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RATIONAL CONJUGACY OF TORSION UNITS IN INTEGRAL GROUP RINGS OF NON-SOLVABLE GROUPS

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Abstract We introduce a new method to study rational conjugacy of torsion units in integral group rings using integral and modular representation theory. Employing this new method, we verify the first Zassenhaus conjecture for the group $\mathrm{PSL}(2,19)$. We also prove the Zassenhaus conjecture for $\mathrm{PSL}(2,23)$. In a second application we show that there are no normalized units of order 6 in the integral group rings of M_{10} and $\mathrm{PGL}(2,9)$. This completes the proof of a theorem of Kimmerle and Konovalov that shows that the prime graph question has an affirmative answer for all groups having an order divisible by at most three different primes.

Keywords: integral group ring; torsion unit; Zassenhaus conjecture; prime graph question

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1. Introduction

Throughout this paper let G be a finite group, let $\mathbb{Z}G$ be the integral group ring of G and let $\mathrm{V}(\mathbb{Z}G)$ be the group of augmentation one units in $\mathbb{Z}G$, the so-called normalized units. The most famous open conjecture regarding torsion units in $\mathbb{Z}G$ is the (first) Zassenhaus conjecture.

The Zassenhaus conjecture (ZC). *Let $u \in \mathrm{V}(\mathbb{Z}G)$ be a torsion unit. Then there exist a unit $x \in \mathbb{Q}G$ and $g \in G$ such that $x^{-1}ux = g$.*

If for a unit u such x and g exist, we say that u is rationally conjugate to g. There are positive results for the Zassenhaus conjecture for classes of solvable groups (e.g. Weiss proved it for nilpotent groups [**33**] and Caicedo *et al*. established it for all cyclic-by-abelian groups [**5**]). For non-solvable groups it is only known for specific groups, e.g. for A_5 [**25**], S_5 [**26**], A_6 [**16**], or $\mathrm{PSL}(2,p)$ for $p \leqslant 17$ a prime [**11**, **15**, **24**].

Considering the difficulty of the Zassenhaus conjecture, and motivated by the results in [**22**], it was proposed in [**21**, Problem 21] to first study the following question.

The prime graph question (PQ). Let p and q be different primes such that $\mathrm{V}(\mathbb{Z}G)$ has a unit of order pq. Does this imply that G has an element of that order?

This is the same as to ask whether G and $\mathrm{V}(\mathbb{Z}G)$ have the same prime graph. Much more is known here; for example, this has an affirmative answer for all solvable groups [**22**] or the series $\mathrm{PSL}(2,p)$, with p a prime [**15**]. Bovdi and Konovalov with different collaborators obtained positive answers to (PQ) for many of the sporadic simple groups; see, for example, [**3**] for recent results. Recently, substantial progress was made when Kimmerle and Konovalov obtained the first reduction result for the prime graph question [**24**, Proposition 4.1] (see also [**23**, Theorem 2.1]).

Theorem (Kimmerle and Konovalov [24, Proposition 4.1]). *If (PQ) has an affirmative answer for all almost simple homomorphic images of G, then it also has an affirmative answer for G itself.*

Recall that a group A is almost simple if $S \leqslant A \leqslant \mathrm{Aut}(S)$ for a simple group S. Using the above theorem, they proved that the prime graph question has a positive answer for every finite group whose order is divisible by at most three different primes if it has a positive answer for M_{10}, the Mathieu group of degree 10, and $\mathrm{PGL}(2,9)$ [**23**, Theorem 3.1]. Their result also places special emphasis on investigating the prime graph question for almost simple groups.

All proofs of (ZC) for non-solvable groups rely on the so-called Hertweck–Luthar–Passi method [**15, 25**], referred to as the HeLP method, but in many cases this method does not suffice to prove (ZC); for example, it fails for A_6 [**16**], $\mathrm{PSL}(2,19)$ (see below) and M_{11} [**2**]. Sometimes special arguments were considered in such situations, as in [**26**], [**14**, Example 2.6] and [**16**], but these arguments were designed for very specific situations and are hard to generalize or do not seem to give new information in other situations.

In this paper we introduce a new method to study rational conjugacy of torsion units inspired by Hertweck's arguments for proving (ZC) for the alternating group of degree 6 [**16**]. This method is especially interesting for units of mixed order (i.e. not of prime power order) and in combination with the HeLP method. We then give two applications of this method to prove the following results.

Theorem 1.1. *The Zassenhaus conjecture holds for* $\mathrm{PSL}(2,19)$ *and* $\mathrm{PSL}(2,23)$.

Theorem 1.2. *Neither* $\mathrm{V}(\mathbb{Z}M_{10})$ *nor* $\mathrm{V}(\mathbb{Z}\,\mathrm{PGL}(2,9))$ *contain units of order* 6.

Theorem 1.2, together with [**23**, Theorems 2.1 and 3.1] (or [**24**, § 4]), directly yields the following.

Corollary 1.3. *Let G be a finite group. Suppose that the order of all almost simple homomorphic images of G is divisible by at most three different primes. Then the prime graph of the normalized units of $\mathbb{Z}G$ coincides with that of G. In particular, the prime graph question has a positive answer for all groups with an order divisible by at most three different primes.*

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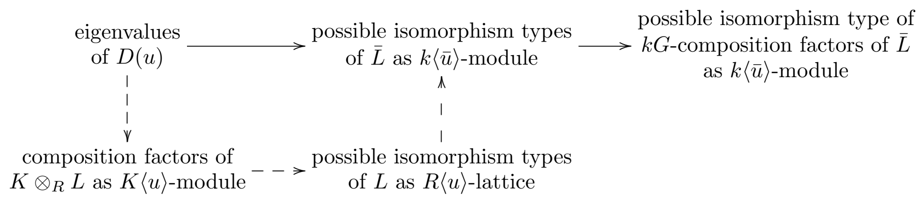

Figure 1. The idea of the method.

2. From eigenvalues under ordinary representations to the modular module structure

Let G be a finite group. The main tool for the study of rational conjugacy of torsion units is the notion of partial augmentations. Let $u = \sum_{g\in G} a_g g \in \mathbb{Z}G$ and let x^G be the conjugacy class of the element $x \in G$ in G. Then $\varepsilon_x(u) = \sum_{g\in x^G} a_g$ is called the partial augmentation of u at x. This relates to (ZC) via the following lemma.

Lemma 2.1 (Marciniak *et al.* [27, Theorem 2.5]). *Let $u \in \mathrm{V}(\mathbb{Z}G)$ be a torsion unit of order n. Then u is rationally conjugate to a group element if and only if $\varepsilon_x(u^d) \geqslant 0$ for all $x \in G$ and all powers u^d of u with $d \mid n$.*

It is well known that if $u \neq 1$ is a torsion unit in $\mathrm{V}(\mathbb{Z}G)$, then $\varepsilon_1(u) = 0$ by the so-called Berman–Higman theorem [**31**, Proposition 1.4]. If $\varepsilon_x(u) \neq 0$, then the order of x divides the order of u (see [**27**, Theorem 2.7], [**14**, Proposition 3.1]). Moreover, the exponents of G and of $\mathrm{V}(\mathbb{Z}G)$ coincide [**6**]. We will use this in the following without further mention.

Let K be a field, let D be a K-representation of G with corresponding character χ and let $u \in \mathrm{V}(\mathbb{Z}G)$ be a torsion unit of order n. If χ and all partial augmentations of u and all its powers are known, and the characteristic of K does not divide n, we can compute the eigenvalues of $D(u)$ in a field extension of K that is large enough (a field that is a splitting field for G and all its subgroups will be a good choice; there are plenty of examples for this kind of calculation in § 3). The HeLP method makes use of the fact that the multiplicity of each nth root of unity as an eigenvalue of $D(u)$ is a non-negative integer.

