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A NOTE ON A-ANNIHILATED GENERATORS OF H_*QX

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Abstract. For a path connected space X, the homology algebra $H_*(QX;\mathbb{Z}/2)$ is a polynomial algebra over certain generators Q^Ix. We reinterpret a technical observation, of Curtis and Wellington, on the action of the Steenrod algebra A on the Λ algebra in our terms. We then introduce a partial order on each grading of H_*QX which allows us to separate terms in a useful way when computing the action of dual Steenrod operations Sq^i_* on $H_*(QX;\mathbb{Z}/2)$. We use these to completely characterise the A-annihilated generators of this polynomial algebra. We then propose a construction for sequences I so that Q^Ix is A-annihilated. As an application, we offer some results on the form of potential spherical classes in H_*QX upon some stability condition under homology suspension. Our computations provide new numerical conditions in the context of hit problem.

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1. Introduction and statement of results.

1.1. Motivation and programme. The aim of this section is to justify the case for studying the action of A^{op} on $\widetilde{H}_*QX$. We work at the prime $p=2$; see Subsection 1.2 for the relevant notation and definitions.

The hit problem. The Steenrod algebra A and its dual A^* are very important objects in algebraic topology, both from structural point of view as an important Hopf algebra and from the point of view of applications. It is the second point of view which we deal with. The algebra A acts on the $\mathbb{Z}/2$-cohomology of any space which allows to codify important information about the stable structure of CW-complexes. This is understood in terms of Adams spectral sequence $\mathrm{Ext}_A(\widetilde{H}^*Y,\mathbb{Z}/2)$ which converges to the 2-component of π^s_*Y [**15**, Theorem 2.1.1]. From this point of view, and in order to construct a minimal projection resolution of $\widetilde{H}^*Y$ over A, having a description of $\widetilde{H}^*Y$ as a module over the Steenrod algebra, or finding a basis for the $\mathbb{Z}/2$-vector space $\mathbb{Z}/2\otimes_A\widetilde{H}^*Y$ becomes very important. This latter problem leads to the hit problem of Peterson which is mainly concerned with the case of $Y=\mathbb{R}P^{\times k}$ where $\mathbb{R}P$ is the infinite dimensional projective space, and $(-)^{\times k}$ takes a space to its k-fold product. The problem in the case of $Y=BO(k)$ is known as the symmetric hit problem (see [**19**, **20**] for a very recent treatment of the subject).

The dual hit problem. Sometimes it is more convenient to study the hit problem in homological settings. The equivalent problem is that given a space Y, we determine the submodule of A-annihilated classes in $\widetilde{H}_*Y$ (see Definition 1.1). This dual problem has already captured interest (see, e.g., [**3**, **7**, **16**]). One of the main motivations and justifications for the present work was formed in Manchester after talks by Grant Walker and

Reg Wood on the hit problem and helpful conversations with Grant Walker; I was hopeful that the action of the Steenrod algebra on the Dyer–Lashof algebra could give interesting information on the hit problem. In a very vague sense, while working with H_*QX when X is an infinite loop space such as $X=\mathbb{R}P^{\times k}$ or $X=BO$, it seemed that the Dyer–Lashof algebra is a kind of 'orthogonal' to $\widetilde{H}_*X$. In this approach, while some numerical conditions as expressed in Theorem 1.2 allow to identify A-annihilated classes Q^Ix in H_*QX, the structure map of X [**14**], say $\theta_X : QX \to X$, allows to send these to A-annihilated classes in H_*X. We have applied this work to obtain new examples of A-annihilated classes in H_*BO [**24**].

Spherical classes and bordism of immersions. For a space Y, determining the image of the Hurewicz homomorphism $h:\pi_*Y\to H_*Y$ is one of the important and often difficult problems in homotopy theory. It is immediate from the definition of spherical class, i.e., a class belong to the image of h, that a spherical class has to be A-annihilated. If $Y=QX=\operatorname{colim}\Omega^i\Sigma^iX$ for some space X, then the adjointness between Ω^∞ and Σ^∞ yields $\pi_*QX\simeq\pi_*^sX$ and determining the image of $h:\pi_*QX\to H_*QX$ even for the case of $X=S^n$ with $n\geqslant 0$ is still an open problem (see, e.g., [**5**, **10**, **11**, **22**]). The interest in computing spherical classes in H_*QX is also justified by its applications in the bordism theory of immersions [**1**] (see also [**2**, **9**]); spherical classes in $H_{n+k}QT(\xi)$ allow to determine normal Stiefel–Whitney numbers, and consequently bordism classes, of codimension k immersion into $\mathbb{R}^{n+k}$ with a ξ-structure on their normal bundle; here ξ is a k-dimensional vector bundle and $T(-)$ is the Thomification functor.

Programme. This paper provides the step which allows to identify A-annihilated monomials Q^Ix in H_*QX. The main observation is to provide necessary and sufficient conditions for $Q^Ix\in H_*QX$ to be A-annihilated which extends the existing results, which prove results only in one direction (see [**5**, **22**]), far beyond the case of $X=S^n$ and generalises it to every path connected space X (see Theorem 1.2 below).

1.2. Statement of results. The focus of this paper is on the action of A^{op} on $\mathbb{Z}/2$-homology of infinite loop spaces of the form $QX=\operatorname{colim}\Omega^i\Sigma^iX$. Consequently, we work at the prime $p=2$ and write H_* for $H_*(-;\mathbb{Z}/2)$ (likewise H^* for $H^*(-;\mathbb{Z}/2)$); we write $\widetilde{H}_*$ for the reduced $\mathbb{Z}/2$-homology and likewise $\widetilde{H}^*$ for the reduced $\mathbb{Z}/2$-cohomology. We use A for the mod 2 Steenrod algebra and A^{op} for its opposite algebra, and R for the Dyer–Lashof algebra (the algebra of Kudo–Araki operations and their iterations). The topological spaces have base points and are localised at 2 if necessary. We write X_+ for X with an added disjoint base point.

A^{op}**-module structures**. For a given space Y, the Steenrod operations $Sq^i:H^*Y\to H^{*+i}Y$ induce an action of the mod 2 Steenrod algebra on H^*Y which furnishes H^*Y with the structure of an A-module. The duality of H^*Y and H_*Y over $\mathbb{Z}/2$, provided by the Universal Coefficient Theorem, allows to consider dual operations $Sq^i_*:H_*Y\to H_{*-i}Y$ which could be evaluated using Kronecker pairing. This induces a right A-action on H_*Y or equivalently a left action of A^{op} on H_*Y which turns H_*Y into a left A^{op}-module.

