sim V(S_{pU}) I_N\bigg(\frac{S_{pU}}{c_p}\bigg)\boldsymbol{P}(L_n \ge 0) \\ &= \frac{V(S_{pU})}{V(c_p)} I_N\bigg(\frac{S_{pU}}{c_p}\bigg) V(c_p)\boldsymbol{P}(L_n \ge 0) \\ &\sim \bigg(\frac{S_{pU}}{c_p}\bigg)^{\alpha(1-\rho)} I_N\bigg(\frac{S_{pU}}{c_p}\bigg)\frac{C_0\boldsymbol{P}(L_n \ge 0)}{\boldsymbol{P}(L_p \ge 0)} \\ &\sim \bigg(\frac{S_{pU}}{c_p}\bigg)^{\alpha(1-\rho)} I_N\bigg(\frac{S_{pU}}{c_p}\bigg)\frac{C_0\boldsymbol{P}(L_n \ge -r)}{\boldsymbol{P}(L_p \ge -r)}. \end{aligned}$$

Hence, after evident but awkward transformations, we have, as $p, n \to \infty$,

$$\begin{aligned} &\boldsymbol{E}\bigg[\phi(\mathcal{Q}^p_U)\phi_1(\mathcal{S}^n) I_N\bigg(\frac{S_{pU}}{c_p}\bigg) \;\bigg|\; L_n \ge -r\bigg] \\ &\qquad \sim C_0\mathbb{E}^{(m)}[\phi_1(\mathcal{B}_1)]\boldsymbol{E}\bigg[\phi(\mathcal{Q}^p_U)\bigg(\frac{S_{pU}}{c_p}\bigg)^{\alpha(1-\rho)} I_N\bigg(\frac{S_{pU}}{c_p}\bigg) \;\bigg|\; L_{pU} \ge -r\bigg]. \end{aligned}$$

By Theorem 1.1 of [13], as $p \to \infty$,

$$\begin{aligned} \boldsymbol{E}\bigg[\phi(\mathcal{Q}^p_U)\bigg(\frac{S_{pU}}{c_p}\bigg)^{\alpha(1-\rho)} I_N\bigg(\frac{S_{pU}}{c_p}\bigg) \;\bigg|\; L_{pU} \ge -r\bigg] &\to \mathbb{E}^{(m)}[\phi(\mathcal{B}^U) B_U^{\alpha(1-\rho)} I_N(B_U)] \\ &= \mathbb{E}^+[\phi(\mathcal{B}^U) I_N(B_U)]. \end{aligned}$$

Thus, under conditions (A1) and (8),

$$\lim \boldsymbol{E}\bigg[\phi(\mathcal{Q}^p_U) I_N\bigg(\frac{S_{pU}}{c_p}\bigg)\phi_1(\mathcal{S}^n) \;\bigg|\; L_n \ge -r\bigg] = C_0\mathbb{E}^+[\phi(\mathcal{B}^U) I_N(B_U)]\mathbb{E}^{(m)}[\phi_1(\mathcal{B}^1)].$$

Letting now $N \to \infty$, we obtain

$$\mathcal{L}((\mathcal{Q}^p_U, \mathcal{S}^n) \mid L_n \ge -r) \xrightarrow{\text{w}} \mathbb{P}^+(\mathcal{B}^U)\mathbb{P}^{(m)}(\mathcal{B}^1) \quad \text{for any } U > 0. \qquad \square$$

Corollary 3. *If conditions (A1) and (8) are valid then*

$$\mathcal{L}(\mathcal{Q}^p \mid L_n \ge -r) \xrightarrow{\text{w}} \mathbb{P}^+(\mathcal{B}) \quad \text{as } n \to \infty.$$

Proof. It follows from Lemma 5 that

$$\mathcal{L}(\mathcal{Q}^p_U \mid L_n \ge -r) \xrightarrow{\text{w}} \mathbb{P}^+(\mathcal{B}^U) \quad \text{for any } U > 0.$$

This fact combined with Theorem 16.7 in [11] completes the proof of the corollary. $\square$

3. Conditional limit theorems

For convenience we introduce the notation

$$A_{\text{u.s.}} = \{Z_n > 0 \text{ for all } n \ge 0\}, \qquad \tau_n := \min\{j : S_j = L_n\},$$