Notation. In this paper p will always denote a prime, $\mathbb{Q}_p$ always the p-adic completion of $\mathbb{Q}$, and $\mathbb{Z}_p$ always the ring of integers of $\mathbb{Q}_p$. By R we denote a complete local ring with maximal ideal P containing p, and by K we denote the field of fractions of R. Furthermore, k denotes a finite field of characteristic p containing the residue class field of R. The reduction modulo P, also with respect to lattices, will be denoted by a bar.

The idea of our method is that if D is an R-representation of a group G with corresponding RG-lattice L and u is a torsion unit in $\mathbb{Z}G$ of order divisible by p, we can reduce D modulo P and obtain restrictions on the isomorphism type of kG-modules considered as $k\langle\bar{u}\rangle$-modules, where k is big enough to allow realizations of all irreducible p-modular representations of G. Note that the Krull–Schmidt–Azumaya theorem holds for finitely generated RG-lattices [**7**, Theorem 30.6]. From these isomorphism types we can then obtain restrictions on the isomorphism types of the kG-composition factors of $\bar{L}$ when viewed as a $k\langle\bar{u}\rangle$-module. Since a simple kG-module may appear in the reduction of several ordinary representations, this may finally yield a contradiction to the existence of u. A rough sketch of the method is given in Figure 1. As the direct path from the eigenvalues of an ordinary representation to the isomorphism types of the corresponding reduced lattice is not always evident, we are sometimes forced to take the detour along the dashed arrows.

The connections between the eigenvalues of ordinary representations and the isomorphism type of the modules in positive characteristic for some cases are collected in the following propositions, which are consequences of known facts from modular and integral representation theory.

The first proposition is a standard fact in modular representation theory and may be found in, for example, [**19**, Theorems 5.3 and 5.5].

Proposition 2.2. *Let $G = \langle g\rangle$ be a cyclic group of order $p^a m$, where p does not divide m. Let k be a field of characteristic p containing a primitive mth root of unity ξ. Then the following hold.*

(a) *Up to isomorphism there are m simple kG-modules. All these modules are one dimensional as k-vector spaces, g^m acts trivially on them and g^{p^a} acts as ξ^i for $1 \leqslant i \leqslant m$. We denote these modules by $k^{\xi}, k^{\xi^2}, \ldots, k^{\xi^m}$.*

(b) *The projective indecomposable kG-modules are of dimension p^a. They are all uniserial and all composition factors of a projective indecomposable kG-module are isomorphic. There are m non-isomorphic projective indecomposable kG-modules.*

(c) *Each indecomposable kG-module is isomorphic to a submodule of a projective indecomposable module. So there are $p^a m$ indecomposable modules, which are all uniserial and all composition factors of an indecomposable kG-module are isomorphic.*

Using Proposition 2.2 and the fact that idempotents can be lifted [**7**, Theorem 30.4], we obtain the next proposition.

Proposition 2.3. *Let $G = \langle g\rangle$ be a cyclic group of order $p^a m$, where p does not divide m. Let R be a complete local ring containing a primitive mth root of unity ζ. Let D be an R-representation of G and let L be an RG-lattice affording this representation.*

Let A_i be sets with multiplicities of p^ath roots of unity such that $\zeta A_1 \cup \zeta^2 A_2 \cup \cdots \cup \zeta^m A_m$ are the complex eigenvalues of $D(g)$, where $A_i = \varnothing$ is possible. Let $V_1, \ldots, V_m$ be KG-modules such that if E_i is a representation of G affording V_i, the eigenvalues of $E_i(g)$ are $\zeta^i A_i$. Then

$$L \cong L^{\zeta^1} \oplus \cdots \oplus L^{\zeta^m} \quad \text{and} \quad \bar{L} \cong \bar{L}^{\zeta^1} \oplus \cdots \oplus \bar{L}^{\zeta^m}$$

such that $\operatorname{rank}_R(L^{\zeta^i}) = \dim_k(\bar{L}^{\zeta^i}) = |A_i|$. (The superscripts ζ^i are merely meant as indices.) Moreover, $K \otimes_R L^{\zeta^i} \cong V_i$ and the only composition factor of $\bar{L}^{\zeta^i}$ is k^{ξ^i} (see the notation in Proposition 2.2).

To understand the full connection between the eigenvalues $\zeta^i A_i$ and the structure of L^{ζ^i}, i.e. to follow the arrow in the second line of Figure 1, one must study the representation theory of $R\langle g^m \rangle$. The representation type of $R\langle g^m \rangle$ may be finite, tame or wild. Roughly speaking, the representation theory gets more complicated with increasing a and increasing ramification index of K over $\mathbb{Q}_p$. A listing of all representation types may be found in [**9**]. Some results concerning the connection between A_i and L^{ζ^i} are recorded in the next propositions. The first one is a consequence of [**13**, Theorem 2.6].

Proposition 2.4. *Let the notation be as in Proposition 2.3. Assume in addition that* $G \cong C_p$ *and that* K *is unramified over* $\mathbb{Q}_p$. *Let* γ *be a primitive pth root of unity. Up to isomorphism, there are three indecomposable RG-lattices* M_1, M_2, M_3. *Each* $\bar{M}_i$ *remains indecomposable. The R-rank and the corresponding eigenvalues of* $D(g)$ *are* $\operatorname{rank}_R(M_1) = 1$ *with eigenvalue* 1, $\operatorname{rank}_R(M_2) = p-1$ *with eigenvalues* $\gamma, \gamma^2, \ldots, \gamma^{p-1}$, *and* $\operatorname{rank}_R(M_3) = p$ *with eigenvalues* $1, \gamma, \gamma^2, \ldots, \gamma^{p-1}$.

Notation. We denote the indecomposable lattices in Proposition 2.4 by their natural names: the trivial lattice $M_1 = R$, the augmentation ideal $M_2 = I(RC_p)$ and the group ring $M_3 = RC_p$.

When considering RC_{2p}-lattices or kC_{2p}-modules as in Proposition 2.3 (i.e. $a = 1$ and $m = 2$), we abbreviate the superscripts 1 and -1 to $+$ and $-$, respectively, i.e. $L^+ = L^1$, $L^- = L^{-1}$, $\bar{L}^+ = \bar{L}^1$ and $\bar{L}^- = \bar{L}^{-1}$ for the direct summands having trivial and non-trivial composition factors, respectively.

Proposition 2.5. *Let the notation be as in Proposition 2.3, let* p *be odd and let* $p \equiv \delta \mod 4$ *with* $\delta \in \{\pm 1\}$. *Assume that* K *is the* p*-adic completion of an extension of* $\mathbb{Q}(\sqrt{\delta p})$ *that is unramified at* p. *Denote by* R *the ring of integers of* K *and let* $G \cong C_p$. *Note that there are exactly three simple RG-lattices up to isomorphism. The two following facts hold for indecomposable RG-lattices:*

(a) *if* L *is an indecomposable RG-lattice, then the non-trivial simple lattices each appear at most once as a composition factor of* L, *and the trivial one at most twice;*

(b) *if* L *is an indecomposable RG-lattice having at most two non-isomorphic composition factors, then each composition factor appears at most once.*

Proof. If K' is the p-adic completion of $\mathbb{Q}(\sqrt{\delta p})$ and R' is its ring of integers, then up to isomorphism all indecomposable $R'G$-lattices are explicitly given in [**12**, Lemma 4.1]. These lattices satisfy the statements of the proposition. If L is an indecomposable RG-lattice, it is a direct summand of $R \otimes_{R'} L'$, where L' is an indecomposable $R'G$-lattice, by the last paragraph of the proof of [**7**, Proposition 33.16]. So the statements of the proposition still hold for indecomposable RG-lattices. □

3. Applications

For a group G we denote by χ_i an ordinary character of G and by D_i a representation of G affording this character. By φ_i we denote a Brauer character and by Θ_i a representation affording φ_i. We write $D_i(u) \sim (\alpha_1, \ldots, \alpha_j)$ or $\Theta_i(u) \sim (\alpha_1, \ldots, \alpha_j)$ to indicate that $\alpha_1, \ldots, \alpha_j$ are the eigenvalues (with multiplicities) of the corresponding matrix. To improve readability, we sometimes group the eigenvalues appearing several times: for example,

$$D_i(u) \sim \left(3 \times \boxed{1}, 2 \times \boxed{\zeta, \zeta^{-1}}\right)$$

indicates that $D_i(u)$ has 1 three times and ζ and ζ^{-1} each twice as eigenvalues. By ζ_n we will denote some fixed primitive complex nth root of unity. In particular, we will use ζ_n to denote the eigenvalues of a matrix of finite order n over a field of characteristic p, where p is coprime to n, in the sense of Brauer, as presented, for example, in [**20**, Chapter 2, § 17].

