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Positive solutions of semilinear elliptic problems with a Hardy potential

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Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\delta(x)$ be the distance of a point $x \in \Omega$ to the boundary. We study the positive solutions of the problem $\Delta u + (\mu/(\delta(x)^2))u = u^p$ in Ω, where $p > 0$, $p \neq 1$ and $\mu \in \mathbb{R}$, $\mu \neq 0$ is smaller than the Hardy constant. The interplay between the singular potential and the nonlinearity leads to interesting structures of the solution sets. In this paper, we first give the complete picture of the radial solutions in balls. In particular, we establish for $p > 1$ the existence of a unique large solution behaving like $\delta^{-(2/(p-1))}$ at the boundary. In general domains, we extend the results of Bandle and Pozio and show that there exists a unique singular solutions u such that $u/\delta^{\beta_-} \to c$ on the boundary for an arbitrary positive function $c \in C^{2+\gamma}(\partial\Omega)\,(\gamma \in (0,1)), c \geqslant 0$. Here β_- is the smaller root of $\beta(\beta-1)+\mu = 0$.

Keywords: Elliptic problems; Hardy potential; power nonlinearities; dead core and blowup solutions; singular boundary data

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1. Introduction

In this paper, we study *positive* solutions of problems of the form

$$L_\mu u = \Delta u + \frac{\mu}{\delta(x)^2} u = u^p \quad \text{in } \Omega, \tag{1.1}$$

where $\mu \in \mathbb{R} \setminus \{0\}$, $\delta(x)$ is the distance of a point $x \in \Omega$ to the boundary, $p \neq 1$ is a positive constant and $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$ is a bounded, connected domain with a boundary $\partial\Omega \in C^{2+\gamma}$, $\gamma \in (0,1)$. The expression

$$\frac{\mu}{\delta(x)^2} =: V_\mu(x)$$

is called the *Hardy potential*. In view of its singularities the boundary data cannot be prescribed arbitrarily. The boundary behaviour depends on the interplay between the linear regime $L_\mu h = 0$ and the nonlinear regime $\Delta U = U^p$. An important ingredient in the study of problem (1.1) is the *Hardy constant*

$$C_H(\Omega) = \inf_{\phi \in W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla \phi|^2 \,\mathrm{d}x}{\int_\Omega \delta^{-2}(x)\phi^2 \,\mathrm{d}x}. \tag{1.2}$$

It is well-known [9] that $0 < C_H(\Omega) \leqslant 1/4$. This implies in particular, that (1.1) cannot have nontrivial solutions belonging to $W_0^{1,2}(\Omega)$ if $\mu < C_H(\Omega)$.

A function h will be called L_μ*-harmonic* or simply *harmonic* if it satisfies $L_\mu h = 0$, *sub-harmonic or super-harmonic* if $L_\mu h \geqslant 0$, or $L_\mu h \leqslant 0$, respectively. In this paper, we shall only be concerned with positive sub- and super-harmonics.

It was shown in [4] that for $\mu \leqslant 1/4$, a local sub-harmonic either dominates every local super-harmonic multiplied by a suitable positive constant, or it is dominated by a multiple of any super-harmonic. This property will be referred to as the *Phragmen-Lindelöf alternative*. It was used in [4] to determine the behaviour of the solutions of (1.1) near the boundary. The singularity of the Hardy potential forces a solution either to vanish or to explode on the boundary.

In the case $p > 1$ the particular feature of the nonlinear problem is the existence of a *maximal* solution which blows up at the boundary, and in the case $p < 1$ the appearance of *dead cores, that is,* regions where the solution vanishes identically.

The structure of the radial solutions in balls is now well-understood. It has been studied in [2] in the case of sublinear nonlinearities. In order to describe the result, we have to introduce some constants which will be crucial in the sequel. For $\mu < 1/4$, we set

$$\beta_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \mu}. \tag{1.3}$$

Furthermore, let

$$\mu^* = \frac{2(p+1)}{(p-1)^2}. \tag{1.4}$$

It turns out that there are essentially two types of positive solutions, those governed by the linear regime and those with a dead core or blowup caused by the nonlinearity. More precisely, we have

THEOREM A. *Assume* $0 < p < 1$, $\mu < 1/4, \mu \neq 0$ *and* $\Omega = B_R := \{x \in \mathbb{R}^N : |x| < R\}$.

(i) *Problem* (1.1) *has a unique radial solution* $u(r)$ *for any* $u(0) > 0$.

(ii) *For any* $\rho \in (0, R)$ *there exists a unique radial solution of problem* (1.1) *such that*

$$u(r) = \begin{cases} 0, & \text{if } r \in (0, \rho], \\ >0, & \text{if } r \in (\rho, R). \end{cases}$$

(iii) *There exists a unique radial solution which vanishes at $r = 0$ and which is positive in $(0, R)$. Near the origin it is of the form*

$$u(r) = r^{2/(1-p)}(c'' + w(r)), \quad w(0) = 0 \text{ and } c'' = \left(\mu^* + \frac{2(N-1)}{1-p}\right)^{1/(p-1)}.$$

(iv) *All radial solutions satisfy $u(r)/((R-r)^{\beta_-}) \to c > 0$ as $r \to R$ and vice versa, for any constant $c > 0$ there exists a unique solution satisfying this condition and it is radial.*

For the superlinear case the situation is slightly different, namely

Theorem B. *Assume $p > 1$, $-\mu^* < \mu < 1/4$, $\mu \neq 0$ and $\Omega = B_R$.*

(i) *There exists a positive number u^* such that problem (1.1) has a unique radial solution for $u(0) \leqslant u^*$ whereas for $u(0) > u^*$ the radial solutions blow up before $r = R$.*

(ii) *If $u(0) < u^*$, $\lim_{r\to R}(u(r)/((R-r)^{\beta_-})) = c > 0$ and vice versa, for any $c > 0$ there exists a unique radial solution satisfying this condition.*

(iii) *If $u(0) = u^*$, then $\lim_{r\to R}(u(r)/((R-r)^{-(2/(p-1))})) = (\mu + \mu^*)^{1/(p-1)}$ (maximal singular solution).*

In short if $u(0) < u^*$ the linear regime prevails, otherwise the nonlinearity dominates.

