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Topological characterisation of rational maps with Siegel disks

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Abstract

We extend Thurston's topological characterisation theorem for postcritically finite rational maps to a class of rational maps which have a fixed bounded type Siegel disk. This makes a small step towards generalizing Thurston's theorem to geometrically infinite rational maps.

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1. *Introduction*

Let $\mathbb{C}$, $\widehat{\mathbb{C}}$, Δ and $\mathbb{T}$ denote the complex plane, the Riemann sphere, the open unit disk and the unit circle respectively. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be an orientation-preserving branched covering map of degree at least two. We call

$$\Omega_f = \left\{x \;\middle|\; \deg_x f > 1\right\}$$

the critical set and

$$P_f = \overline{\bigcup_{1 \le k < \infty} f^k(\Omega_f)}$$

the postcritical set of f. We say f is PCF (*postcritically finite*) if P_f is a finite set. In 1982 Thurston established the topological characterisation for PCF rational maps. The reader may refer to [3] for the details of this theory. It has since been generalised to sub-hyperbolic rational maps [2, 7] and post-singularly finite exponential maps [6]. There are also some other types of generalisation of the original theorem by constraining just some of the critical points [15]. The reader may refer to Rees' survey article [16] for a more detailed introduction of the development in this aspect. In this work we will extend this theory to a class of rational maps with a Siegel disk of bounded type rotation number. Recall that an irrational number is of *bounded type* if all the coefficients of its continued fraction have a finite upper bound. *Let* $0 < \theta < 1$ *be an irrational number of bounded type and be fixed throughout.*

Definition 1·1. Let R_θ^{geom} denote the class of all the rational maps g such that:

(1) g has a fixed Siegel disk D_g with rotation number θ;
(2) the forward orbit of each critical point of g either intersects $\overline{D_g}$, or is eventually periodic, or belongs to the basin of some attracting cycle.

The main purpose of this paper is to give a topological characterisation of the maps in R_θ^{geom}. Before we state the main result, let us introduce some terminologies first. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be an orientation-preserving branched covering map. Suppose $\mathcal{O} = \{x_1, \ldots, x_p\}$ is a periodic cycle of f with period p. We say $\mathcal{O}$ is a *holomorphic attracting cycle* of f if f is holomorphic in an open neighbourhood of $\mathcal{O}$, $|Df^p(x_1)| < 1$, and moreover, $\mathcal{O}$ attracts at least one infinite critical orbit of f.

Definition 1·2. Let R_θ^{top} denote the class of all the orientation-preserving branched covering maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree at least two such that:

(i) $f|\Delta : \Delta \to \Delta$ is the rigid rotation $R_\theta : z \mapsto e^{2\pi i\theta} z$ given by θ;
(ii) $\mathbb{T} \cap \Omega_f \neq \emptyset$;
(iii) the forward orbit of each critical point of f either intersects $\overline{\Delta}$, or is eventually periodic, or converges to some holomorphic attracting cycle.

Definition 1·3. We say a map $f \in R_\theta^{top}$ is combinatorially equivalent to a map $g \in R_\theta^{geom}$ if there exist a pair of homeomorphisms $\phi, \psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that:

(i) $\phi \circ f = g \circ \psi$;
(ii) $\phi|\Delta = \psi|\Delta$ is holomorphic;
(iii) for each point x_i in the holomorphic attracting cycles of f, there is a Jordan disk D_i containing x_i such that $\phi|D_i = \psi|D_i$ is holomorphic, and moreover, ϕ is isotopic to ψ rel $P_f \cup \cup_i \overline{D_i}$.

Since a Siegel disk of a rational map with bounded type rotation number is a Jordan domain (actually a quasi-disk) with the boundary containing at least one critical point [18], it follows that every $g \in R_\theta^{geom}$ is modeled by some $f \in R_\theta^{top}$ in the sense of Definition 1·3. The main result of the paper is to prove the converse.

MAIN THEOREM. *Suppose* $f \in R_\theta^{top}$. *Then* f *is combinatorially equivalent to some* $g \in R_\theta^{geom}$ *if and only if* f *has no Thurston obstructions in* $\widehat{\mathbb{C}} \setminus \overline{\Delta}$, *and moreover, if it exists,* g *must be unique up to Möbius conjugation.*

The necessity part of the existence is an immediate consequence of a theorem of McMullen, see [10, Appendix B]. The proof of the sufficiency is the main task of this paper. The general idea is to construct a Blaschke product G such that G and f act in a similar way in the outside of the unit disk, and when restricted on the unit circle, G is a critical circle homeomorphism of rotation number θ. The candidate rational map g is then obtained

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by performing a quasiconformal surgery on G. The following is a sketch of the line of the proof.

Let d denote the covering degree of $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$. Since $\Omega_f\neq\emptyset$ by definition, we have $d\geq 2$. We first construct a branched covering map F of degree $2d-1$ which is symmetric about $\mathbb{T}$ such that when restricted on $\widehat{\mathbb{C}}\setminus\overline{\Delta}$, F acts in a similar way with f. We then perturb F and get a sequence of orientation-preserving branched covering maps $F_n:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ such that $F_n\to F$ uniformly, F_n is symmetric about $\mathbb{T}$, and moreover, every critical orbit of F_n either converges to some holomorphic attracting cycle or is eventually periodic.

The next part is the core of the proof where we need to deal with two main issues. Firstly, we will show that for all n large enough, the maps F_n have no Thurston obstructions. Hence by Thurston's theorem and its extension to sub-hyperbolic case, F_n is combinatorially equivalent to some rational map G_n. Since F_n is symmetric about $\mathbb{T}$, G_n is a Blaschke product. Secondly, we will prove that the geometry of P_{G_n} is uniformly bounded, and therefore, as $n\to\infty$, G_n converges to some Blaschke product G. The main tool in solving both the two issues is Lemma 4·3. The lemma asserts that, as one iterates the Thurston operator induced by F_n, the length of certain groups of simple closed geodesics has positive lower bounds.

From the uniform boundedness of the geometry of P_{G_n} we will construct a combinatorial equivalence between F and G. The map $g\in R_\theta^{geom}$ is then obtained by performing a quasiconformal surgery on G. The existence part of the main theorem then follows. To prove the uniqueness we first prove that the Julia set of g has zero Lebesgue measure. This, together with the fact that the boundaries of bounded type Siegel disks of rational maps are quasicircles [**18**], implies that if f is combinatorially equivalent to both g and h in R_θ^{geom}, then g and h are quasiconformally equivalent to each other. By lifting the quasiconformal equivalence, we will get a quasiconformal conjugation between g and h which is conformal in the Fatou set of g. Since the Julia set of g has zero Lebesgue measure, the conjugation map must be a Möbius map. This implies the uniqueness part of the main theorem.

Here is the structure of the paper. In Section 2 we give a brief introduction of Thurston's theory on the topological characterisation of PCF rational maps and its generalization to sub-hyperbolic rational maps. In Section 3 we construct the branched covering map $F:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ by symmetrizing f about $\mathbb{T}$ and construct the sequence F_n by perturbing F. In Section 4 by assuming Lemma 4·3 we prove that F_n has no Thurston obstructions and P_{G_n} has uniformly bounded geometry. In Sections 5 and 6 we prove the existence and the uniqueness of the main theorem respectively. The proof of Lemma 4·3 is the heart of this work and is presented in Section 7. The appendix contains some known lemmas from [**3**] and [**7**], all of which are used in Section 7, and a technical lemma on homotopy, which is used in Section 5.

2. *Background*

In this section we will give a brief introduction of Thurston's theory on the topological characterisation of PCF rational maps and its generalisation to sub-hyperbolic rational maps. The reader may refer to [**2**, **3**, **7**, **14**] for relative details. Suppose $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is an orientation-preserving branched covering map. We say a simple closed curve $\gamma\subset\widehat{\mathbb{C}}\setminus P_f$ is *non-peripheral* if each component of $\widehat{\mathbb{C}}\setminus\gamma$ contains at least two points in P_f. Let $\Gamma=\{\gamma_1,\cdots,\gamma_n\}$ be a family of disjoint non-peripheral curves which are not homotopic to each other. We call such Γ a *multi-curve*. We say a multi-curve Γ is *stable* if for each $\gamma_j\in\Gamma$, all the non-peripheral components of $f^{-1}(\gamma_j)$ are homotopic to the elements in Γ. Associated to each stable multi-curve Γ, there is a linear transformation $f_\Gamma:\mathbb{R}^\Gamma\to\mathbb{R}^\Gamma$ with non-negative entries: let $\gamma_{i,j,\alpha}$ denote all the non-peripheral components of $f^{-1}(\gamma_j)$

homotopic to γ_i, and let $d_{i,j,\alpha}$ denote the covering degree of $f:\gamma_{i,j,\alpha}\to\gamma_j$, define

$$f_\Gamma(\gamma_j)=\sum_i a_{i,j}\gamma_i,$$

where $a_{i,j}=\sum_\alpha 1/d_{i,j,\alpha}$. Let λ_Γ denote the maximal eigenvalue of f_Γ. We call Γ a *Thurston obstruction* if $\lambda_\Gamma\geq 1$. Most importantly, for any $k\geq 1$ we have

$$\left(f_\Gamma\right)^k=\left(f^k\right)_\Gamma. \tag{2·1}$$

Remark 2·1. In the above setup all the concepts are still meaningful if we replace P_f by a compact subset $E\subset\widehat{\mathbb{C}}$ satisfying $P_f\subset E$ and $f(E)\subset E$. In particular, in Section 7, for $f\in R_\theta^{top}$, we will consider the f-stable multi-curves in $\widehat{\mathbb{C}}\setminus(\overline{\Delta}\cup P_f\cup\cup_i\overline{D_i})$; and for the maps F_n constructed in Section 3, we will consider the F_n-stable multi-curves in $\widehat{\mathbb{C}}\setminus(\mathbb{T}\cup P_{F_n}\cup\cup_i\overline{D_i})$.

2·1. *Topological characterisation of PCF rational maps*

In the PCF case, we define the orbifold $\mathcal{O}_f$ associated to f to be the pair $(\widehat{\mathbb{C}},\nu(x))$ where $\nu:\widehat{\mathbb{C}}\to\mathbb{N}^+\cup\{\infty\}$ is the smallest function such that $\nu(x)\deg_x(f)$ is a divisor of $\nu(f(x))$. We say $\mathcal{O}_f$ is hyperbolic if

$$2-\sum_x\left(1-\frac{1}{\nu(x)}\right)<0.$$

We say f is *combinatorially equivalent* to a rational function g if there is a pair of homeomorphisms $\phi,\psi:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ such that ψ is isotopic to ϕ rel P_f and $\phi\circ f=g\circ\psi$.

THEOREM 2·1 (Thurston). *Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a PCF and orientation-preserving branched covering map. Suppose O_f is hyperbolic. Then f is combinatorially equivalent to a rational map if and only if f has no Thurston obstructions.*

Let T_f denote the Teichmuller space modelled on $(\widehat{\mathbb{C}},P_f)$. The proof of Theorem 2.1 is by considering the iteration of the analytic operator $\sigma_f:T_f\to T_f$ induced by f, which is now called Thurston operator. The existence of the candidate rational map g is equivalent to the existence of a fixed point of σ_f. Let $\tau_0\in T_f$ be an arbitrary point and $\tau_i=\sigma_f^i(\tau_0)$. It turns out that the non-existence of Thurson obstructions implies the convergence of τ_i, which then implies the existence of a fixed point of σ_f.

Here are some more details. As we iterate the Thurston operator σ_f, we get a sequence of Riemann surfaces R_m. The non-existence of Thurston obstructions implies the existence of a positive lower bound of the length of all simple closed geodesics in R_m. More precisely, let $\phi_m:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be the homeomorphism which represents τ_m. For each non-peripheral curve $\gamma\subset\widehat{\mathbb{C}}\setminus P_f$, let $l(\gamma,\tau_m)$ denote the length of the simple closed geodesic in $\widehat{\mathbb{C}}\setminus\phi_m(P_f)$ which is homotopic to $\phi_m(\gamma)$. The non-existence of Thurston obstructions implies that there is a constant $C>0$ depending only on τ_0 such that $l(\gamma,\tau_m)>C$ for all $m\geq 1$. From this one can further deduce that $d_{T_f}(\tau_m,\tau_{m+1})$ decay exponentially, which then implies the existence of a fixed point of σ_f.

2·2. *Topological characterisation of sub-hyperbolic rational maps*

Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be an orientation-preserving branched covering map. We call f a *sub-hyperbolic semi-rational* branched covering map if every critical orbit of f either converges

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to some holomorphic attracting cycle or is eventually periodic. By [7, Lemma 2·1], for each x_i in the holomorphic attracting cycles of f, which is necessarily an accumulation point of P_f by definition, we can take a topological disk D_i containing x_i such that:

(i) $\overline{D_i} \cap \overline{D_j} = \emptyset$ for $i \neq j$ and all ∂D_i are real analytic curves;
(ii) for each D_i, there is an annulus H_i surrounding it with ∂D_i being the inner component of ∂H_i and $\overline{H_i} \cap P_f = \emptyset$;
(iii) f is holomorphic in $\overline{D_i} \cup H_i$ and maps $\overline{D_i} \cup H_i$ into some $D_{i'}$.

These D_i are called holomorphic disks of f. Since $P_f \cup \cup_i \overline{D_i}$ is a forward invariant compact set, in $\widehat{\mathbb{C}} \setminus (P_f \cup \cup_i \overline{D_i})$ one can define the non-peripheral curves, the f-stable multi-curves and the Thurston obstructions in the same way as in the PCF case. We say f is *combinatorially equivalent* to a rational map g if there is a pair of homeomorphisms $\phi, \psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that ψ is isotopic to ϕ rel P_f and $\phi \circ f = g \circ \psi$, and moreover, $\phi|D_i = \psi|D_i$ is holomorphic for all D_i. With different approaches in [2] and [7] the authors prove the following independently.

THEOREM 2·2 (Cui–Tan, Jiang–Zhang). *Suppose f is a sub-hyperbolic semi-rational branched covering map. Then f is combinatorially equivalent to a rational map if and only if f has no Thurston obstructions.*

The proof of [7, Theorem 2·2] has the same spirit as in the PCF case, namely, by considering the iteration of the analytic map $\sigma_f : T_f \to T_f$ induced by f where T_f is the Teichmuller space modelled on $(\widehat{\mathbb{C}}, P_f \cup \cup_i \overline{D_i})$. The existence of the candidate rational map g is equivalent to the existence of a fixed point of σ_f. There are two steps in the proof. In the first step, the proof of the existence of the fixed point is reduced to the proof of certain bounded geometry. In the second step, it is shown that the non-existence of Thurston obstructions implies the bounded geometry. More precisely, using the same notations as in Section 2·1, it is shown that there is a constant $C > 0$ depending only on τ_0 such that for any non-peripheral $\gamma \subset (\widehat{\mathbb{C}}, P_f \cup \cup_i \overline{D_i})$, if η is the geodesic in $\widehat{\mathbb{C}} \setminus \phi_m(P_f \cup \cup_i \overline{D_i})$ which is homotopic to $\phi_m(\gamma)$, then the length of η is greater than C for all $m \geq 0$.

The proof in the PCF case relies essentially on the finiteness of P_f and the invariance property $f(P_f) \subset P_f$ (see [3]). To adapt the argument in the PCF case, a key trick in [7] is to take a pair of distinct points $\{a_i, b_i\} \subset D_i \cap P_f$ for each D_i and replace P_f by the finite set $E = P_1 \cup \cup_i \{a_i, b_i\}$ where

$$P_1 = P_f \setminus \cup_i \overline{D_i}.$$

Note that E is necessarily not forward invariant. In [7] we overcame this non-invariance problem by introducing certain intermediate larger sets, see the proof of [7, Lemma 7·5]. Such idea will be systematically used in this work, see Lemma 4·2 and the proof of Lemma 4·3 in Section 7.

3. *Constructing F_n*

3·1. *Notations*

Given a point $w \in \widehat{\mathbb{C}}$, let w^* denote the symmetric image of w about $\mathbb{T}$, i.e., $w^* = 1/\overline{w}$. For a set $W \subset \widehat{\mathbb{C}}$, let $W^* = \{w^* \mid w \in W\}$ and $|W|$ denote the cardinality of W, and $W^c = \widehat{\mathbb{C}} \setminus W$

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denote its complement in $\widehat{\mathbb{C}}$. We say W is symmetric about $\mathbb{T}$ if $W = W^*$. For any $\alpha \in \mathbb{R}$, let R_α denote the rotation given by $z \mapsto e^{2\pi i\alpha}z$. Let id denote the identity map.

3·2. *Construction of F*

Suppose $f \in R_\theta^{top}$ has no Thurston obstructions in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$. Let $d \geq 2$ be the degree of f. We may assume that $\infty \in P_f$. In fact, if $P_f \setminus \overline{\Delta} \neq \emptyset$, we take $P \in P_f \setminus \overline{\Delta}$. Let $\Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a homeomorphism such that $\Phi(P) = \infty$, $\Phi|\Delta = \text{id}$ and Φ is holomorphic in an open neighborhood of all holomorphic attracting cycles of f. Let $\tilde{f} = \Phi \circ f \circ \Phi^{-1}$. It is clear that $\tilde{f} \in R_\theta^{top}$ has no Thurston obstructions in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$ and $\infty \in P_{\tilde{f}}$. Then we need only consider $\tilde{f}$. If $P_f \setminus \overline{\Delta} = \emptyset$, we take $P \in \widehat{\mathbb{C}} \setminus \overline{\Delta}$ such that $f(P) = 1$ and define Φ and $\tilde{f}$ in the same way as before. Then $\tilde{f}(\infty) \in P_{\tilde{f}}$. In this case we consider the forward invariant compact subset $P_{\tilde{f}} \cup \{\infty\}$ instead of $P_{\tilde{f}}$, see Remark 2·1. Note that with respect to this larger set $\tilde{f}$ still has no non-peripheral curves in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$ and thus has no Thurston obstructions in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$. In this case, to simplify the notations, we still use $P_{\tilde{f}}$ to refer to the larger set $P_{\tilde{f}} \cup \{\infty\}$.

We construct F by symmetrising f. Before that we need to make some preliminary preparation. Firstly, if f has holomorphic attracting cycles, as in Section 2·2, for each point x_i in the holomorphic attracting cycles we take a disk D_i and an annulus H_i surrounding D_i so that the three properties there are satisfied. Secondly, we need make a slight modification of f in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$. Since $\Omega_f \cap \mathbb{T} \neq \emptyset$, we may assume that $1 \in \Omega_f$. It follows that there is a curve segment, say Λ_f, which is attached to 1 from the outside of Δ, such that $f(\Lambda_f) \subset \mathbb{T}$. Let

$$X = \{z \in \Omega_f - \overline{\Delta} \mid f^i(z) \in \Delta \text{ for some } i \geq 1\}.$$

For each $z \in X$, let $i_z \geq 1$ be the smallest integer such that $f^{i_z}(z) \in \Delta$. Let $\widetilde{X} = \{f^{i_z}(z) \mid z \in X\}$ and $\sigma : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a homeomorphism such that $\sigma|(\widehat{\mathbb{C}} - \Delta) = \text{id}$ and $\sigma(\widetilde{X}) \subset \Lambda_f^*$. Since $\widetilde{X}$ is a finite set contained in Δ, such map obviously exists. Let $f_1 = \sigma \circ f$. Define

$$F(z) = \begin{cases} f_1(z) & \text{if } |z| \geq 1, \\ (f_1(z^*))^* & \text{for otherwise.} \end{cases} \tag{3·1}$$

Since $1 \in \Omega_f$ and $f|\mathbb{T} = R_\theta$, by the construction it follows directly that $1 \in \Omega_F$ and $F|\mathbb{T} = R_\theta$.