DEFINITION 1.1. An element $y\in H_*Y$ is called A-annihilated if and only if $Sq^i_*y=0$ for all $i>0$.

For a given space X, let $QX=\operatorname{colim}\Omega^i\Sigma^iX$ be the infinite loop space associated with the suspension spectrum $\Sigma^\infty X$, and Q_0X its base point component corresponding to $0\in\pi_0QX\simeq\pi_0^sX$; in particular if X is path connected, then $Q_0X=QX$. If X is path connected, then H_*QX is a polynomial algebra over generators of the form Q^Ix where $x\in\widetilde{H}_*X$ is a homogeneous basis element and $I=(i_1,\ldots,i_s)$ ranges over certain sequence

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of positive integers, called admissible sequences which includes the empty sequence [4, Part I] (see Section 3 for precise definitions). For a nonempty sequence $I=(i_1,\ldots,i_s)$ define its excess by $\mathrm{excess}(I)=i_1-(i_2+\cdots+i_s)$ and let $\mathrm{excess}(Q^Ix)=\mathrm{excess}(I)-\dim x$. Also define length of I by $l(I)=s$. For the empty sequence ϕ we set $\mathrm{excess}(\phi)=+\infty$ and $l(\phi)=0$. Define $\rho:\mathbb{N}\longrightarrow\mathbb{N}\cup\{0\}$ by $\rho(n)=\min\{i:n_i=0\}$ for $n=\sum_{i=0}^{\infty}n_i2^i$ with $n_i\in\{0,1\}$. As our main observation, we give a complete characterisation of those classes Q^Ix which are A-annihilated, which reads as follows.

THEOREM 1.2. *Suppose X is path connected. Let Q^Ix be a generator of H_*QX with $I=(i_1,\ldots,i_s)$. The class Q^Ix is A-annihilated if and only if* (1) *$x\in\widetilde{H}_*X$ is A-annihilated,* (2) $\mathrm{excess}(Q^Ix)<2^{\rho(i_1)}$ *and* (3) $0\leqslant 2i_{j+1}-i_j<2^{\rho(i_{j+1})}$, $1\leqslant j\leqslant s-1$. *If $s=1$, then the first two conditions determine all A-annihilated classes of the form Q^ix of positive excess.*

Theorem 1.2 generalises work of Curtis and Wellington on iterated loop spaces of spheres to iterated loop spaces of any path connected space.

REMARK 1.3. As we have mentioned earlier, Theorem 1.2 has applications to the symmetric and unsymmetric hit problems. We have showed in [24, Theorem 1.1] that if $I=(i_1,\ldots,i_s)$ is a sequence satisfying conditions of Theorem 1.2, then we obtain an A-annihilated class in $H_*(\mathbb{Z}\times BO)$ whose leading term is given by

$$e_0e_{\mathrm{excess}_0}e^2_{\mathrm{excess}_1}\cdots e^{2^{s-1}}_{\mathrm{excess}_{s-1}}.$$

Here, for I as above we have $\mathrm{excess}_j=i_{j+1}-(i_{j+2}+\cdots+i_s)$ for $0\leqslant j<s$. Moreover, this class pulls back to an A-annihilated class in $H_*BO(2^s-1)$. The technical outcome is that we obtain a new family of A-annihilated classes of elements in $H_*(\mathbb{Z}\times BO)$ as well as H_*BO which depends on only three combinatorial conditions. Moreover, these classes can be pushed forward to $H_*\mathbb{R}P^{2^s-1}$ using transfer map [24, Theorem 1.4].

Next, recall that if X is path connected, then $H_*Q_0(X_+)$ is a polynomial algebra over certain generators $Q^Ix*[-2^{l(I)}]$ where $x\in\widetilde{H}_*X$ is an element of an additive basis, $*$ denotes Pontrjagin product in $H_*Q(X_+)$ and $[-2^{l(I)}]\in H_0Q(X_+)$ [4, Part I, Lemma 4.10]. The above theorem has the following corollary.

COROLLARY 1.4. *Let $Q^Ix*[-2^{l(I)}]$ be a monomial generator of $H_*Q_0(X_+)$ with $I=(i_1,\ldots,i_s)$. This class is A-annihilated if and only if the following conditions are satisfied:* (1) *$x\in\widetilde{H}_*X$ is A-annihilated;* (2) $\mathrm{excess}(Q^Ix)<2^{\rho(i_1)}$ *and* (3) $0\leqslant 2i_{j+1}-i_j<2^{\rho(i_{j+1})}$, $1\leqslant j\leqslant s-1$. *If $s=1$, then the first two conditions determine all A-annihilated classes of the form $Q^ix*[-2]$ of positive excess.*

The proof is very short, so we include it here.

Proof. If X is path connected, then $\pi_0Q(X_+)\simeq\mathbb{Z}$. We may label the path components of QX accordingly, writing Q_mX for the path component of $Q(X_+)$ corresponding to $m\in\pi_0Q(X_+)$, i.e., $Q_m(X_+)$ is the space of all elements $S^0\to X_+$ which induce multiplication by m on $H_0(-;\mathbb{Z})$. This induces certain translation maps in homology $*[m]:H_*Q_n(X_+)\to H_*Q_{n+m}(X_+)$ [4, Part I]. Using these translation maps, $H_*Q_0(X_+)$ is a polynomial algebra and has generators of the form $Q^Ix*[-2^{l(I)}]$ with x being a homogeneous basis element of $\widetilde{H}_*X_+\simeq H_*X$ [4, Part I, Lemma 4.10] (see also [6, Section 3.4] for a description of $H_*Q_0(X_+)$). The claim now follows from Cartan formula for Sq^i_* operations (see Section 3 below) and Theorem 1.2. □

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It is possible to use Theorem 1.2 to derive a construction for sequence I considered in the theorem. The following provides an example of such construction. For this purpose, our method below provides a method to construct sequences $I = (i_1, \ldots, i_s)$ which satisfy condition (3) of Theorem 1.2.