Let K be an algebraically closed field, let D be a K-representation of G with character χ and let u be a torsion unit in $\mathrm{V}(\mathbb{Z}G)$ such that the characteristic of K does not divide the order of u. Let m and n be natural numbers such that $u^{m+n} = u$. Let $D(u^m) \sim (\alpha_1, \ldots, \alpha_\ell)$ and $D(u^n) \sim (\beta_1, \ldots, \beta_\ell)$. As $D(u^m)$ and $D(u^n)$ are simultaneously diagonalizable over K, this means that $D(u) \sim (\alpha_1\beta_{i_1}, \ldots, \alpha_\ell\beta_{i_\ell})$ with $\{i_1, \ldots, i_\ell\} = \{1, \ldots, \ell\}$. On the other hand, if X denotes a set of representatives of conjugacy classes of G, then $\chi(u) = \sum_{x \in X} \varepsilon_x(u)\chi(x)$. Comparing the values for $\chi(u)$ obtained in these two ways is the basic idea of the HeLP method.

3.1. The groups $\mathrm{PSL}(2, q)$ and proof of Theorem 1.1

Rational conjugacy of torsion units in integral group rings of the groups $\mathrm{PSL}(2, p^f)$ were studied by Hertweck in [**15**]. The next proposition summarizes some results from that article.

Proposition 3.1 (Hertweck [15]). *Let $G = \mathrm{PSL}(2, p^f)$ and let u be a torsion unit in $\mathrm{V}(\mathbb{Z}G)$.*

(a) *If u is of order prime to p, there exists an element in G of the same order as u. If, moreover, the order of u is prime, u is rationally conjugate to an element in G.*

(b) *If $f = 1$ and p divides the order of u, then u is of order p and rationally conjugate to an element in G.*

(c) *Assume that $p \notin \{2, 3\}$ and u is of order 6. Then u is rationally conjugate to an element in G.*

Proof. See [**15**, Propositions 6.1, 6.3, 6.4, 6.6 and 6.7]. □

The HeLP method verifies the Zassenhaus conjecture for $\mathrm{PSL}(2, p)$ if $p \leqslant 17$. We give a quick account. (ZC) was already verified for $p = 2$ in [**17**], $p = 3$ in [**1**], $p = 5$ in [**25**], $p = 7$ in [**14**], $p \in \{11, 13\}$ in [**15**] and $p = 17$ independently in [**24**] and [**11**]. The HeLP method also suffices to prove (ZC) for $p = 23$ (see below), but not for $p = 19$. We will always use the character tables and Brauer tables from ATLAS [**34**]*. We will use throughout the GAP notation for conjugacy classes.

For $G = \mathrm{PSL}(2, p^f)$ and $p > 2$ we have $|G| = (p^f - 1)p^f(p^f + 1)/2$, there are cyclic subgroups of order $(p^f - 1)/2$, p and $(p^f + 1)/2$ in G, and every cyclic subgroup of G lies in a conjugate of such a subgroup. There are two conjugacy classes of elements of order p and, if g is an element of order prime to p, the only conjugate of g in $\langle g \rangle$ is g^{-1}. All of this follows from a result of Dickson [**18**, Satz 8.27].

We first list the results that can be obtained using solely the HeLP method to distinguish those from the results involving the new method.

Lemma 3.2. *Let p be a prime, let $f \in \mathbb{N}$ and set $G = \mathrm{PSL}(2, p^f)$.*

(a) *If $p \notin \{2, 3\}$, then elements in $\mathrm{V}(\mathbb{Z}G)$ of order 4, 9 or 12 are rationally conjugate to group elements.*

(b) *If $p \notin \{2, 5\}$ and $u \in \mathrm{V}(\mathbb{Z}G)$ is of order 10, then either u is rationally conjugate to a group element or it has the following partial augmentations:*

$$(\varepsilon_{\mathtt{2a}}(u), \varepsilon_{\mathtt{5a}}(u), \varepsilon_{\mathtt{5b}}(u), \varepsilon_{\mathtt{10a}}(u), \varepsilon_{\mathtt{10b}}(u)) = (0, 1, -1, 1, 0)$$

if u^2 is rationally conjugate to an element in `5a`*, or*

$$(\varepsilon_{\mathtt{2a}}(u), \varepsilon_{\mathtt{5a}}(u), \varepsilon_{\mathtt{5b}}(u), \varepsilon_{\mathtt{10a}}(u), \varepsilon_{\mathtt{10b}}(u)) = (0, -1, 1, 0, 1)$$

if u^2 is rationally conjugate to an element in `5b`*. The conjugacy classes are listed in such a way that squares of elements in* `10b` *are lying in* `5a`*.*

Proof. We will use the representations given in [**15**] and explicitly proved in [**28**, Lemma 1.2], namely, if a is an element of order $(p^f + 1)/2$ and b is an element of order $(p^f - 1)/2$ in G, then there is a primitive $(p^f + 1)/2$th root of unity α and a primitive $(p^f - 1)/2$th root of unity β such that for every $i \in \mathbb{N}_0$ there exists a p-modular representation Θ_i of G with character φ_i such that

$$\Theta_i(a) \sim (1, \alpha, \alpha^{-1}, \alpha^2, \alpha^{-2}, \ldots, \alpha^i, \alpha^{-i}),$$
$$\Theta_i(b) \sim (1, \beta, \beta^{-1}, \beta^2, \beta^{-2}, \ldots, \beta^i, \beta^{-i}).$$

For convenience, the relevant parts of the characters φ_i are collected in Tables 1 and 2, where dashes indicate zeroes.

Let $p \notin \{2, 3\}$. If $u \in \mathrm{V}(\mathbb{Z}G)$ is of order 4, then $\varepsilon_{\mathtt{2a}}(u) = 0$ by [**15**, Proposition 6.5]. Thus, $\varepsilon_{\mathtt{4a}}(u) = 1$ and u is rationally conjugate to a group element.

Assume that $u \in \mathrm{V}(\mathbb{Z}G)$ is of order 9. Then $\varepsilon_{\mathtt{3a}}(u) = 0$ by [**15**, Proposition 6.5]. Let `9a`, `9b` and `9c` denote the conjugacy classes of elements of order 9 in G and $x \in G$ such that $x \in$ `9a`, $x^2 \in$ `9b` and $x^4 \in$ `9c`. Then

$$\varepsilon_{\mathtt{9a}}(u) + \varepsilon_{\mathtt{9b}}(u) + \varepsilon_{\mathtt{9c}}(u) = 1.$$

* All tables used in this paper are accessible in GAP [**10**] via the package [**4**] using the commands `CharacterTable("G");` and `CharacterTable("G") mod p;`, where `G` is the identifier of the group, e.g. `PSL(2,19)` or `M10`. The corresponding decomposition matrix for a Brauer table can then be obtained by using `DecompositionMatrix`.