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where 'u.s.' denotes ultimate survival. Recall that $\boldsymbol{P}^+(A_{\mathrm{u.s.}}) > 0$ by Proposition 3.1 in [8] and that, as $n \to \infty$,

$$\boldsymbol{P}(Z_n > 0) \sim \theta \boldsymbol{P}(L_n \geq 0) \sim \theta n^{-(1-\rho)} l(n) \sim \frac{\theta C_0}{V(c_n)} \tag{14}$$

by Corollary 1.2 in [8], (3), and (7), where

$$\theta = \sum_{k=0}^{\infty} \boldsymbol{E}[\boldsymbol{P}^+_{Z_k}(A_{\mathrm{u.s.}});\ \tau_k = k]. \tag{15}$$

Let

$$\hat{L}_{k,n} := \min_{0 \leq j \leq n-k} (S_{k+j} - S_k)$$

and let $\tilde{\mathcal{F}}_k$ be the σ-algebra generated by the tuple $\{Z_0, Z_1, \dots, Z_k;\ Q_1, Q_2, \dots, Q_k\}$; see (1). For further references we formulate two statements borrowed from [8].

Lemma 6. (See Lemma 2.5 in [8].) *Assume that condition (A1) holds. Let $Y_1, Y_2, \dots$ be a uniformly bounded sequence of real-valued random variables adapted to the filtration $\tilde{\mathcal{F}} = \{\tilde{\mathcal{F}}_k,\ k \in \mathbb{N}\}$, which converges $\boldsymbol{P}^+$-a.s. to some random variable Y_∞. Then, as $n \to \infty$,*

$$\boldsymbol{E}[Y_n \mid L_n \geq 0] \to \boldsymbol{E}^+[Y_\infty].$$

Lemma 7. (See Lemma 4.1 in [8].) *Assume that conditions (A1) hold and let $l \in \mathbb{N}_0$. Suppose that $\zeta_1, \zeta_2, \dots$ is a uniformly bounded sequence of real-valued random variables, which, for every $k \geq 0$, meets the equality*

$$\boldsymbol{E}[\zeta_n;\ Z_{k+l} > 0, \hat{L}_{k,n} \geq 0 \mid \tilde{\mathcal{F}}_k] = \boldsymbol{P}(L_n \geq 0)(\zeta_{k,\infty} + o(1)), \qquad \boldsymbol{P}\text{-a.s. as } n \to \infty$$

with random variables $\zeta_{1,\infty} = \zeta_{1,\infty}(l)$ and $\zeta_{2,\infty} = \zeta_{2,\infty}(l), \dots$. Then

$$\boldsymbol{E}[\zeta_n;\ Z_{\tau_n+l} > 0] = \boldsymbol{P}(L_n \geq 0)\bigg(\sum_{k=0}^{\infty} \boldsymbol{E}[\zeta_{k,\infty};\ \tau_k = k] + o(1)\bigg) \quad \text{as } n \to \infty,$$

where the right-hand side series is absolutely convergent.

For $U > 0$ and $q \leq p$, $pU \leq n$, let

$$\mathcal{X}^{q,p}_U := \{X^{q,p}_u = \mathrm{e}^{-S_{q+[u(p-q)]}} Z_{q+[u(p-q)]},\ 0 \leq u \leq U\},$$
$$\mathcal{X}^{q,p} := \{X^{q,p}_u = \mathrm{e}^{-S_{q+[u(p-q)]}} Z_{q+[u(p-q)]},\ 0 \leq u < \infty\},$$
$$\mathcal{Y}^{p,n}_U := \{Y^{p,n}_t = \mathrm{e}^{-S_{pU+[(n-pU)t]}} Z_{pU+[(n-pU)t]},\ 0 \leq t \leq 1\}, \qquad \mathcal{Y}^{p,n} := \mathcal{Y}^{p,n}_0.$$

The next statement is an evident corollary of Theorem 1.3 in [8] and we present its proof for completeness only.