We shall see later that with such modification the symmetrization does not produce Thurston obstructions. From the construction we have

PROPOSITION 3·1. *For $z \in X$, $F^{i_z}(z) \in \Lambda_f^*$, and hence $F^{i_z+1}(z) \in \mathbb{T}$. By symmetry, for $z \in X^*$, $F^{i_z}(z) \in \Lambda_f$, and hence $F^{i_z+1}(z) \in \mathbb{T}$.*

Note that symmetrisation may produce more post-critical points in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$. In Figure 1, the point C^* is such a point. Note also that more post-critical points will cause more non-peripheral curves, and that for any non-peripheral curve γ in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$, $F^{-1}(\gamma)$ may have non-peripheral components in Δ. Thus, a priori, F and its perturbations F_n, which will be constructed in the following, may have Thurston obstructions. But we shall see later that this will not happen.

3·3. *Construction of F_n*

Let $\theta_n = p_n/q_n$ be any sequence of rational numbers such that $(p_n, q_n) = 1$ and $\theta_n \to \theta$ as $n \to \infty$. Let

$$O_n = \left\{e^{2\pi i k\theta_n} \,\middle|\, 0 \leq k < q_n\right\}.$$

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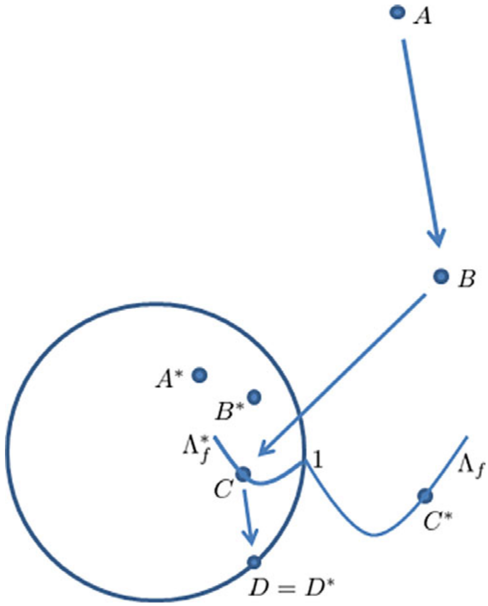

Fig. 1. Symmetrisation produces more post-critical points in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$.

For $0 < b < a$ let $A(a, b) = \{z : b < |z| < a\}$. By the construction of F, any infinite critical orbit of F which does not intersect $\mathbb{T}$ must converge to some holomorphic attracting cycle. So there is an $0 < r_0 < 1$ such that

$$(A(1 + r_0, 1 - r_0) \setminus \mathbb{T}) \cap \left(\Omega_F \cup P_F \cup \cup_i \overline{D_i}\right) = \emptyset. \tag{3·2}$$

Set

$$Y = \left\{z \in (\Omega_F \cup P_F) \setminus \mathbb{T} \;\middle|\; F(z) \in \mathbb{T}\right\}, \tag{3·3}$$

and

$$Z = (\Omega_F \cap \mathbb{T}) \cup F(Y). \tag{3·4}$$

Clearly, Z is a finite set.

PROPOSITION 3·2. *There is a sequence of sub-hyperbolic semi-rational branched covering maps $F_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $F_n \to F$ uniformly with respect to the spherical metric, and moreover, the following properties hold for all n large enough.*

(i) $F_n(z)^* = F_n(z^*)$ *for all* $z \in \widehat{\mathbb{C}}$, *and* $F_n|\mathbb{T} = R_{\theta_n}$.

(ii) If $\{z, F_n(z)\}$ *or* $\{z, F(z)\}$ *is contained in* $\widehat{\mathbb{C}} \setminus A(1 + r_0/2, 1 - r_0/2)$, *then* $F_n(z) = F(z)$.

(iii) $1 \in \Omega_{F_n}$, $|\Omega_{F_n}| = |\Omega_F|$, $\Omega_{F_n} \setminus \mathbb{T} = \Omega_F \setminus \mathbb{T}$*čň and* $P_{F_n} \setminus \mathbb{T} = P_F \setminus \mathbb{T}$.

(iv) $Y = \{z \in (\Omega_{F_n} \cup P_{F_n}) \setminus \mathbb{T} \;\big|\; F_n(z) \in \mathbb{T}\}$.

(v) Let $Z_n = (\Omega_{F_n} \cap \mathbb{T}) \cup F_n(Y)$. *Then* $Z_n \subset O_n$ *and* $P_{F_n} \cap \mathbb{T} = O_n$. *Moreover,* $|Z_n| = |Z|$ *and* $Z_n \to Z$ *as* $n \to \infty$. *In addition, if* $F^m(x) = y$ *for some* $x, y \in Z$ *and integer* $m \geq 0$, *then* $F_n^m(x_n) = y_n$ *with* x_n *and* y_n *being respectively the corresponding points of x and y in* Z_n.

(vi) There is a curve segment Λ_n *attached to* 1 *from the outside of* Δ *such that* $F_n(\Lambda_n) \subset \mathbb{T}$, *and moreover, if* $F_n(z) \in \widehat{\mathbb{C}} \setminus \overline{\Delta}$ *holds for some* $z \in (\Omega_{F_n} \cup P_{F_n}) \cap \Delta$, *then* $F_n(z) \in \Lambda_n$. *In particular, this implies that* $(P_{F_n} \setminus P_f) \cap (\widehat{\mathbb{C}} \setminus \overline{\Delta}) \subset \Lambda_n$.

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Proof. Since Z is a finite set and $\theta_n \to \theta$, we can define $\tau_n : Z \to O_n$ for all n large enough so that:

(a) $\tau_n(1) = 1$;
(b) $\tau_n(z) \to z$ for all $z \in Z$; and
(c) τ_n preserves the orbit relations among the points in Z in the following sense: if there exist an $m \geq 0$ and $x, y \in Z$ such that $R_\theta^m(x) = y$ then $R_{\theta_n}^m \circ \tau_n(x) = \tau_n(y)$.

Then we define the map $\sigma_n : (\Omega_F \cap \mathbb{T}) \cup R_\theta^{-1}(F(Y)) \to \mathbb{T}$ as follows.

$$\sigma_n(z) = \begin{cases} \tau_n(z) & \text{for } z \in \Omega_F \cap \mathbb{T}, \\ R_{\theta_n}^{-1} \circ \tau_n \circ R_\theta(z) & \text{for } z \in R_\theta^{-1}(F(Y)). \end{cases} \tag{3·5}$$

Note that σ_n is well defined when $(\Omega_F \cap \mathbb{T}) \cap R_\theta^{-1}(F(Y)) \neq \emptyset$. To see this, suppose $z \in (\Omega_F \cap \mathbb{T}) \cap R_\theta^{-1}(F(Y))$. Then $w = R_\theta(z) \in F(Y)$. By (c) it follows that $R_{\theta_n}(\tau_n(z)) = \tau_n(w) = \tau_n(R_\theta(z))$. That is, $\tau_n(z) = R_{\theta_n}^{-1} \circ \tau_n \circ R_\theta(z)$.

By (b) it follows that $\sigma_n \to \text{id}$ on the finite set $(\Omega_F \cap \mathbb{T}) \cup R_\theta^{-1}(F(Y))$. So we can extend σ_n to a homeomorphism $\sigma_n : \mathbb{T} \to \mathbb{T}$ so that:

(d) $\sigma_n(z) \to z$ uniformly for $z \in \mathbb{T}$.

Now we extend σ_n to a homeomorphism of the sphere to itself, which is still denoted by σ_n, such that:

(e) $\sigma_n = \text{id}$ in $\widehat{\mathbb{C}} \setminus A(1 + r_0/2, 1 - r_0/2)$;
(f) $\sigma_n(z)^* = \sigma_n(z^*)$;
(g) as $n \to \infty$, $\sigma_n \to \text{id}$ uniformly with respect to the spherical metric.

Next we define a homeomorphism $h_n : \mathbb{T} \to \mathbb{T}$ by

$$h_n(z) = R_{\theta_n} \circ \sigma_n \circ R_\theta^{-1}(z), \tag{3·6}$$

and extend it to a homeomorphism of the sphere to itself, which is still denoted by h_n, such that:

(h) $h_n = \text{id}$ in $\widehat{\mathbb{C}} \setminus A(1 + r_0/2, 1 - r_0/2)$;
(i) $h_n(z)^* = h_n(z^*)$;
(j) as $n \to \infty$, $h_n(z) \to \text{id}$ uniformly with respect to the spherical metric.

Define

$$F_n = h_n \circ F \circ \sigma_n^{-1}. \tag{3·7}$$

Now let us show that the properties (i-vi) hold for F_n. Since h_n, F and σ_n are all symmetric about $\mathbb{T}$, $F_n^*(z) = F_n(z^*)$ holds by (3·7). By (3·6), (3·7) and that $F|\mathbb{T} = R_\theta$, it follows that $F_n|\mathbb{T} = R_{\theta_n}$. This proves (i).

(ii) follows from (e), (h) and (3·7).

Since $1 \in \Omega_F$ and $\sigma_n(1) = \tau_n(1) = 1$, it follows that $1 \in \Omega_{F_n}$. By (3·7) we have $\Omega_{F_n} = \sigma_n(\Omega_F)$. So $|\Omega_{F_n}| = |\Omega_F|$. From (e) and (3·2) and that $\sigma_n(\mathbb{T}) = \mathbb{T}$, it follows that

$\Omega_{F_n} \setminus \mathbb{T} = \Omega_F \setminus \mathbb{T}$. This, together with (3·2) and the property (ii) we just proved, implies that $P_{F_n} \setminus \mathbb{T} = P_F \setminus \mathbb{T}$. This proves (iii).

By (iii) we have $(\Omega_{F_n} \cup P_{F_n}) \setminus \mathbb{T} = (\Omega_F \cup P_F) \setminus \mathbb{T}$. For $z \in (\Omega_{F_n} \cup P_{F_n}) \setminus \mathbb{T}$, it follows from (3·2), (*e*) and (3·7) that $F_n(z) = h_n(F(z))$. Since h_n is a homeomorphism and $h(\mathbb{T}) = \mathbb{T}$, $F_n(z) \in \mathbb{T}$ if and only if $F(z) \in \mathbb{T}$. This proves (iv).

To prove (v) note that $(\Omega_{F_n} \cap \mathbb{T}) = \sigma_n(\Omega_F \cap \mathbb{T}) = \tau_n(\Omega_F \cap \mathbb{T})$. By (*e*) and (3·2) we have $\sigma_n^{-1}(Y) = Y$, and thus

$$F_n(Y) = h_n \circ F \circ \sigma_n^{-1}(Y) = h_n(F(Y)) = R_{\theta_n} \circ \sigma_n \circ R_\theta^{-1}(F(Y)) = \tau_n(F(Y)).$$

So $Z_n = (\Omega_{F_n} \cap \mathbb{T}) \cup F_n(Y) = \tau_n(Z) \subset O_n$ by the definition of τ_n. This, together with the fact that O_n is a periodic orbit of F_n, implies that $P_{F_n} \cap \mathbb{T} = O_n$. Since τ_n is injective and $\tau_n(z) \to z$ for all $z \in Z$, it follows that $|Z_n| = |Z|$ and $Z_n \to Z$. The last statement of (v) holds since τ_n preserves the orbit relations among the points in Z.

Since $\sigma_n(1) = 1$, $F(\Lambda_f) \subset \mathbb{T}$ and $h_n(\mathbb{T}) = \mathbb{T}$, it follows from (3·7) that the curve segment $\Lambda_n = \sigma_n(\Lambda_f)$ is attached to 1 from the outside of Δ and $F_n(\Lambda_n) \subset \mathbb{T}$. Suppose $F_n(z) \in \widehat{\mathbb{C}} \setminus \overline{\Delta}$ for some $z \in (\Omega_{F_n} \cup P_{F_n}) \cap \Delta$. Then $F_n(z) \in P_{F_n} \setminus \mathbb{T}$. From (3·2) and property (ii), we have $F_n(z) = F(z)$. Since $z \in (\Omega_{F_n} \cup P_{F_n}) \cap \Delta = (\Omega_F \cup P_F) \cap \Delta$ by property (iii), $F(z) \in \Lambda_f$ by Proposition 3·1. From (3·2) we have $F(z) \notin A(1 + r_0, 1 - r_0)$. From (*e*) and $F(z) \in \Lambda_f$ we have $F_n(z) = F(z) = \sigma_n(F(z)) \in \sigma_n(\Lambda_f) = \Lambda_n$.

If $w \in (P_{F_n} \setminus P_f) \cap (\widehat{\mathbb{C}} \setminus \overline{\Delta})$, then $w = F_n(z)$ for some $z \in (\Omega_{F_n} \cup P_{F_n}) \cap \Delta$. According to what we have just proved, we get $w = F_n(z) \in \Lambda_n$. The property (vi) follows.

Remark 3·1. Since $P_f \setminus \overline{\Delta}$ is a finite set, by deforming f in its combinatorial equivalence class if necessary, we may assume that f and thus F are both quasiregular maps. From the construction we see that both σ_n and h_n can be taken to be K-quasiconformal with $K > 1$ being some constant independent of n. So besides the six assertions claimed in Proposition 3·2 we may further assume that F_n is K_0-quasiregular with $K_0 > 1$ being independent of n.

Remark 3·2. By the continuity of F we can take $r_0 > 0$ in (3·2) small and an $r_1 > r_0$ such that

$$\{z \mid 1 < |z| < 1 + 2r_1\} \cap \left(P_{F_n} \cup \cup_i \overline{D_i}\right) = \emptyset,$$

and moreover, for any $z \in A(1 + r_0, 1 - r_0)$, $F(z) \in A(1 + r_1, 1 - r_1)$. So for any w with $|w| > 1 + r_1$, by (*e*), (*h*), (3·7) and (3·1) we have

$$F_n^{-1}(w) \cap \left(\widehat{\mathbb{C}} \setminus \overline{\Delta}\right) = F^{-1}(w) \cap \left(\widehat{\mathbb{C}} \setminus \overline{\Delta}\right) = f^{-1}(w).$$

4. *Proving that F_n has no Thurston obstructions and P_{G_n} has uniformly bounded geometry by assuming Lemma 4·3*

For $n \geq 1$, let $\tau_{0,n}$ denote the standard complex structure on $\widehat{\mathbb{C}}$. For $m, n \geq 1$, let $\tau_{m,n}$ denote the complex structure on $\widehat{\mathbb{C}}$ which is obtained by pulling back $\tau_{0,n}$ by F_n^m. Associated to each $\tau_{m,n}$ is a quasiconformal homeomorphism $\phi_{m,n} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which fixes 0, 1 and ∞. Then

$$G_{m,n} = \phi_{m,n} \circ F_n \circ \phi_{m+1,n}^{-1}$$

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is a rational map. By the symmetry of F_n, it follows that $\tau_{m,n}$ is symmetric about $\mathbb{T}$. By symmetry again, both $\phi_{m,n}$ and $(\phi_{m,n}(z^*))^*$ fix 0, 1 and ∞, and are associated to the same complex structure $\tau_{m,n}$. By the uniqueness we have $\phi_{m,n}(z) = (\phi_{m,n}(z^*))^*$. It follows that

$$\begin{aligned} G_{m,n}(z^*) &= \phi_{m,n} \circ F_n \circ \phi_{m+1,n}^{-1}(z^*) \\ &= \phi_{m,n} \circ F_n \left(\left(\phi_{m+1,n}^{-1}(z) \right)^* \right) \\ &= \phi_{m,n} \left(\left(F_n \left(\phi_{m+1,n}^{-1}(z) \right) \right)^* \right) \\ &= \left(\phi_{m,n} \left(F_n \left(\phi_{m+1,n}^{-1}(z) \right) \right) \right)^* \\ &= \left(G_{m,n}(z) \right)^* . \end{aligned} \tag{4·1}$$

This implies that all $G_{m,n}$ are Blaschke products.

Definition 4·1. Suppose $P \subset \widehat{\mathbb{C}}$ is a proper closed subset with $|P| \geq 3$ and $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homeomorphism. Suppose $\gamma \subset \widehat{\mathbb{C}} \setminus P$ is a non-peripheral curve. Let $\eta \subset \widehat{\mathbb{C}} \setminus \phi(P)$ be the unique simple closed geodesic which is homotopic to $\phi(\gamma)$ in $\widehat{\mathbb{C}} \setminus \phi(P)$. We use $l_\phi(\gamma, P)$ to denote the length of η with respect to the hyperbolic metric of $\widehat{\mathbb{C}} \setminus \phi(P)$. We say γ is a (ϕ, P)-geodesic if $\eta = \phi(\gamma)$.

By (3) of Proposition 3·2 we have $P_F \setminus \mathbb{T} = P_{F_n} \setminus \mathbb{T}$. Suppose F_n has holomorphic disks. Otherwise we just omit this step. By Section 2·2 we may assume that

> each D_i satisfies the properties (i-iii) given in Section 2·2, in particular, by (ii) $\partial D_i \cap P_{F_n} = \emptyset$, and by (iii) $\mathbb{T} \cap (H_i \cup \overline{D_i}) = D_j \cap (H_i \cup \overline{D_i}) = \emptyset$ for $i \neq j$.

We may assume that the holomorphic disks are presented in pairs which are symmetric with each other about $\mathbb{T}$, that is

> D is a holomorphic disk of F_n if and only if D^* is a holomorphic disk of F_n.

We may further assume that each D_i is small enough so that

> each D_i contains at most one critical value of F_n.

For each holomorphic disk D_i, we take a pair of distinct points $\{a_i, b_i\} \subset D_i \cap P_{F_n} = D_i \cap P_F$. It is easy to see that we can take the pair $\{a_i, b_i\} \subset D_i$ so that the following three properties hold:

(i) If $\infty \in D_i$, then $\infty \in \{a_i, b_i\}$.
(ii) If $v \in D_i$ is the only critical value contained in D_i, then $v \in \{a_i, b_i\}$.
(iii) If $D_j = D_i^*$, then $\{a_j, b_j\} = \{a_i^*, b_i^*\}$.

Let Y and Z be the sets defined in (3·3) and (3·4) respectively. Set

$$\begin{aligned} P_1 &= P_F \setminus \left(\mathbb{T} \cup \cup_i \overline{D_i} \right), \\ Z_n &= \left(\Omega_{F_n} \cap \mathbb{T} \right) \cup F_n(Y), \\ A_n &= P_1 \cup Z_n \cup \cup_i \{a_i, b_i\}, \\ B_n &= P_1 \cup O_n \cup \cup_i \{a_i, b_i\} \end{aligned}$$

and

$$A = P_1 \cup Z \cup \cup_i \{a_i, b_i\}.$$

For $k \geq 0$, let

$$Z_n^k = \bigcup_{i=0}^{k} F_n^i(Z_n)\ ,\ A_n^k = \bigcup_{i=0}^{k} F_n^i(A_n)\ ,\ B_n^k = \bigcup_{i=0}^{k} F_n^i(B_n). \tag{4·2}$$

and

$$Z^k = \bigcup_{i=0}^{k} F^i(Z) \text{ and } A^k = \bigcup_{i=0}^{k} F^i(A).$$

Then $A_n^k \cap O_n = Z_n^k \subset O_n$ and $B_n^k \cap O_n = O_n$.