Table 1. *Parts of some p-Brauer-characters of $G = \mathrm{PSL}(2, p^f)$ for $p \notin \{2, 3\}$ and $24 \mid |G|$.*

	1a	2a	3a	4a	6a	12a	12b
φ_1	3	-1	—	1	2	$1+\zeta_{12}+\zeta_{12}^{-1}$	$1-\zeta_{12}-\zeta_{12}^{-1}$
φ_2	5	1	-1	-1	1	$2+\zeta_{12}+\zeta_{12}^{-1}$	$2-\zeta_{12}-\zeta_{12}^{-1}$
φ_3	7	-1	1	-1	-1	$2+\zeta_{12}+\zeta_{12}^{-1}$	$2-\zeta_{12}-\zeta_{12}^{-1}$
φ_5	11	-1	-1	1	-1	1	1

Table 2. *Part of some p-Brauer-characters of $G = \mathrm{PSL}(2, p^f)$ with $p \notin \{2, 5\}$ and $20 \mid |G|$ with $\alpha = \zeta_5 + \zeta_5^4$, $\beta = \zeta_5^2 + \zeta_5^3$.*

	1a	2a	5a	5b	10a	10b
φ_1	3	-1	$-\beta$	$-\alpha$	$-2\alpha-\beta$	$-\alpha-2\beta$
φ_2	5	1	—	—	-2α	-2β

Let ζ be a primitive complex 9th root of unity such that $\Theta_1(x) \sim (1, \zeta, \zeta^8)$. Since $\Theta_1(u^3) \sim (1, \zeta^3, \zeta^6)$ and φ_1 is real valued, we get $\Theta_1(u) \sim (1, \gamma, \delta)$ with $(\gamma, \delta) \in \{(\zeta, \zeta^8), (\zeta^2, \zeta^7), (\zeta^4, \zeta^5)\}$. So

$$(1+\zeta+\zeta^8)\varepsilon_{\mathtt{9a}}(u) + (1+\zeta^2+\zeta^7)\varepsilon_{\mathtt{9b}}(u) + (1+\zeta^4+\zeta^5)\varepsilon_{\mathtt{9c}}(u) \in \{1+\zeta+\zeta^8, 1+\zeta^2+\zeta^7, 1+\zeta^4+\zeta^5\}.$$

Using ζ^2, ζ^3, ζ^4, ζ^5, ζ^6, ζ^7 as a $\mathbb{Z}$-basis of $\mathbb{Z}[\zeta]$ (see [**29**, Chapter 1, (10.2) Proposition]), this gives

$$(-\varepsilon_{\mathtt{9b}}(u) + \varepsilon_{\mathtt{9c}}(u), \varepsilon_{\mathtt{9a}}(u) - \varepsilon_{\mathtt{9b}}(u)) \in \{(-1, -1), (1, 0), (0, 1)\}.$$

Combining each of these possibilities with $\varepsilon_{9a}(u) + \varepsilon_{9b}(u) + \varepsilon_{9c}(u) = 1$, we get

$$(\varepsilon_{\mathtt{9a}}(u), \varepsilon_{\mathtt{9b}}(u), \varepsilon_{\mathtt{9c}}(u)) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

Thus u is rationally conjugate to a group element. This is also a consequence of [**28**, Theorem 1].

Now assume that u is of order 12. Then G contains an element of order 12 by Proposition 3.1. So let 2a, 3a, 4a, 6a, 12a, 12b be the conjugacy classes with potentially non-vanishing partial augmentations for u. Let ζ be a primitive 12th root of unity such that $\Theta_1(\mathtt{12a}) \sim (1, \zeta, \zeta^{11})$.

We will use ζ, ζ^4, ζ^8, ζ^{11} as a $\mathbb{Z}$-basis of $\mathbb{Z}[\zeta]$. (This is a basis since $\varphi(12) = 4$, where φ denotes Euler's totient function, and $1 = -\zeta^4 - \zeta^8$, $\zeta^2 = -\zeta^8$, $\zeta^3 = \zeta - \zeta^{11}$, $\zeta^5 = -\zeta^{11}$, $\zeta^6 = -1 = \zeta^4 + \zeta^8$, $\zeta^7 = -\zeta$, $\zeta^9 = -\zeta + \zeta^{11}$ and $\zeta^{10} = -\zeta^4$.) We have

$$\varepsilon_{\mathtt{2a}}(u) + \varepsilon_{\mathtt{3a}}(u) + \varepsilon_{\mathtt{4a}}(u) + \varepsilon_{\mathtt{6a}}(u) + \varepsilon_{12a}(u) + \varepsilon_{12b}(u) = 1. \quad (3.1)$$

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Furthermore, $\Theta_1(u^9) \sim (1, \zeta^3, \zeta^9)$ and $\Theta_1(u^4) \sim (1, \zeta^4, \zeta^8)$. Thus, as φ_1 has only real values, $\Theta_1(u) \sim X$ with $X \in \{(1, \zeta^5, \zeta^7), (1, \zeta, \zeta^{11})\} = \{(1, -\zeta^{11}, -\zeta), (1, \zeta, \zeta^{11})\}$. Hence, using Table 1, we obtain

$$- \varepsilon_{2a}(u) + \varepsilon_{4a}(u) + 2\varepsilon_{6a}(u) + (1 + \zeta + \zeta^{11})\varepsilon_{12a}(u) + (1 - \zeta - \zeta^{11})\varepsilon_{12b}(u) \in \{1 + \zeta + \zeta^{11}, 1 - \zeta - \zeta^{11}\}.$$

Again using ζ, ζ^4, ζ^8, ζ^{11} as a basis of $\mathbb{Z}[\zeta]$, this gives

$$\varepsilon_{12a}(u) - \varepsilon_{12b}(u) = \pm 1, \tag{3.2}$$
$$-\varepsilon_{2a}(u) + \varepsilon_{4a}(u) + 2\varepsilon_{6a}(u) + \varepsilon_{12a}(u) + \varepsilon_{12b}(u) = 1. \tag{3.3}$$

Proceeding the same way, we have

$$\Theta_2(u^9) \sim (1, -1, -1, \zeta^3, \zeta^9), \quad \Theta_2(u^4) \sim (1, \zeta^4, \zeta^8, \zeta^4, \zeta^8) \quad \text{and} \quad \Theta_2(u) \sim X$$

with $X \in \{(1, \zeta^2, \zeta^{10}, \zeta, \zeta^{11}), (1, \zeta^2, \zeta^{10}, \zeta^5, \zeta^7)\}$. So, by Table 1 and $\zeta^2 + \zeta^{10} = 1$, we get

$$\varepsilon_{2a}(u) - \varepsilon_{3a}(u) - \varepsilon_{4a}(u) + \varepsilon_{6a}(u) + (2 + \zeta + \zeta^{11})\varepsilon_{12a}(u) + (2 - \zeta - \zeta^{11})\varepsilon_{12b}(u) \in \{2 + \zeta + \zeta^{11}, 2 - \zeta - \zeta^{11}\}.$$

Comparing coefficients of ζ^4 gives

$$\varepsilon_{2a}(u) - \varepsilon_{3a}(u) - \varepsilon_{4a}(u) + \varepsilon_{6a}(u) + 2\varepsilon_{12a}(u) + 2\varepsilon_{12b}(u) = 2. \tag{3.4}$$

Applying the same for φ_3, we obtain $\Theta_3(u^9) \sim (1, -1, -1, \zeta^3, \zeta^9, \zeta^3, \zeta^9)$, $\Theta_3(u^4) \sim (1, \zeta^4, \zeta^8, 1, \zeta^4, \zeta^8, 1)$ and $\Theta_3(u) \sim (X)$ with

$$X \in \{(1, -1, -1, \zeta, \zeta^{11}, \zeta, \zeta^{11}), (1, -1, -1, \zeta, \zeta^{11}, \zeta^5, \zeta^7), (1, -1, -1, \zeta^5, \zeta^7, \zeta^5, \zeta^7), (1, \zeta^2, \zeta^{10}, \zeta^3, \zeta^9, \zeta, \zeta^{11}), (1, \zeta^2, \zeta^{10}, \zeta^3, \zeta^9, \zeta^5, \zeta^7)\}.$$