Lemma 8. *Assume that conditions (A1) and (A2) hold. Let $(q_1, p_1), (q_2, p_2), \dots$ be a sequence of pairs of positive integers such that $q_n \ll p_n$ as $n \to \infty$. If $p_n \ll n$ then, for any $U > 0$,*

$$\mathcal{L}((\mathcal{X}^{q_n,p_n}_U, \mathcal{Y}^{p_n,n}_U) \mid Z_n > 0, Z_0 = 1) \xrightarrow{\mathrm{w}} \mathcal{L}((W_u,\ 0 \leq u \leq U), (\check{W}_t,\ 0 \leq t \leq 1))$$

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as $n \to \infty$, *where*

$$P(W_u = \check{W}_t = W,\ 0 \le u \le U, 0 \le t \le 1) = 1$$

for some random variable W *such that*

$$P(0 < W < \infty) = 1.$$

Proof. We follow (with minor changes) the line of the proof of Theorem 1.3 in [8]. According to Proposition 3.1 in [8], there exists a strictly positive and finite random variable W^+ such that, as $n \to \infty$,

$$\mathrm{e}^{-S_n} Z_n \to W^+, \qquad P^+\text{-a.s.} \tag{16}$$

and

$$\{W^+ > 0\} = \{Z_n > 0 \text{ for all } n\}, \qquad P^+\text{-a.s.} \tag{17}$$

Fix $U > 0$ and let ϕ be a bounded continuous function on the space $\Omega_U = D([0, U], \mathbb{R})$ of càdlàg functions and let ϕ_1 be a bounded continuous function on the space Ω_1. For $s \in \mathbb{R}$, let $\mathcal{W}_U^s := \{W_u^s,\ 0 \le u \le U\}$ and $\check{\mathcal{W}}^s := \{\check{W}_t^s,\ 0 \le t \le 1\}$ denote the processes with constant paths coinciding (formally) within the time-interval $[0, \min\{U, 1\}]$, i.e.

$$W_u^s := \mathrm{e}^{-s} W^+, \quad 0 \le u \le U, \qquad \check{W}_t^s := \mathrm{e}^{-s} W^+, \quad 0 \le t \le 1.$$

From (16), it follows that, for fixed $s \in \mathbb{R}$, the two-dimensional process

$$(\mathrm{e}^{-s} \mathcal{X}_U^{q_n, p_n}, \mathrm{e}^{-s} \mathcal{Y}_U^{p_n, n})$$

converges, as $n, p_n \to \infty$ with $q_n \le p_n \ll n$, to $(\mathcal{W}_U^s, \check{\mathcal{W}}^s)$ in the metric of uniform convergence and, consequently, in the Skorokhod metric on the space $\Omega_U \times \Omega_1$ P^+-a.s., and

$$\begin{aligned} \mathcal{K}_n &:= \phi(\mathrm{e}^{-s} \mathcal{X}_U^{q_n, p_n}) \phi_1(\mathrm{e}^{-s} \mathcal{Y}_U^{p_n, n})\, \mathbf{1}(Z_n > 0) \\ &\to \mathcal{K}_\infty := \phi(\mathcal{W}_U^s) \phi_1(\check{\mathcal{W}}^s)\, \mathbf{1}\{W^+ > 0\}, \qquad P^+\text{-a.s.} \end{aligned}$$

For $q \le p \le n$ and $z \in \mathbb{N}_0$, define

$$\begin{aligned} \psi(z, s, q, p, n) &:= E_z[\phi(\mathrm{e}^{-s} \mathcal{X}_U^{q,p}) \phi_1(\mathrm{e}^{-s} \mathcal{Y}_U^{p,n});\ Z_n > 0, L_n \ge 0] \\ &= E_z[\phi(\mathrm{e}^{-s} \mathcal{X}_U^{q,p}) \phi_1(\mathrm{e}^{-s} \mathcal{Y}_U^{p,n})\, \mathbf{1}(Z_n > 0) \mid L_n \ge 0] P(L_n \ge 0). \end{aligned}$$

Since $\mathcal{K}_n \to \mathcal{K}_\infty$, P^+-a.s. as $n \to \infty$, from Lemma 6, it follows that

$$\psi(z, s, q_n, p_n, n) = P(L_n \ge 0)(E_z^+[\phi(\mathcal{W}_U^s) \phi_1(\check{\mathcal{W}}^s);\ W^+ > 0] + o(1)).$$