Remark 4·1. If there are no holomorphic disks, we just define $A_n = P_1 \cup Z_n$, $B_n = P_1 \cup O_n = P_{F_n}$ and $A = P_1 \cup Z$. In addition, from the construction it follows that $\{0, 1, \infty\} \subset A_n$, $A_n^k = (A_n^k)^*$ and $B_n^k = (B_n^k)^*$.

PROPOSITION 4·1. *Let $k \geq 0$. Then $|Z_n^k| = |Z^k|$ and $|A_n^k| = |A^k|$ hold for all n large enough. Moreover, $Z_n^k \to Z^k$ and $A_n^k \to A^k$ as $n \to \infty$. In particular, there is a $\mu > 0$ depending only on k such that the spherical distance between any two distinct points in A_n^k is greater than μ for all n large enough.*

Proof. By (v) of Proposition 3·2 $|Z_n| = |Z|$ and $Z_n \to Z$, and moreover the orbit relations are preserved. Since $F_n \to F$, this implies that $|Z_n^k| = |Z^k|$ for all n large enough and $Z_n^k \to Z^k$ as $n \to \infty$. These, together with (ii) of Proposition 3·2 and the definitions of A_n^k and A^k, implies that $|A_n^k| = |A^k|$ and $A_n^k \to A^k$. The last assertion is then obvious.

LEMMA 4·2. *Let $N_0 \geq 0$. For $0 \leq i \leq N_0$, define*

$$\tilde{A}_n^i = F_n^{-i}\left(A_n^{N_0}\right) \text{ and } \tilde{B}_n^i = F_n^{-i}\left(B_n^{N_0}\right). \tag{4·3}$$

Then for all $0 \leq i \leq N_0$, $A_n \subset \tilde{A}_n^i \cap A_n^{N_0}$, $B_n \subset \tilde{B}_n^i \cap B_n^{N_0}$, and the maps

$$F_n^i : \widehat{\mathbb{C}} \setminus \tilde{A}_n^i \to \widehat{\mathbb{C}} \setminus A_n^{N_0} \text{ and } F_n^i : \widehat{\mathbb{C}} \setminus \tilde{B}_n^i \to \widehat{\mathbb{C}} \setminus B_n^{N_0}$$

are covering maps.

Proof. For $0 \leq i \leq N_0$, by the definition of $A_n^{N_0}$ we have $F_n^i(A_n) \subset A_n^{N_0}$ and thus $A_n \subset F_n^{-i}(A_n^{N_0}) = \tilde{A}_n^i$ (here $F_n^0 = \text{id}$ and $\tilde{A}_n^0 = A_n^{N_0}$). Since $A_n \subset A_n^{N_0}$ we have $A_n \subset \tilde{A}_n^i \cap A_n^{N_0}$. The same argument implies that $B_n \subset \tilde{B}_n^i \cap B_n^{N_0}$.

Note that for each critical point c of F_n, either $c \in A_n$ or $F_n(c) \in A_n$. From this and the definition of $A_n^{N_0}$, it follows that for each $1 \leq i \leq N_0$, all the critical values of F_n^i are contained in $A_n^{N_0}$. Thus $F_n^i : \widehat{\mathbb{C}} \setminus \tilde{A}_n^i \to \widehat{\mathbb{C}} \setminus A_n^{N_0}$ is a covering map. The same argument implies that $F_n^i : \widehat{\mathbb{C}} \setminus \tilde{B}_n^i \to \widehat{\mathbb{C}} \setminus B_n^{N_0}$ is also a covering map.

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LEMMA 4·3. *The following two assertions hold*:

(i) *for any* $k \geq 0$*, there exist a* $\delta > 0$ *and* $N = N(k) \geq 1$ *depending only on* k *such that for any non-peripheral curve* $\gamma \subset \widehat{\mathbb{C}} \setminus A_n^k$, $l_{\phi_{m,n}}(\gamma, A_n^k) \geq \delta$ *holds for all* $m \geq 0$ *and* $n \geq N$;

(ii) *there exists an* $N \geq 1$ *such that for all* $n \geq N$*, there is a* $\delta_n > 0$ *depending only on* n *such that for any non-peripheral curve* $\gamma \subset \widehat{\mathbb{C}} \setminus B_n$, $l_{\phi_{m,n}}(\gamma, B_n) \geq \delta_n$ *holds for all* $m \geq 0$.

Remark 4·2. To save the notations of the constants, from now on when we say an assertion holds for all $m \geq 0$ and all n large enough, we mean that there is an $N \geq 0$ such that the assertion holds for all m, n with $m \geq 0$ and $n \geq N$.

The proof of Lemma 4·3 is the heart of the paper and is postponed until Section 7.

COROLLARY 4·4. F_n *has no Thurston obstructions for all* n *large enough.*

Proof. According to [**14**] (or [**1**] if F_n has holomorphic disks) the canonical Thurston obstruction for F_n is defined to be the set of all homotopy classes of non-peripheral curves γ with $l_{\phi_{m,n}}(\gamma, P_{F_n}) \to 0$ (or $l_{\phi_{m,n}}(\gamma, P_{F_n} \cup \cup_i \overline{D_i}) \to 0$) as $m \to \infty$. By [**14**, Theorem 1·1] (or by [**1**, theorem 1]), if F_n has Thurston obstructions, then F_n has a canonical Thurston obstruction. So it suffices to prove that F_n has no canonical Thurston obstructions for all n large enough. In the case that there are no holomorphic disks, $P_{F_n} = B_n$. By the second assertion of Lemma 4·3, F_n has no canonical Thurston obstructions. In the case that there are holomorphic disks, since each D_i contains two points $\{a_i, b_i\} \subset B_n$, any $(\phi_{m,n}, P_{F_n} \cup \cup_i \overline{D_i})$-geodesic ξ is non-peripheral in $\widehat{\mathbb{C}} \setminus B_n$, and therefore, is homotopic to some $(\phi_{m,n}, B_n)$-geodesic γ in $\widehat{\mathbb{C}} \setminus B_n$. Since $B_n \subset P_{F_n} \cup \cup_i \overline{D_i}$, by the second assertion of Lemma 4·3, we have

$$l_{\phi_{m,n}}(\xi, P_{F_n} \cup \cup_i \overline{D_i}) > l_{\phi_{m,n}}(\gamma, B_n) > \delta_n.$$

Again F_n has no canonical Thurston obstructions. The proof of Corollary 4·4 is completed.

Let T_{F_n} denote the Teichmüller space modelled on $(\widehat{\mathbb{C}}, P_{F_n})$ (or modelled on $(\widehat{\mathbb{C}}, P_{F_n} \cup \cup_i \overline{D_i})$ if there are holomorphic disks). By Corollary 4·4 F_n has no Thurston obstructions for all n large enough. Now let us fix an n large enough. According to the proofs of Thurston's characterisation theorem and its extension to sub-hyperbolic case (see [**3**] and [**7**] or Section 2 of this paper), $[\tau_{m,n}]$ converges to $[\tau_n]$ as $m \to \infty$, where $[\tau_{m,n}]$ denotes the point in T_{F_n} represented by $\tau_{m,n}$, and $[\tau_n]$ denotes a point in T_{F_n} which is fixed by σ_{F_n}. Let $\lambda_{m,n} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ denote the Teichmüller map of $[\tau_{m,n}]$ which fixes 0, 1 and ∞ and is conformal in all the D_i if there are holomorphic disks. By the uniqueness of the Techmüller map and the symmetry of F_n, $\lambda_{m,n}$ must be symmetric about $\mathbb{T}$, that is, $\lambda_{m,n}(z)^* = \lambda_{m,n}(z^*)$. Since $[\tau_{m,n}]$ converges, all $\lambda_{m,n}$ must be $K(n)$-quasiconformal with $K(n) > 1$ depending only on n. Since $\phi_{m,n}$ is isotopic to $\lambda_{m,n}$ rel P_{F_n} (or rel $P_{F_n} \cup \cup_i \overline{D_i}$), we can lift the isotopy between $\phi_{m,n}$ and $\lambda_{m,n}$ to the isotopy between $\phi_{m+1,n}$ and some quasiconformal homeomorphism of the sphere, say $\eta_{m,n}$, through $\phi_{m,n} \circ F_n = G_{m,n} \circ \phi_{m+1,n}$. In this way we get

$$G_{m,n} = \lambda_{m,n} \circ F_n \circ \eta_{m,n}^{-1}.$$

Since $\lambda_{m,n}$ is $K(n)$-quasiconformal and F_n is K_0-quasiregular (see Remark 3·1), it follows that $\eta_{m,n}$ is $K_0K(n)$-quasiconformal. By the symmetry of $\lambda_{m,n}$ and F_n, the complex structure

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associated to $\eta_{m,n}$ is symmetric about $\mathbb{T}$. Since $\eta_{m,n}$ is isotopic to $\phi_{m+1,n}$ rel P_{F_n} (or rel $P_{F_n} \cup \cup_i \overline{D_i}$) and $\phi_{m+1,n}$ fixes 0, 1 and ∞, $\eta_{m,n}$ fixes 0, 1 and ∞ also. It follows that $\eta_{m,n}$ must also be symmetric about $\mathbb{T}$. By taking subsequence if necessary, we may assume that $\lambda_{m,n}$ and $\eta_{m,n}$ converge uniformly to ϕ_n and ψ_n, both of which are quasiconformal homeomorphisms of the sphere and are symmetric about $\mathbb{T}$ and fix 0, 1 and ∞. It follows that

$$G_n = \phi_n \circ F_n \circ \psi_n^{-1} \tag{4·4}$$

is a Blaschke product. Since ψ_n and ϕ_n represent the same point $[\tau_n]$ in T_{F_n}, they are isotopic to each other rel P_{F_n} (or rel $P_{F_n} \cup \cup_i \overline{D_i}$). Note that by symmetry $\phi_n(\mathbb{T}) = \psi_n(\mathbb{T}) = \mathbb{T}$.

Let $\mathrm{diam}_{\widehat{\mathbb{C}}}(\cdot)$ and $\mathrm{dist}_{\widehat{\mathbb{C}}}(\cdot, \cdot)$ denote respectively the diameter and the distance with respect to the spherical metric.

COROLLARY 4·5. *There exists a $K > 1$ independent of n such that the following assertions hold for all n large enough (In the case that F_n has no holomorphic disks, the first four assertions are just disregarded).*

(i) For every D_i, $\mathrm{diam}_{\widehat{\mathbb{C}}}(\phi_n(D_i)) > 1/K$.
(ii) For every $z \in P_1$ and every D_i, $\mathrm{dist}_{\widehat{\mathbb{C}}}(\phi_n(z), \phi_n(D_i)) > 1/K$.
(iii) For every D_i, $\mathrm{dist}_{\widehat{\mathbb{C}}}(\phi_n(D_i), \mathbb{T}) > 1/K$.
(iv) For every two distinct D_i and D_j, $\mathrm{dist}_{\widehat{\mathbb{C}}}(\phi_n(D_i), \phi_n(D_j)) > 1/K$.
(v) For every $z \in P_1$, $\mathrm{dist}_{\widehat{\mathbb{C}}}(\phi_n(z), \mathbb{T}) > 1/K$.
(vi) For every two distinct z and w in P_1, $\mathrm{dist}_{\widehat{\mathbb{C}}}(\phi_n(z), \phi_n(w)) > 1/K$.

Proof. Let us assume that F_n has holomorphic disks. Otherwise, the proof is the same and is even simpler. Note that $\lambda_{m,n}|P_{F_n} \cup \cup_i \overline{D_i} = \phi_{m,n}|P_{F_n} \cup \cup_i \overline{D_i}$ and $\lambda_{m,n} \to \phi_n$ uniformly with respect to the spherical metric. So we need only to prove the existence of some $K > 1$ independent of m and n so that the assertions hold for the maps $\phi_{m,n}$ with all $m \ge 0$ and all n large enough. The lemma then follows by letting $m \to \infty$ and taking the limit. But this can be easily deduced from the following two facts.

Fact (1) By the first assertion of Lemma 4·3, there is a $\delta > 0$ independent of n and m such that the length of any simple closed geodesic in $\widehat{\mathbb{C}} \setminus \phi_{m,n}(A_n)$ is greater than δ.

Fact (2) All the maps $\phi_{m,n}$ are conformal in $H_i \cup \overline{D_i}$ for each D_i. Thus for any $z \notin H_i \cup \overline{D_i}$, by Koebe's distortion theorem, we have

$$\mathrm{dist}_{\widehat{\mathbb{C}}}(\phi_{m,n}(a_i), \phi_{m,n}(b_i)) \asymp \mathrm{diam}_{\widehat{\mathbb{C}}}(\phi_{m,n}(D_i)) \preceq \mathrm{dist}_{\widehat{\mathbb{C}}}(\phi_{m,n}(z), \phi_{m,n}(D_i)).$$

Since $\mathbb{T} \cap (H_i \cup \overline{D_i}) = D_j \cap (H_i \cup \overline{D_i}) = \emptyset$ for $i \neq j$, by Koebe's distortion theorem again, we have

$$\mathrm{dist}_{\widehat{\mathbb{C}}}(\phi_{m,n}(a_i), \phi_{m,n}(b_i)) \preceq \mathrm{dist}_{\widehat{\mathbb{C}}}(\phi_{m,n}(D_i), \mathbb{T})$$

and

$$\mathrm{dist}_{\widehat{\mathbb{C}}}(\phi_{m,n}(a_i), \phi_{m,n}(b_i)) \preceq \mathrm{dist}_{\widehat{\mathbb{C}}}(\phi_{m,n}(D_i), \phi_{m,n}(D_j)).$$

If any of the above assertions were nor true, there would be a pair of points in $\phi_{m,n}(A_n)$ whose spherical distance could be arbitrarily small. This is obvious for (vi). For (i), (ii), (iii) and (iv) we use Fact (2). For (v) the two points are just $\phi_{m,n}(z)$ and $\phi_{m,n}(z^*)$

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since $\mathrm{dist}_{\widehat{\mathbb{C}}}(\phi_{m,n}(z), \phi_{m,n}(z^*)) \preceq \mathrm{dist}_{\widehat{\mathbb{C}}}(\phi_{m,n}(z), \mathbb{T})$. Since $\{0, 1, \infty\} \subset \phi_{m,n}(A_n)$, (Because $\{0, 1, \infty\} \subset A_n$ by Remark 4·1 and $\phi_{m,n}$ fixes 0, 1 and ∞), this will produce a simple closed geodesic in $\widehat{\mathbb{C}} \setminus \phi_{m,n}(A_n)$ whose length could be arbitrarily small. This contradicts Fact (1).

COROLLARY 4·6. *Let $\mathcal{R}_{2d-1}$ denote the space of all the rational maps of degree $2d-1$ endowed with the metric $d(f, g) = \max_{z\in\widehat{\mathbb{C}}} \mathrm{dist}_{\widehat{\mathbb{C}}}(f(z), g(z))$. Then the sequence $\{G_n\}$ obtained in (4·4) lies in a compact subset of $\mathcal{R}_{2d-1}$.*

Proof. Note that each G_n has the following form

$$\lambda \prod_{i=1}^{d} \frac{z - p_i}{1 - \bar{p}_i z} \prod_{j=1}^{d-1} \frac{z - q_j}{1 - \bar{q}_j z}$$

with $|p_i| < 1$ for $1 \le i \le d$ and $|q_j| > 1$ for $1 \le j \le d-1$ and $|\lambda| = 1$, see [18, section 2]. So it suffices to prove that the spherical distance between any pole and any zero of G_n has a positive lower bound independent of n. Since $d(G_{m,n}, G_n) \to 0$, it suffices to prove that there is a $\delta > 0$ so that the spherical distance between any pole and any zero of $G_{m,n}$ is greater than δ for all $m \ge 0$ and all n large enough. In Lemma 4·2 let $N_0 = 1$ and define the set $\tilde{A}_n^1 = F_n^{-1}(A_n^1)$. Let $\mathcal{P}_{m,n}$ and $\mathcal{Z}_{m,n}$ denote the sets of poles and zeros of $G_{m,n}$ respectively. Since

$$\{0, 1, \infty\} \subset A_n \subset A_n^1$$

by Remark 4·1 and Lemma 4·2, we have $F_n^{-1}(\{0, \infty\}) \subset F_n^{-1}(A_n^1) = \tilde{A}_n^1$. Since $\phi_{m,n}$ fixes 0, 1 and ∞, from $\phi_{m,n} \circ F_n = G_{m,n} \circ \phi_{m+1,n}$, we have

$$\mathcal{P}_{m,n} \cup \mathcal{Z}_{m,n} = \phi_{m+1,n}(F_n^{-1}(\{0, \infty\}) \subset \phi_{m+1,n}(\tilde{A}_n^1).$$

Since $\{0, 1, \infty\} \subset A_n \subset \tilde{A}_n^1$ by Lemma 4·2 and since $\phi_{m+1,n}$ fixes 0, 1 and ∞, we have $\{0, 1, \infty\} \subset \phi_{m+1,n}(\tilde{A}_n^1)$. Suppose there were no such positive lower bound. Then there would be two points in $\mathcal{P}_{m,n} \cup \mathcal{Z}_{m,n}$ whose spherical distance could be arbitrarily small. This would produce a $(\phi_{m+1,n}, \tilde{A}_n^1)$-geodesic η whose length could be arbitrarily small. By Lemma 4·2

$$F_n : \widehat{\mathbb{C}} \setminus \tilde{A}_n^1 \longrightarrow \widehat{\mathbb{C}} \setminus A_n^1$$

is a covering map. This, together with Lemma 8·2, implies that $\gamma = F_n(\eta)$ must be a $(\phi_{m,n}, A_n^1)$-geodesic whose length whose length could be arbitrarily small. This contradicts the first assertion of Lemma 4·3.

By Corollary 4·6 $\{G_n\}$ lies in a compact set of $\mathcal{R}_{2d-1}$. We can thus assume that

$$G_n \longrightarrow G \tag{4·5}$$

by taking a subsequence if necessary. Since the rotation number of $G_n|\mathbb{T}$ is θ_n and $\theta_n \to \theta$, $G|\mathbb{T}$ is a critical circle homeomorphism with rotation number θ ([8, Proposition 11·1·6]). Since θ is of bounded type, by Herman–Swiatek's theorem [12], there is a quasi-symmetric circle homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h(1) = 1$ and

$$G|\mathbb{T} = h^{-1} \circ R_\theta \circ h. \tag{4·6}$$

Let $\phi_n, \psi_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the pair of homeomorphisms in (4·4).

LEMMA 4·7. *$\phi_n|\mathbb{T} \to h^{-1}$ and $\psi_n|\mathbb{T} \to h^{-1}$ uniformly.*

Proof. Let us simply write $\phi_n|\mathbb{T}$ and $\psi_n|\mathbb{T}$ as ϕ_n and ψ_n respectively. We first prove Lemma 4·7 by assuming that ϕ_n and ψ_n converge uniformly. Let ϕ and ψ be the limit maps respectively. Since the convergence is uniform, it follows that both ϕ and ψ are continuous. Note that $1 \in O_n$ and $\phi_n(1) = \psi_n(1) = 1$. Since F_n is conjugate to G_n on O_n through $\phi_n|O_n = \psi_n|O_n$, it follows that $\phi_n(F_n^k(1)) = \psi_n(F_n^k(1)) = G_n^k(1)$ for all $k \geq 0$. Since $F_n|\mathbb{T} \to R_\theta$ and $G_n \to G$, by letting $n \to \infty$ we get $\phi(e^{2k\pi i\theta}) = \psi(e^{2k\pi i\theta}) = G^k(1)$ for all $k \geq 0$. But on the other hand, from $G|\mathbb{T} = h^{-1} \circ R_\theta \circ h$ and $h(1) = 1$ we get $h^{-1}(e^{2k\pi i\theta}) = G^k(1)$ for all $k \geq 0$. It follows that ϕ, ψ and h^{-1} coincide on $\{e^{2\pi ik\theta}\}_{k=0}^{\infty}$, which is a dense subset of $\mathbb{T}$. This implies that $\phi = \psi = h^{-1}$ by the continuity.