THEOREM 1.5. *Let $s > 1$ and let $(\rho_1, \ldots, \rho_s)$ be a sequence of integers $0 \leqslant \rho_1 \leqslant \rho_2 \leqslant \cdots \leqslant \rho_s$. Let N_s be a nonnegative integer, and inductively, having chosen N_{j+1}, choose N_j, for $1 \leqslant j < s$, such that*

$$2^{\rho_{j+1}-\rho_j+1}N_{j+1} + 2^{\rho_{j+1}-\rho_j-1} \leqslant N_j < 2^{\rho_{j+1}-\rho_j+1}N_{j+1} + 2^{\rho_{j+1}-\rho_j}.$$

For $1 \leqslant j \leqslant s$, let $i_j = 2^{\rho_j+1}N_j + 2^{\rho_j} - 1$. Then $I = (i_1, \ldots, i_s)$ satisfies condition (3) of Theorem 1.2, i.e.,

$$0 \leqslant 2i_{j+1} - i_j < 2^{\rho(i_{j+1})},$$

for all $1 \leqslant j \leqslant s - 1$.

The above construction is possible for all $s > 1$. If we are given a path connected space X which we know the A-annihilated elements of $\widetilde{H}_*X$, then in order to construct an A-annihilated generator $Q^I x$ with $l(I) = s > 1$, one first has to use the above construction to construct all possible sequences I, and then check conditions (1) and (3). On the other hand, we may consider condition (2) of Theorem 1.2 as a condition which tells us when to terminate the process. The above theorem to provide some algorithms to find A-annihilated monomials $Q^I x$ which is under investigation [**8**].

On similar previous results. Our motivation for studying A-annihilated classes comes from the problem of determining spherical classes in H_*QX. A spherical class in particular should be A-annihilated and primitive [**2**, Lemma 2.5]. Hence, determining the submodule of A-annihilated classes in H_*QX is helpful in this direction. The latter problem has been studied by means of unstable Adams spectral sequences (ASS), most notably by Curtis [**5**] and Wellington [**22**] (see also [**21**]); in this approach, the edge homomorphism on the 0-line of a suitable unstable ASS was meant to determine the Hurewicz homomorphism $\pi_*QS^0 \to H_*QS^0$ [**5**, Section 4]. By certain computations in the Λ-algebra, conditions of our Theorem 1.2 have first appeared in [**5**, Lemma 6.2 and Theorem 6.3] for loop spaces on spheres $\Omega^i S^{i+n}$. The result was also generated to odd primes by Wellington as well [**21**, Theorem 5.6]. We note that we do not claim to provide a basis for the submodule of A-annihilated classes since as noted by Wellington there are counter examples of classes of the form say $x + y$ where the sum is A-annihilated but neither of the terms are [**22**, Section 11]. Finally, we note that taking Wellington's counter example into account, the work of Wellington and Curtis seems to offer a proof of our Theorem 1.2 only in one direction by showing that having conditions of Theorem 1.2 we have annihilation under the action of A. We must also mention the work of Snaith and Tornehave [**17**] (see also [**18**, Chapter 2 and Lemma 1.5]) where a so-called $H(j)$ condition is introduced: if $Q^I x$ satisfies $H(j)$, then it is annihilated by Sq^t for all $t < 2^{j+1}$. The conditions therein do not seem so explicit as ours, and we believe our description is more applicable. Here, we include a full proof which we believe to be an economic one compared to what we offered in [**26**, Theorem 2]; our proof in the reverse direction partly depends on the partial order that we have defined which could be of independent interest. We mostly rely on the computations of Curtis and Wellington, and our main computational tool, namely Lemma 2.1, is derived using their computations and we do not provide a direct proof which we believe would be quite a mess, and not as neat as one would hope for. Finally, we note that studying the submodule of A-annihilated classes, in cohomological setting, is equivalent to the famous hit problem of Wood [23], and our computations have direct applications to the hit problem for H^*QX providing new numerical conditions in the context of hit problem; we refer the reader to [24] for more applications of these results.

2. The action of the A on R.

The aim of this section is to record a proof of the following lemma; the lemma has to be well known to Curtis [5] and Wellington [22], although not expressed in terms we do, and the proof we offer is built upon their observations.

LEMMA 2.1. *Suppose $I=(i_1,\ldots,i_s)$ is an admissible sequence such that $2i_{j+1}-i_j<2^{\rho(i_{j+1})}$ for all $1\leqslant j\leqslant s-1$. Let $N(Sq^a_*,Q^I)=\sum_{K\text{admissible}}Q^K$. Then*

$$\mathrm{excess}(K)\leqslant \mathrm{excess}(I)-2^{\rho(i_1)}.$$

Moreover, $\rho(i_1)\leqslant\rho(i_2)\leqslant\cdots\leqslant\rho(i_s)$.

Here, $N(Sq^a_*,Q^I)$ refer to Nishida relations for the action of A^{op} on R which is described below. The proof that we offer follows from the interaction between the differential of the Λ algebra and the action of A^{op} on Λ. For this purpose, we include a brief description of these algebras and recall the desired properties. We assume the reader has a basic knowledge on the Steenrod algebra A. We mostly follow Wellington [22, Chapter 7] for the material on the subject (see also [5]).

The Λ and Dyer–Lashof algebras. Let Θ be the free-graded associated algebra over $\mathbb{Z}/2$ generated with generators λ_i in grading $i\geqslant 0$. For $a>2b$, let

$$R_\Lambda(a,b)=\lambda_b\lambda_a+\sum_{a+b\leqslant 3t}\binom{t-b-1}{2t-a}\lambda_t\lambda_{a+b-t}. \tag{2.1}$$

We define the Λ algebra by $\Lambda:=\Theta/\langle R_\Lambda(a,b):a>2b\rangle$. We keep using λ_i for the image of λ_i in Λ. Hence, whenever $a>2b$ we have the relations

$$\lambda_b\lambda_a=\sum_{a+b\leqslant 3t}\binom{t-b-1}{2t-a}\lambda_t\lambda_{a+b-t}$$

in Λ which we refer to them as the Adem relations for the Λ algebra. For a sequence of nonnegative integers $I=(i_1,\ldots,i_s)$, we write λ_I for $\lambda_{i_s}\cdots\lambda_{i_1}$. We shall refer to I as admissible if $i_j\leqslant 2i_{j+1}$ for all $1\leqslant j\leqslant s-1$. We shall refer to $l(I)=s$ and $\mathrm{excess}(I)=i_1-(i_2+\cdots+i_s)$ as length and excess of I respectively. The Dyer–Lashof algebra R is defined by

$$R:=\Lambda/\langle\lambda_I:\mathrm{excess}(I)<0\rangle.$$

The elements of R are known as the Kudo–Araki or Kudo–Araki–Dyer–Lashof operations. We write $Q^{i_1}\cdots Q^{i_s}$ for the image of $\lambda_{i_s}\cdots\lambda_{i_1}$ in R under the natural projection $\Lambda\to R$. We set $\mathrm{excess}(Q^I)=\mathrm{excess}(I)$ and $l(Q^I)=l(I)$. In this algebra, whenever $a>2b$, we have Adem relations as

$$Q^aQ^b=\sum_{a+b\leqslant 3t}\binom{t-b-1}{2t-a}Q^{a+b-t}Q^t. \tag{2.2}$$

We shall refer to Q^I as admissible whenever $I=(i_1,\ldots,i_s)$ is admissible. If I is not admissible, then the Adem relations (2.2) allow to write Q^I as a sum of admissible terms.