Observe now that, for $k \le q \le p \le n$,

$$E[\phi(\mathrm{e}^{-s} \mathcal{X}_U^{q,p}) \phi_1(\mathrm{e}^{-s} \mathcal{Y}_U^{p,n});\ Z_n > 0, \hat{L}_{k,n} \ge 0 \mid \mathcal{F}_k] = \psi(Z_k, S_k, q - k, p - k, n - k).$$

Therefore, we may apply Lemma 7 to the random variables

$$\zeta_n = \phi(\mathrm{e}^{-s} \mathcal{X}_U^{q_n, p_n}) \phi_1(\mathrm{e}^{-s} \mathcal{Y}_U^{p_n, n})\, \mathbf{1}\{Z_n > 0\}, \qquad \zeta_{k,\infty} = E_{Z_k}^+[\phi(\mathcal{W}_U^{S_k}) \phi_1(\check{\mathcal{W}}^{S_k});\ W^+ > 0]$$

with $l = 0$.

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Using (14), we obtain

$$\boldsymbol{E}[\phi(\mathcal{X}_U^{q_n,p_n})\phi_1(\mathcal{Y}_U^{p_n,n}) \mid Z_n > 0] \to \int \phi(\mathfrak{m})\phi_1(\mathfrak{n})\lambda(\mathrm{d}\mathfrak{m} \times \mathrm{d}\mathfrak{n}) \quad \text{as } n \to \infty,$$

where λ is the measure on the product space of càdlàg functions on $\Omega_U \times \Omega_1$ specified by

$$\lambda(\mathrm{d}\mathfrak{m} \times \mathrm{d}\mathfrak{n}) := \frac{1}{\theta} \sum_{k=0}^{\infty} \boldsymbol{E}[\lambda_{Z_k,S_k}(\mathrm{d}\mathfrak{m} \times \mathrm{d}\mathfrak{n});\ Z_k > 0, \tau_k = k]$$

with

$$\lambda_{z,s}(\mathrm{d}\mathfrak{m} \times \mathrm{d}\mathfrak{n}) := \boldsymbol{P}_z^+[\mathcal{W}_U^s \in \mathrm{d}\mathfrak{m},\ \check{\mathcal{W}}^s \in \mathrm{d}\mathfrak{n},\ W^+ > 0].$$

By (17), the total mass of $\lambda_{z,s}$ is equal to $\boldsymbol{P}_z^+(Z_n > 0$ for all $n \geq 0)$. Therefore, the representation of θ in (15) shows that λ is a probability measure. Again using (17), we see that $\lambda_{z,s}$ is concentrated on strictly positive constant functions only. Hence, the same is true for the measure λ. □

Corollary 4. *Assume that conditions (A1) and (A2) hold. Let* $(q_1, p_1), (q_2, p_2), \ldots$ *be a sequence of pairs of positive integers such that* $q_n \ll p_n \ll n$ *and* $q_n \to \infty$ *as* $n \to \infty$. *Then*

$$\mathcal{L}(\mathcal{X}^{q_n,p_n} \mid Z_n > 0, Z_0 = 1) \xrightarrow{\mathrm{w}} \mathcal{L}(\{W_u,\ 0 \leq u < \infty\}).$$

Proof of Theorem 2. Let $U > 0$ be fixed. Consider the processes

$$\mathcal{Q}_U^{q,p} = \{S_u^{q,p},\ 0 \leq u \leq U\}, \qquad \tilde{\mathcal{Q}}_U^{q,p} = \{\tilde{S}_u^{q,p},\ 0 \leq u \leq U\}, \quad 0 \leq q \leq pU,$$

given by

$$S_u^{q,p} := \frac{S_{[pu]\wedge q}}{c_p}, \qquad \tilde{S}_u^{q,p} := \frac{1}{c_p}(S_{[pu]} - S_{[pu]\wedge q}), \quad 0 \leq u \leq U.$$