In the previous papers [**2**, **3**] the authors considered also *general domains* and constructed solutions which behaved like $c_0\delta^{\beta_-} \leqslant u(x) \leqslant c_1\delta^{\beta_-}$ near the boundary. In this paper, we prove *existence and uniqueness* of solutions with a *prescribed boundary behaviour*. The existence results in theorem A (*iv*) and theorem B (*ii*) are special cases of the following more general result

Theorem C. *Let $p > 0$, $p \neq 1$, $\mu < C_H(\Omega)$ and $\mu \neq 0$. If $p > 1$ we assume in addition that $\mu > -\mu^*$. Let $\partial\Omega \in C^{2+\gamma}$ for some $\gamma \in (0,1)$. For any $c \in C^{2+\gamma}(\partial\Omega)$, $c \geqslant 0$, there exists a unique solution of (1.1) such that*

$$\lim_{\delta(x)\to 0}\left(\frac{u(x)}{\delta(x)^{\beta_-}} - c(x^*)\right) = 0, \tag{1.5}$$

where $x^* \in \partial\Omega$ is the projection of x on the boundary.

It was shown in [**3**] that for $p > 1$ and $\mu < -\mu^*$ problem (1.1) has no nontrivial solutions. Related results are found in [**2**, **5**, **6**, **8**].

This paper ends our long collaboration. Maria Assunta Pozio passed away a few days before the paper was accepted. The first author mourns for a dear friend and for a valuable collaborator.

2. Preliminaries

We recall some lemmas which will be used in the proofs of our theorems.

LEMMA 2.1 (Maximum principle). *Let $\mu < C_H(\Omega)$ and $\omega \subseteq \Omega$. If $\Delta u + V_\mu u \geqslant 0$ in ω and $u \in W_0^{1,2}(\omega)$ then $u \leqslant 0$ in ω.*

The proof simply follows from the definition of the Hardy constant in (1.2).

DEFINITION 2.2. A function $\underline{u} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is called a subsolution of (1.1) if

$$\int_\Omega \nabla \underline{u} \cdot \nabla \varphi \, \mathrm{d}x - \mu \int_\Omega \frac{\underline{u}\varphi}{\delta^2} \, \mathrm{d}x + \int_\Omega \underline{u}^p \varphi \, \mathrm{d}x \leqslant 0,$$

for all positive $\varphi \in W^{1,2}(\Omega)$ with compact support in Ω. A supersolution $\overline{u}$ is defined similarly with the inequality sign $\leqslant$ replaced by $\geqslant$.

From lemma 2.1 we deduce immediately

LEMMA 2.3 (Comparison principle). *Let $G \subset \Omega$ be an open set such that $\overline{G} \subset \Omega$, and let $0 \leqslant \underline{u}, \overline{u} \in W^{1,2}_{loc}(G) \cap C(G)$ be sub- and supersolutions of (1.1). Assume that $\underline{u} \leqslant \overline{u}$ on ∂G.*

(i) *If $\mu < C_H(\Omega)$, then $\underline{u} \leqslant \overline{u}$ in G.*

(ii) *If $p > 1$ and $\overline{u} > 0$ in G, the same statement holds without restriction on μ.*

The fact that (ii) is valid for any $\mu \in \mathbb{R}$, was observed in [**3**]. Also the next result is taken from [**3**, theorem 2.6].

LEMMA 2.4 (Phragmen–Lindelöf alternative). *Let $\mu \leqslant 1/4$ and let $\underline{h}$ be a local sub–harmonic. Then either of the following alternatives holds:*

(i) *for every local super–harmonic $\overline{h} > 0$,*

$$\limsup_{x \to \partial\Omega} \frac{\underline{h}}{\overline{h}} > 0,$$

or

(ii) *for every local super–harmonic $\overline{h} > 0$,*

$$\limsup_{x \to \partial\Omega} \frac{\underline{h}}{\overline{h}} < +\infty.$$

M. Marcus and P-T. Nguyen [**8**] have shown that for $0 < \mu < C_H(\Omega)$ every harmonic function can be represented by the Martin kernel $K^\Omega_\mu(x, y)$, $(x, y) \in \Omega \times \partial\Omega$. For fixed $y \in \partial\Omega$, $K^\Omega_\mu(x, y)$, is an L_μ-harmonic function vanishing on $\partial\Omega \setminus y$ and equal to one at an arbitrary, but fixed point $x_0 \in \Omega$. Based on estimates by Filipas, Moschini and Tertikas, Marcus and Nguyen showed that there exists a constant $c_K > 1$ such that $\forall x \in \Omega, y \in \partial\Omega$,

$$c_K^{-1} \delta^{\beta_+}(x) |x - y|^{2\beta_- - N} \leqslant K^\Omega_\mu(x, y) \leqslant c_K \delta^{\beta_+}(x) |x - y|^{2\beta_- - N}. \tag{2.1}$$

A further important tool for establishing the existence of solutions is (cf. [**3**])

LEMMA 2.5. *Let $\mu < C_H(\Omega)$ and $p \neq 1$. If there exists a sub and a supersolution $0 \leqslant \underline{u} \leqslant \overline{u}$ in Ω, then problem (1.1) admits a solution U in Ω such that $\underline{u} \leqslant U \leqslant \overline{u}$. If $p > 1$, the condition $\mu < C_H(\Omega)$ can be replaced by $\mu \leqslant 1/4$.*

The parallel set $\Omega_\rho := \{x \in \Omega : \delta(x) < \rho\}$ will be used to determine the behaviour of the solutions near the boundary. If Ω is of class C^k, $k \geqslant 2$, then δ is in $C^k(\Omega_{\rho_0})$ for $\rho_0 > 0$ sufficiently small [7]. Denote as before by x^* the nearest point to x on $\partial\Omega$. Let $K_i(x^*)$, $i = 1, \ldots, N-1$ be the principal curvatures of $\partial\Omega$ at x^*. For any $x \in \Omega_{\rho_0}$ we have

$$|\nabla\delta(x)| = 1,$$
$$-\frac{N-1}{\rho_0 - \delta(x)} \leqslant \Delta\delta(x) = -\sum_{i=1}^{N-1} \frac{K_i}{1 - K_i\delta(x)} \leqslant \frac{N-1}{\rho_0 + \delta(x)}. \tag{2.2}$$

The maximal distance ρ_0 for which $\partial\Omega_{\rho_0}$ is in C^2 can be estimated by means of the principal curvatures

$$0 < \rho_0 \leqslant \frac{1}{K_{\max}}, \quad K_{\max} := \max\{|K_i(x)|, i = 1, \ldots, N-1, x \in \partial\Omega\}. \tag{2.3}$$

3. Ball

Theorem A (*i*), (*ii*), (*iii*) have been proved in [2]. The existence result (*iv*) is a consequence of theorem C. The proof of theorem B is based on ode techniques partly developed in [2].