It remains to show that ϕ_n and ψ_n converge uniformly. Let us do this only for ϕ_n since the same argument works for ψ_n. For each $N \geq 0$, define the orbit segment $\mathcal{O}_N(G)$ by

$$\mathcal{O}_N(G) = \{1, G^1(1), \ldots, G^N(1)\}.$$

In the same way one can define $\mathcal{O}_N(G_n)$, $\mathcal{O}_N(F)$ and $\mathcal{O}_N(F_n)$. Let $\epsilon > 0$ be an arbitrary small number. Since $G|\mathbb{T}$ is a circle homeomorphism with irrational rotation number, there exists an $N = N(\epsilon)$ such that the length of each component of $\mathbb{T} \setminus \mathcal{O}_N(G)$ is less than $\epsilon/4$. Since $F_n \to F$ and $G_n \to G$ uniformly, there exists an $M > 1$ such that the following three assertions hold:

(1) for all $m \geq M$, the length of each component of $\mathbb{T} \setminus \mathcal{O}_N(G_m)$ is less than $\epsilon/3$;
(2) for all $n \geq M$, the points in the orbit segments, $\mathcal{O}_N(G)$, $\mathcal{O}_N(G_n)$, $\mathcal{O}_N(F)$ and $\mathcal{O}_N(F_n)$ have the same order;
(3) suppose $m, n \geq M$. Then for any component I of $\mathbb{T} \setminus \mathcal{O}_N(F_n)$, let $\tilde{I}$ be the corresponding component of $\mathbb{T} \setminus \mathcal{O}_N(F_m)$, then $\overline{I} \subset \overline{\tilde{I}^l \cup \tilde{I} \cup \tilde{I}^r}$, where $\tilde{I}^l$ and $\tilde{I}^r$ denote the two components of $\mathbb{T} \setminus \mathcal{O}_N(F_m)$ which are adjacent to $\tilde{I}$ from the left and right respectively. Similarly, for any component J of $\mathbb{T} \setminus \mathcal{O}_N(G_n)$, let $\tilde{J}$ be the corresponding component of $\mathbb{T} \setminus \mathcal{O}_N(G_m)$, then $\overline{J} \subset \overline{\tilde{J}^l \cup \tilde{J} \cup \tilde{J}^r}$, where $\tilde{J}^l$ and $\tilde{J}^r$ denote the two components of $\mathbb{T} \setminus \mathcal{O}_N(G_m)$ which are adjacent to $\tilde{J}$ from the left and right respectively.

Now for any $z \in \mathbb{T}$ and any $m, n \geq M$, let I be the component of $\mathbb{T} \setminus \mathcal{O}_N(F_n)$ such that $z \in \overline{I}$. Since $\overline{I} \subset \overline{\tilde{I}^l \cup \tilde{I} \cup \tilde{I}^r}$, we have $z \in \overline{\tilde{I}^l \cup \tilde{I} \cup \tilde{I}^r}$. Let $J = \phi_n(I)$ and $\tilde{J} = \phi_m(\tilde{I})$. It follows that $\phi_n(z) \in \overline{J} \subset \overline{\tilde{J}^l \cup \tilde{J} \cup \tilde{J}^r}$. On the other hand, $\phi_m(z) \in \phi_m(\overline{\tilde{I}^l \cup \tilde{I} \cup \tilde{I}^r}) = \overline{\tilde{J}^l \cup \tilde{J} \cup \tilde{J}^r}$. Then $|\phi_n(z) - \phi_m(z)| \leq |\tilde{J}^l| + |\tilde{J}| + |\tilde{J}^r| < \epsilon$. This proves that ϕ_n converges uniformly. The proof of Lemma 4·7 is completed.

5. *Proof of the existence part of the main theorem*

Let G be the Blaschke product obtained in (4·5).

LEMMA 5·1. *There exist a pair of homeomorphisms $\phi, \psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which fix 0, 1 and ∞ such that:*

(i) $\phi \circ F = G \circ \psi$;
(ii) ϕ is isotopic to ψ rel $P_F \cup \cup_i \overline{D_i}$ and $\phi|D_i = \psi|D_i$ is holomorphic for each D_i.

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Proof. Let ϕ_n, ψ_n and G_n be the maps in (4·4). Note that for each D_i there is an annulus H_i surrounding it such that $\phi_n|\overline{D_i} \cup H_i = \psi_n|\overline{D_i} \cup H_i$ is conformal. By Corollary 4·5 and by taking a subsequence if necessary, we may assume that for each D_i, there is a domain containing $\overline{D_i}$ on which $\phi_n = \psi_n$ converges uniformly to a univalent map. Let $U_i = \lim_{n\to\infty} \phi_n(D_i)$. Then by Corollary 4·5 and Lemma 4·7, it follows that when restricted to the set $P_F \cup \cup_i \overline{D_i}$ (Note that $P_{F_n} \setminus \mathbb{T} = P_F \setminus \mathbb{T}$), ϕ_n and ψ_n converge uniformly to some homeomorphism

$$\sigma : P_F \cup \cup_i \overline{D_i} \longrightarrow P_G \cup \cup_i \overline{U_i}. \tag{5·1}$$

Note that $\sigma|\mathbb{T} = h^{-1}$ and σ is holomorphic in each D_i, and moreover, $\sigma \circ F = G \circ \sigma$ holds on $P_F \cup \cup_i \overline{D_i}$.

The maps ϕ and ψ are constructed by perturbing ϕ_n and ψ_n for a large n. Take $\delta > \eta > 0$ small such that $\overline{B_{\mathbb{T}}(\delta)}$, $\overline{B_{D_i}(\delta)}$ and $\overline{B_z(\delta)}$, $z \in P_1$, are all disjoint with each other. Here $B_*(\delta)$ denotes the δ-neighbourhood of the object. Let Λ and Ξ denote the δ-neighbourhood and η-neighbourhood of $P_F \cup \cup_i \overline{D_i}$ respectively. Then $P_{F_n} \cup \cup_i \overline{D_i} \subset \Xi \subset \overline{\Xi} \subset \Lambda$. Since $F_n \to F$ uniformly, $\overline{F_n(F^{-1}(\Xi))} = F_n(F^{-1}(\overline{\Xi})) \subset \Lambda$ for all n large enough. By deforming ϕ_n through homotopy rel $P_{F_n} \cup \cup_i \overline{D_i}$, we can make $\phi_n|\Lambda$ converge uniformly to a homeomorphism $\chi : \Lambda \to \chi(\Lambda)$. It follows that $\{\phi_n\}$ is equicontinuous on $\overline{F_n(F^{-1}(\Xi))}$. This, together with $\phi_n \circ F_n = G_n \circ \psi_n$, $F_n \to F$ and $G_n \to G$ uniformly with respect to the spherical metric, implies that $\{\psi_n\}$ must be equicontinuous on $F^{-1}(\Xi)$. By taking a subsequence if necessary we can assume that $\psi_n|F^{-1}(\Xi)$ converges uniformly to a continuous map $\tau : F^{-1}(\Xi) \to \tau(F^{-1}(\Xi))$, and moreover, the following diagram commutes.

$$\begin{array}{ccc} F^{-1}(\Xi) & \xrightarrow{\ \tau\ } & \tau(F^{-1}(\Xi)) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \Xi & \xrightarrow{\ \chi\ } & \chi(\Xi). \end{array}$$

Since ψ_n maps any critical point of F_n to the corresponding critical point of G_n with the same local degree, τ maps any critical point of F to the corresponding critical point of G with the same local degree. This, together with the last diagram, implies that τ is locally homeomorphic. Since $0, 1, \infty \in F^{-1}(\Xi)$ and ψ_n fixes 0, 1 and ∞, τ fixes 0, 1 and ∞. It is also clear that $\chi|P_F \cup \cup_i \overline{D_i} = \tau|P_F \cup \cup_i \overline{D_i} = \sigma$.

Let Π denote the set of the critical values of F. Take $0 < \rho < \eta$. Then all the closed disks $\overline{B_v(\rho)}$, $v \in \Pi$, are disjoint with each other. Let $X = \bigcup_{v\in\Pi} B_v(\rho)$. Then $\overline{X} \subset \Xi$. Let $Y = F^{-1}(X)$. Then $Y \subset F^{-1}(\Xi)$ and ∂Y is the union of finitely many disjoint Jordan curves.

Take a small $\epsilon > 0$ and a large n, which, in each of the following steps, will be required to be even smaller and larger respectively. By perturbing ϕ_n we can construct a homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which fixes 0, 1 and ∞, such that $\phi = \chi$ holds on Ξ and dist $(\phi, \phi_n) < \epsilon$.

Now let us construct the homeomorphism $\psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Let $x \in \widehat{\mathbb{C}} \setminus Y$ be an arbitrary point. Then $F(x) \notin \Pi$ and $\phi \circ F(x)$ is not a critical value of G. Since the degree of G is $2d-1$, $G^{-1}(\phi \circ F(x))$ contains exactly $2d-1$ points. Define $\psi(x)$ to be the one in $G^{-1}(\phi \circ F(x))$ which is closest to $\psi_n(x) \in G_n^{-1}(\phi_n \circ F_n(x))$. Such definition does not cause any ambiguity provided that $\epsilon > 0$ is small and n is large enough. This is because by taking $\epsilon > 0$ small and n large, the set $G^{-1}(\phi \circ F(x))$ can be arbitrarily close to the set $G_n^{-1}(\phi_n \circ F_n(x))$, and on the other hand, for $x \in \widehat{\mathbb{C}} \setminus Y$ any two points in $G^{-1}(\phi \circ F(x))$ are uniformly bounded away

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from each other. So ψ is well defined on $\widehat{\mathbb{C}} \setminus Y$ and is locally homeomorphic on $\widehat{\mathbb{C}} \setminus Y$. For $x \in \overline{Y}$, let $\psi(x) = \tau(x)$. Since τ is the limit of ψ_n, the two definitions of ψ coincide on ∂Y. Since τ is locally homeomorphic on $F^{-1}(\Xi)$ and $\overline{Y} \subset F^{-1}(\Xi)$, it follows that $\psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is locally homeomorphic on $\widehat{\mathbb{C}}$ and is thus a covering map, and moreover, $\phi \circ F = G \circ \psi$ holds on the sphere. By Riemann–Hurwitz formula, the degree of a covering map from the sphere to itself must be equal to one. So $\psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homeomorphism. Since $\psi | F^{-1}(\Xi) = \tau$ and τ fixes 0, 1 and ∞, ψ fixes 0, 1 and ∞. By the construction of ψ, it follows that ψ can be arbitrarily close to ψ_n provided that $\epsilon > 0$ is small and n is large enough.

By the construction $\phi | P_F \cup \cup_i \overline{D_i} = \psi | P_F \cup \cup_i \overline{D_i} = \sigma$ and when restricted to each D_i, ϕ and ψ are holomorphic. It remains to show that there is an isotopy between ϕ and ψ rel $P_F \cup \cup_i \overline{D_i}$. Since for orientation-preserving surface homeomorphisms, homotopy implies isotopy (see [5, appendix C3]), it suffices to show the existence of a homotopy between ϕ and ψ rel $P_F \cup \cup_i \overline{D_i}$.

Since $\phi_n | \mathcal{O}_n = \psi_n | \mathcal{O}_n$ and since each component of $\mathbb{T} \setminus \mathcal{O}_n$ can be arbitrarily small provided that n is large enough, we can construct a homeomorphism $\omega_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by deforming ψ_n in a small neighborhood of $\mathbb{T}$ so that $\omega_n | \mathbb{T} = \phi_n | \mathbb{T}$ and ω_n is homotopic to ψ_n rel $P_{F_n} \cup \cup_i \overline{D_i}$, and moreover, ω_n can be arbitrarily close to ψ_n and thus arbitrarily close to ψ provided that $\epsilon > 0$ is small and n is large enough.

Let $H(t, \cdot)$, $0 \le t \le 1$, be the homotopy between ϕ_n and ω_n. Since $\phi_n | \mathbb{T} = \omega_n | \mathbb{T}$, H can be constructed such that $H(t, \cdot) | \mathbb{T} = \omega_n | \mathbb{T} = \phi_n | \mathbb{T}$ for $0 \le t \le 1$. Thus ω_n and ϕ_n are homotopic to each other rel $P_F \cup \cup_i \overline{D_i}$. Let $\xi = \phi \circ \phi_n^{-1}$. By taking $\epsilon > 0$ small and n large enough, ξ and ω_n can be arbitrarily close to the identity and ψ respectively, and thus $(\xi \circ \omega_n) \circ \psi^{-1}$ can be arbitrarily close to the identity. Since $(\xi \circ \omega_n) \circ \psi^{-1} | P_G \cup \cup_i \overline{U_i} = \mathrm{id}$, from Lemma 8·5 it follows that $(\xi \circ \omega_n) \circ \psi^{-1}$ is homotopic to id rel $P_G \cup \cup_i \overline{U_i}$. So ψ is homotopic to $\xi \circ \omega_n$ rel $P_F \cup \cup_i \overline{D_i}$, which is homotopic to $\xi \circ \phi_n = \phi$ rel $P_F \cup \cup_i \overline{D_i}$, since ω_n is homotopic to ϕ_n rel $P_F \cup \cup_i \overline{D_i}$.

The proof of Lemma 5·1 is completed.

Remark 5·1. Note that $P_F \setminus (\mathbb{T} \cup \cup_i \overline{D_i})$ is a finite set and each ∂D_i is a real analytic curve. Since ϕ is quasisymmetric on $\mathbb{T}$ and conformal in an open neighborhood containing each D_i, we can assume that ϕ is quasiconformal by deforming it rel $P_F \setminus (\mathbb{T} \cup \cup_i \overline{D_i})$, and thus ψ is also quasiconformal by lifting ϕ through $\phi \circ F = G \circ \psi$.

Let $h : \mathbb{T} \to \mathbb{T}$ be the quasisymmetric circle homeomorphism in Lemma 4·7. Let $H : \Delta \to \Delta$ be the Ahlfors–Bers quasiconformal extension of h. Define

$$\widehat{G}(z) = \begin{cases} G(z) & \text{for } |z| \ge 1, \\ H^{-1} \circ R_\theta \circ H(z) & \text{for } |z| < 1. \end{cases} \tag{5·2}$$

Let U_i, $1 \le i \le l$, denote the components of $\widehat{G}^{-1}(\Delta)$ other than Δ. Let ϕ and ψ be the homeomorphisms guaranteed by Lemma 5·1. Define

$$\widetilde{G}(z) = \begin{cases} H^{-1} \circ f \circ \psi^{-1} & \text{for } z \in U_i,\ 1 \le i \le l, \\ \widehat{G}(z) & \text{otherwise.} \end{cases} \tag{5·3}$$

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Define homeomorphisms λ, $\rho : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by setting

$$\lambda(z) = \begin{cases} \phi(z) & \text{for } |z| \ge 1 \\ H^{-1}(z) & \text{for } |z| < 1, \end{cases} \tag{5·4}$$

and

$$\rho(z) = \begin{cases} \psi(z) & \text{for } |z| \ge 1 \\ H^{-1}(z) & \text{for } |z| < 1, \end{cases} \tag{5·5}$$

By (3·1), (5·2-5·5) and $\phi \circ F = G \circ \psi$, we have

$$\lambda \circ f = \widetilde{G} \circ \rho. \tag{5·6}$$

Let μ_0 denote the complex structure in Δ obtained by pulling back the standard one by H. We then pull back μ_0 to the whole sphere by the iterations of $\widetilde{G}$. Let Φ be the qc homeomorphism of the sphere which fixes 0, 1 and ∞ and solves the Beltrami equation given by μ. It follows that $g = \Phi \circ \widetilde{G} \circ \Phi^{-1}$ is a rational map and belongs to R_θ^{geom}. From (5·6) we have

$$(\Phi \circ \lambda) \circ f = g \circ (\Phi \circ \rho).$$

Since λ is isotopic to ρ rel $P_f \cup \cup_i \overline{D_i}$ by Lemma 5·1, $\Phi \circ \lambda$ is isotopic to $\Phi \circ \rho$ rel $P_f \cup \cup_i \overline{D_i}$. Note also that $\Phi \circ \lambda|\Delta = \Phi \circ \rho|\Delta$ and $\Phi \circ \lambda|D_i = \Phi \circ \rho|D_i$ are all holomorphic. This completes the proof of the existence part.

6. *Proof of the uniqueness part of the main theorem*

Let $\widetilde{G}$, Φ and g be the maps obtained at the end of Section 5. Let J_g denote the Julia set of g.

LEMMA 6·1. *J_g has zero Lebesgue measure.*

Let us first prove the uniqueness by assuming Lemma 6·1. Suppose f is also combinatorially equivalent to some $h \in R_\theta^{geom}$. By Definition 1·3 we get the following diagram where ϕ and ϕ' are isotopic to ψ and ψ' respectively rel $P_f \cup \cup_i \overline{\Delta}$, and all of which are holomorphic on $\Delta \cup \cup_i D_i$.

$$\begin{array}{ccccc} \widehat{\mathbb{C}} & \xleftarrow{\psi} & \widehat{\mathbb{C}} & \xrightarrow{\psi'} & \widehat{\mathbb{C}} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\ \widehat{\mathbb{C}} & \xleftarrow{\phi} & \widehat{\mathbb{C}} & \xrightarrow{\phi'} & \widehat{\mathbb{C}} \end{array}$$

Let P_g and P_h, D_g and D_h denote the post-critical sets and the Siegel disks of f and g respectively. Let $\phi_1 = \phi' \circ \phi^{-1}$ and $\phi_2 = \psi' \circ \psi^{-1}$. Then

(1) $\phi_1 \circ g = h \circ \phi_2$, and
(2) ϕ_1 is isotopic to ϕ_2 rel P_g, and
(3) when restricted to D_g and an open neighbourhood $U = \cup_i \phi(D_i)$ of all attracting cycles of g, $\phi_1 = \phi_2$ is holomorphic.

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Note that ϕ and ψ can be chosen quasiconformal by the proof of the existence part. Since ϕ' can be chosen quasiconformal by C^0 perturbation, and since f can be assumed quasi-regular by Remark 3·1, ψ' is therefore quasiconformal by pulling back ϕ' by f.

Suppose $\phi_k : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasiconformal homeomorphism which is isotopic to ϕ_1 rel $P_g \cup \overline{(D_g \cup U)}$ with $k \geq 2$. Since $\phi_1 \circ g = h \circ \phi_2$, we can define ϕ_{k+1} by lifting ϕ_k through

$$\phi_k \circ g = h \circ \phi_{k+1}. \tag{6·1}$$

Then ϕ_{k+1} is isotopic to ϕ_2 and is thus isotopic to ϕ_1 rel $P_g \cup \overline{(D_g \cup U)}$. By induction we get a sequence of quasiconformal homeomorphisms $\phi_k : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ all of which are isotopic to ϕ_1 rel $P_g \cup \overline{(D_g \cup U)}$ and satisfy (6·1). Since g and h are holomorphic, the quasiconformal constant of each ϕ_k is bounded by that of ϕ_1. Let μ_k be the Beltrami coefficient of ϕ_k. Note that the forward orbit of any point in the Fatou set of g either converges to some attracting cycle of g, or eventually enters D_g. Since ϕ_1 is conformal in $D_g \cup U$, it follows that $\mu_k \to 0$ on the Fatou set of g. Since J_g has zero Lebesgure measure by Lemma 6·1, $\mu_k \to 0$ a.e. It follows that ϕ_k converges to some Möbius map uniformly in the sphere. From (6·1) it follows that g is conjugate to h through a Möbius map. This implies the uniqueness assertion of the main theorem.