So, by Table 1, $\zeta^2 + \zeta^{10} = 1$ and $\zeta^3 + \zeta^9 = 0$, and we get

$$\varepsilon_{2a}(u) + \varepsilon_{3a}(u) - \varepsilon_{4a}(u) - \varepsilon_{6a}(u) + (2 + \zeta + \zeta^{11})\varepsilon_{12a}(u) + (2 - \zeta - \zeta^{11})\varepsilon_{12b}(u) \in \{-1 + 2\zeta + 2\zeta^{11}, -1, -1 - 2\zeta - 2\zeta^{11}, 2 + \zeta + \zeta^{11}, 2 - \zeta - \zeta^{11}\}.$$

As the first three possibilities would give $\varepsilon_{12a}(u) - \varepsilon_{12b}(u) \in \{-2, 0, 2\}$, contradicting (3.2), only the last two remain and give

$$-\varepsilon_{2a}(u) + \varepsilon_{3a}(u) - \varepsilon_{4a}(u) - \varepsilon_{6a}(u) + 2\varepsilon_{12a}(u) + 2\varepsilon_{12b}(u) = 2. \tag{3.5}$$

In the same way,

$$\Theta_5(u^6) \sim \left(5 \times \boxed{1}, 6 \times \boxed{-1}\right), \qquad \Theta_5(u^9) \sim \left(3 \times \boxed{1}, 2 \times \boxed{-1}, 3 \times \boxed{\zeta^3, \zeta^9,}\right),$$
$$\Theta_5(u^4) \sim \left(3 \times \boxed{1}, 4 \times \boxed{\zeta^4, \zeta^8}\right), \qquad \Theta_5(u) \sim (1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5, \zeta^7, \zeta^8, \zeta^9, \zeta^{10}, \zeta^{11})$$

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(note that $\varphi_5(u)$ has to be even rational, as φ_5 has only rational values). Thus $-\varepsilon_{2a}(u) - \varepsilon_{3a}(u) + \varepsilon_{4a}(u) - \varepsilon_{6a}(u) + \varepsilon_{12a}(u) + \varepsilon_{12b}(u) = 1$, giving

$$-\varepsilon_{2a}(u) - \varepsilon_{3a}(u) + \varepsilon_{4a}(u) - \varepsilon_{6a}(u) + \varepsilon_{12a}(u) + \varepsilon_{12b}(u) = 1. \tag{3.6}$$

Now, subtracting (3.1) from (3.6) gives $\varepsilon_{2a}(u) + \varepsilon_{3a}(u) + \varepsilon_{6a}(u) = 0$, while subtracting (3.4) from (3.5) gives $\varepsilon_{2a}(u) - \varepsilon_{3a}(u) + \varepsilon_{6a}(u) = 0$. Thus $\varepsilon_{3a}(u) = 0$. Then subtracting (3.1) from (3.3) gives $-2\varepsilon_{2a}(u) + \varepsilon_{6a}(u) = 0$, so $\varepsilon_{2a}(u) = \varepsilon_{6a}(u) = 0$. Now multiplying (3.1) by 2 and subtracting it from (3.4) gives $\varepsilon_{4a}(u) = 0$. Using (3.1) and (3.2), this leaves only the trivial possibilities $(\varepsilon_{2a}(u), \varepsilon_{3a}(u), \varepsilon_{4a}(u), \varepsilon_{6a}(u), \varepsilon_{12a}(u), \varepsilon_{12b}(u)) \in \{(0,0,0,0,1,0), (0,0,0,0,0,1)\}$.

For part (b) assume that $p \notin \{2,5\}$, $u \in \mathrm{V}(\mathbb{Z}G)$ is of order 10 and let ζ be a primitive 5th root of unity such that $\Theta_1(\mathtt{10a}) \sim (1, -\zeta, -\zeta^4)$. Assume furthermore that u^2 is rationally conjugate to an element in 5a.

We have

$$\varepsilon_{2a}(u) + \varepsilon_{5a}(u) + \varepsilon_{5b}(u) + \varepsilon_{10a}(u) + \varepsilon_{10b}(u) = 1.$$

Furthermore, $\Theta_1(u^5) \sim (1, -1, -1)$ and $\Theta_1(u^6) \sim (1, \zeta^2, \zeta^3)$. As φ_1 has only real values, we get $\Theta_1(u) \sim (1, -\zeta^2, -\zeta^3)$. Thus

$$\begin{aligned} -\varepsilon_{2a}(u) + (-\zeta^2 - \zeta^3)\varepsilon_{5a}(u) + (-2\zeta - \zeta^2 - \zeta^3 - 2\zeta^4)\varepsilon_{10a}(u) \\ + (-\zeta - \zeta^4)\varepsilon_{5b}(u) + (-\zeta - 2\zeta^2 - 2\zeta^3 - \zeta^4)\varepsilon_{10b}(u) = 1 - \zeta^2 - \zeta^3. \end{aligned}$$

Using ζ, ζ^2, ζ^3, ζ^4 as a $\mathbb{Z}$-basis of $\mathbb{Z}[\zeta]$, we obtain

$$\begin{aligned} \varepsilon_{2a}(u) - \varepsilon_{5b}(u) - 2\varepsilon_{10a}(u) - \varepsilon_{10b}(u) &= -1, \\ \varepsilon_{2a}(u) - \varepsilon_{5a}(u) - \varepsilon_{10a}(u) - 2\varepsilon_{10b}(u) &= -2. \end{aligned}$$

In the same way we get $\Theta_2(u^5) \sim (1,1,1,-1,-1)$, $\Theta_2(u^6) \sim (1, \zeta, \zeta^2, \zeta^3, \zeta^4)$ and $\Theta_2(u) \sim X$ with $X \in \{(1, -\zeta, \zeta^2, \zeta^3, -\zeta^4), (1, \zeta, -\zeta^2, -\zeta^3, \zeta^4)\}$. We have $\varphi_2(u) = \varepsilon_{2a}(u) - 2(\zeta + \zeta^4)\varepsilon_{10a}(u) - 2(\zeta^2 + \zeta^3)\varepsilon_{10b}(u)$. Hence

$$(-\varepsilon_{2a}(u) - 2\varepsilon_{10a}(u), -\varepsilon_{2a}(u) - 2\varepsilon_{10b}(u)) \in \{(-2, 0), (0, -2)\}.$$

Combining these equations with the equations obtained above, we get

$$(\varepsilon_{2a}(u), \varepsilon_{5a}(u), \varepsilon_{5b}(u), \varepsilon_{10a}(u), \varepsilon_{10b}(u)) \in \{(0, 1, -1, 1, 0), (0, 0, 0, 0, 1)\}.$$

If u^2 is rationally conjugate to an element in 5b, then replacing every ζ^i with ζ^{2i} and doing the same computations as above gives the result. □

Proof of Theorem 1.1. By Proposition 3.1, to obtain the Zassenhaus conjecture for $G = \mathrm{PSL}(2, 23)$ only elements of orders 4 and 12 in $\mathrm{V}(\mathbb{Z}G)$ need to be checked and these are rationally conjugate to group elements by Lemma 3.2. So from now on let $G = \mathrm{PSL}(2, 19)$. Then by Proposition 3.1 only elements of orders 9 and 10 need to be checked, but elements of order 9 were already handled in Lemma 3.2. So assume that

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Table 3. *Part of the ordinary character table of* PSL(2, 19) *with* $\alpha = \zeta_5 + \zeta_5^4$, $\beta = \zeta_5^2 + \zeta_5^3$.