3.1. Proof of theorem B

The radial solutions of (1.1) in B_R satisfy the ordinary differential equation ($' := \mathrm{d}/\mathrm{d}r$)

$$u'' + \frac{(N-1)}{r}u' + \frac{\mu}{(R-r)^2}u = u^p \quad \text{in } (0, R), \quad u'(0) = 0. \tag{3.1}$$

This equation has given $u(0) > 0$ a unique local solution. It can be continued until one of the following cases occurs:

1. The solution vanishes before $r = R$,
2. It blows up before $r = R$,
3. It exists and it is positive in the whole interval $[0, R)$.

The first case is excluded by the fact that $\mu < 1/4 = C_H(B_R)$. It is well-known [1] that for any $0 < \rho < R$ there exists a unique large solution U_ρ of (3.1) in B_ρ which blows up at the boundary at the rate $(\mu^*)^{1/(p-1)}(\rho - r)^{-(2/(p-1))}$. Clearly, by lemma 2.3, $U_\rho(0)$ decreases as ρ increases. Define $u^* := \lim_{\rho\to R} U_\rho(0)$. Obviously, $u(r)$ blows up at $\rho < R$ if $u(0) > u^*$. This establishes the nonexistence part in theorem B (*i*).

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Next, we consider the solution $U^*(r)$ of (3.1) satisfying $U^*(0) = u^*$. We fix $r_0 \in (0, R)$ and by means of suitable super and subsolutions we shall show that there exist positive constants $0 < c_1 < c_2$ depending only on r_0, such that $c_1(R - r)^{-(2/(p-1))} \leqslant U^*(r) \leqslant c_2(R - r)^{-(2/(p-1))}$ for $r \geqslant r_0$.

We start with the supersolution. Let $w_+ = c_+(R - r)^{-(2/(p-1))}$. It satisfies

$$L_\mu w_+ - w_+^p = \left[\mu^* + \mu - c_+^{p-1} + \frac{2(N-1)}{(p-1)r}(R-r)\right] c_+(R-r)^{-(2p/(p-1))}. \tag{3.2}$$

If

$$c_+^{p-1} > \mu^* + \mu + \frac{2(N-1)}{(p-1)r_0}(R - r_0),$$

then w_+ is a supersolution of (3.1) in (r_0, R).

Let $U_{r_1}(r)$ be the solution which blows up at $r_1 \in (r_0, R)$. Choose c_+ so large that $w_+(r_0) > U_{r_1}(r_0)$. Let $\tilde{r}$ be the largest number such that $w_+(\tilde{r}) = U_{r_1}(\tilde{r})$. Then the function

$$\overline{u}(r) = \begin{cases} U_{r_1}(r) & \text{in } [0, \tilde{r}], \\ w_+(r) & \text{in } [\tilde{r}, R) \end{cases}$$

is a weak supersolution with $\overline{u}(0) > u^*$.

Next, we construct a subsolution. From (3.2) it follows that for small $c_- \leqslant (\mu + \mu^*)^{1/(p-1)}$, $w_- = c_-(R - r)^{-(2/(p-1))}$ is a subsolution in $(0, R)$. Here we need the assumption $-\mu^* < \mu$. Since $w_-'(0) = c_- \frac{2}{p-1} R^{-(2/(p-1))-1} > 0$, w_- is a radial subsolution in B_R.

Applying lemma 2.5 we deduce that there exists a solution $\underline{u} \leqslant U^* \leqslant \overline{u}$ in $[0, R)$. The assertion (iii) of theorem B will be established if we prove the following

Claim: The function

$$w^*(r) := U^*(r)/(R - r)^{-(2/(p-1))}$$

that is bounded from below and above by positive constants satisfies

$$w^*(R) = \lim_{r \to R^-} w^*(r) = (\mu + \mu^*)^{1/(p-1)}$$

In $[0, R)$, w^* satisfies the equation

$$(w_r^* \sigma)_r = \frac{\sigma}{(R-r)^2}\left[w^{*p} - \left(\mu^* + \mu + \frac{2(N-1)}{(p-1)r}(R-r)\right) w^*\right], \tag{3.3}$$

where $\sigma = (R - r)^{-(4/(p-1))} r^{N-1}$.

Put

$$f(w, r) := w^p - [\mu^* + \mu + \eta(r)]w, \; w > 0, \; r \in (0, R],$$

$$\eta(r) = \frac{2(N-1)}{(p-1)r}(R - r).$$

For any $r \in [0, R]$, the equation $f(w, r) = 0$ has two zeros, namely $w = 0$ and $w_0(r) = (\mu^* + \mu + \eta(r))^{1/(p-1)}$. The function $w_0(r)$ is monotone decreasing with

$\lim_{r\to 0} w_0(r) = +\infty$ and $\lim_{r\to R} w_0(r) = (\mu^* + \mu)^{1/(p-1)}$. It is important to point out that all the local maxima of the solutions of (3.3) are below $w_0(r)$ and the local minima are above $w_0(r)$.