Proof of Lemma 6·1 Let $J_{\widetilde{G}} = \Phi^{-1}(J_g)$. Since Φ is quasiconformal, it suffices to prove that $J_{\widetilde{G}}$ has zero measure. Suppose it were not true.

Let z_0 be a Lebesgue point of $J_{\widetilde{G}}$. For $n \geq 1$ let $z_n = G^n(z_0)$ (Note that $\widetilde{G} = G$ on $J_{\widetilde{G}}$). By [**9**, proposition 1·14], $\{z_n\}$ accumulates to $P_{\widetilde{G}} \cap J_{\widetilde{G}}$. Note that $(P_{\widetilde{G}} \cap J_{\widetilde{G}}) \setminus \mathbb{T}$ is either empty or contains finitely many points whose forward orbits eventually enter into some repelling periodic cycles or intersect $\mathbb{T}$. It follows that $\{z_n\}$ accumulates to $\mathbb{T}$. The idea of the proof is adapted from [**11**] and [**17**]. If L is a ray with one end point $z \in \mathbb{T}$, let us use $\angle(L, \mathbb{T})$ to denote the smaller angle formed by L and $\mathbb{T}$ at z. Then $0 \leq \angle(L, \mathbb{T}) \leq \pi/2$.

Claim. There exist a critical point $c \in \mathbb{T}$ of G, an $0 < \alpha < 1/2$, and a cone Ω attached to $\mathbb{T}$ at c from the outside of Δ and bounded by two rays L and R starting from c, and a subsequence z_{n_j}, such that $\angle(L, \mathbb{T}) = \angle(R, \mathbb{T}) = \alpha\pi$, and $z_{n_j} \in \Omega$ for all $j \geq 1$.

Let us first prove the lemma by assuming the claim. In the following proof, we abuse the notations A_j and B_j to denote topological disks, which have different meanings in other sections. Since c is a critical point, there is another cone Ω' attached at c from the outside of Δ and bounded by two rays L' and R' such that $\angle(L', \mathbb{T}), \angle(R', \mathbb{T}) > \beta\pi$ with $0 < \beta < 1/2$, and moreover, all the points in $B_c(r) \cap \Omega'$ are mapped into Δ with $r > 0$ being some small number. Here and in the following we use diam$(\cdot)$ and dist$(\cdot, \cdot)$ to denote the diameter and the distance with respect to the Euclidean metric. Then for each n_j large enough, we can take a small disk B_j, such that:

(1) $B_j \subset B_c(r) \cap \Omega'$ and thus $G(B_j) \subset \Delta$;
(2) $\mathrm{dist}(B_j, z_{n_j}) \preceq \mathrm{dist}(B_j, \mathbb{T}) \asymp \mathrm{diam}(B_j) \asymp \mathrm{dist}(z_{n_j}, \mathbb{T})$.

From (2) we can take a Jordan domain A_j which is disjoint with P_G and contains both B_j and z_{n_j} such that

$$\mathrm{diam}(A_j) \asymp \mathrm{dist}(A_j, \mathbb{T}) \asymp \mathrm{diam}(B_j). \tag{6·2}$$

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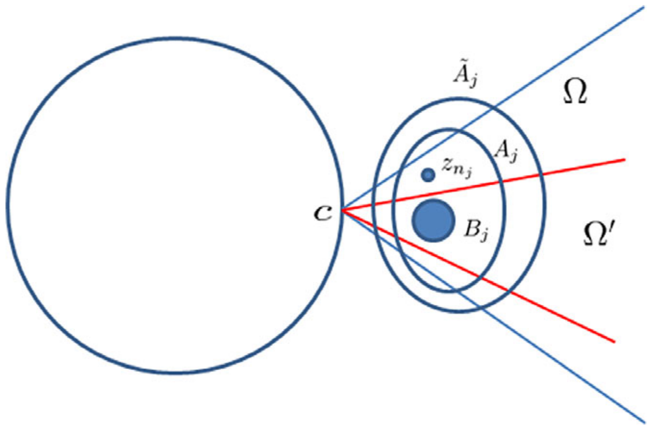

Fig. 2. Ω, Ω', z_{n_j}, B_j, A_j and $\tilde{A}_j$.

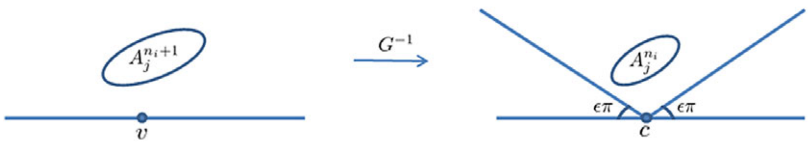

Fig. 3. $\mathrm{diam}_X(A_j^{n_i}) < \delta \cdot \mathrm{diam}_X(A_j^{n_i+1})$.

Please see Figure 2 for an illustration of the above construction. Let X denote the component of $\widehat{\mathbb{C}} \setminus P_G$ which contains z_{n_j}. It follows that $\mathrm{diam}_X(A_j) \asymp 1$ by (6·2) where $\mathrm{diam}_X(\cdot)$ denotes the diameter with respect to the hyperbolic metric in X. Now we pull back A_j along the orbit $z_0, \ldots, z_{n_j}$, and denote the component of $G^{-k}(A_j)$ which contains z_{n_j-k} by $A_j^{n_j-k}$. In particular, $A_j^{n_j} = A_j$.

Let X_k be the component of $G^{-k}(X)$ containing z_{n_j-k}. Then $G^k : X_k \to X$ is a holomorphic covering map preserving the hyperbolic metric and $X_k \subset X$. Note that $A_j^{n_j-k} \subset X_k$. It follows that

$$\mathrm{diam}_X\left(A_j^{n_j-k}\right) \le \mathrm{diam}_{X_k}\left(A_j^{n_j-k}\right) = \mathrm{diam}_X\left(A_j\right) \asymp 1 \tag{6·3}$$

and

$$\mathrm{diam}_X\left(A_j^{n_j-k}\right) \le \mathrm{diam}_{X_1}\left(A_j^{n_j-k}\right) = \mathrm{diam}_X\left(A_j^{n_j-k+1}\right). \tag{6·4}$$

For each $1 \le i \le j$, from (6·3) we get $\mathrm{diam}_X(A_j^{n_i}) \preceq 1$.

Since $A_j^{n_i}$ contains z_{n_i} and z_{n_i} is contained in the cone Ω by the claim, it follows from $\mathrm{diam}_X(A_j^{n_i}) \preceq 1$ that there is some $0 < \epsilon < 1/2$ such that $A_j^{n_i}$ is contained in a cone spanned at c so that the angles formed by the two boundary rays and $\mathbb{T}$ are equal to $\epsilon\pi$. Please see Figure 3 for an illustration. By [**13**, Lemma 1·11] or [**17**, Lemma 3·2], the inclusion of X_1 in X contracts the hyperbolic metric in $A_j^{n_i}$ by by a definite factor $0 < \delta < 1$ depending only on ϵ. Since $G : X_1 \to X$ is holomorphic covering which preserves the hyperbolic metric, we thus have

$$\mathrm{diam}_X\left(A_j^{n_i}\right) < \delta \cdot \mathrm{diam}_{X_1}\left(A_j^{n_i}\right) = \delta \cdot \mathrm{diam}_X\left(A_j^{n_i+1}\right).$$

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This, together with (6·4), implies that

$$\operatorname{diam}_X\left(A_j^0\right) < \delta^{j-1}\cdot \operatorname{diam}_X\left(A_j\right) \asymp \delta^{j-1} \longrightarrow 0$$

as $j\to\infty$ and thus $\operatorname{diam}(A_j^0)\to 0$ as $j\to\infty$. Since $\operatorname{dist}(A_j, P_G)\asymp \operatorname{diam}(A_j)$, we can take a larger disk $\tilde{A}_j\supset\overline{A_j}$ so that $\tilde{A}_j\cap P_G=\emptyset$ and $\operatorname{mod}(\tilde{A}_j\setminus\overline{A_j}\geq\mu$ with $\mu>0$ being independent of j. Then by Koebe's distortion theorem the distortion of any branch of G^{-n_j} on A_j is uniformly bounded. Let B_j^0 denote the component of $G^{-n_j}(B_j)$ which is contained in A_j^0. We have

$$area\left(B_j^0\right)\succeq \operatorname{diam}\left(A_j^0\right)^2.$$

Since B_j is disjoint from $J_{\tilde{G}}$, it follows that B_j^0 is disjoint from $J_{\tilde{G}}$. This is a contradiction with the assumption that z_0 is a Lebesgue point of $J_{\tilde{G}}$.

Now it suffices to prove the Claim. For each open arc $I\subset\mathbb{T}$, Let $\Omega_I=\widehat{\mathbb{C}}\setminus(P_G\setminus I)$. Let $d_{\Omega_I}(\cdot,\cdot)$ denote the hyperbolic distance in Ω_I. For $d_0>0$, let $\Omega_{d_0}(I)=\{z\in\Omega_I\mid d_{\Omega_I}(z,I)<d_0\}$. When $|I|$ is small, $\Omega_{d_0}(I)$ is almost like the domain bounded by two arcs of Euclidean circles which are symmetric about $\mathbb{T}$ and such that the four exterior angles formed by the two arcs and $\mathbb{T}$ are all equal to σ with $d_0=\ln\cot(\sigma/4)$ (see [**19**, Lemma 2·2]). Define

$$H_{d_0}(I)=\{z\mid |z|>1 \text{ and } z\in\Omega_{d_0}(I)\}.$$

Take $d_0>0$ such that

$$\sigma=\left(1-\frac{1}{4(2d-1)}\right)\pi.$$

So if V is a cone spanned at some $z\in\mathbb{T}\setminus I$ and $\angle(\partial V,\mathbb{T})=\pi/3(2d-1)$, then $V\cap H_{d_0}(I)=\emptyset$.

Now let $h:\mathbb{T}\to\mathbb{T}$ be the quasi-symmetric circle homeomorphism in (4·6). For each z_n, let $I_n\subset\mathbb{T}$ be the arc such that $z_n\in\overline{H_{d_0}(I_n)}$ and moreover, I_n has the minimal property in the following sense

$$|h(I_n)|=\min\{|h(I)|\mid I\subset\mathbb{T} \text{ and } z_n\in\overline{H_{d_0}(I)}\}.$$

Since $z_n\to\mathbb{T}$, we have $|I_n|\to 0$ and thus $|h(I_n)|\to 0$ as $n\to\infty$. So there is an increasing subsequence of integers, say m_j, such that $|h(I_{m_j})|<|h(I_n)|$ for all $1\leq n<m_j$.

Let $n_j=m_j-1$. Let us prove that there is a critical point $c\in\mathbb{T}$ and a cone Ω spanned at c such that z_{n_j} and Ω satisfy the requirement in the Claim. Since $|I_{m_j}|\to 0$, by disregarding finitely many m_j we may assume that $\overline{H_{d_0}(I_{m_j})}\setminus\overline{I_{m_j}}$ contains no critical value of G and $\overline{I_{m_j}}$ contains at most one critical value of G. Let $J\subset\mathbb{T}$ be the arc such that $G(J)=I_{m_j}$. Then we have the following two cases.

In the first case, $\overline{I_{m_j}}$ contains no critical values. By assumption $\overline{H_{d_0}(I_{m_j})}\setminus\overline{I_{m_j}}$ contains no critical values of G. Let K be the component of $G^{-1}(\overline{H_{d_0}(I_{m_j})})$ which is attached to $\overline{J}$. Let X denote the component of $G^{-1}(\Omega_{I_{m_j}})$ which contains J. Since $\Omega_{I_{m_j}}$ does not contain critical values of G, $G:X\to\Omega_{I_{m_j}}$ is a holomorphic covering map which preserves the hyperbolic metric. So K is contained in the d_0-neighbourhood of J with respect to the hyperbolic metric in X. Note that $X\subset\Omega_J$, it follows that $K\subset\overline{H_{d_0}(J)}$. Since $|h(I)|=|h(J)|$, by the minimal property of I_{m_j}, it follows that $z_{m_j-1}=z_{n_j}\notin\overline{H_{d_0}(J)}$. This means that z_{m_j} has at least two

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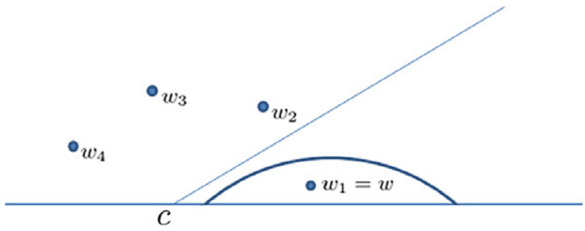

Fig. 4. $\angle([c, z_{n_j}], \mathbb{T}) \geq \angle([c, w_{\frac{m+1}{2}}], \mathbb{T})$ with $m = 7$.

pre-images near $\mathbb{T}$. So z_{m_j} must be close to a critical value in $\mathbb{T}$, and $\overline{H_{d_0}(J)}$ is close to some critical point c in $\mathbb{T}$. Let $3 \leq m \leq 2d - 1$ be the local degree of G at c. Since G is a Blaschke product m must be an odd integer by symmetry. In a small neighbourhood of c, we may regard G approximately as the map $z \mapsto \lambda \cdot (z - c)^m + v$ where $\lambda \neq 0$ is some constant and $v = G(c)$. By the choice of d_0, $\overline{H_{d_0}(J)}$ belongs to the angle domain bounded by $\mathbb{T}$ and a ray L starting from c with $\angle(L, \mathbb{T}) = \pi/3(2d - 1)$. Let $w \in \overline{H_{d_0}(J)}$ be the pre-image of z_{m_j}. Then $\angle([c, w], \mathbb{T}) < \pi/3(2d - 1)$. Note that z_{m_j} has exactly $(m + 1)/2$ preimages in a small neighborhood of c and belonging to the outside of Δ. These $(m + 1)/2$ preimages are distributed in the "half circle" which has center at c and radius $|w - c|$ and which belongs to the outside of Δ, and moreover, the angle between any two such pre-images, which are adjacent to each other, is approximately equal to $2\pi/m$. If we label these $m + 1/2$ preimages anticlockwise as $w_1, \ldots, w_{\frac{m+1}{2}}$, then $\angle([c, w_1], \mathbb{T}) + \angle(c, w_{\frac{m+1}{2}}], \mathbb{T}) \approx \pi/m$, and for $1 < i < (m + 1)/2$, $\angle([c, w_i], \mathbb{T}) \geq \pi/m$ approximately. Since $\angle([c, w], \mathbb{T}) < \pi/3(2d - 1) < \pi/m$, we may assume that $w = w_1$. Since z_{n_j} is one of such pre-images except w_1, we have

$$\angle([c, z_{nj}], \mathbb{T}) \geq \angle([c, w_{\frac{m+1}{2}}], \mathbb{T}) \approx \frac{\pi}{m} - \angle([c, w_1], \mathbb{T}) > \frac{\pi}{2(2d - 1)}. \tag{6·5}$$

Please see Figure 4 for an illustration. Let V be the cone spanned at c such that $\angle(\partial V, \mathbb{T}) = \pi/2(2d - 1)$. Then $z_{n_j} \in V$.

In the second case, $\overline{I_{m_j}}$ contains exactly one critical value v. Let c be the critical point in $\mathbb{T}$ such that $G(c) = v$. Then $z_{n_j} = z_{m_j - 1}$ is near c. By [**17**, Lemma 4·9], there is a $0 < \delta < 1/2$ such that if $\angle([v, z_{m_j}], \mathbb{T}) < \delta\pi$, we would have an arc $I \subset \mathbb{T}$ such that $z_{m_j} \in H_{d_0}(I)$ and $|I| < |I_{m_j}|$, which would contradict the minimality of I_{m_j}. So we must have $\angle([v, z_{m_j}], \mathbb{T}) \geq \delta\pi$. Please see Figure 5 for an illustration. Since G behaves almost like the map $z \mapsto \lambda \cdot (z - c)^m + v$, $\lambda \neq 0$ where $3 \leq m \leq 2d - 1$, we have

$$\angle([c, z_{n_j}], \mathbb{T}) > \delta\pi/2(2d - 1). \tag{6·6}$$

Since there are only finitely many critical points in $\mathbb{T}$, from (6·5) and (6·6) the Claim follows by taking $\alpha = \delta/2(2d - 1)$ and taking a subsequence of $\{n_j\}$ if necessary so that z_{n_j} converges to some critical point $c \in \mathbb{T}$.

This completes the proof of the uniqueness part of the main theorem.

7. *Proof of Lemma* 4·3

We need a few lemmas before the proof of Lemma 4·3.

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LEMMA 7·1. *There is a constant $C_0 > 0$ independent of m and n such that if $P \subset (P_{F_n} \cup \cup_i \overline{D_i})$ is a closed subset and γ is a non-peripheral curve in $\widehat{\mathbb{C}} \setminus P$, then*

$$e^{-2C_0} \cdot l_{\phi_{m,n}}(\gamma, P) \leq l_{\phi_{m+1,n}}(\gamma, P) \leq e^{2C_0} \cdot l_{\phi_{m,n}}(\gamma, P). \tag{7·1}$$

Proof. The proof is an adaptation of the proof of [3, Proposition 7·2]. Assume that F_n has holomorphic disks. Otherwise the proof is the same and is even simpler. Let T_{F_n} be the Teichmüller space modelled on $(\widehat{\mathbb{C}}, P_{F_n} \cup \cup_i \overline{D_i})$. Then $\sigma_{F_n} : T_{F_n} \to T_{F_n}$ is analytic and does not increase the Teichmüller metric on T_{F_n}. By Remark 3·1 F_n is K_0-quasiregular with $K_0 > 1$ being a constant independent of n. Since $\tau_{0,n}$ is the standard complex structure and $\tau_{1,n}$ is the pull back of $\tau_{0,n}$ by F_n, $\phi_{1,n}$ is K_0-quasiconformal. It follows that

$$d_{T_{F_n}}(\tau_{m+1,n}, \tau_{m,n}) \leq d_{T_{F_n}}(\tau_{1,n}, \tau_{0,n}) \leq C_0$$

with $C_0 = \ln K_0/2$. So there is a K_0-quasiconformal homeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which maps $\phi_{m,n}(P_{F_n} \cup \cup_i \overline{D_i})$ to $\phi_{m+1,n}(P_{F_n} \cup \cup_i \overline{D_i})$, and when restricted to $\phi_{m,n}(\cup_i \overline{D_i})$, $h = \phi_{m+1,n} \circ \phi_{m,n}^{-1}$ is conformal, and moreover, $\phi_{m+1,n}$ is isotopic to $h \circ \phi_{m,n}$ rel $P_{F_n} \cup \cup_i \overline{D_i}$. Since $P \subset P_{F_n} \cup \cup_i \overline{D_i}$, $h(\phi_{m,n}(\gamma))$ is homotopic to $\phi_{m+1,n}(\gamma)$ in $\widehat{\mathbb{C}} \setminus \phi_{m+1,n}(P)$. The map h can then be lifted to a K_0-quasiconformal homeomorphism between the annular covering surfaces of $\widehat{\mathbb{C}} \setminus \phi_{m,n}(P)$ and $\widehat{\mathbb{C}} \setminus \phi_{m+1,n}(P)$ associated to the homotopy classes of $\phi_{m,n}(\gamma)$ and $\phi_{m+1,n}(\gamma)$ respectively. This implies (7·1) and the lemma follows.