	1a	2a	5a	5b	10a	10b
χ_{18}	18	-2	$-\alpha$	$-\beta$	$-\alpha$	$-\beta$
χ_{19}	19	-1	-1	-1	-1	-1

Table 4. *Part of the Brauer table and decomposition matrix of* PSL(2, 19) *for the prime* 5.

(a) *Part of the Brauer table*

	1a	2a
φ_1	1	1
φ_{18}	18	-2

(b) *Part of the decomposition matrix*

	φ_1	φ_{18}
χ_1	1	—
χ_{18}	—	1
χ_{19}	1	1

$u \in \mathrm{V}(\mathbb{Z}G)$ is of order 10 and not rationally conjugate to a group element. If u is not rationally conjugate to an element of G, then also u^3 is not rationally conjugate to an element in G. Furthermore, if u^2 is rationally conjugate to an element in 5a, then u^6 is rationally conjugate to an element in 5b. So we may assume that u^2 is conjugate to an element in 5a, and by Lemma 3.2 we get

$$(\varepsilon_{2\mathtt{a}}(u), \varepsilon_{5\mathtt{a}}(u), \varepsilon_{5\mathtt{b}}(u), \varepsilon_{10\mathtt{a}}(u), \varepsilon_{10\mathtt{b}}(u)) = (0, 1, -1, 1, 0).$$

We give the parts of the character tables relevant for our proof in Tables 3 and 4.

Let D_{18} and D_{19} be representations affording the characters χ_{18} and χ_{19} given in Table 3. We compute the eigenvalues of $D_{18}(u)$ and $D_{19}(u)$ using the character table in the way demonstrated above. We have

$$D_{18}(u^5) \sim \left(8 \times \boxed{1}, 10 \times \boxed{-1}\right) \quad \text{and} \quad D_{18}(u^6) \sim \left(3 \times \boxed{1, \zeta, \zeta^2, \zeta^3, \zeta^4}, 1 \times \boxed{1, \zeta, \zeta^4}\right).$$

Since $\chi_{18}(u) = \chi_{18}(\mathtt{5a}) - \chi_{18}(\mathtt{5b}) + \chi_{18}(\mathtt{10a}) = -2(\zeta + \zeta^4) + (\zeta^2 + \zeta^3)$, we obtain

$$D_{18}(u) \sim (1, \zeta, \zeta^2, \zeta^3, \zeta^4, 1, \zeta^2, \zeta^3, -1, -\zeta, -\zeta^2, -\zeta^3, -\zeta^4, -1, -\zeta, -\zeta^4, -\zeta, -\zeta^4).$$

Moreover,

$$D_{19}(u^5) \sim \left(9 \times \boxed{1}, 10 \times \boxed{-1}\right) \quad \text{and} \quad D_{19}(u^6) \sim \left(3 \times \boxed{1}, 4 \times \boxed{\zeta, \zeta^2, \zeta^3, \zeta^4}\right).$$

Since $\chi_{19}(u) = -1$, we get

$$D_{19}(u) \sim (1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta, \zeta^2, \zeta^3, \zeta^4, -1, -\zeta, -\zeta^2, -\zeta^3, -\zeta^4, -1, -\zeta, -\zeta^2, -\zeta^3, -\zeta^4).$$

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By [**32**], the Schur indices of all irreducible representations of G are 1 and we may thus assume that D_{19} is a $\mathbb{Q}_5$-representation while D_{18} is a K-representation, where K is the 5-adic completion of $\mathbb{Q}(\zeta_9 + \zeta_9^{-1}, \zeta_5 + \zeta_5^{-1}) = \mathbb{Q}(\zeta_9 + \zeta_9^{-1}, \sqrt{5})$. For both these fields, the rings of integers are principal ideal domains, so by [**7**, Proposition 23.16] we may assume that D_{19} is a $\mathbb{Z}_5$-representation and D_{18} is an R-representation, where R denotes the ring of integers of K. Let L_{19} and L_{18} respectively be a $\mathbb{Z}_5G$-lattice and an RG-lattice affording these representations. As usual, denote by a bar the reduction modulo the maximal ideal of $\mathbb{Z}_5$ and the maximal ideal of R. Denote by k a field of characteristic 5 that contains $\bar{\mathbb{Z}}_5$ and $\bar{R}$ and affords all irreducible 5-modular representations of G.

We may assume that $\bar{L}_{19}$ contains $\bar{L}_{18}$ as a submodule (multiplying a module by the augmentation ideal $I(kG)$ annihilates precisely the trivial kG-submodules). $\bar{L}_{19}/\bar{L}_{18}$ is a trivial kG-module, and so also a trivial $k\langle\bar{u}\rangle$-module. By Proposition 2.3, as $\mathbb{Z}_5\langle u\rangle$-lattice and as $R\langle u\rangle$-lattice we may respectively write $L_{19} \cong L_{19}^{+} \oplus L_{19}^{-}$ and $L_{18} \cong L_{18}^{+} \oplus L_{18}^{-}$ such that the composition factors of $\bar{L}_{\mathtt{i}}^{+}$ are all trivial and the composition factors of $\bar{L}_{\mathtt{i}}^{-}$ are all non-trivial as $k\langle\bar{u}\rangle$-modules for $\mathtt{i} \in \{18, 19\}$. As $\bar{L}_{19}/\bar{L}_{18}$ is a trivial module, we have $\bar{L}_{18}^{-} \cong \bar{L}_{19}^{-}$ (as $k\langle\bar{u}\rangle$-modules).

By the computations above, the eigenvalues of $D_{19}(u)$, which are not 5th roots of unity, i.e. which contribute to L_{19}^{-} by Proposition 2.3, are

$$(-1, -\zeta, -\zeta^2, -\zeta^3, -\zeta^4, -1, -\zeta, -\zeta^2, -\zeta^3, -\zeta^4).$$

Recall that we denote by $(\mathbb{Z}_5)^{-}$, $I(\mathbb{Z}_5C_5)^{-}$ and $(\mathbb{Z}_5C_5)^{-}$ the indecomposable $\mathbb{Z}_5C_{10}$-lattices of rank 1, 4 and 5 respectively, which have non-trivial composition factors (see Propositions 2.3 and 2.4). By Proposition 2.4 the eigenvalues imply that $L_{19}^{-} \cong X$ with

$$X \in \{2(\mathbb{Z}_5)^{-} \oplus 2I(\mathbb{Z}_5C_5)^{-}, (\mathbb{Z}_5)^{-} \oplus I(\mathbb{Z}_5C_5)^{-} \oplus (\mathbb{Z}_5C_5)^{-}, 2(\mathbb{Z}_5C_5)^{-}\}.$$

In any case, $\bar{L}_{19}^{-}$ has two indecomposable summands of k-dimension at least 4, as indecomposable summands of X stay indecomposable after reduction by Proposition 2.4.