The radial symmetry implies that $U_r^*(0) = 0$ and consequently,

$$w_r^*(0) = -\frac{2}{p-1} U^*(0) R^{(3-p)/(p-1)} < 0.$$

Thus w^* decreases in a neighbourhood of $r = 0$. If it has a local minimum in $r = r_0$, by the previous remark $w^*(r_0) \geqslant w_0(r_0)$. Then $w^*(r)$ cannot have a first local maximum $w^*(r_1)$ for some $r_1 \in (r_0, R)$, since we get $w_0(r_1) \geqslant w^*(r_1) > w^*(r_0) \geqslant w_0(r_0)$, which is impossible because $w_0(r)$ is decreasing. Therefore $w^*(r)$ is either decreasing in $(0, R)$ or it has one local minimum.

Thus $w^*(r)$ is monotone in a neighbourhood of $r = R$. Its boundedness implies that $\lim_{r\to R} w^*(r) = w^*(R)$ and in addition, $L := \lim_{r\to R} f(w^*(r), r)$ with $|L| < \infty$.

Integration of (3.3) leads to

$$w_r^*(r)\sigma(r) - w_r^*(r_0)\sigma(r_0) = \int_{r_0}^{r} \frac{\sigma}{(R-s)^2} f(w^*, s)\,\mathrm{d}s. \tag{3.4}$$

Next, it will be shown that $L = 0$. We divide (3.4) by σ and integrate. We then obtain

$$\begin{aligned} &w^*(r) - w^*(r_0) - w_r^*(r_0)\sigma(r_0) \int_{r_0}^{r} \sigma^{-1}(s)\,\mathrm{d}s \\ &= \int_{r_0}^{r} \frac{\sigma(s)}{(R-s)^2} f(w^*, s) \left(\int_{s}^{r} \sigma^{-1}(y)\,\mathrm{d}y \right) \mathrm{d}s. \end{aligned}$$

The left-hand side has a limit as $r \to R$. Near $r = R$ the integrand at the right-hand side behaves like $(R-r)^{-1}$. The integral exists therefore for $r \to R$ only if $f(w^*(r), r) \to 0$ as $r \to R$. Recall that $f(w, r)$ has only two zeros $w = 0$ and $w = w_0$. The case $w^*(R) = 0$ is excluded because $w^*(r) > c_- > 0$. Consequently, $w^*(R) = w_0(R)$ which establishes theorem B (*iii*).

Moreover, theorem B (*iii*) implies that w^* decreases in $[0, R]$. Namely if w^* has a local minimum at $r_0 \in (0, R)$, then $w(R) = (\mu + \mu^*)^{1/(p-1)} > w(r_0) > (\mu + \mu^*)^{1/(p-1)}$, which is impossible as observed above.

Next, we proceed to the proof theorem B (*ii*). Consider the function $w(r) = u(r)(R-r)^{2/(p-1)}$ where $u(r)$ solves (3.1) and $u(0) < u^*$.

We claim that $w < w^*$ in $(0, R)$. In fact w cannot intersect w^*. Namely, if $w(r_1) = w^*(r_1)$ and $w < w^*$ in $(0, r_1)$, then $U^* > u$ in B_{r_1}. This implies $L_\mu(U^* - u) > 0$. From lemma 2.1 it follows that $U^* < u$, which is obviously a contradiction. This establishes the assertion. The same arguments as for w^* show that $w < w^*$ decreases in the whole interval $(0, R)$. Exactly in the same way as for w^* we prove that $\lim_{r\to R} f(w(r), r) = 0$. Thus either $w(R) = (\mu + \mu^*)^{1/(p-1)}$ or $w(R) = 0$.

Suppose that $w(R) = w^*(R)$. Define $m(x)$ by $u(r) = m(r)U^*(r)$. By the previous observations, we have $m \leqslant 1$ in B_R and $m = 1$ on ∂B_R. Since u solves (1.1) in B_R, the differential equation for $m(r)$ is

$$U^*\Delta m + 2\nabla m \cdot \nabla U^* + m\Delta U^* + \frac{\mu}{\delta^2} mU^* = m^p (U^*)^p. \tag{3.5}$$

Hence

$$U^*\Delta m + 2\nabla m \cdot \nabla U^* = m^p (U^*)^p - m(U^*)^p \leqslant 0. \tag{3.6}$$

The maximum principle implies that m attains its minimum on the boundary. Consequently, $m(r) \equiv 1$ and $u = U^*$. This contradicts the fact that $u(r) < U^*(r)$. Therefore we must have $w(R) = 0$. This argument shows that there is only one radial solution which behaves like $\delta(x)^{-(2/(p-1))}$ near the boundary. Notice that this uniqueness result holds also for nonradial solutions in general domains.

We now need to prove the boundary behaviour in (ii).
The function $u(r) = w(r)(R-r)^{-(2/(p-1))}$ is a solution of (3.1) which can be written as follows

$$u'' + \frac{N-1}{r}u' + \frac{\mu - w^{p-1}}{(R-r)^2}u = 0, \quad u'(0) = 0. \tag{3.7}$$

A radial harmonic in B_R solves

$$h''(r) + \frac{N-1}{r}h'(r) + \frac{\mu}{(R-r)^2}h(r) = 0 \quad \text{in } [0, R),\ h'(0) = 0.$$

The difference $z = u - h$ is positive in $[0, \tilde{r})$ and satisfies $z'' + ((N-1)/r)z' + \mu(z/((R-r)^2)) \geqslant 0$ in $(0, R)$. If $\tilde{r} < R$ we can test this inequality with z and obtain that $\mu > C_H(B_R) = 1/4$. This contradicts the assumption $\mu < 1/4$. We can now apply the Fuchsian theory to h. Since $\mu < 1/4$, $\lim_{r\to R} h(r)/(R-r)^{\beta_-} = c_0 > 0$. Consequently, $u(r) \geqslant c_1(R-r)^{\beta_-}$ for some sufficiently small constant c_1.