Take $k, l \geq 0$ with $k + l \leq pU$. We may decompose the stochastic process $\mathcal{Q}_U^p$ as

$$\mathcal{Q}_U^p := \mathcal{Q}_U^{k+l,p} + \tilde{\mathcal{Q}}_U^{k+l,p}.$$

Let ϕ be a bounded continuous functional on Ω_U. Define

$$\psi(\mathfrak{m}, r) := \boldsymbol{E}[\phi(\mathfrak{m} + \tilde{\mathcal{Q}}_U^{k+l,p});\ \hat{L}_{k+l,n} \geq -r]$$

for $\mathfrak{m} \in D[0, U]$ and $r \geq 0$. If $p, n \to \infty$ in such a way that $pn^{-1} \to 0$ then, according to Corollary 3,

$$\mathcal{L}(\{S_u^{k+l,p},\ 0 \leq u < \infty\} \mid \hat{L}_{k+p,n} \geq -r) \xrightarrow{\mathrm{w}} \mathbb{P}^+(\{B_u,\ 0 \leq u < \infty\})$$

for each fixed pair k and l. Hence, if the càdlàg functions $\mathfrak{m}^p \in \Omega_U$ converge uniformly to the zero function as $p \to \infty$, then, given (8),

$$\begin{aligned}\psi(\mathfrak{m}^p, r) &= \boldsymbol{P}(L_{n-(k+l)} \geq -r)(\mathbb{E}^+[\phi(\mathcal{B}^U)] + o(1)) \\ &= V(r)\boldsymbol{P}(L_n \geq 0)(\mathbb{E}^+[\phi(\mathcal{B}^U)] + o(1)) \quad \text{as } p, n \to \infty,\end{aligned}$$

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where for the second equality we have applied (10). Using the representation

$$\{\hat{L}_{k,n} \geq 0\} = \{\hat{L}_{k,k+l} \geq 0\} \cap \{\hat{L}_{k+l,n} \geq -(S_{k+l} - S_k)\} \tag{18}$$

and taking into account that $\mathcal{Q}_U^{k+l,p}$ converges uniformly to 0, $\boldsymbol{P}$-a.s. as $p \to \infty$, we have, under condition (A),

$$\begin{aligned}\boldsymbol{E}[\phi(\mathcal{Q}_U^p);\ Z_{k+l} > 0, \hat{L}_{k,n} \geq 0 \mid \mathcal{F}_{k+l}] \\ &= \psi(\mathcal{Q}_U^{k+l,p},\ S_{k+l} - S_k)\, \mathbf{1}\{Z_{k+l} > 0, \hat{L}_{k,k+l} \geq 0\} \\ &= V(S_{k+l} - S_k)\boldsymbol{P}(L_n \geq 0)(\mathbb{E}^+[\phi(\mathcal{B}^U)] + o(1))\, \mathbf{1}\{Z_{k+l} > 0, \hat{L}_{k,k+l} \geq 0\}, \quad \boldsymbol{P}\text{-a.s.}\end{aligned} \tag{19}$$

This representation combined with (11) and (18) allows us to deduce the chain of estimates

$$\begin{aligned}|\boldsymbol{E}[\phi(\mathcal{Q}_U^p);\ Z_{k+l} > 0, \hat{L}_{k,n} \geq 0 \mid \mathcal{F}_{k+l}]| \\ &\leq \sup|\phi|\boldsymbol{P}(\hat{L}_{k,n} \geq 0 \mid \mathcal{F}_{k+l}) \\ &= \sup|\phi|\boldsymbol{P}(\hat{L}_{k+l,n} \geq -(S_{k+l} - S_k) \mid \mathcal{F}_{k+l})\, \mathbf{1}\{\hat{L}_{k,k+l} \geq 0\} \\ &\leq K_1 V(S_{k+l} - S_k)\boldsymbol{P}(L_{n-(k+l)} \geq 0)\, \mathbf{1}\{\hat{L}_{k,k+l} \geq 0\}, \qquad \boldsymbol{P}\text{-a.s. for some } K_1 > 0.\end{aligned}$$

Observe that, according to (5),

$$\boldsymbol{E}[V(S_{k+l} - S_k);\ \hat{L}_{k,k+l} \geq 0 \mid \mathcal{F}_k] = V(0) < \infty, \qquad \boldsymbol{P}\text{-a.s.}$$