LEMMA 7·2. *For any $0 < \kappa < 1$, there is an integer k_1 depending only on κ and θ such that for any two distinct points $a, b \in \mathbb{T}$, if the length of each components of $\mathbb{T} \setminus \{a, b\}$ is greater than $2\pi\kappa$, then a, b and their images under the map $R_\theta^{k_1}$ appear as a, $R_\theta^{k_1}(a)$, b, $R_\theta^{k_1}(b)$ anticlockwise, and moreover, the length of each component of $\mathbb{T} \setminus \{a, R_\theta^{k_1}(a), b, R_\theta^{k_1}(b)\}$ is greater than $\pi\kappa/2$.*

Proof. For any integer $k \geq 1$, let $[a, R_\theta^k(a)]$ denote the the arc from a to $R_\theta^k(a)$ anticlockwise. Since θ is irrational, $\{R_\theta^i(a), i \geq 0\}$ is dense in $\mathbb{T}$. So there is a smallest integer $k_1 \geq 1$ depending only on κ and θ such that the length of $[a, R_\theta^{k_1}(a)]$ is between $\pi\kappa/2$ and $3\pi\kappa/2$. It is clear that such k_1 satisfies the requirement in the lemma.

LEMMA 7·3. *Suppose $k \geq 0$ is an integer. Then there is a $\lambda = \lambda(k) > 0$ depending only on k such that for all $m \geq 0$ and all n large enough, if γ is a $(\phi_{m,n}, A_n^k)$-geodesic (or a $(\phi_{m,n}, B_n^k)$-geodesic) with $l_{\phi_{m,n}}(\gamma, A_n^k) < \lambda$ (or $l_{\phi_{m,n}}(\gamma, B_n^k) < \lambda$), then γ does not intersect the closure of any holomorphic disk D_i.*

Proof. By the construction of F_n, for each holomorphic disk D_i, there is an annulus H_i around D_i so that $\overline{H_i} \cap P_{F_n} = \emptyset$, and the inner component of ∂H_i coincides with ∂D_i, and moreover, F_n is holomorphic in the disk $H_i \cup \overline{D_i}$, and maps each $H_i \cup \overline{D_i}$ into some D_j. Since $\tau_{m,n}$ is the pull back of the standard complex structure by F_n^m, it follows that on all the disks $H_i \cup \overline{D_i}$ $\tau_{m,n} = 0$ and thus $\phi_{m,n}$ is conformal. Suppose γ is a $(\phi_{m,n}, A_n^k)$-geodesic which intersects $\overline{D_i}$. The same argument works if γ is a $(\phi_{m,n}, B_n^k)$-geodesic which intersects $\overline{D_i}$. By composing with a Möbius map, we may assume that $\phi_{m,n}$ maps a_i, b_i and 1 to 0, 1 and ∞ respectively. Take a Jordan domain C_i such that $\overline{D_i} \subset C_i \subset \overline{C_i} \subset \overline{D_i} \cup H_i$. Then there are two cases.

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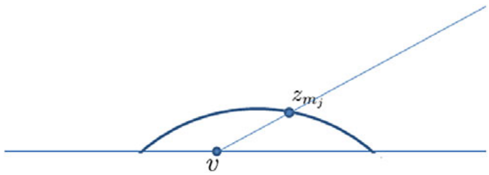

Fig. 5. $\angle([v, z_{m_j}], \mathbb{T}) \geq \delta\pi$.

In the first case, $\gamma \subset C_i$. In this case, Since $\overline{H_i} \cap A_n^k \subset \overline{H_i} \cap P_{F_n} = \emptyset$, γ surrounds at least two points in $A_n^k \cap D_i$. Since $\phi_{m,n}(\{a_i, b_i\}) = \{0, 1\}$ and since the spherical distance between any two points in $A_n^k \cap D_i$ has a positive lower bound depending only on k (Proposition 4·1), by Koebe's distortion theorem, the Euclidean distance between any two points in $\phi_{m,n}(A_n^k \cap D_i)$ has a positive lower bound depending only on k. Thus the Euclidean length of $\phi_{m,n}(\gamma)$ has a positive lower bound depending only on k. Note that $\phi_{m,n}(\gamma) \subset \phi_{m,n}(\overline{C_i})$. By Koebe's distortion theorem, there is an $R > 0$ depending only on D_i, C_i and H_i such that $\phi_{m,n}(\overline{C_i}) \subset \{z \mid |z| \leq R\} = \overline{B_R}$. Since the density of the hyperbolic metric of $\widehat{\mathbb{C}} \setminus \phi_{m,n}(A_n^k)$ is bounded from below by that of $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$, which has a positive lower bound on $\overline{B_R}$ with respect to the Euclidean metric, it follows that $l_{\phi_{m,n}}(\gamma, A_n^k)$ must have a positive lower bound depending only k.

In the second case, γ is not contained in C_i. Since γ intersects D_i, $\phi_{m,n}(\gamma)$ contains an curve segment Γ connecting $\phi_{m,n}(\partial D_i)$ and $\phi_{m,n}(\partial C_i)$. Since $\{0, 1\} \subset \phi_{m,n}(D_i)$ and $\mathrm{mod}(\phi_{m,n}(C_i \setminus \overline{D_i})) = \mathrm{mod}(C_i \setminus \overline{D_i})$, the Euclidean length of Γ must have a positive lower bound depending only on D_i and C_i. Since $\Gamma \subset \overline{\phi_{m,n}(C_i)} \subset \overline{B_R}$, as in the first case, the density of the hyperbolic metric in $\widehat{\mathbb{C}} \setminus \phi_{m,n}(A_n^k)$ with respect to the Euclidean metric, when restricted to Γ, is bounded from below by a positive constant depending only on D_i, C_i and H_i. It follows that $l_{\phi_{m,n}}(\gamma, A_n^k)$ must have a positive lower bound depending only on D_i, C_i and H_i.

Since there are only finitely many D_i and since C_i and H_i are all fixed, the lemma follows by taking λ to be the minimum of all the above lower bounds which clearly depends only on k.

LEMMA 7·4. *Let $P \subset \widehat{\mathbb{C}}$ be a proper closed subset such that $P^* = P$ and $|P| \geq 3$. Suppose γ is a $(\phi_{m,n}, P)$-geodesic such that $l_{\phi_{m,n}}(\gamma, P) < \log(\sqrt{2}+1)$ and $\gamma \cap \mathbb{T} \neq \emptyset$. Then γ is symmetric about $\mathbb{T}$, and in particular, γ intersects $\mathbb{T}$ at exactly two points.*

Proof. By symmetry γ^* is also a $(\phi_{m,n}, P)$-geodesic with the same length. Since $\gamma \cap \mathbb{T} \neq \emptyset$, we have $\gamma \cap \gamma^* \supset \gamma \cap \mathbb{T} \neq \emptyset$. Since the length of both γ and γ^* is less than $\log(\sqrt{2}+1)$, by Lemma 8·1 we must have $\gamma = \gamma^*$.

Note that the sets A_n^k and B_n^k are all symmetric about $\mathbb{T}$. From Lemma 7·4 it follows that if γ is a $(\phi_{m,n}, A_n^k)$-geodesic or $(\phi_{m,n}, B_n^k)$-geodesic with length less than $\log(\sqrt{2}+1)$ and $\gamma \cap \mathbb{T} \neq \emptyset$, then γ is symmetric about $\mathbb{T}$ and $\gamma \cap \mathbb{T}$ contains exactly two points. In this case, by symmetry 0 and ∞ belong to one component of $\widehat{\mathbb{C}} \setminus \gamma$. For such γ we use $D(\gamma)$ to denote the other component of $\widehat{\mathbb{C}} \setminus \gamma$ which does not contain 0 and ∞ and use $D(\gamma)^c$ to denote the complement of $D(\gamma)$ in $\widehat{\mathbb{C}}$, that is, $D(\gamma)^c = \widehat{\mathbb{C}} \setminus D(\gamma)$.

Proof of Lemma 4·3. Without loss of generality we may assume that both $l_{\phi_{m,n}}(\gamma, A_n^k)$ and $l_{\phi_{m,n}}(\gamma, B_n)$ are less than $\log(\sqrt{2}+1)$. The proof is divided into five steps. In the first four

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steps, we assume that $\gamma \cap \mathbb{T} \neq \emptyset$. By Lemma 7·4 γ is symmetric about $\mathbb{T}$ and intersects $\mathbb{T}$ at exactly two points. Recall that $D(\gamma)$ denotes the component of $\widehat{\mathbb{C}} \setminus \gamma$ which does not contain 0 and ∞ and $D(\gamma)^c = \widehat{\mathbb{C}} \setminus D(\gamma)$. Also recall that $Z_n^k = A_n^k \cap \mathbb{T}$ and $O_n = B_n \cap \mathbb{T}$.

Step 1. Prove the assertion (i) under the assumption that both $D(\gamma) \cap Z_n^k$ and $D(\gamma)^c \cap Z_n^k$ contain at least two points.

Step 2. Prove the assertion (ii) under the assumption that both $D(\gamma) \cap O_n$ and $D(\gamma)^c \cap O_n$ contain at least two points.

Step 3. Prove the assertion (i) under the assumption that either $D(\gamma) \cap Z_n^k$ or $D(\gamma)^c \cap Z_n^k$ contains at most one point.

Step 4. Prove the assertion (ii) under the assumption that either $D(\gamma) \cap O_n$ or $D(\gamma)^c \cap O_n$ contains at most one point.

Step 5. Prove the assertions (i) and (ii) under the assumption that $\gamma \cap \mathbb{T} = \emptyset$.

***Step* 1.** Suppose γ is a $(\phi_{m,n}, A_n^k)$-geodesic which intersects $\mathbb{T}$ so that both $D(\gamma) \cap Z_n^k$ and $D(\gamma)^c \cap Z_n^k$ contain at least two points. By Proposition 4·1 there is a $\kappa > 0$ depending only on k such that the length of each component of $\mathbb{T} \setminus A_n^k$ is greater than $2\pi\kappa$ for all n large enough. For such κ, let k_1 be the integer guaranteed by Lemma 7·2. Let $k_2 \geq 0$ be an integer depending only on κ such that for any open arc $I \subset \mathbb{T}$ with $|I| > \pi\kappa/3$, $I \cap A_n^{k_2}$ contains at least one point for all n large enough. Note that k_1 and k_2 depend only on κ which depends only on k. So k_1 and k_2 are bounded by some constant $M(k) \geq 1$ depending only on k.

In Lemma 4·2 let $N_0 = k + k_1 + k_2$ and define the set $\tilde{A}_n^{k_1}$. By Lemma 7·1,

$$l_{\phi_{m+k_1,n}}\left(\gamma, A_n^k\right) \leq e^{2k_1 C_0} \cdot l_{\phi_{m,n}}\left(\gamma, A_n^k\right) \tag{7·2}$$

with $C_0 > 0$ being the constant in Lemma 7·1. Note that $A_n^k \subset \tilde{A}_n^{k_1}$ (Since $F_n^{k_1}(A_n^k) \subset A_n^{N_0}$ by (4·2)) and the number of the points in $\tilde{A}_n^{k_1} \setminus A_n^k$ is bounded by some constant depending only on k. By Lemma 8·3 there is a $(\phi_{m+k_1,n}, \tilde{A}_n^{k_1})$-geodesic γ_{k_1} which is homotopic to γ in $\widehat{\mathbb{C}} \setminus A_n^k$ so that

$$l_{\phi_{m+k_1,n}}\left(\gamma_{k_1}, \tilde{A}_n^{k_1}\right) < C(k) \cdot l_{\phi_{m+k_1,n}}\left(\gamma, A_n^k\right) \tag{7·3}$$

provided that $l_{\phi_{m+k_1,n}}(\gamma, A_n^k) < C(k)^{-1}$ with $C(k) > 1$ being some constant depending only on k. By Lemma 4·2 $F_n^{k_1} : \widehat{\mathbb{C}} \setminus \tilde{A}_n^{k_1} \to \widehat{\mathbb{C}} \setminus A_n^{N_0}$ is a covering map. So by lemma 8·2 $\eta_{k_1} = F_n^{k_1}(\gamma_{k_1})$ is a $(\phi_{m,n}, A_n^{N_0})$-geodesic with

$$l_{\phi_{m,n}}\left(\eta_{k_1}, A_n^{N_0}\right) \leq l_{\phi_{m+k_1,n}}\left(\gamma_{k_1}, \tilde{A}_n^{k_1}\right) \tag{7·4}$$

provided that $l_{\phi_{m+k_1,n}}(\gamma_{k_1}, \tilde{A}_n^{k_1}) < \log(\sqrt{2} + 1)$. Since both $D(\gamma) \cap Z_n^k$ and $D(\gamma)^c \cap Z_n^k$ contain at least two points by assumption and since γ_{k_1} is homotopic to γ in $\widehat{\mathbb{C}} \setminus A_n^k$, it follows that both $D(\gamma_{k_1}) \cap Z_n^k$ and $D(\gamma_{k_1})^c \cap Z_n^k$ contain at least two points also. Thus the length of both the two arc components of $\mathbb{T} \setminus \gamma_{k_1}$ is greater than $2\pi\kappa$. Note that $F_n|\mathbb{T}$ is the rigid rotation given by θ_n and $\eta_{k_1} = F_n^{k_1}(\gamma_{k_1})$. Since $\theta_n \to \theta$, from Lemma 7·2 γ_{k_1} and η_{k_1} well intersect each other in the following sense for all n large enough. Let $\{a, b\} = \gamma_{k_1} \cap \mathbb{T}$ and $\{c, d\} = \eta_{k_1} \cap \mathbb{T}$. Then the four intersection points can be labeled so that they appear in the order a, c, b, d anticlockwise and moreover, the length of each component of $\mathbb{T} \setminus \{a, c, b, d\}$

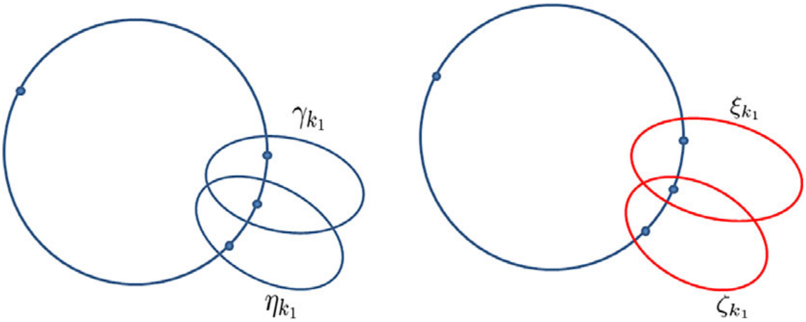

Fig. 6. $\gamma_{k_1}, \eta_{k_1}, \xi_{k_1}, \zeta_{k_1}$ and points in A_n^{k2}.

is greater than $\pi\kappa/3$. By the choice of k_2, each component of $\mathbb{T}\setminus\{a, c, b, d\}$ contains at least one point in $A_n^{k_2}$. Since $k_2 + k_1 \le N_0$, we have

$$A_n^{k_2} \subset \tilde{A}_n^{k_1} \cap A_n^{N_0}. \tag{7·5}$$

Since each component of $\mathbb{T}\setminus\{a, c, b, d\}$ contains at least one point in $A_n^{k_2}$, γ_{k_1} and η_{k_1} are both non-peripheral in $\widehat{\mathbb{C}}\setminus A_n^{k_2}$, see Figure 4. Let ξ_{k_1} and ζ_{k_1} be the $(\phi_{m,n}, A_n^{k_2})$-geodesics which are homotopic to γ_{k_1} and η_{k_1} in $\widehat{\mathbb{C}}\setminus A_n^{k_2}$ respectively. Let $\{a', b'\} = \xi_{k_1}\cap\mathbb{T}$ and $\{c', d'\} = \zeta_{k_1}\cap\mathbb{T}$. Then a', c', b', d' have the same order as a, c, b, d, and each component of $\mathbb{T}\setminus\{a', c', b', d'\}$ contains at least one point in $A_n^{k_2}$. This implies that ξ_{k_1} and ζ_{k_1} are two distinct simple closed geodesics which intersect with each other. But on the other hand,

$$\begin{aligned} l_{\phi_{m,n}}\left(\xi_{k_1}, A_n^{k_2}\right) &\le e^{2k_1C_0}\cdot l_{\phi_{m+k_1,n}}\left(\xi_{k_1}, A_n^{k_2}\right)\\ &= e^{2k_1C_0}\cdot l_{\phi_{m+k_1,n}}\left(\gamma_{k_1}, A_n^{k_2}\right)\\ &\le e^{2k_1C_0}\cdot l_{\phi_{m+k_1,n}}\left(\gamma_{k_1}, \tilde{A}_n^{k_1}\right)\\ &< C(k)\, e^{4k_1C_0}\cdot l_{\phi_{m,n}}\left(\gamma, A_n^{k}\right)\end{aligned} \tag{7·6}$$

and

$$l_{\phi_{m,n}}\left(\zeta_{k_1}, A_n^{k_2}\right) = l_{\phi_{m,n}}\left(\eta_{k_1}, A_n^{k_2}\right) \le l_{\phi_{m,n}}\left(\eta_{k_1}, A_n^{N_0}\right) < C(k)\, e^{2k_1C_0}\cdot l_{\phi_{m,n}}\left(\gamma, A_n^{k}\right). \tag{7·7}$$

The first inequality of (7·6) comes from Lemma 7·1. Both the equalities in (7·6) and (7·7) hold because, by Definition 4·1, $l_{\phi_{m,n}}(\xi, P)$ depends only on the homotopy class of ξ in $\widehat{\mathbb{C}}\setminus P$. The second inequality of both (7·6) and (7·7) hold because of (7·5). The last inequalities of both (7·6) and (7·7) hold by combining the inequalities (7·2-7·4). Now the assertion in Step 1 follows by taking

$$\delta = C(k)^{-1}e^{-4M(k)C_0}\cdot\log\left(\sqrt{2}+1\right).$$

In fact, if $l_{\phi_{m,n}}(\gamma, A_n^k) < \delta$, by (7·6), (7·7) and $k_1 \le M(k)$, we get $l_{\phi_{m,n}}(\xi_{k_1}, A_n^{k_2}) < \log(\sqrt{2}+1)$ and $l_{\phi_{m,n}}(\zeta_{k_1}, A_n^{k_2}) < \log(\sqrt{2}+1)$. But we have seen $\xi_{k_1}\cap\zeta_{k_1}\ne\emptyset$ and $\xi_k \ne \zeta_k$, which contradicts Lemma 8·1.