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The action of the Steenrod algebra. We follow Wellington [22, Chapter 7], and begin by formally introducing the Nishida relations which allows to define a right A-module structure on Λ, hence a left A^{op}-module structure on Λ. Formally, define the Nishida relations for the Λ algebra by

$$\lambda_b Sq^a = \sum_{t\geqslant 0} \binom{b-a}{a-2t} Sq^t \lambda_{b-a+r}. \tag{2.3}$$

We may use Nishida relations to define a right action $N_\Lambda : \Lambda \otimes A \to \Lambda$ by

$$N_\Lambda(\lambda_i, Sq^j) := \tbinom{i-j}{j}\lambda_{i-j},$$

$$N_\Lambda(\lambda_I, Sq^a) := \sum \tbinom{i_1-a}{a-2t}\lambda_{i_1-a+t} N_\Lambda(\lambda_{I_1}, Sq^t),$$

where $I = (i_1, I_1)$. That is, if iterated application of Nishida relation above yields $\lambda_I Sq^a_* = \sum Sq^{a^K}\lambda_K$ with $a^K \in \mathbb{Z}_{\geqslant 0}$, then

$$N_\Lambda(\lambda_I, Sq^a) = \sum_{a^K=0} \lambda_K.$$

In an obvious manner, we may define a left A^{op}-module structure on Λ which by abuse of notion we denote by $N_\Lambda : A^{\mathrm{op}} \otimes \Lambda \to \Lambda$. On the other hand, Λ admits a boundary map ∂ which on the generators is defined by

$$\partial\lambda_i = \sum_{j\geqslant 1} \binom{i-j}{j} \lambda_{i-j}\lambda_{j-1}.$$

Note that by the above definition we have

$$\partial\lambda_i = \sum_{j\geqslant 1} N_\Lambda(\lambda_i, Sq^j)\lambda_{j-1}.$$

This appears to be true in general if λ_i is replaced by I with $l(I) > 1$. Reformulating [22, Theorem 7.11(i)] in terms of the action N_Λ we have the following.

THEOREM 2.2. *The differential ∂ of the Λ algebra is related to the Steenrod operations when* excess$(I) \geqslant 0$ *and I is admissible by*

$$\partial\lambda_I = \sum_{j\geqslant 1} N_\Lambda(\lambda_I, Sq^j)\lambda_{j-1}.$$

Next, we describe a left A^{op}-module structure for R. For the Kudo–Araki operations, formally set the Nishida relations to be [4, Part I, Theorem 1.1]

$$Sq^a_* Q^b = \sum_{t\geqslant 0} \binom{b-a}{a-2t} Q^{b-a+r} Sq^t_*. \tag{2.4}$$

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According to Madsen [13, Equation (3.2)], we may use Nishida relations to define a left action $N : A^{\mathrm{op}} \otimes R \longrightarrow R$ by

$$N(Sq^a_*, Q^b) = \binom{b-a}{a} Q^{b-a}, \tag{2.5}$$

$$N(Sq^a_*, Q^I) = \sum \binom{i_1 - a}{a - 2t} Q^{i_1 - a + t} N(Sq^t_*, Q^{I_1}), \tag{2.6}$$

where $I = (i_1, I_1)$. In other words, if $Sq^a_* Q^I = \sum Q^K Sq^{a^K}_*$ with K admissible and $a^K \in \mathbb{Z}$, then

$$N(Sq^a_*, Q^I) = \sum_{a^K = 0} Q^K.$$

If we write $q : \Lambda \to R$ for the natural projection, and if I is given with $\mathrm{excess}(I) \geqslant 0$, then

$$N(Sq^a_*, Q^I) = qN_\Lambda(\lambda_I, Sq^a).$$

The second result that we recall from [22, Theorem 7.12] is on the relation between the differential of the Λ algebra and the A^{op}-module structure of R. We state the result in Λ bearing in mind that the element $\lambda_I \in \Lambda$ with $\mathrm{excess}(I) \geqslant 0$ and I admissible, projects onto the nontrivial element $Q^I \in R$.

THEOREM 2.3. *Let I be admissible,* $\mathrm{excess}(I) \geqslant 0$ *and suppose that*

$$\partial \lambda_I = \sum_{K \text{admissible}} \alpha_K \lambda_K,$$

where $\alpha_K \in \mathbb{Z}/2$. Then

$$N_\Lambda(\lambda_I, Sq^j) = \sum \alpha_K \lambda_{K'},$$

where $K = (K', j-1)$ and $\mathrm{excess}(K') \geqslant 0$. *In particular, K' is admissible.*

The final result that we need is the following that we recall from [5, Lemma 6.2] and [22, Lemma 12.5].

LEMMA 2.4. *Let λ_I be given with* $\mathrm{excess}(I) \geqslant 0$ *such that $I = (i_1, \ldots, i_s)$ is an admissible sequence such that $2i_{j+1} - i_j < 2^{\rho(i_{j+1})}$ for all $1 \leqslant j \leqslant s-1$. Assume*

$$\partial \lambda_I = \sum_{K \text{admissible}} \alpha_K \lambda_K$$

with $\alpha_K \in \mathbb{Z}/2$. Then for those $K = (K', k)$ with $\mathrm{excess}(K') \geqslant 0$*, we have that*

$$\mathrm{excess}(K) \leqslant \mathrm{excess}(I) - 2^{\rho(i_1)}.$$

Moreover, $\rho(i_1) \leqslant \rho(i_2) \leqslant \cdots \leqslant \rho(i_s)$.

Let us note that in [5] for $I = (i_1, \ldots, i_s)$, λ_I is written for $\lambda_{i_1} \cdots \lambda_{i_s}$, whereas we write λ_I for $\lambda_{i_s} \cdots \lambda_{i_1}$. The part $\rho(i_1) \leqslant \rho(i_2) \leqslant \cdots \leqslant \rho(i_s)$ is also implicit in Curtis's proof. These considerations are helpful while comparing the above lemma to [5, Lemma 6.2]. Lemma 2.1 follows immediately, but we include a short proof.