In order to construct an upper bound we take the solution u_ϵ of (3.7) with w replaced by $w_\epsilon = \max\{w, \epsilon\}$. It is a subsolution to (3.7) and if $u_\epsilon(0) > u(0)$ it is an upper bound for u. Since w is decreasing and $w(R) = 0$, we have $w_\epsilon = \epsilon$ near $r = R$. We can now apply the Fuchsian theory to u_ϵ. The indicial equation for u_ϵ implies that $u_\epsilon(r) = (R-r)^{\beta_\epsilon} f(R-r)$ where β_ϵ is the smaller root of $\beta_\epsilon(\beta_\epsilon - 1) + \mu - \epsilon^{p-1} = 0$ and f is an analytic function near $r = R$ such that $f(0) = b > 0$. Hence for some $\delta_0 > 0$ we have

$$0 < c_0(R-r)^{\beta_-} \leqslant u(r) \leqslant u_\epsilon(r) \leqslant k(R-r)^{\beta_\epsilon}, \quad \forall r \in (R-\delta_0, R). \tag{3.8}$$

We replace r by $\delta = R - r$ and consider the function $v(\delta) = u(R-\delta)\delta^{-\beta_-}$. A straightforward computation leads to

$$(\sigma v')' = \sigma\left(v^p \delta^{\beta(p-1)} + \beta \frac{N-1}{(R-\delta)\delta} v\right), \tag{3.9}$$

$$\text{where } \sigma(\delta) = \delta^{2\beta}(R-\delta)^{N-1} \text{ and } \beta = \beta_-.$$

We take $0 < \delta < \delta_0$ and we integrate (3.9) to get:

$$v(\delta) - v(\delta_0) + \sigma(\delta_0)v'(\delta_0)\int_\delta^{\delta_0} \sigma^{-1}\,\mathrm{d}s \qquad (3.10)$$
$$= \int_\delta^{\delta_0} \sigma\left(v^p s^{\beta(p-1)} + \beta\frac{N-1}{(R-s)s}v\right)\left(\int_\delta^s \sigma^{-1}\,\mathrm{d}\xi\right)\mathrm{d}s.$$

Keeping in mind that $\beta < 1/2$, we have for $s \in (\delta, \delta_0]$

$$\int_\delta^s \sigma^{-1}\,\mathrm{d}\xi \leqslant \frac{s^{1-2\beta}}{(1-2\beta)(R-\delta_0)^{N-1}} < \infty.$$

From (3.8) and the definition of v it follows that $v \leqslant k\delta^{\beta_\epsilon - \beta}$. Hence there exist some positive constants k_1, k_2, k_3 independent of δ such that

$$\int_\delta^{\delta_0} \sigma\left(v^p s^{\beta(p-1)} + \beta\frac{N-1}{(R-s)s}v\right)\left(\int_\delta^s \sigma^{-1}\,\mathrm{d}\xi\right)\mathrm{d}s$$
$$\leqslant k_1\int_\delta^{\delta_0} s^{2\beta+\beta_\epsilon-\beta-1+1-2\beta}\,\mathrm{d}s + k_2\int_\delta^{\delta_0} s^{2\beta+p(\beta_\epsilon-\beta)+\beta(p-1)+1-2\beta}\,\mathrm{d}s \leqslant k_3,$$

for sufficiently small ϵ. By our assumption $\mu > -\mu^*$ we have $\beta > -(2/(p-1))$. It is therefore possible to choose ϵ so small that $p(\beta_\epsilon - \beta) + \beta(p-1) + 1 > -1$, and $\beta_\epsilon - \beta > -1$. Hence the integrals above converge as $\delta \to 0$ and v is uniformly bounded.

Next, we want to show that $v(\delta)$ has a limit as δ tends to 0.
If $\beta = \beta_- > 0$, (3.9) implies that v cannot have a local maximum and since $v'(R) = u'(0)R^{-\beta} - \beta u(0)\delta^{-\beta-1} = -\beta u(0)\delta^{-\beta-1} < 0$, it decreases in the whole interval. Then $u(r)/(R-r)^{\beta_-}$ is an increasing bounded function for $r = R - \delta \in (0, R)$, hence it has a limit as $r \to R^-$.
Assume $\beta = \beta_- < 0$. Recall that v is uniformly bounded. Since by assumption $\beta_- > -(2/(p-1))$ all integrals in (3.10) exist for $\delta = 0$. Consequently, we can pass to the limit which shows that $\lim_{\delta\to 0} v(\delta)$ exists.

For the last assertion of (*ii*) we refer to the proof of theorem C. This completes the proof of theorem B. □

4. General domains

4.1. Proof of theorem C

The existence of a solution u of (1.1) satisfying (1.5) will be proved by means of a supersolution $\overline{u}$ and a subsolution $\underline{u}$ of (1.1), which both satisfy (1.5) and which are such that $\underline{u} \leqslant \overline{u}$.

We start with an important observation. For $\delta \leqslant \delta_0 \leqslant \rho_0/2$, ρ_0 defined in § 2, set

$$u(x) = \delta^\beta w(x), \quad (\beta = \beta_-). \qquad (4.1)$$

Then u is a local solution of (1.1) in Ω_{δ_0}, if and only if w satisfies

$$\mathcal{A}(w) := \Delta w + \tfrac{2\beta}{\delta} < \nabla\delta, \quad \nabla w > + \tfrac{\beta\Delta\delta}{\delta}w = \delta^{\beta(p-1)}w^p. \qquad (4.2)$$

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The function $\underline{u} = \delta^\beta \underline{w}(x)$ is a local subsolution of (1.1) in Ω_{δ_0} if (4.2) holds with the equality sign replaced by $\geqslant$. Analogously $\overline{u} = \delta^\beta \overline{w}$ is a super solution if the inequality sign is reversed.
We shall construct a supersolution as the minimum between a local supersolution satisfying (1.5) and a global one which satisfies (1.5) with $\geqslant$ instead of equality.