COROLLARY 7·5. *Suppose $l \ge 0$ is an integer. Then there is a $\delta = \delta(l) < \log(\sqrt{2}+1)$ depending only on l such that for all $m \ge 0$ and all n large enough, if γ is a*

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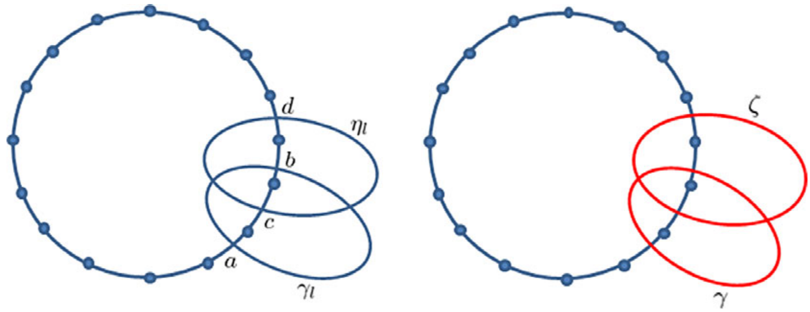

Fig. 7. The construction of ζ.

$(\phi_{m,n}, A_n^l)$-geodesic such that $\gamma \cap \mathbb{T} \neq \emptyset$ and $l_{\phi_{m,n}}(\gamma, A_n^l) < \delta$, then either $D(\gamma) \cap Z_n^l$ or $D(\gamma)^c \cap Z_n^l$ contains at most one point.

***Step* 2**. Suppose γ is a $(\phi_{m,n}, B_n)$-geodesic which intersects $\mathbb{T}$ so that both $D(\gamma) \cap O_n$ and $D(\gamma)^c \cap O_n$ contain at least two points. The idea of the proof is almost the same as in the Step 1. Since O_n is forward invariant, the argument is even simpler.

Let $l = l(n) \geq 1$ be the least integer such that F_n^l maps any point in O_n to the one which is on the right and adjacent to it. Then l depends only on n. In Lemma 4·2 let $N_0 = l$ and define the set $\tilde{B}_n^l$. By Lemma 7·1 it follows that

$$l_{\phi_{m+l,n}}(\gamma, B_n) \leq e^{2lC_0} \cdot l_{\phi_{m,n}}(\gamma, B_n). \tag{7·8}$$

Note that $B_n \subset \tilde{B}_n^l$ and the number of the points in $\tilde{B}_n^l \setminus B_n$ depends only on n. By Lemma 8·3, there is a $(\phi_{m+l,n}, \tilde{B}_n^l)$-geodesic γ_l which is homotopic to γ in $\widehat{\mathbb{C}} \setminus B_n$ such that

$$l_{\phi_{m+l,n}}(\gamma_l, \tilde{B}_n^l) \leq C(n) \cdot l_{\phi_{m+l,n}}(\gamma, B_n) \tag{7·9}$$

provided that $l_{\phi_{m+l,n}}(\gamma, B_n) < C(n)^{-1}$ where $C(n) > 1$ is a constant depending only on n. Since γ_l is homotopic to γ in $\widehat{\mathbb{C}} \setminus B_n$, both $D(\gamma_l) \cap O_n$ and $D(\gamma_l)^c \cap O_n$ contain at least two points. Then, by Lemmas 4·2 and 8·2, $\eta_l = F_n^l(\gamma_l)$ is a $(\phi_{m,n}, B_n^l)$-geodesic provided that $l_{\phi_{m+l,n}}(\gamma_l, \tilde{B}_n^l) < \log(\sqrt{2} + 1)$. By the choice of l, γ_l and η_l well intersect each other in the following sense: *Let $\gamma_l \cap \mathbb{T} = \{a, b\}$ and $\eta_l \cap \mathbb{T} = \{c, d\}$. Then the four intersection points can be labeled so that they appear in the order a, c, b, d anticlockwise, and each component of $\mathbb{T} \setminus \{a, c, b, d\}$ contains at least one point in O_n.*

Since $F_n|\mathbb{T}$ is the rotation given by θ_n and O_n is periodic under F_n, and since both $D(\gamma_l) \cap O_n$ and $D(\gamma_l)^c \cap O_n$ contain at least two points, from $\eta_l = F_n^l(\gamma_l)$ it follows that both $D(\eta_l) \cap O_n$ and $D(\eta_l)^c \cap O_n$ contain at least two points. In particular, η_l is non-peripheral in $\widehat{\mathbb{C}} \setminus B_n$. Let ζ be the $(\phi_{m,n}, B_n)$-geodesic which is homotopic to η_l in $\widehat{\mathbb{C}} \setminus B_n$. Then

$$\begin{aligned} l_{\phi_{m,n}}(\zeta, B_n) &= l_{\phi_{m,n}}(\eta_l, B_n) \\ &\leq l_{\phi_{m,n}}\left(\eta_l, B_n^l\right) \\ &\leq l_{\phi_{m+l,n}}\left(\gamma_l, \tilde{B}_n^l\right) \\ &\leq e^{2lC_0} C(n) \cdot l_{\phi_{m,n}}(\gamma, B_n)\,. \end{aligned} \tag{7·10}$$

The equality holds since, by Definition 4·1, $l_{\phi_{m,n}}(\zeta, B_n)$ depends only on the homotopy class of ζ in $\widehat{\mathbb{C}} \setminus B_n$. The first inequality holds because $B_n \subset B_n^l$. The second inequality holds by Lemma 8·2. The last equality follows from (7·8) and (7·9). Since γ and ζ are homotopic respectively to γ_l and η_l in $\widehat{\mathbb{C}} \setminus B_n$, γ and ζ well intersect each other in the same sense as above. In particular, it follows that $\gamma \neq \zeta$ and $\gamma \cap \zeta \neq \emptyset$. Please see Figure 7 for an illustration.

Now the assertion follows by taking

$$\delta_n = e^{-2lC_0} C(n)^{-1} \cdot \log\left(\sqrt{2}+1\right).$$

In fact, if $l_{\phi_{m,n}}(\gamma, B_n) < \delta_n$, then both ζ and γ are $(\phi_{m,n}, B_n)$-geodesics with length less than $\log(\sqrt{2}+1)$ by (7·10). But we have seen that ζ and γ are distinct and intersect with each other. This contradicts Lemma 8·1.

***Step* 3**. Suppose γ is a $(\phi_{m,n}, A_n^k)$-geodesic which intersects $\mathbb{T}$ so that either $D(\gamma) \cap Z_n^k$ or $D(\gamma)^c \cap Z_n^k$ contains at most one point.

Let N_1 be the number of the points in $P_1 \setminus \overline{\Delta}$ and N_2 be the number of the holomorphic disks in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$. Since $F_n|\mathbb{T} = R_{\theta_n}$ and $\theta_n \to \theta$ which is irrational, there is a smallest integer $i_0 \geq k$ which depends only on k such that the following holds for all n large enough: if I and J are any two adjacent components of $\mathbb{T} \setminus Z_n^{i_0}$ and $S = \overline{I \cup J}$, then

$$S,\ F_n(S),\ \ldots,\ F_n^{N_1+N_2+1}(S)$$

are disjoint, and moreover, each component of $\mathbb{T} \setminus \cup_{i=0}^{N_1+N_2+1} F_n^i(S)$ contains at least one point in $Z_n^{i_0}$.

In Lemma 4·2 let $N_0 = N_1 + N_2 + i_0 + 1$ and for $0 \leq i \leq N_1 + N_2 + 1$ define the sets $\tilde{A}_n^i$. Since $A_n^k \subset A_n^{i_0}$ (because $i_0 \geq k$) and the number of the points in $A_n^{i_0} \setminus A_n^k$ depends only on k (because i_0 depends only on k), by Lemma 8·3 there is a $(\phi_{m,n}, A_n^{i_0})$-geodesic $\tilde{\gamma}$ which is homotopic to γ in $\widehat{\mathbb{C}} \setminus A_n^k$ such that

$$l_{\phi_{m,n}}(\tilde{\gamma}, A_n^{i_0}) \leq C_1(k) \cdot l_{\phi_{m,n}}\left(\gamma, A_n^k\right) \tag{7·11}$$

provided that $l_{\phi_{m,n}}(\gamma, A_n^k) < C_1(k)^{-1}$, where $C_1(k) > 1$ is a constant depending only on k. Now for $0 \leq i \leq N_1 + N_2 + 1$, by Lemma 7·1,

$$l_{\phi_{m+i,n}}\left(\tilde{\gamma}, A_n^{i_0}\right) \leq e^{2iC_0} \cdot l_{\phi_{m,n}}\left(\tilde{\gamma}, A_n^{i_0}\right). \tag{7·12}$$

Since $A_n^{i_0} \subset \tilde{A}_n^i$, $0 \leq i \leq N_1 + N_2 + 1$, and the number of the points in $\tilde{A}_n^i \setminus A_n^{i_0}$ is bounded by some constant depending only on k, by Lemma 8·3 again there is a $(\phi_{m+i,n}, \tilde{A}_n^i)$-geodesic γ_i which is homotopic to $\tilde{\gamma}$ in $\widehat{\mathbb{C}} \setminus A_n^{i_0}$ such that

$$l_{\phi_{m+i,n}}\left(\gamma_i, \tilde{A}_n^i\right) \leq C_2(k) \cdot l_{\phi_{m+i,n}}\left(\tilde{\gamma}, A_n^{i_0}\right). \tag{7·13}$$

provided that $l_{\phi_{m+i,n}}(\tilde{\gamma}, A_n^{i_0}) < C_2(k)^{-1}$ where $C_2(k) > 1$ is some constant depending only on k.

Let us show that the assertion holds by taking

$$\delta = C_1(k)^{-1} C_2(k)^{-1} e^{-2(N_1+N_2+1)C_0} \cdot \min\left\{\delta(N_0), \delta(i_0), \lambda(N_0)\right\},$$

where $\delta(N_0)$, $\delta(i_0)$ are respectively the constants guaranteed by Corollary 7·5 by taking $l = N_0$, i_0, and $\lambda(N_0)$ is the constant guaranteed by Lemma 7·3 by taking $k = N_0$. Note that $\delta(i_0) < \log(\sqrt{2}+1)$ by Corollary 7·5.

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Assume $l_{\phi_{m,n}}(\gamma, A_n^k) < \delta$. By (7·11) it follows that

$$l_{\phi_{m,n}}\left(\tilde{\gamma}, A_n^{i_0}\right) < \delta(i_0) < \log\left(\sqrt{2}+1\right). \tag{7·14}$$

By (7·14) and Corollary 7·5 it follows that $D(\tilde{\gamma}) \cap Z_n^{i_0}$ or $D(\tilde{\gamma})^c \cap Z_n^{i_0}$ contains at most one point. For $0 \le i \le N_1 + N_2 + 1$, from (7·11-7·13) we have

$$l_{\phi_{m+i,n}}\left(\gamma_i, \tilde{A}_n^i\right) < \min\left\{\delta(N_0), \lambda(N_0)\right\} < \log\left(\sqrt{2}+1\right). \tag{7·15}$$

By (7·14), (7·15) and Lemma 7·4 all γ_i and $\tilde{\gamma}$ are symmetric about $\mathbb{T}$ and intersect $\mathbb{T}$ at exactly two points. For $0 \le i \le N_1 + N_2 + 1$, let $\{a^i, b^i\} = \gamma_i \cap \mathbb{T}$ and $\{a, b\} = \tilde{\gamma} \cap \mathbb{T}$. Since all γ_i are homotopic to $\tilde{\gamma}$ in $\widehat{\mathbb{C}} \setminus A_n^{i_0}$ and since $D(\tilde{\gamma}) \cap Z_n^{i_0}$ or $D(\tilde{\gamma})^c \cap Z_n^{i_0}$ contains at most one point, it follows from $A_n^{i_0} \cap \mathbb{T} = Z_n^{i_0}$ that all the points in $\{a, b, a^i, b^i, 0 \le i \le N_1 + N_2 + 1\}$ are contained in either one component of $\mathbb{T} \setminus Z_n^{i_0}$ or the union of two adjacent components of $\mathbb{T} \setminus Z_n^{i_0}$. For all $0 \le i \ne j \le N_1 + N_2 + 1$ and all n large enough, by the definition of i_0, we have

$$\left\{F_n^i\left(a^i\right), F_n^i\left(b^i\right)\right\} \cap \left\{F_n^j\left(a^j\right), F_n^j\left(b^j\right)\right\} = \emptyset, \tag{7·16}$$

and moreover, for any $p \in \{F_n^i(a^i), F_n^i(b^i)\}$ and $q \in \{F_n^j(a^j), F_n^j(b^j)\}$, if I is a component of $\mathbb{T} \setminus \{p, q\}$, then I contains at least one point in $Z_n^{i_0}$.

From Lemma 8·2 and that $F_n^i : \widehat{\mathbb{C}} \setminus \tilde{A}_n^i \to \widehat{\mathbb{C}} \setminus A_n^{N_0}$ is a covering map, and that $l_{\phi_{m+i,n}}(\gamma_i, \tilde{A}_n^i) < \log(\sqrt{2}+1)$, it follows that $\eta_i = F_n^i(\gamma_i)$ is a $(\phi_{m,n}, A_n^{N_0})$-geodesic so that

$$l_{\phi_{m,n}}\left(\eta_i, A_n^{N_0}\right) \le l_{\phi_{m+i,n}}\left(\gamma_i, \tilde{A}_n^i\right) < \min\left\{\delta\left(N_0\right), \lambda\left(N_0\right)\right\} < \log\left(\sqrt{2}+1\right). \tag{7·17}$$

From (7·17) and Corollary 7·5 it follows that:

fact 1. for $0 \le i \le N_1 + N_2 + 1$, either $D(\eta_i) \cap Z_n^{N_0}$ or $D(\eta_i)^c \cap Z_n^{N_0}$ contains at most one point.

Note that $\eta_i \cap \mathbb{T} = \{F_n^i(a^i), F_n^i(b^i)\}$ for $0 \le i \le N_1 + N_2 + 1$. From (7·16) all η_i are distinct with each other. By Lemma 8·1 and (7·17) we have:

fact 2. all η_i, $0 \le i \le N_1 + N_2 + 1$, are disjoint with each other.

Since $Z_n^{i_0} \subset Z_n^{N_0}$ and $\eta_i \cap \mathbb{T} = \{F_n^i(a^i), F_n^i(b^i)\}$ for $0 \le i \le N_1 + N_2 + 1$, as we have seen before,

fact 3. for any $0 \le i \ne j \le N_1 + N_2 + 1$, if $p \in \eta_i \cap \mathbb{T}$ and $q \in \eta_j \cap \mathbb{T}$, then each arc component of $\mathbb{T} \setminus \{p, q\}$ contains at least one point in $Z_n^{i_0}$, and therefore, must contain at least one point in $Z_n^{N_0}$.

Now by Fact 1 we have two cases.

Case I. $D(\eta_i) \cap Z_n^{N_0}$ contains at most one point for all $0 \le i \le N_1 + N_2 + 1$. In this case, all $D(\eta_i)$ must be disjoint with each other. Since otherwise, by Fact 2 we must have $D(\eta_i) \subset D(\eta_j)$ for some $i \ne j$. But by Fact 3, this implies that $D(\eta_j)$ would contain at least two points in $Z_n^{N_0}$. This is a contradiction. Since η_i is non-peripheral in $\widehat{\mathbb{C}} \setminus A_n^{N_0}$ and $Z_n^{N_0} = A_n^{N_0} \cap \mathbb{T}$, by symmetry each $D(\eta_i)$ contains at least one point in $P_1 \setminus \Delta$ or one holomorphic

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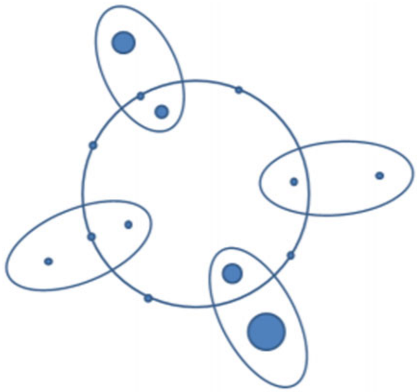

Fig. 8. Case I. all $D(\eta_i)$, $0 \le i \le N_1 + N_2 + 1$, are disjoint.

disk in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$ (Note that from (7·17) and Lemma 7·3 η_i does not intersect the holomorphic disks). But this is impossible since there are $N_1 + N_2 + 2$ disjoint domains $D(\eta_i)$, $0 \le i \le N_1 + N_2 + 1$, but only N_1 points in $P_1 \setminus \overline{\Delta}$ and N_2 holomorphic disks in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$. Please see Figure 8 for an illustration.

Case II. There is some $0 \le j \le N_1 + N_2 + 1$ such that $D(\eta_j) \cap Z_n^{N_0}$ contains at least two points. Thus, by Fact 1, $D(\eta_j)^c \cap Z_n^{N_0}$ contains at most one point. By Facts 2 and 3, for each $0 \le i \ne j \le N_1 + N_2 + 1$, $D(\eta_i) \subset D(\eta_j)$ and $D(\eta_i)^c \cap Z_n^{N_0}$ contains more than one point. This, together with Fact 1, implies that for $0 \le i \ne j \le N_1 + N_2 + 1$, $D(\eta_i) \cap Z_n^{N_0}$ contains at most one point. Now using the same argument as in the first case, one can first deduce that all these $N_1 + N_2 + 1$ domains $D(\eta_i)$ are disjoint and then get a contradiction by the symmetry of $D(\eta_i)$ and by counting the number of the points in $P_1 \setminus \overline{\Delta}$ and the number of the holomorphic disks in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$. Please see Figure 9 for an illustration.

From the assertions in Steps 1 and 3 we have:

COROLLARY 7·6. *For any $l \ge 0$, there exists a $0 < \delta = \delta(l) < \log(\sqrt{2} + 1)$ depending only on l such that for all $m \ge 0$ and all n large enough, if γ is a $(\phi_{m,n}, A_n^l)$-geodesic which intersect $\mathbb{T}$, then $l_{\phi_{m,n}}(\gamma, A_n^l) \ge \delta$.*

***Step* 4**. The argument is similar with that used in the Step 3. Again, since O_n is forward invariant, the argument is even simpler. Suppose γ is a $(\phi_{m,n}, B_n)$-geodesic which intersect $\mathbb{T}$ so that either $D(\gamma) \cap O_n$ or $D(\gamma)^c \cap O_n$ contains at most one point. Let N_1 and N_2 be the integers defined in the Step 3. In Lemma 4·2 let $N_0 = N_1 + N_2 + 1$ and define the sets $\tilde{B}_n^i$, $0 \le i \le N_0$. For $0 \le i \le N_0$, by Lemma 7·1 we have

$$l_{\phi_{m+i,n}}(\gamma, B_n) \le e^{2iC_0} \cdot l_{\phi_{m,n}}(\gamma, B_n)\,. \tag{7·18}$$

For $0 \le i \le N_0$, from Lemma 8·3, $B_n \subset \tilde{B}_n^i$ and that the number of the points in $\tilde{B}_n^i \setminus B_n$ is bounded by some number depending only on n, it follows that there is a $(\phi_{m+i,n}, \tilde{B}_n^i)$-geodesic γ_i which is homotopic to γ in $\widehat{\mathbb{C}} \setminus B_n$ such that

$$l_{\phi_{m+i,n}}\left(\gamma_i, \tilde{B}_n^i\right) \le C_1(n) \cdot l_{\phi_{m+i,n}}(\gamma, B_n) \tag{7·19}$$

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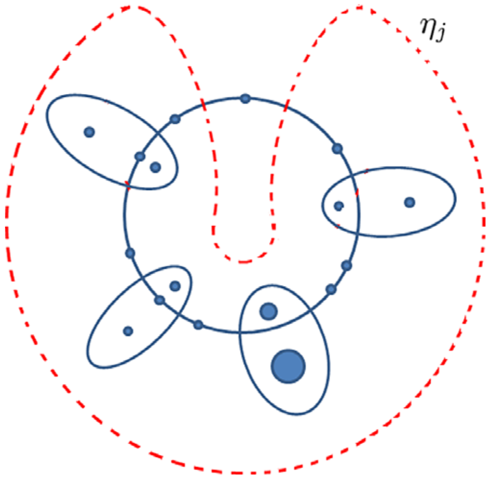

Fig. 9. Case II. all $D(\eta_i)$, $0 \le i \ne j \le N_1 + N_2 + 1$, are disjoint and are contained in $D(\eta_j)$.

provided that $l_{\phi_{m+i,n}}(\gamma, B_n) < C_1(n)^{-1}$ where $C_1(n) > 1$ is some constant depending only on n.