Proof of Lemma 2.1. From the equality $N(Sq^a_*, Q^I) = qN_\Lambda(\lambda_I, Sq^a)$ we deduce that if $N_\Lambda(\lambda_I, Sq^a) = \sum_{K \text{admissible}} \lambda_K$, then

$$N(Sq^a_*, Q^I) = \sum_{K \text{admissible}, \text{excess}(K)>0} \lambda_K.$$

Now, the lemma follows from Theorem 2.3 and Lemma 2.4. □

3. The A^{op}-module H_*QX. The space QX is an infinite loop space, so its homology is a commutative algebra. If X is path connected, then as an algebra it is described as [4, Part I, Lemma 4.10]

$$H_*QX \simeq \mathbb{Z}/2[Q^I x_\mu : I \text{ is admissible , excess}(I) > \dim(x_\mu)],$$

where $\{x_\mu\}$ is an additive basis for $\widetilde{H}_*X$ and $I = (i_1, \ldots, i_s)$ is admissible if $i_j \leqslant 2i_{j+1}$ for all $0 < j < s-1$; the empty sequence $I = \phi$ is declared to be admissible with $\text{excess}(\phi) = +\infty$ and $Q^\phi x = x$.

Moreover, the infinite loop structure of QX furnishes H_*QX with an R-module structure which is described by the following requirements: For any $i \geqslant 0$, the Kudo–Araki operation $Q^i : H_*QX \to H_{*+i}QX$ is an additive homomorphism so that
(1) $Q^i\xi = \xi^2$, if $i = \dim \xi$;
(2) $Q^i\xi = 0$, if $i < \dim \xi$ for any $\xi \in H_*QX$;
(3) Q^i satisfies Cartan formula, so that $Q^i(xy) = \sum_{i+j=k}(Q^jx)(Q^ky)$.
Condition (2) is to be interpreted that $Q^I x_\mu = 0$, if $\text{excess}(I) < \dim x_\mu$. Moreover, the action $R \otimes H_*QX \to H_*QX$ sends $(Q^i, Q^I x_\mu)$ to $Q^{(i,I)}x_\mu$ modulo the above requirements and the Adem relations for the Kudo–Araki operations [4, Part I, Theorem 1.1]. These together describe the R-module structure of H_*QX. The action of A^{op} on H_*QX is described as follows. On the generators $Q^I x_\mu \in H_*QX$, the evaluation of $Sq^t_* Q^I x_\mu$ is done by (iterated) application of Nishida relations (see Nishida relations for the Dyer–Lashof algebra (2.4)) together with the action of A on $\widetilde{H}_*X$. The action on decomposable elements of H_*QX is determined by Nishida relations together with Cartan formula [22, Chapter 5]

$$Sq^r_*(\xi\eta) = \sum (Sq^{r-i}_*\xi)(Sq^i_*\eta).$$

Noting that Sq^t_* is a group homomorphism, the above relations completely determine the action of A^{op} on H_*QX.

4. Ordering monomials. Let X be path connected. Since H_*QX is a polynomial algebra, then it is more convenient to fix some partial order on the monomial generators of this algebra. More precisely, we define a partial order on H_iQX for a given $i > 0$ as follows. For an additive basis $\{x_\mu\}$ of $\widetilde{H}_*X$, given generators $Q^I x_\mu$ and $Q^J x_{\mu'}$, define $Q^I x_\mu > Q^J x_{\mu'}$ if and only if $\text{excess}(Q^I x_\mu) > \text{excess}(Q^J x_{\mu'})$. Moreover, if $\text{excess}(Q^I x_\mu) = \text{excess}(Q^K x_\nu)$ define $Q^I x_\mu > Q^K x_\nu$ if $l(I) < l(K)$. Finally, if $\text{excess}(Q^I x_\mu) = \text{excess}(Q^K x_\nu)$ and $l(I) = l(K)$, then writing the operations in lower indices, say $Q^I x_\mu = Q_E x_\mu$ and $Q^K x_\nu = Q_F x_\nu$, we define $Q^I x_\mu > Q^K x_\nu$ if the first nonzero entry of $E - F$ from left is positive. Here, the lower indexed operations Q_i is defined by $Q_i x = Q^{i+\dim x}x$ and $Q_E = Q_{e_1}\cdots Q_{e_s}$ for $E = (e_1, \ldots, e_s)$. We refer to this order, as the total-partial order on H_*QX.

REMARK 4.1. It may seem for terms $Q^I x_\mu$ as $l(I)$ increases the excess will decrease. This does not hold in general, however. As an example, for $Q^{15}Q^{13}g_1, Q^{16}Q^8Q^4g_1 \in$

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$H_{29}QS^1$ we see that the term of shorter length is also of lower excess. One could construct counter examples for other similar statements to the above.

5. Proof of Theorem 1.2. We break the proof into separate lemmata.

LEMMA 5.1. *Let $x \in \widetilde{H}_*X$ be A-annihilated, and I an admissible sequence such that* (1) $0 < \text{excess}(Q^I x) < 2^{\rho(i_1)}$; (2) $2i_{j+1} - i_j < 2^{\rho(i_{j+1})}$ *for all* $1 \leqslant j \leqslant s-1$. *Then $Q^I x$ is A-annihilated.*

Proof. Let $a > 0$. Since x is A-annihilated, then

$$Sq^a_* Q^I x = \sum Q^K Sq^{a^K}_* x = \sum\nolimits_{a^K=0} Q^K x,$$

where K is admissible. But, notice that according to Lemma 2.1

$$\text{excess}(Q^K x) \leqslant \text{excess}(Q^I x) - 2^{\rho(i_1)} < 0.$$

Hence the above sum is trivial, and we are done. □

This proves the Theorem 1.2 in one direction. Next, we show if any of the conditions (1)–(3) of Theorem 1.2 does not hold, then $Q^I x$ will be not-A-annihilated. Note that it is enough to work with operations of the form Sq^{2^s} with $s \geqslant 0$.