Local supersolution. Let $\alpha \in (0,1)$ be such that

$$\alpha \in \begin{cases} (0,1) & \text{if } \beta_- < 0, \\ (0, 1-2\beta) & \text{if } \beta_- > 0. \end{cases} \tag{4.3}$$

For any given $c \in C^{2+\gamma}(\partial\Omega)$, $c \geqslant 0$, let $h \in C^{2+\gamma}(\overline{\Omega})$ be the solution of

$$\begin{cases} \Delta h = 0, & x \in \Omega, \\ h(x) = c(x), & x \in \partial\Omega. \end{cases} \tag{4.4}$$

Set $\overline{w} = h(x) + A\delta^\alpha$ where $A \geqslant 1$ will be fixed below. Then for $x \in \Omega_{\delta_0}$

$$\mathcal{A}(\overline{w}) = -A\alpha(1-\alpha)\delta^{\alpha-2} + A\alpha\delta^{\alpha-1}\Delta\delta + \frac{2\beta}{\delta} < \nabla\delta, \nabla h > \\ + 2A\beta\alpha\delta^{\alpha-2} + \frac{\beta\Delta\delta}{\delta}\overline{w}.$$

Since $|\Delta\delta| \leqslant c$ and $|\nabla\delta| \equiv 1$ in Ω_{δ_0} we have

$$\mathcal{A}(\overline{w}) \leqslant A\alpha(\alpha + 2\beta - 1)\delta^{\alpha-2} + A(\alpha + |\beta|)c\delta^{\alpha-1} + \frac{2|\beta|}{\delta}|\nabla h|_\infty + \frac{|\beta|c}{\delta}|h|_\infty.$$

By our assumptions on the regularity of $\partial\Omega$ and c, $|h|$ and $|\nabla h|$ are uniformly bounded in Ω, cf. [**10**, page 161]. Hence $2|\beta||\nabla h|_\infty + |\beta|c|h|_\infty < C$ and consequently,

$$\mathcal{A}(\overline{w}) \leqslant \delta^{\alpha-2}A\left[\alpha(\alpha + 2\beta - 1) + \delta(\alpha + |\beta|)c + \frac{C}{A}\delta^{1-\alpha}\right].$$

Since α satisfies (4.3), it is possible to choose $\delta_0 < \rho_0$ sufficiently small such that

$$\mathcal{A}(\overline{w}) \leqslant 0 \leqslant \delta^{\beta(p-1)}\overline{w}^p, \quad x \in \Omega_{\delta_0}, \quad \forall\, A > 1. \tag{4.5}$$

Thus $\hat{u} := \delta^\beta \bar{w}$ is a local super-harmonic function and therefore the desired local supersolution.

Next, we construct a supersolution in the whole domain. We shall treat the case $\mu > 0$ and $\mu < 0$ separately.

Global supersolution for $\mu < 0$. Let $\eta = \eta(r)$ be the solution of

$$\begin{cases} \eta'' + \frac{(N-1)}{r}\eta' + \frac{\mu}{(\delta_0 - r)^2}\eta = 0, & r \in (\delta_0/2, \delta_0), \\ \eta(\delta_0/2) = 1, \quad \eta'(\delta_0/2) = 0. \end{cases} \tag{4.6}$$

Since $\mu < 0$ the function $\eta(r)$ is increasing in a neighbourhood of $\delta_0/2$ and it has no local maximum. Thus $\eta(r)$ is a positive increasing solution in $(\delta_0/2, \delta_0)$.

We claim that

$$\lim_{r\to\delta_0} \frac{\eta(r)}{(\delta_0 - r)^{\beta_-}} = C_\eta > 0. \tag{4.7}$$

For the proof of this claim we proceed as in [**2**] where a similar result has been derived for the nonlinear equation. We choose $\delta = \delta_0 - r$ instead of r as the new variable and set $\eta(\delta_0 - \delta) = \delta^\beta v$ where $\beta = \beta_- < 0$. From (4.6) we obtain

$$v'' + \left(2\frac{\beta}{\delta} - \frac{N-1}{\delta_0 - \delta}\right) v' - \beta\frac{N-1}{(\delta_0-\delta)\delta} v = 0, \quad \delta \in (0, \delta_0/2). \tag{4.8}$$

This equation can be written in the form

$$(\sigma v')' = \beta\sigma\frac{N-1}{(\delta_0-\delta)\delta} v, \text{ where } \sigma(\delta) = \delta^{2\beta}(\delta_0 - \delta)^{N-1}. \tag{4.9}$$

Since $\beta < 0$ it follows that $(\sigma v')' \leqslant 0$ which implies that for $\epsilon < \delta$

$$\sigma(\epsilon)v'(\epsilon) \geqslant \sigma(\delta)v'(\delta)\,.$$

Dividing the last inequality by $\sigma(\epsilon)$ and integrating in the same interval we get

$$0 < v(\epsilon) \leqslant v(\delta) - \sigma(\delta)v'(\delta)\int_\epsilon^\delta \sigma^{-1}(s)\,\mathrm{d}s.$$

This implies that v is bounded as $\epsilon \to 0$.
Integrating (4.9) we get

$$v'(\epsilon) = \sigma^{-1}(\epsilon)\sigma(\delta)v'(\delta) - \beta\sigma^{-1}(\epsilon)\int_\epsilon^\delta \sigma(s)\frac{N-1}{(\delta_0 - s)s} v\,\mathrm{d}s.$$

We can pass to the limit $\epsilon \to 0$ and conclude, since β_- is negative, that $|v'|$ is bounded. Hence there exists

$$\lim_{\delta\to 0} v(\delta) = v(0).$$

Suppose that our claim (4.7) is not true and that $v(0) = 0$. Following the proof of lemma 2.4 in [**2**], we define w by $\eta = \delta^{\beta_-}v = \delta^{\beta_+}w$. It satisfies the same equation (4.9) as v with β_- replaced by β_+,

$$\begin{gathered}(\sigma_+ w')' = \beta_+\sigma_+\frac{N-1}{(\delta_0-\delta)\delta} w (\geqslant 0),\\ \text{where } \sigma_+(\delta) = \delta^{2\beta_+}(\delta_0 - \delta)^{N-1}.\end{gathered} \tag{4.10}$$

We integrate (4.10) over $[\epsilon, \delta]$ for some $\delta \in (0, \delta_0/2]$ and we get

$$\begin{aligned}\sigma_+(\delta)w'(\delta) - \sigma_+(\epsilon)w'(\epsilon) &= \beta_+\int_\epsilon^\delta \sigma_+(s)\frac{N-1}{(\delta_0 - s)s} w(s)\,\mathrm{d}s\\ &= \beta_+(N-1)\int_\epsilon^\delta (\delta_0 - s)^{N-2} v(s)\,\mathrm{d}s.\end{aligned}$$