On the other hand, the eigenvalues of $D_{18}(u)$ that are not 5th roots of unity are

$$(-1, -\zeta, -\zeta^2, -\zeta^3, -\zeta^4, -1, -\zeta, -\zeta^4, -\zeta, -\zeta^4).$$

Note that the simple $R\langle u\rangle$-lattice S affording the eigenvalues $(-\zeta^2, -\zeta^3)$ appears exactly once as a composition factor of L_{18}^{-}. Let $L_{18}^{-} \cong Y \oplus Z$ such that Y is indecomposable and S is a composition factor of Y. There are at most two non-isomorphic simple $R\langle u\rangle$-lattices involved in Z, namely, the one affording eigenvalues $(-\zeta, -\zeta^4)$ and the one affording the eigenvalue -1. Hence, by Proposition 2.5, the maximal R-rank of an indecomposable summand of Z is 3. Again by Proposition 2.5 both simple lattices corresponding to the eigenvalues $(-\zeta, -\zeta^4)$ and $(-\zeta^2, -\zeta^3)$, which both have R-rank 2, appear each at most once as a composition factor of Y, while the simple lattice corresponding to the eigenvalue -1 appears at most twice. Thus the maximal R-rank of Y is 6. So in any case $\bar{L}_{18}^{-}$ does not posses two indecomposable direct summands of dimension at least 4, but since the Krull–Schmidt–Azumaya theorem holds, we obtain a contradiction to $\bar{L}_{18}^{-} \cong \bar{L}_{19}^{-}$ and the above paragraph. □

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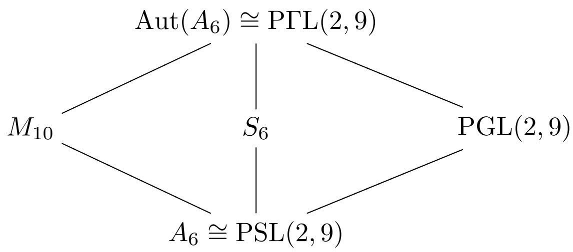

Figure 2. Almost simple groups containing A_6. Indices of consecutive subgroups are 2.

Remark 3.3. Note that the proof of the Zassenhaus conjecture for PSL(2, 23) only uses the HeLP method. Naturally, the next candidate to study the Zassenhaus conjecture for PSL(2, p) would be PSL(2, 29). After applying the HeLP method for this group, units of order 14 remain critical. Using the method from above would in any case involve RC_7-lattices, where R denotes the ring of integers of $\mathbb{Q}_7(\zeta_7 + \zeta_7^{-1})$. Since the representation type of RC_7 is wild (see [**9**]) this seems hopeless using only theoretical arguments. However, the degrees of the representations involved are at most 30, so a computational approach seems to be feasible. This is, however, not a part of this paper.

3.2. Proof of Theorem 1.2

Let $G = \mathrm{Aut}(A_6)$, the automorphism group of the alternating group of degree 6. Both M_{10} and PGL(2, 9) are subgroups of index 2 in G (see Figure 2). There is a unique conjugacy class `3a` of elements of order 3 in G (it is the union of the two conjugacy classes of elements of order 3 in A_6 and has length 80). This class is also the conjugacy class of elements of order 3 in M_{10} and PGL(2, 9). Furthermore, there is a conjugacy class of G consisting of all the involutions in A_6 of length 45, which we will denote by `2a`. This is clearly also a conjugacy class of M_{10} and PGL(2, 9). Let u be a unit of order 6 in $\mathrm{V}(\mathbb{Z}M_{10})$ or in $\mathrm{V}(\mathbb{Z}\,\mathrm{PGL}(2,9))$. By [**23**, Proof of Theorem 3.1], if such a unit exists, then all partial augmentations of u vanish except at the classes `2a` and `3a`, and then $(\varepsilon_{\mathtt{2a}}(u), \varepsilon_{\mathtt{3a}}(u)) = (-2, 3)$. We will first compute in Aut(A_6) and then obtain contradictions to the existence of u via restriction to M_{10} and PGL(2, 9).

The relevant parts of the character tables* of G are given in Table 5, the corresponding decomposition matrix in Table 6. By $\chi_{\mathtt{10}}$ we denote the irreducible character of degree 10 that contains a trivial constituent after reduction modulo 3.

Denote by ζ a complex primitive 3rd root of unity. Using the HeLP method and the fact that each $\chi_{\mathtt{i}}$ is real valued, we obtain

$$D_{\mathtt{10}}(u) \sim \left(2 \times \boxed{1}, 2 \times \boxed{\zeta, \zeta^2}, 2 \times \boxed{-1}, 1 \times \boxed{-\zeta, -\zeta^2}\right),$$
$$D_{\mathtt{20}}(u) \sim \left(8 \times \boxed{1}, 6 \times \boxed{-\zeta, -\zeta^2}\right).$$

* These tables can be obtained in their entirety in GAP by calling `CharacterTable("A6.2^2");` and `CharacterTable("A6.2^2") mod 3;` respectively.

Table 5. *Parts of the ordinary character table and Brauer table for the prime* 3 *for the group* Aut(A_6).

(a) *Part of the ordinary character table*

	1a	2a	3a
χ_{1a}	1	1	1
χ_{1b}	1	1	1
χ_{10}	10	2	1
χ_{20}	20	−4	2

(b) *Part of the Brauer table for* $p = 3$

	1a	2a	5a	2b
φ_{1a}	1	1	1	1
φ_{1b}	1	1	1	−1
φ_{6a}	6	−2	1	—
φ_{6b}	6	−2	1	—
φ_{8}	8	—	−2	—

Table 6. *Part of the decomposition matrix of* Aut(A_6) *for the prime* 3.

	φ_{1a}	φ_{1b}	φ_{6a}	φ_{6b}	φ_{8}
χ_{1a}	1	—	—	—	—
χ_{1b}	—	1	—	—	—
χ_{10}	1	1	—	—	1
χ_{20}	—	—	1	1	1

This can be computed in the way demonstrated above. As u^4 is rationally conjugate to an element in 3a and u^3 is rationally conjugate to an element in 2a, we have $\chi_{10}(u^4) = \chi_{10}(\mathtt{3a}) = 1$ and $\chi_{10}(u^3) = \chi_{10}(\mathtt{2a}) = 2$. This gives

$$D_{10}(u^4) \sim \left(4 \times \boxed{1}, 3 \times \boxed{\zeta, \zeta^2}\right) \quad \text{and} \quad D_{10}(u^3) \sim \left(6 \times \boxed{1}, 4 \times \boxed{-1}\right).$$

Now $\chi_{10}(u) = \varepsilon_{\mathtt{2a}}(u)\chi_{10}(\mathtt{2a}) + \varepsilon_{\mathtt{3a}}(u)\chi_{10}(\mathtt{3a}) = -1$ and, as the eigenvalues of $D_{10}(u)$ are products of the eigenvalues of $D_{10}(u^4)$ and $D_{10}(u^3)$, this gives the stated eigenvalues.

Moreover, we have $\chi_{20}(u^4) = \chi_{20}(\mathtt{3a}) = 2$ and $\chi_{20}(u^3) = \chi_{20}(\mathtt{2a}) = -4$. So

$$D_{20}(u^4) \sim \left(8 \times \boxed{1}, 6 \times \boxed{\zeta, \zeta^2}\right) \quad \text{and} \quad D_{20}(u^3) \sim \left(8 \times \boxed{1}, 12 \times \boxed{-1}\right).$$

Since $\chi_{20}(u) = \varepsilon_{\mathtt{2a}}(u)\chi_{20}(\mathtt{2a}) + \varepsilon_{\mathtt{3a}}(u)\chi_{20}(\mathtt{3a}) = 14$ we obtain the claimed eigenvalues.