REMARK 5.2. By looking at the binary expansions, it is easy to see that given a positive integer n, then $\rho(n)$ is the least integer t such that

$$\binom{n-2^t}{2^t} \equiv 1 \bmod 2.$$

LEMMA 5.3. *Let X be path connected. Suppose $I = (i_1, \ldots, i_s)$ is an admissible sequence, and let $Q^I x$ be given with* $\text{excess}(Q^I x) > 0$ *with j being the least positive integer such that* $2i_{j+1} - i_j \geqslant 2^{\rho(i_{j+1})}$. *Then such a class is not A-annihilated, and we have*

$$Sq_*^{2^{\rho(i_{j+1})+j}} Q^I x = Q^{i_1 - 2^{\rho+j-1}} Q^{i_2 - 2^{\rho+j-2}} \cdots Q^{i_j - 2^\rho} Q^{i_{j+1} - 2^\rho} Q^{i_{j+2}} \cdots Q^{i_s} x$$

modulo terms of lower excess and total order.

Proof. Assume that $Q^I x$ satisfies the condition above. We may write this condition as

$$i_j - 2^\rho \leqslant 2i_{j+1} - 2^{\rho+1} = 2(i_{j+1} - 2^\rho),$$

where $\rho = \rho(i_{j+1})$. This is the same as the admissibility condition for the pair $(i_j - 2^\rho, i_{j+1} - 2^\rho)$. In this case we use $Sq_*^{2^{\rho+j}}$ where we get

$$Sq_*^{2^{\rho+j}} Q^I x = \underbrace{Q^{i_1 - 2^{\rho+j-1}} Q^{i_2 - 2^{\rho+j-2}} \cdots Q^{i_j - 2^\rho} Q^{i_{j+1} - 2^\rho} Q^{i_{j+2}} \cdots Q^{i_s} x}_{A} + O.$$

Here, O denotes other terms which after being written in terms of admissible sequences, using Adem relations, is given by a sum of terms of lower excess, hence of lower order (note that these are all in the same dimension). The term A in the right-hand side of the

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above equality is admissible. Moreover,

$$\begin{aligned}\text{excess}(A) &= (i_1 - 2^{\rho+j-1}) - (i_2 - 2^{\rho+j-2}) - (i_j - 2^{\rho}) - (i_{j+1} - 2^{\rho})\\ &\quad -(i_{j+2} + \cdots + k_s + \dim x)\\ &= i_1 - (i_2 + \cdots + k_s + \dim x)\\ &= \text{excess}(Q^I x) > 0.\end{aligned}$$

First, this implies that A is nontrivial. Second, being of higher excess and total order shows that A will not be equal to any of the terms in O. This implies that $Sq_*^{2^{\rho+j}} Q^I x \neq 0$ and hence completes the proof. □

Notice that choosing the least j is necessary, as otherwise we may not get nontrivial action. Now, assume that the above condition does hold, but condition (2) in Theorem 1.2 fails. This case is resolved in the following theorem.

LEMMA 5.4. *Let X be path connected. Suppose $I = (i_1, \ldots, i_s)$ is an admissible sequence, such that* $\text{excess}(Q^I x) \geqslant 2^{\rho(i_1)}$*, and* $2i_{j+1} - i_j < 2^{\rho(i_{j+1})}$ *for all* $1 \leqslant j \leqslant s-1$*. Then such a class is not A-annihilated.*

Proof. We use $Sq_*^{2^\rho}$ with $\rho = \rho(i_1)$ which gives

$$Sq_*^{2^\rho} Q^I x = Q^{i_1 - 2^\rho} Q^{i_2} \cdots Q^{i_s} x + O,$$

where O denotes other terms given by

$$O = \sum_{t>0} \binom{i_1 - 2^\rho}{2^\rho - 2t} Q^{i_1 - 2^\rho + t} Sq_*^t Q^{i_2} \cdots Q^{i_s} x.$$

Notice that $\text{excess}(Q^I x) \geqslant 2^{\rho(i_1)}$ ensures that i_1 is not of the form 2^ρ. By iterated application of the Nishida relations, we may write

$$O = \sum_{\alpha \leqslant s} \epsilon_1 \cdots \epsilon_\alpha Q^{i_1 - 2^\rho + r_1} Q^{i_2 - r_2 + r_3} \cdots Q^{i_\alpha - r_\alpha} Q^{i_{\alpha+1}} \cdots Q^{i_s} x,$$

where

$$\epsilon_1 = \binom{i_1 - 2^\rho}{2^\rho - 2r_1},\ \epsilon_2 = \binom{i_2 - r_1}{r_1 - 2r_2}, \ldots, \epsilon_{\alpha-1} = \binom{i_{\alpha-1} - r_{\alpha-2}}{2r_{\alpha-2} - 2r_{\alpha-1}},\ \epsilon_\alpha = \binom{i_\alpha - r_{\alpha-1}}{r_{\alpha-1}},$$

such that $2r_k \leqslant r_{k-1}$ for all $k \leqslant \alpha$. The sequence I satisfies the condition of Lemma 2.1 which in particular implies that $\rho(i_1) \leqslant \cdots \leqslant \rho(i_\alpha) \leqslant \cdots \leqslant \rho(i_s)$. Notice that $r_{\alpha-1} < 2^{\rho(i_\alpha) - \alpha + 1} < 2^{\rho(i_\alpha)}$ which together with Remark 5.2 implies that $\epsilon_\alpha = 0$ and therefore $O = 0$. This then shows that

$$Sq_*^{2^\rho} Q^I x = Q^{i_1 - 2^\rho} Q^{i_2} \cdots Q^{i_s} x \neq 0.$$

This completes the proof. □

Now we show that the condition (1) is also necessary in the proof of the Theorem 1.2.

LEMMA 5.5. *Let X be path connected, and let $Q^I x \in H_* QX$ be a term of positive excess with I admissible such that $x \in \widetilde{H}_* X$ is not A-annihilated. Then $Q^I x$ is not A-annihilated.*

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Proof. Let t be the least nonnegative integer such that $Sq_*^{2^t}x \neq 0$. For $I = (i_1, \ldots, i_s)$, we have

$$Sq_*^{2^{s+t}}Q^I x = Q^{i_1-2^{s+t-1}} \cdots Q^{i_s-2^s}Sq_*^{2^t}x + O,$$

where O denotes sum of the other terms which are of the form $Q^J x$. The obvious inequality $Sq_*^{2^t}x \neq x$ implies that the first term in the above equality will not cancel with any of the other terms. Notice that the first term in the above expression is admissible, and $\text{excess}(Q^{i_1-2^{s+t-1}} \cdots Q^{i_t-2^s}Sq_*^{2^s}x) = \text{excess}(Q^I x) > 0$. Hence $Sq_*^{2^{s+t}}Q^I x \neq 0$. □

6. Constructing A-classes in H_*QX. The aim of this section is to give a construction of sequences which satisfy condition (3) of Theorem 1.2. This construction, at least in theory, will determine all such sequences. One can see that our construction here is the most general one, obtained by properties of sequences I satisfying condition (3) of Theorem 1.2. One observes that condition (2), i.e., $\text{excess}(Q^I x) < 2^{\rho(i_1)}$, tells us when the construction has to terminate. Hence, having conditions (2) and (3) of Theorem 1.2, one can for example construct A-annihilated classes in H_*BO as well as H_*QBO.