Since v is bounded we can pass to the limit as $\epsilon \to 0$ and deduce that $\lim_{\epsilon\to 0}\sigma_+(\epsilon)w'(\epsilon) = M$. and hence $w(\delta) = O(\delta^{1-2\beta_+})$. By assumption $v(0) = 0$ and

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therefore $\delta^{2\beta_+-1}w = v \to 0$ as $\delta \to 0$, that is, $M = 0$. Thus

$$w'(\delta) = \beta_+\sigma_+^{-1}(\delta)\int_0^\delta \sigma_+(s)\frac{N-1}{(\delta_0-s)s}w(s)\,\mathrm{d}s \geqslant 0.$$

Hence $w(\delta) \geqslant w(\epsilon) \geqslant 0$, that is, $w(\epsilon)$ is bounded in a neighbourhood of $\delta = 0$. This implies $\eta(\delta_0-\delta) \leqslant c\delta^{\beta_+} \to 0$ as $\delta \to 0$. This is impossible since η increases as $\delta \to 0^+$. Consequently, $v(0) > 0$ and (4.7) follows.

Choose

$$M > \frac{|c(\cdot)|_\infty}{C_\eta}, \tag{4.11}$$

where C_η is defined in (4.7). Then the function

$$\tilde{u}(x) := \begin{cases} M\eta(\delta_0-\delta(x)), & x \in \Omega_{\delta_0/2}, \\ M, & x \in \Omega\setminus\Omega_{\delta_0/2}. \end{cases} \tag{4.12}$$

is in $C^1(\Omega)$ and is a (weak) supersolution of (1.1) satisfying

$$\liminf_{\delta(x)\to 0}\frac{\tilde{u}}{\delta(x)^{\beta_-}} > |c(\cdot)|_\infty. \tag{4.13}$$

Indeed since $\mu < 0$, any nonnegative constant is a supersolution. Moreover, since $\eta'(r) \geqslant 0$ and $\delta_0 \leqslant \rho_0/2$, by (2.2) we have $-\eta'\Delta\delta \leqslant ((N-1)/(\delta_0-\delta))\eta'$. Hence

$$\Delta\tilde{u} + \frac{\mu}{\delta^2}\tilde{u} = M\left(\eta'' - \eta'\Delta\delta + \frac{\mu}{\delta^2}\eta\right) \leqslant M\left(\eta' + \frac{(N-1)}{\delta_0-\delta}\eta' + \frac{\mu}{\delta^2}\eta\right) = 0 \leqslant \tilde{u}^p.$$

Global supersolution for $\mu \in (0, C_H(\Omega))$. We consider the function $z(x) = \int_{\partial\Omega} K_\mu^\Omega(x,y)\,\mathrm{d}S_y$, where K_μ^Ω is the Martin Kernel introduced in § 2. By [8] we have that z is harmonic in Ω. Next, we will prove that estimate (2.1) implies

$$\liminf_{d(x,\partial\Omega)\to 0}\frac{z}{\delta(x)^{\beta_-}} > c_0 > 0. \tag{4.14}$$

For any $x \in \Omega_{\delta_0}$, and $y \in \partial\Omega \cap B_{\delta(x)/2}(x^*(x))$, there holds

$$|x-y| \leqslant |x - x^*(x)| + |x^*(x) - y| \leqslant \delta(x) + \delta(x)/2 = 3\delta(x)/2,$$

where $x^*(x) \in \partial\Omega$ is the nearest point to x. Obviously,

$$z(x) \geqslant \int_{\partial\Omega\cap B_{\delta(x)/2}(x^*(x))} K_\mu^\Omega(x,y)\,\mathrm{d}S_y.$$

The boundary regularity together with the fact that $x \in \Omega_{\delta_0}$, $(\delta_0 < \rho_0/2)$, imply that $|\partial\Omega \cap B_{\delta(x^*(x))/2}| \geqslant \epsilon_0\delta(x)^{N-1}$ for some $\epsilon_0 > 0$. Then by (2.1) there exists a constant $c_0 > 0$ independent of x such that

$$z(x) \geqslant c_K^{-1}\delta(x)^{\beta_+}\big(2\delta(x)/3\big)^{2\beta_- - N}\epsilon_0\delta(x)^{N-1} > c_0\delta(x)^{\beta_-}. \tag{4.15}$$

For a given $M \geqslant (|c|_\infty/c_0)$ define

$$\tilde{u}(x) := Mz(x).$$

Then $\tilde{u}(x)$ satisfies (4.13) and

$$L_\mu\tilde{u}(x) = 0 \leqslant \tilde{u}(x)^p.$$

Hence $\tilde{u}(x)$ is a supersolution of (1.1) satisfying $\tilde{u}/(\delta(x)^{\beta_-}) \geqslant c(x)$ on $\partial\Omega$.

Next we construct supersolutions in the whole domain with boundary values $c(x)$, cf (1.5).

Supersolutions satisfying (1.5). Consider the local supersolution $\hat{u} = \delta^\beta\overline{w}$ constructed at the beginning of § 4.1. Choose

$$A > (\delta_0/2)^{-\beta_- - \alpha}\max\{\tilde{u}(x):\ x \in \Omega,\ \delta(x) = \delta_0/2\},$$

where $\tilde{u}$ is the global supersolution satisfying $\tilde{u}/(\delta(x)^{\beta_-}) \geqslant c(x)$ on $\partial\Omega$. Then for any $x \in \Omega$ such that $\delta(x) = \delta_0/2$ we have $\tilde{u}(x) \leqslant \delta_0^{\beta_- + \alpha}A \leqslant \hat{u}(x)$, while $\tilde{u} > \hat{u}$ in a neighbourhood of $\partial\Omega$. Then

$$\overline{u}(x) := \begin{cases} \min\{\hat{u}(x),\, \tilde{u}(x)\}, & x \in \Omega_{\delta_0/2}, \\ \tilde{u}(x), & x \in \Omega \setminus \Omega_{\delta_0/2}. \end{cases} \tag{4.16}$$

is the desired supersolution which satisfies (1.5).