Hence, using the dominated convergence theorem, (6), and the definition of $\boldsymbol{P}^+$, we obtain, by (19),

$$\begin{aligned}\boldsymbol{E}[\phi(\mathcal{Q}_U^p);\ Z_{k+l} > 0, \hat{L}_{k,n} \geq 0 \mid \mathcal{F}_k] \\ &= (\mathbb{E}^+[\phi(\mathcal{B}^U)] + o(1))\boldsymbol{P}(L_n \geq 0)\boldsymbol{E}[V(S_{k+l} - S_k);\ Z_{k+l} > 0, \hat{L}_{k,k+l} \geq 0 \mid \mathcal{F}_k] \\ &= (\mathbb{E}^+[\phi(\mathcal{B}^U)] + o(1))\boldsymbol{P}(L_n \geq 0)\boldsymbol{P}^+_{Z_k}(Z_l > 0), \qquad \boldsymbol{P}\text{-a.s.}\end{aligned}$$

Applying Lemma 7 to $\zeta_n := \phi(\mathcal{Q}_U^p)$ and $\zeta_{k,\infty}(l) := \mathbb{E}^+[\phi(\mathcal{B}^U)]\boldsymbol{P}^+_{Z_k}(Z_l > 0)$ and letting $n \gg p = p(n) \to \infty$ yields

$$\boldsymbol{E}[\phi(\mathcal{Q}_U^p);\ Z_{\tau_n+l} > 0] = (\mathbb{E}^+[\phi(\mathcal{B}^U)] + o(1))\boldsymbol{P}(L_n \geq 0)\sum_{k=0}^{\infty} \boldsymbol{E}[\boldsymbol{P}^+_{Z_k}(Z_l > 0);\ \tau_k = k].$$

Therefore,

$$\boldsymbol{P}(Z_{\tau_n+l} > 0) \sim \boldsymbol{P}(L_n \geq 0)\sum_{k=0}^{\infty} \boldsymbol{E}[\boldsymbol{P}^+_{Z_k}(Z_l > 0);\ \tau_k = k] \quad \text{as } n \to \infty,$$

where the right-hand side series is convergent. Observe that

$$\begin{aligned}|\mathbb{E}^+[\phi(\mathcal{B}^U)]\boldsymbol{P}(Z_n > 0) - \boldsymbol{E}[\phi(\mathcal{Q}_U^p);\ Z_n > 0]| \\ &\leq |\mathbb{E}^+[\phi(\mathcal{B}^U)]\boldsymbol{P}(Z_n > 0) - \boldsymbol{E}[\phi(\mathcal{Q}_U^p);\ Z_{\tau_n+l} > 0]| \\ &\quad + \sup|\phi|\boldsymbol{E}|\, \mathbf{1}\{Z_n > 0\} - \mathbf{1}\{Z_{\tau_n+l} > 0\}|\end{aligned}$$

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and

$$\boldsymbol{E}|\mathbf{1}\{Z_n > 0\} - \mathbf{1}\{Z_{\tau_n+l} > 0\}| \\ \le (\boldsymbol{P}(Z_n > 0) - \boldsymbol{P}(Z_{n+l} > 0)) + (\boldsymbol{P}(Z_{\tau_n+l} > 0) - \boldsymbol{P}(Z_{n+l} > 0)).$$

These estimates and (14) lead to the inequality

$$|\mathbb{E}^+[\phi(\mathcal{B}^U)] - \boldsymbol{E}[\phi(\mathcal{Q}_U^p) \mid Z_n > 0]| \\ \le 2\sup|\phi|\bigg(\frac{1}{\theta}\sum_{k=0}^{\infty}\boldsymbol{E}[\boldsymbol{P}_{Z_k}^+(Z_l > 0);\ \tau_k = k] - 1\bigg) + \varepsilon(p,n), \tag{20}$$

where $\lim \varepsilon\,(p,n) = 0$. By the dominated convergence theorem and the definition of θ in (15), we conclude that

$$\sum_{k=0}^{\infty}\boldsymbol{E}[\boldsymbol{P}_{Z_k}^+(Z_l > 0);\ \tau_k = k] \downarrow \theta \quad \text{as } l \to \infty.$$

Since the left-hand side of (20) does not depend on l, this gives the assertion of Theorem 2 for an arbitrary interval $0 \le u \le U$. To complete the proof of the theorem it remains to apply Theorem 16.7 of [11]. □