Let $I = (i_1, \ldots, i_r)$ be a sequence satisfying condition (3), i.e.,

$$0 \leqslant 2i_{j+1} - i_j \leqslant 2^{\rho(i_{j+1})}.$$

Note that a given positive integer n maybe written as $n = 2^{\rho(n)+1}N_n + 2^{\rho(n)} - 1$ for some $N_n \geqslant 0$. Suppose we are given a pair of integers (m, n), $m > n$, such that

$$0 \leqslant 2n - m < 2^{\rho(n)},$$

which is the same as assuming $2n - 2^{\rho(n)} < m \leqslant 2n$. From this, by looking at the binary expansions for m and n, we deduce that

$$\rho(m) \leqslant \rho(n).$$

To construct a sequence I of length r, consider an r-tuple of nondecreasing positive integers,

$$\rho_1 \leqslant \rho_2 \leqslant \cdots \leqslant \rho_r.$$

Choose a nonnegative integer N_r, and let $i_r = 2^{\rho_r+1}N_r + 2^{\rho_r} - 1$. We want to find $i_{r-1} = 2^{\rho_{r-1}+1}N_{r-1} + 2^{\rho_{r-1}} - 1$ such that

$$2i_r - 2^{\rho_r} < i_{r-1} \leqslant 2i_r.$$

Plugging in the value of i_{r-1}, i_r, gives the boundary conditions on N_{r-1},

$$2^{\rho_r+2}N_r + 2^{\rho_r} - 1 < 2^{\rho_{r-1}+1}N_{r-1} + 2^{\rho_{r-1}} \leqslant 2^{\rho_r+2}N_r + 2^{\rho_r+1} - 1.$$

This can be refined as

$$2^{\rho_r+2}N_r + 2^{\rho_r} \leqslant 2^{\rho_{r-1}+1}N_{r-1} + 2^{\rho_{r-1}} < 2^{\rho_r+2}N_r + 2^{\rho_r+1}.$$

Hence we have

$$2^{\rho_r-\rho_{r-1}+1}N_r + 2^{\rho_r-\rho_{r-1}-1} \leqslant N_{r-1} + \frac{1}{2} < 2^{\rho_r-\rho_{r-1}+1}N_r + 2^{\rho_r-\rho_{r-1}}.$$

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As N_{r-1} is an integer, hence one has

$$2^{\rho_r-\rho_{r-1}+1}N_r+2^{\rho_r-\rho_{r-1}-1}\leqslant N_{r-1}<2^{\rho_r-\rho_{r-1}+1}N_r+2^{\rho_r-\rho_{r-1}}.$$

This means that there are $2^{\rho_r-\rho_{r-1}-1}$ choices for N_{r-1}. By proceeding in this way, we can construct such sequences which only will satisfy condition (3) of Theorem 1.2. Notice that

$$2^{\rho_i-\rho_{i-1}+1}N_i+2^{\rho_i-\rho_{i-1}-1}\leqslant N_{i-1}$$

for any $1\leqslant i<r$. This implies that having fixed a nondecreasing r-tuple of positive integers,

$$\rho:\rho_1\leqslant\rho_2\leqslant\cdots\leqslant\rho_r,$$

then different choices for N_i will give different sequences in different dimensions. However, it is possible to have two different sequences, say ρ, ρ', but giving two r-tuples in the same dimensions. As an example, let $r=2$. Then (17, 15) and (21, 11) both are sequences satisfying condition (3), and both are in dimension 32. Notice that

$$\rho(17)=1<\rho(15)=4,$$
$$\rho(21)=1\leqslant\rho(11)=3.$$

Now we give some specific examples of constructing such sequences which seem to be more applicable.

EXAMPLE 6.1. This is the simplest possible case when we choose

$$\rho:\rho_1=\rho_2=\cdots=\rho_r.$$

Let us choose a specific fixed value for ρ_i, say $\rho_i=2$. However, in this case we don't restrict ourselves to some specific length. We have $i_r=2^3N_r+(2^3-1)$. Let us choose $N_r=1$, then $i_r=11$. Now set $i_{r-1}=2i_r-(2^2-1)$, and inductively set $i_{r-j}=2i_{i-j+1}-(2^2-1)$. Then it is easy to see that $i_j\equiv 2^2-1 \bmod 2^3$. For example, continuing in this way for 3 times we obtain the sequence

$$(67,35,19,11).$$

This automatically satisfies conditions (2) and (3) of Theorem 1.2, i.e., $Q^{67}Q^{35}Q^{19}Q^{11}$ is an A-annihilated class in the Dyer–Lashof algebra R. This also implies that

$$Q^{67}Q^{35}Q^{19}p'_{11}$$

is an A-annihilated class in $H_*Q_0S^0$. Notice that $Q^{67}Q^{35}Q^{19}p'_{11}$ is a primitive A-annihilated class (see [12] and [26, Chapter 5 and Page 98] for the definition of class p'_n).

As another example, let us choose $\rho_i=\rho=3$, then $i_j=2^4N_j+(2^3-1)$. Let us choose $N_r=2$, then $i_r=39$. Now let $i_j=2i_{j+1}-(2^3-1)$. If we look for a sequence I such that $\text{excess}(I)<2^\rho=8$, we then obtain the sequence

$$(1031,519,263,135,71,39),$$

which means $Q^{1031}Q^{519}Q^{263}Q^{135}Q^{71}Q^{39}$ is an A-annihilated class in the Dyer–Lashof algebra. This is the sequence used in [22, Remark 11.26] to construct a sum of even degree which is A-annihilated, but its terms are not.