Subsolution satisfying (1.5). Set $\underline{w} = (h(x) - a\delta^\alpha)_+$ where α is defined in (4.3), h solves (4.4) and $a > 0$ will be fixed below so that the support of $\underline{w}$ is contained in Ω_{δ_0}. This is the case if $a \geqslant a_0 > 0$ for a suitable a_0. If $\mathcal{A}(\underline{w}) \geqslant \delta^{\beta(p-1)}\underline{w}^p$, at points where $\underline{w} > 0$ and since 0 is a solution of (1.1) then,

$$\underline{u} = \begin{cases} \delta^\beta\underline{w}, & x \in \Omega_{\delta_0}, \\ 0, & x \in \Omega \setminus \Omega_{\delta_0}, \end{cases} \tag{4.17}$$

is the desired subsolution. Since $a\delta^\alpha \leqslant |h|_\infty$ where $\underline{w} > 0$, we have

$$\begin{aligned} \mathcal{A}(\underline{w}) &= a\alpha(1-\alpha)\delta^{\alpha-2} - a\alpha\delta^{\alpha-1}\Delta\delta + \frac{2\beta}{\delta} < \nabla\delta, \nabla h > -2a\beta\alpha\delta^{\alpha-2} + \frac{\beta\Delta\delta}{\delta}\underline{w} \\ &\geqslant a\alpha(1-\alpha-2\beta)\delta^{\alpha-2} - \delta^{-1}[(\alpha + |\beta|)|h|_\infty|\Delta\delta|_\infty + 2|\beta||\nabla h|_\infty]. \end{aligned} \tag{4.18}$$

In $\Omega_{\delta_0/2}$, where $\underline{w} > 0$ we want to have

$$\delta^{2-\alpha}\mathcal{A}(\underline{w}) \geqslant \delta^{\beta(p-1)+2-\alpha}\underline{w}^p,$$

If $p < 1$ it suffices to choose a sufficiently large, whereas in the superlinear case $p > 1$ we have to require in addition that $\beta(p-1) + 2 - \alpha > 0$. In view of our assumption $\mu > -\mu^*$ this condition can always be satisfied by choosing α very small. By construction the subsolution is below the supersolution. Hence the existence of a solution satisfying (1.5) follows.

Uniqueness. For any given boundary data $c \in C^{2+\gamma}(\partial\Omega)$, $c \geqslant 0$, let u_1, u_2 solve (1.1) and (1.5). We argue by contradiction. Suppose that $\underline{h} := (u_1 - u_2)_+ \not\equiv 0$ (or $\underline{h} := (u_2 - u_1)_+ \not\equiv 0$). Clearly, $\underline{h}$ is a sub-harmonic function, and it satisfies $\lim_{x\to\partial\Omega}(\underline{h}/\delta^{\beta_-}) = 0$.

In [**3**] simple local super-harmonic functions $\overline{H}$ and $\overline{h}$ have been constructed with the property that $\lim_{d(x,\partial\Omega)\to 0}(\overline{H}/\delta^{\beta_-}) = 1$ and $\lim_{d(x,\partial\Omega)\to 0}(\overline{h}/\delta^{\beta_+}) = 1$. Since $\limsup_{x\to\partial\Omega}(\underline{h}/\overline{H}) = 0$ the Phragmen–Lindelöf alternative (*ii*) (see lemma 2.4) applies and we get $\underline{h} < c\delta^{\beta_+}$. Consequently, $(u_1 - u_2)_+$ belongs to $W_0^{1,2}(\Omega)$. Since $\mu < C_H(\Omega)$ the comparison principle implies that $(u_1 - u_2)_+ \equiv 0$ in contradiction to our assumption.

This completes the proof of theorem C. □

Open problem. It is not known what is the behaviour of the solution in theorem C at the points where $c(x^*) = 0$. According to a local version of the Phragmen-Lindelöf alternative, which is not yet available, it should be $\lim_{\delta(x)\to 0} u(x)/\delta(x)^{\beta_+} = c$.

References

1 C. Bandle and M. Marcus. Asymptotic behaviour of solutions and their derivatives, for semilinear elliptic problems with blowup on the boundary. *Ann. Inst. Henri Poincaré* **12** (1995), 155–171.

2 C. Bandle and M. A. Pozio. Sublinear elliptic problems with a Hardy potential. *Nonlinear Analysis* **119** (2015), 149–166.

3 C. Bandle, V. Moroz and W. Reichel. 'Boundary blowup' type subsolutions to semilinear elliptic equations with Hardy potential. *J. London Math. Soc.* **77** (2008), 503–523.

4 C. Bandle, V. Moroz and W. Reichel. Large solutions to semilinear elliptic equations with Hardy potential and exponential nonlinearity. In *Around the research of Vladimir Maz'ya II. Partial differential equations* (ed. A. Laptev), International Series, vol. 12, pp. 1–22 (New York/Dordrecht/Heidelberg/London: Springer, 2010).

5 C. Bandle, M. Marcus and V. Moroz. Boundary singularities of solutions of semilinear ellipticequations with Hardy potential, boundary singularities of solutions of semilinear elliptic equations in the half-space with a Hardy potential. *Israel J. Math.* **222** (2017), 487–514.

6 Y. Du and L. Wei. Boundary behavior of positive solutions to nonlinear elliptic equations with Hardy potential. *J. London Math. Soc.* **91** (2015), 731–749.

7 R. L. Foote. Regularity of distance function. *Proc. AMS* **92** (1984), 153–155.

8 M. Marcus and P.-T. Nguyen. Moderate solutions of semilinear elliptic equations with Hardy potential. *Annales de l'Institut Henri Poincare (C) Non Linear Anal.* **34** (2014), 69–88.

9 M. Marcus, V. J. Mizel and Y. Pinchover. On the best constant for Hardy's inequality in $\mathbb{R}^n$. *Trans. A.M.S.* **350** (1998), 3237–3255.

10 C. Miranda. *Partial differential equations of elliptic type*. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 2 (Berlin/Heidelberg/New York: Springer, 1970).