As all the character values of all ordinary characters of G are integers on all conjugacy classes of G, we may assume by a theorem of Fong [**20**, Corollary 10.13] that all ordinary representations mentioned above are K-representations, where K is the 3-adic completion

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Table 7. *Dimensions of T^+ and T^- for certain $k\langle\bar{u}\rangle$-modules, where $*$ takes all possible values in $\{\mathtt{a},\mathtt{b}\}$.*

$k\langle\bar{u}\rangle$-module T	k-dimension of T^+	k-dimension of T^-
$\bar{L}_{10}$	6	4
$\bar{L}_{20}$	8	12
T_{1*}	1	0
T_{6*}	2	4
T_{8}	4	4

Table 8. *Decomposition factors of certain reduced $R\,\mathrm{Aut}(A_6)$-lattices, where $(\mathtt{i},\mathtt{j})$ takes a value in $\{(\mathtt{a},\mathtt{b}),(\mathtt{b},\mathtt{a})\}$.*

kG-module T	Socle of T	Head of T
$\bar{L}_{10}$	$T_{1\mathtt{i}}$	$T_{1\mathtt{j}}$
$\bar{L}_{20}$	$T_{6\mathtt{a}}\oplus T_{6\mathtt{b}}$	T_{8}

of an extension of $\mathbb{Q}$ that is unramified at 3. So if R is the ring of integers of K we may assume that they are even R-representations. Let P be the maximal ideal of R and let a bar denote the reduction modulo P. Let k be a finite field of characteristic 3 containing the residue class field of R and affording all irreducible 3-modular representations of $M_{10}, \mathrm{PGL}(2,9)$ and $\mathrm{Aut}(A_6)$. Denote by L_* an RG-lattice affording the representation D_*. Recall that k, $I(kC_3)$ and kC_3 respectively denote the indecomposable kC_6-modules of k-dimension 1, 2 and 3 having trivial composition factors, and $(k)^-$, $I(kC_3)^-$ and $(kC_3)^-$ respectively denote the indecomposable kC_6-modules of k-dimension 1, 2 and 3 having non-trivial composition factors (see Propositions 2.2 and 2.4). We will write T_* for a simple kG-module having character φ_*.

Regarded as $k\langle\bar{u}\rangle$-modules, using Propositions 2.2 and 2.3 we may write $\bar{L}_* \cong \bar{L}_*^+ \oplus \bar{L}_*^-$ and $T_* \cong T_*^+ \oplus T_*^-$, where all the composition factors of T_*^+ and $\bar{L}_*^+$ are trivial and all the composition factors of T_*^- and $\bar{L}_*^-$ are non-trivial. As u^3 is rationally conjugate to an element in $2\mathtt{a}$, it is also 3-adically conjugate to this element by [**14**, Lemma 2.9]. Thus the k-dimensions of T_*^+ and T_*^- can be deduced from the Brauer table above. The k-dimensions of $\bar{L}_*^+$ and $\bar{L}_*^-$ can be deduced from the eigenvalues given above using Proposition 2.3. The dimensions are given in Table 7.

The Krull–Schmidt–Azumaya theorem will be used without further mention. We will use decomposition series of $\bar{L}_*$ as kG-module, which we obtained using the MeatAxe algorithm [**30**] as implemented in GAP [**10**]*, as shown in Table 8.

* The representations of irreducible modules are available in GAP through use of the command `IrreducibleRepresentationsDixon` or `IrreducibleAffordingRepresentation` once the character is given. The latter is a function of the package Repsn [**8**]

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Table 9. *Part of the Brauer table of M_{10} for the prime 3, including all characters up to degree 6.*

	1a	2a
ψ_{1a}	1	1
ψ_{1b}	1	1
ψ_{4a}	4	—
ψ_{4b}	4	—
ψ_{6}	6	−2

In this paragraph all modules will be $k\langle\bar{u}\rangle$-modules. With the eigenvalues of $D_{20}(u)$ as above, using Propositions 2.3 and 2.4 we get $\bar{L}_{20}^{-} \cong 6I(kC_3)^{-}$. As T_{1a} and T_{1b} are trivial $k\langle\bar{u}\rangle$-modules by the Brauer table given above, using the eigenvalues of $D_{10}(u)$ and Proposition 2.4, we obtain $T_8^{-} \cong \bar{L}_{10}^{-} \cong X$ with $X \in \{(k)^{-} \oplus (kC_3)^{-}, 2(k)^{-} \oplus I(kC_3)^{-}\}$. Moreover, as $T_8^{-} \cong \bar{L}_{20}^{-}/(T_{6a}^{-} \oplus T_{6b}^{-})$, i.e. T_8^{-} is also a quotient of $\bar{L}_{20}^{-} \cong 6I(kC_3)^{-}$, we get

$$T_8^{-} \cong 2(k)^{-} \oplus I(kC_3)^{-}.$$

So $6I(kC_3)^{-}/(T_{6a}^{-} \oplus T_{6b}^{-}) \cong \bar{L}_{20}^{-}/(T_{6a}^{-} \oplus T_{6b}^{-}) \cong T_8^{-} \cong 2(k)^{-} \oplus I(kC_3)^{-}$ and this implies that

$$T_{6a}^{-} \oplus T_{6b}^{-} \cong 2(k)^{-} \oplus 3I(kC_3)^{-}.$$

As $\dim_k(T_{6a}^{-}) = \dim_k(T_{6b}^{-}) = 4$, this gives either

$$T_{6a}^{-} \cong 2(k)^{-} \oplus I(kC_3)^{-}, \qquad T_{6b}^{-} \cong 2I(kC_3)^{-}$$

or

$$T_{6a}^{-} \cong 2I(kC_3)^{-}, \qquad T_{6b}^{-} \cong 2(k)^{-} \oplus I(kC3)^{-}.$$

We will now apply restriction. First consider $u \in \mathrm{V}(\mathbb{Z}M_{10})$.

Looking at the Brauer table of $\mathrm{Aut}(A_6)$ and the Brauer table of M_{10} stated in Table 9, we obtain that T_{6a} and T_{6b} are isomorphic as kM_{10}-modules. So if u lies in $\mathbb{Z}M_{10}$, they must also be isomorphic as $k\langle\bar{u}\rangle$-modules, contradicting the above.

Now assume that u lies in $\mathbb{Z}\,\mathrm{PGL}(2,9)$. Let T be T_{6a} or T_{6b} such that as a $k\langle\bar{u}\rangle$-module $T^{-} \cong 2(k)^{-} \oplus I(kC_3)^{-}$. Looking at the Brauer table of $\mathrm{PGL}(2,9)$ given in Table 10 we obtain that, considered as a $k\,\mathrm{PGL}(2,9)$-module, T has two three-dimensional composition factors, say S_{3x} and S_{3y}. Moreover, by Clifford's theorem [**7**, Theorem 11.1], T is the direct sum of its two composition factors. The characters belonging to S_{3x} and S_{3y} both have the value -1 on 2a, so as $k\langle\bar{u}\rangle$-modules S_{3x}^{-} and S_{3y}^{-} are both two dimensional, and thus one of them is isomorphic to $2(k)^{-}$ while the other one is isomorphic to $I(kC_3)^{-}$.

Let $D_{3x}\colon \mathrm{PGL}(2,9) \to \mathrm{GL}(3,k)$ be a representation of $\mathrm{PGL}(2,9)$ affording S_{3x}. Let α be the Frobenius automorphism of k applied to every entry of a 3×3-matrix over k. Then $\alpha \circ D_{3x}$ is also a k-representation of $\mathrm{PGL}(2,9)$ and, looking at the Brauer table, we obtain that this representation affords S_{3y}. As α is linear on the full matrix ring, S_{3x}

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Table 10. *Part of the Brauer table of* PGL(2, 9) *for the prime* 3, *including all characters up to degree 6 with* $\alpha = \zeta_5 + \zeta_5^4$.

	1a	2a	4a	5a	5b	2b
τ_{1a}	1	1	1	1	1	1
τ_{1b}	1	1	1	1	1	-1
τ_{3a}	3	-1	1	$-\alpha$	$1+\alpha$	-1
τ_{3b}	3	-1	1	$1+\alpha$	$-\alpha$	-1
τ_{3c}	3	-1	1	$-\alpha$	$1+\alpha$	1
τ_{3d}	3	-1	1	$1+\alpha$	$-\alpha$	1
τ_{4a}	4	—	-2	-1	-1	—
τ_{4b}	4	—	-2	-1	-1	—

is also sent to S_{3y} as a $k\langle\bar{u}\rangle$-module via α, and hence S_{3x}^- is sent to S_{3y}^- via α. However, since α preserves the dimensions of eigenspaces of a matrix, S_{3x}^- and S_{3y}^- must in fact be isomorphic as $k\langle\bar{u}\rangle$-modules. This contradicts the above and thus the existence of u. □

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