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7. Discussion. We wish to include a short discussion on the possible applications to the problem of determining spherical classes in H_*QX where the case of $X = S^n$ with $n > 0$ is of special interest. First, recall that the evaluation map $\Sigma QX \to Q\Sigma X$ induces homology suspension homomorphism $\sigma_* : H_*QX \to H_{*+1}Q\Sigma X$. We write $\sigma_*^k : H_*QX \to H_{*+k}Q\Sigma^k X$ for its iteration, and $\sigma_*^\infty : H_*QX \to H_*X$ for the stable suspension homomorphism. The homomorphism σ_* is characterised by $\sigma_* Q^I x = Q^I \Sigma x$ [4, Part I, Page 47] and the fact that it acts trivially on decomposable elements. It is known that the kernel of $\sigma_* : H_*QX \to H_{*+1}Q\Sigma X$ consists of only decomposable elements [27, Lemma 3.4] (see also [25, Lemma 2.3]). As we noted earlier, not any A-annihilated class is a sum of A-annihilated terms. However, those A-annihilated classes that are stable under homology suspension show some nicer behaviour. Suppose X is a path connected space and $f : S^n \to QX$ is given so that $\sigma_*^2 h(f) \neq 0$. Since $\sigma_* h(f) \neq 0$, then we may write $h(f) = \sum Q^I x$ modulo decomposable terms where x may vary in an additive basis for $\widetilde{H}_*X$, and consequently $\sigma_* h(f) = \sum Q^I \Sigma x$ where the sum involved the possible decomposable terms corresponding to terms $Q^I \Sigma x$ with $\mathrm{excess}(Q^I \Sigma x) = 0$ or equivalently $\mathrm{excess}(Q^I x) = 1$ and no other decomposable term is involved. The fact that Nishida relations respect length implies the following.

LEMMA 7.1. *Suppose $\xi = \sum Q^I x$ modulo decomposable terms is an A-annihilated class in H_*QX with $\sigma_*^2 \xi \neq 0$. Then, for any $l \geqslant 0$ the sum*

$$\sum_{l(I)=l} Q^I \Sigma x$$

is A-annihilated.

In some cases of interest, such as $X = S^n$ with $n > 0$ or $X = \mathbb{R}P^\infty, \mathbb{C}P^\infty$, it is possible to use computations of Section 1.2 together with Lemma 7.1 to make more eliminations to the possible expressions of $\sigma_* h(f)$ or equivalently to the 'indecomposable part' of $h(f)$. We have the following partial observation.

LEMMA 7.2. *Suppose X is a space so that $\widetilde{H}_*X$ has at most one generator in each dimension. Suppose $f : S^n \to QX$ is given so that $\sigma_* h(f) \neq 0$. Then $\sigma_* h(f) = \sum Q^I \Sigma x$ with I running over certain admissible terms, and x running over elements of an additive basis for $\widetilde{H}_*X$, such that if x_m^l is the element being of least dimension among other x's involved in $Q^I x$ with $l(I) = l$, then x_m^l is A-annihilated.*

Proof. Since $\sigma_* h(f) \neq 0$, then we may write $h(f) = \sum Q^I x$ modulo decomposable terms where x_n may vary in an additive basis for $\widetilde{H}_*X$. Hence, $\sigma_* h(f) = \sum Q^I \Sigma x$. If x_m^l is not A-annihilated, then choose $t \geqslant 0$ to be the least nonnegative integer so that $Sq_*^{2^t} x_m^l \neq 0$. By the above comments, since Nishida relations respect length, we may assume that $l(I) = l$ is fixed. By computations of Section 1.2 we may write

$$Sq_*^{2^{l+t}} \sigma_* h(f) = \sum Sq_*^{2^{l+t}} Q^I x + \sum Q^{i_1 - 2^{l+t-1}} \cdots Q^{i_l - 2^s} Sq_*^{2^t} x + O,$$

if $I = (i_1, \ldots, i_l)$. Note that the second sum may involve more than one term. However, the terms of the first sum as well as O would be of the form $Q^J y$ with $\dim y > \dim Sq_*^{2^t} x_m$. So, none of these terms would cancel the terms of the middle sum. On the other hand, for two terms $Q^I x_m$ and $Q^{I'} x_m$ with I and I' admissible, obviously the leading terms for $Sq_*^{2^{l+t}} Q^I x_m$ and $Sq_*^{2^{l+t}} Q^{I'} x_m$ recorded in the middle sum are distinct admissible terms, so they do not cancel each other. Consequently, $Sq_*^{2^{l+t}} \sigma_* h(f) \neq 0$ which contradicts the fact that $\sigma_* h(f)$ as a spherical class should be A-annihilated. This completes the proof. $\square$

Next, we turn to our total-partial order. If $Q^I x$ is written as $Q_E x$ in lower indexed operations with $E = (e_1, \ldots, e_s)$, then

$$\sigma_* Q_E x = Q_{E-\mathbf{1}} \Sigma x,$$

where $E - \mathbf{1} = (e_1 - 1, \ldots, e_s - 1)$. Also, note that obviously, σ_* preserves the length of E. An immediate consequence of this is the following.

LEMMA 7.3. *The total-partial of Section 4 is stable under homology suspension in the sense that if $Q^I x \leqslant Q^J x'$, then $\sigma_* Q^I x \leqslant \sigma_* Q^J x'$.*

Now, let X be a path connected space and $f : S^n \to QX$ is given so that $\sigma_*^2 h(f) \neq 0$. Since $\sigma_* h(f) \neq 0$, then we may write $h(f) = \sum Q^I x$ modulo decomposable terms where x may vary in an additive basis for $\widetilde{H}_* X$. Let $e = \max\{\text{excess}(I)\}$. Then

$$\sigma_*^{e-1} h(f) = \sum_{\text{excess}(I)=e} Q^I \Sigma^{e-1} x.$$

Note that all the terms of the above sum are of excess equal to 1. This class is spherical, since for the $(e-1)$-th adjoint of f as $f' : S^{n+e-1} \to Q\Sigma^{e-1} X$ we have $h(f') = \sigma_*^{e-1} h(f)$. Therefore, the above sum has to be A-annihilated. On the other hand, we observe that Nishida relations respect length. This results in the following observation.

LEMMA 7.4. *Suppose $f : S^n \to QX$ is given with $\sigma_*^\infty h(f) = 0$. For $e = \max\{\text{excess}(I)\}$, and f given as above, for any $l > 0$,*

$$\sum_{\text{excess}(I)=e, l(I)=l} Q^I \Sigma^{e-1} x$$

is A-annihilated.

The condition $\sigma_*^\infty h(f) = 0$ guarantees that $h(f)$ does not have any term of the form $Q^I x$ with $I = \phi$.

The important question here which is of a more combinatorial nature is that if there exists a partial order on $H_* QX$ which is both stable under homology suspension and under the action of the opposite Steenrod algebra A^{op}. We have not checked the answer to this question with respect to our total-partial order. But, having such an order, even for a class of spaces X, would result in more powerful elimination results on the form of potential spherical classes in $H_* QX$.

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