Proof of Corollary 2. We use the notation of Lemma 5 and define

$$\psi^*(\mathfrak{m},\mathfrak{n},r) := \boldsymbol{E}[\phi(\mathfrak{m} + \tilde{\mathcal{Q}}_U^{k+l,p})\phi_1(\mathfrak{n} + \tilde{\mathcal{S}}^{k+l,p});\ \hat{L}_{k+l,n} \ge -r]$$

for $(\mathfrak{m},\mathfrak{n}) \in \Omega_U \times \Omega_1$ and $r \ge 0$. If a two-dimensional vector of càdlàg functions $(\mathfrak{m}^p, \mathfrak{n}^n) \in \Omega_U \times \Omega_1$ converges uniformly to the two-dimensional vector of zero functions as $p = p(n) \to \infty$ as $n \to \infty$, and condition (8) is valid, then, according to Lemma 5,

$$\psi^*(\mathfrak{m}^p,\mathfrak{n}^n,r) = \boldsymbol{P}(L_{n-(k+l)} \ge -r)(\mathbb{E}^+[\phi(\mathcal{B}^U)]\mathbb{E}^{(m)}[\phi_1(\mathcal{B}^1)] + o(1)) \\ = V(r)\boldsymbol{P}(L_n \ge 0)(\mathbb{E}^+[\phi(\mathcal{B}^U)]\mathbb{E}^{(m)}[\phi_1(\mathcal{B}^1)] + o(1)).$$

Let k and l be fixed. We know that the pair $(\mathcal{Q}^{k+l,p}, S^{k+l,n})$ converges uniformly, as $p, n \to \infty$ to the two-dimensional vector of zero functions $\boldsymbol{P}$-a.s. Hence, we obtain

$$\boldsymbol{E}[\phi(\mathcal{Q}_U^p)\phi_1(S^n);\ Z_{k+l} > 0, \hat{L}_{k,n} \ge 0 \mid \mathcal{F}_{k+l}] \\ = \psi^*(\mathcal{Q}_U^{k+l,p}, \mathcal{S}^{k+l,n}, S_{k+l} - S_k)\,\mathbf{1}\{Z_{k+l} > 0, \hat{L}_{k,k+l} \ge 0\} \\ = V(S_{k+l} - S_k)\boldsymbol{P}(L_n \ge 0)(\mathbb{E}^+[\phi(\mathcal{B}^U)]\mathbb{E}^{(m)}[\phi_1(\mathcal{B}^1)] + o(1)) \\ \times \mathbf{1}\{Z_{k+l} > 0, \hat{L}_{k,k+l} \ge 0\}, \qquad \boldsymbol{P}\text{-a.s.}$$

Repeating now almost literally (with evident changes) the proof of Theorem 2 one can check the validity of Corollary 2. □

Proof of Theorem 1. For each $U > 0$, we have

$$\mathcal{L}\bigg(\bigg\{\frac{\log Z_{q+up}}{c_p}, 0 \le u \le U\bigg\} \bigg| Z_n > 0, Z_0 = 1\bigg) \\ = \mathcal{L}\bigg(\bigg\{\frac{\log X_u^{q,p}}{c_p} + \frac{S_{pu}}{c_p},\ 0 \le u \le U\bigg\} \bigg| Z_n > 0, Z_0 = 1\bigg).$$

This equality, Theorem 2, and Lemma 8, combined with Theorem 16.7 of [11] justify the desired statement. □

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Proof of Corollary 1. The needed statement follows from the representation

$$\mathcal{L}\left(\left\{\frac{\log Z_{q+up}}{c_p},\ 0\le u\le U;\ \frac{\log Z_{pU+[(n-pU)t]}}{c_n},\ 0\le t\le 1\right\}\ \Big|\ Z_n>0,\ Z_0=1\right)$$
$$=\mathcal{L}\left(\left\{\frac{S_{pu}+\log X_u^{q,p}}{c_p},\ 0\le u\le U;\right.\right.$$
$$\left.\left.\frac{S_{pU+[(n-pU)t]}+\log Y_t^{p,n}}{c_n},\ 0\le t\le 1\right\}\ \Big|\ Z_n>0,\ Z_0=1\right),$$

Lemma 8, and Corollary 2. □

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