Let $\lambda = \lambda(N_0)$ be the constant guaranteed by Lemma 7·3. Let us show that the assertion holds by taking

$$\delta_n = C_1(n)^{-1} e^{-2N_0C_0} \cdot \min\left\{\lambda, \log\left(\sqrt{2}+1\right)\right\}.$$

Assume that $l_{\phi_{m,n}}(\gamma, B_n) < \delta_n$. Let us get the contradiction. Since either $D(\gamma) \cap O_n$ or $D(\gamma)^c \cap O_n$ contains at most one point, so does $D(\gamma_i) \cap O_n$ or $D(\gamma_i)^c \cap O_n$. It follows that $(\gamma \cap \mathbb{T}) \cup \bigcup_{i=0}^{N_0} (\gamma_i \cap \mathbb{T})$ either belongs to the same component of $\mathbb{T} \setminus O_n$ or belongs to the union of two adjacent components of $\mathbb{T} \setminus O_n$. So the points in $(\gamma \cap \mathbb{T}) \cup \bigcup_{i=0}^{N_0} (\gamma_i \cap \mathbb{T})$ can be arbitrarily close to a single point in $\mathbb{T}$, say z_0, provided that n is large enough. Let $z_i = F_n^i(z_0)$, $0 \le i \le N_0$. Since $F_n|\mathbb{T} = R_{\theta_n}$ and $\theta_n \to \theta$ which is irrational, as n is large enough, all z_i, $0 \le i \le N_0$, are bounded away from each other with the bound independent of n. Let $\eta_i = F_n^i(\gamma_i)$, $0 \le i \le N_0$. Since $F_n^i : \widehat{\mathbb{C}} \setminus \tilde{B}_n^i \to \widehat{\mathbb{C}} \setminus B_n^{N_0}$ is a covering map and $l_{\phi_{m+i,n}}(\gamma_i, \tilde{B}_n^i) < \log(\sqrt{2}+1)$ by (7·18-7·19), from Lemma 8·2 it follows that all η_i, $0 \le i \le N_0$, are $(\phi_{m,n}, B_n^{N_0})$-geodesics, and moreover,

$$l_{\phi_{m,n}}\left(\eta_i, B_n^{N_0}\right) \le l_{\phi_{m+i,n}}\left(\gamma_i, \tilde{B}_n^i\right). \tag{7·20}$$

Since both the two points in $\gamma_i \cap \mathbb{T}$ are close to z_0, it follows that both the two points in $\eta_i \cap \mathbb{T}$ are close to z_i, $0 \le i \le N_0$. Since all z_i are bounded away from each other, it follows that all η_i are distinct with other provided that n is large enough.

Since either $D(\gamma_i) \cap O_n$ or $D(\gamma_i)^c \cap O_n$ contains at most one point, and since $\eta_i = F_n^i(\gamma_i)$ and O_n is F_n-invariant, we have:

fact 1. Either $D(\eta_i) \cap O_n$ or $D(\eta_i)^c \cap O_n$ contains at most one point.

From (7·18-7·20) we have $l_{\phi_{m,n}}(\eta_i, B_n^{N_0}) < \min\{\lambda, \log(\sqrt{2}+1)\}$. Since all η_i are distinct with each other, by Lemma 8·1 we have:

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fact 2. all η_i, $0 \le i \le N_1 + N_2 + 1$, are disjoint with each other.

Since for each $0 \le i \le N_0$, both the two points in $\eta_i \cap \mathbb{T}$ are close to z_i, and since all z_i are bounded away from each other with the bound independent of n, it follows that, as n is large enough,

fact 3. For any $0 \le i \ne j \le N_1 + N_2 + 1$, if $p \in \eta_i \cap \mathbb{T}$ and $q \in \eta_j \cap \mathbb{T}$, then each arc component of $\mathbb{T} \setminus \{p, q\}$ contains at least one point in O_n.

Note that by Lemma 7·3 and $l_{\phi_{m,n}}(\eta_i, B_n^{N_0}) < \lambda$, all η_i do not intersect the holomorphic disks. Now the situation is the same as the one that we met at the end of Step 3. The remaining argument is completely the same and we leave it to the reader.

***Step* 5**. Instead of proving the two assertions claimed in Lemma 4·3 under the assumption that $\gamma \cap \mathbb{T} = \emptyset$, it suffices to prove the following one assertion:

Assertion. There exists a $\delta > 0$ such that for all $m \ge 0$ and all n large enough, if γ is a $(\phi_{m,n}, A_n)$-geodesic with $\gamma \cap \mathbb{T} = \emptyset$, then $l_{\phi_{m,n}}(\gamma, A_n) \ge \delta$.

To see this, first suppose that γ is a $(\phi_{m,n}, A_n^k)$-geodesic for some integer $k \ge 1$ such that $\gamma \cap \mathbb{T} = \emptyset$. We may assume that γ dose not intersect the holomorphic disks since otherwise we have $l_{\phi_{m,n}}(\gamma, A_n^k) \ge \lambda(k)$ with $\lambda(k) > 0$ being the constant guaranteed by Lemma 7·3. Since $A_n^k \setminus A_n \subset \mathbb{T} \cup \cup_i \overline{D_i}$ and $\gamma \cap (\mathbb{T} \cup \cup_i \overline{D_i}) = \emptyset$, and since each D_i contains two points a_i and b_i in A_n, γ must be non-peripheral in $\widehat{\mathbb{C}} \setminus A_n$. Since $\widehat{\mathbb{C}} \setminus A_n^k \subset \widehat{\mathbb{C}} \setminus A_n$ we have $l_{\phi_{m,n}}(\gamma, A_n^k) \ge l_{\phi_{m,n}}(\gamma, A_n) \ge \delta$. This implies that

$$l_{\phi_{m,n}}\left(\gamma, A_n^k\right) \ge \min\{\delta, \lambda(k)\}.$$

Now suppose that γ is a $(\phi_{m,n}, B_n)$-geodesic such that $\gamma \cap \mathbb{T} = \emptyset$. Since the points in $B_n \setminus A_n$ are contained in $\mathbb{T}$ and since $\gamma \cap \mathbb{T} = \emptyset$, it follows that γ must be non-peripheral in $\widehat{\mathbb{C}} \setminus A_n$. Since $\widehat{\mathbb{C}} \setminus B_n \subset \widehat{\mathbb{C}} \setminus A_n$ it follows that

$$l_{\phi_{m,n}}(\gamma, B_n) \ge l_{\phi_{m,n}}(\gamma, A_n) \ge \delta.$$

Now let us prove the assertion claimed above. We first define two constants, M and m_0, which depend only on f. Define

$$M = |A_n| - 3. \tag{7·21}$$

Note that $|A_n|$ is independent of n by Proposition 4·1. By Lemma 8·1 it follows that the number of $(\phi_{m,n}, A_n)$-geodesics with length less than $\log(\sqrt{2}+1)$ is not greater than M. Since f has no Thurston obstructions in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$, the maximal eigenvalues of all the linear transformation matrices induced by the f-stable multi-curves in $\widehat{\mathbb{C}} \setminus (\overline{\Delta} \cup P_f \cup \cup_i \overline{D_i})$ are less than 1 (see Remark 2·1). Note that the number of the elements in any f-stable multi-curves in $\widehat{\mathbb{C}} \setminus (\overline{\Delta} \cup P_f \cup \cup_i \overline{D_i})$ and the number of the possible values taken by the entries of the linear transformation matrices are both bounded by some constants depending only on

f. It follows that there are only finitely many such linear transformation matrices. So there is an $m_0 \geq 1$ depending only on f such that for any such matrix A, let $A^{m_0} = (b_{i,j})$, then

$$\max_j \sum_i b_{i,j} < (2M)^{-1}. \tag{7·22}$$

Claim 1. There is a $\tau > 0$ such that for all $m \geq 0$ and all n large enough, if η is a $(\phi_{m,n}, A_n)$-geodesic such that $l_{\phi_{m,n}}(\eta, A_n) < \tau$, then $\eta \subset \widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$, and moreover, for each $1 \leq j \leq m_0$, there is a unique $(\phi_{m,n}, A_n^j)$-geodesic ξ which is homotopic in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$ to η. Moreover, $l_{\phi_{m,n}}(\xi, A_n^j) < 2l_{\phi_{m,n}}(\eta, A_n)$.

Proof. For $0 \leq j \leq m_0$, let $\lambda(j) > 0$ and $\delta(j) > 0$ be respectively the constants guaranteed by Lemma 7·3 and Corollary 7·6. In Lemma 8·3 Let us take $X = \widehat{\mathbb{C}} \setminus \phi_{m,n}(A_n)$, $P = \phi_{m,n}(A_n^j \setminus A_n)$, $l = l_{\phi_{m,n}}(\eta, A_n)$ and

$$K = \min\{\lambda(j), \delta(j), 0 \leq j \leq m_0\}.$$

Note that the number the points in $A_n^j \setminus A_n$ is bounded by some constant $p(m_0)$ depending only on m_0. By considering all the $(\phi_{m,n}, A_n^j)$-geodesics η_i which have length $l_i < K$ and which are homotopic to η in $\widehat{\mathbb{C}} \setminus A_n$ and applying Lemma 8·3, we get

$$\frac{1}{l} - \frac{2}{\pi} - \frac{p(m_0)+1}{K} < \sum_i \frac{1}{l_i}. \tag{7·23}$$

Now let us prove Claim 1 by taking

$$\tau = \left(\frac{4}{\pi} + \frac{2(p(m_0)+1)}{K}\right)^{-1}.$$

It is clear that $\tau < K/2$. Suppose $l < \tau$. Then $l < K/2$. By the definition of K, Lemma 7·3 and Corollary 7·6, η does not intersect $\mathbb{T} \cup \cup_i \overline{D_i}$, and so $\eta \subset \widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \overline{D_i})$. From $l < \tau$ and the definition of τ, the left-hand side of (7·23) is greater than $1/2l$. Thus the sum on the right-hand side contains at least one term, that is, there is at least one $(\phi_{m,n}, A_n^j)$-geodesic η_i which is homotopic to η in $\widehat{\mathbb{C}} \setminus A_n$ so that $l_{\phi_{m,n}}(\eta_i, A_n^j) < K$. From Lemma 7·3, Corollary 7·6 and the definition of K, such η_i do not intersect holomorphic disks and $\mathbb{T}$, and thus contained in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \overline{D_i})$ also. Since $\mathbb{T} \cap A_n \neq \emptyset$ and $\overline{D_i} \cap A_n \neq \emptyset$, such η_i are actually homotopic to η in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup A_n \cup \cup_i \overline{D_i}) = \widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$. It is clear that there is at most one such $(\phi_{m,n}, A_n^j)$-geodesic, because any two such ones must be homotopic to each other in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$, and thus in $\widehat{\mathbb{C}} \setminus A_n^j$, and are thus identified with each other. This means, there is exactly one term in the sum on the right-hand side of (7·23). Let ξ denote this η_i. Then $l_{\phi_{m,n}}(\xi, A_n^j) < 2l_{\phi_{m,n}}(\eta, A_n)$. The proof of Claim 1 is completed.

Claim 2. There exists a constant $\kappa > 0$ such that for all $m \geq 0$ and all n large enough, if $\gamma \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}$ is a $(\phi_{m,n}, A_n)$-geodesic with $l_{\phi_{m,n}}(\gamma, A_n) < \kappa$, then all components of $F_n^{-1}(\gamma)$ which are non-peripheral in $\widehat{\mathbb{C}} \setminus A_n$ are contained in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$.

Proof. Suppose $\eta \subset \Delta$ is a component of $F_n^{-1}(\gamma)$ which is non-peripheral in $\widehat{\mathbb{C}} \setminus A_n$. Let Ω denote the component of $\widehat{\mathbb{C}} \setminus \gamma$ which does not contain the origin. Let us deform γ in Ω into a single point. Then we have two cases. In the first case, the deformation of γ can be

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lifted to a deformation of η by F_n^{-1} so that η is deformed into a point. In this case we must meet a point $z \in P_{F_n} \cap \Delta$ so that $F_n(z) \in \Omega \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}$. In the second case, the deformation of γ can not be lifted to a deformation of η by F_n^{-1}. In this case we must meet some critical point $z \in \Delta$ so that $F_n(z) \in \Omega \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}$. By (vi) of Proposition 3·2 in both the cases we have $F_n(z) \in \Lambda_n$ where $\Lambda_n \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}$ is a curve segment connecting $F_n(z)$ to the point 1 such that $F_n(\Lambda_n) \subset \mathbb{T}$. Since $F_n(z) \in \Omega$, γ must separate 1 and $F_n(z)$. In Lemma 4·2 let $N_0 = 1$ and define the set $\tilde{A}_n^1$. By Lemma 7·1

$$l_{\phi_{m+1,n}}(\gamma, A_n) \le e^{2C_0} \cdot l_{\phi_{m,n}}(\gamma, A_n)\,. \tag{7·24}$$

Since $A_n \subset \tilde{A}_n^1$ and the number of the points in $\tilde{A}_n^1 \setminus A_n$ does not depend on n, by Lemma 8·3 there is a $(\phi_{m+1,n}, \tilde{A}_n^1)$-geodesic γ_1 which is homotopic to γ in $\widehat{\mathbb{C}} \setminus A_n$ such that

$$l_{\phi_{m+1,n}}\left(\gamma_1, \tilde{A}_n^1\right) \le C_1 \cdot l_{\phi_{m+1,n}}(\gamma, A_n) \tag{7·25}$$

provided that $l_{\phi_{m+1,n}}(\gamma, A_n) < C^{-1}$ with $C_1 > 1$ being some constant independent of m and n. In particular, γ_1 must separate 1 and $F_n(z)$ and thus intersect Λ_n. Let $\delta = \delta(1) < \log(\sqrt{2}+1)$ be the constant guaranteed by Corollary 7·6. Now take

$$\kappa = C_1^{-1} e^{-2C_0} \delta.$$

Let us show that the κ is the desired constant by contradiction. Assume that $l_{\phi_{m,n}}(\gamma, A_n) < \kappa$. Then $l_{\phi_{m+1,n}}(\gamma_1, \tilde{A}_n^1) < \delta < \log(\sqrt{2}+1)$ by (7·24-7·25). Since $F_n : \widehat{\mathbb{C}} \setminus \tilde{A}_n^1 \to \widehat{\mathbb{C}} \setminus A_n^1$ is a covering map (see Lemma 4·2) and γ_1 intersects Λ_n, by Lemma 8·2, $\eta_1 = F_n(\gamma_1)$ is a $(\phi_{m,n}, A_n^1)$-geodesic which intersects $\mathbb{T}$ such that

$$l_{\phi_{m,n}}\left(\eta_1, A_n^1\right) \le l_{\phi_{m+1,n}}\left(\gamma_1, \tilde{A}_n^1\right) < \delta.$$

This contradicts Corollary 7·6 and Claim 2 has been proved.

Recall that any point in $P_{F_n} \setminus (P_f \cup \overline{\Delta})$ is connected to the point 1 by the curve segment Λ_n, see assertion (vi) of Proposition 3·2. So the proof of Claim 2 actually proves:

Claim 3. Let $\kappa > 0$ be the constant in Claim 2. Then for all $m \ge 0$ and all n large enough, if $\gamma \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}$ is a $(\phi_{m,n}, A_n)$-geodesic with $l_{\phi_{m,n}}(\gamma, A_n) < \kappa$, then γ does not separate the point 1 and the points in $P_{F_n} \setminus (P_f \cup \overline{\Delta})$.

Let

$$L = \max\left\{(\tau A)^{-1}, (\kappa A)^{-1}, 2(2d-1)e^{2C_0}\right\} \tag{7·26}$$

with and $A = 1/\log(\sqrt{2}+1)$, and $\tau, \kappa, C_0 > 0$ being respectively the constants determined in the Claims 1 and 2 and Lemma 7·1.

Claim 4. Let $m \ge 0$ and n be large enough. Suppose the following two conditions hold for some $a \ge A = 1/\log(\sqrt{2}+1)$:

(i) there is no $(\phi_{m,n}, A_n)$-geodesic γ satisfying $a < 1/l_{\phi_{m,n}}(\gamma, A_n) \le La$;
(ii) $\Sigma = \{\gamma : \gamma \text{ is a } (\phi_{m,n}, A_n)\text{-geodesic with } 1/l_{\phi_{m,n}}(\gamma, A_n) > La\} \neq \emptyset$.

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Let $\Gamma = \{\gamma \in \Sigma : \gamma \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}\}$ and $\Gamma^* = \{\gamma \in \Sigma : \gamma \subset \Delta\}$. Then $\Sigma = \Gamma \cup \Gamma^*$ and moreover, the following two assertions hold:

(1) both Γ and Γ^* are F_n-stable multi-curves in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$;
(2) for every $\gamma \in \Gamma$, γ does not separate the point 1 and the points in $P_{F_n} \setminus (P_f \cup \overline{\Delta})$.

Proof. Suppose (i) and (ii) hold. Let $\gamma \in \Sigma$. From $1/l_{\phi_{m,n}}(\gamma, A_n) > La$ and (7·26) we have

$$l_{\phi_{m,n}}(\gamma, A_n) < \min\{\tau, \kappa\} < \log\left(\sqrt{2}+1\right). \tag{7·27}$$

By (7·27), Claim 1 and Lemma 8·1 all the elements in Σ are contained in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$ and are disjoint with other. Since $A_n \subset (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$, all the elements in Σ are non-peripheral and non-homotopic to each other in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$. So $\Sigma = \Gamma \cup \Gamma^*$ and both Γ and Γ^* are multi-curves in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$.

To prove that Γ is F_n-stable, let $N_0 = 1$ in Lemma 4·2 and define the set $\tilde{A}_n^1$. For each $\gamma \in \Gamma$, by Claim 1, there is a unique $(\phi_{m,n}, A_n^1)$-geodesic ξ which is homotopic to γ in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$ with

$$l_{\phi_{m,n}}\left(\xi, A_n^1\right) < 2 \cdot l_{\phi_{m,n}}(\gamma, A_n). \tag{7·28}$$

Suppose $\eta \subset \widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$ is a non-peripheral component of $F_n^{-1}(\gamma)$. Since each D_i contains two points a_i and b_i in A_n, η is also non-peripheral in $\widehat{\mathbb{C}} \setminus A_n$. By (7·27) and Claim 2, we have

$$\eta \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}. \tag{7·29}$$

Since ξ is homotopic to γ in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$, and since $F_n : \widehat{\mathbb{C}} \setminus \tilde{A}_n^1 \to \widehat{\mathbb{C}} \setminus A_n^1$ is a covering map and $A_n^1 \subset (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$, there is a $(\phi_{m+1,n}, \tilde{A}_n^1)$-geodesic ζ which is homotopic to η in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$ such that $\xi = F_n(\zeta)$. In particular, by Definition 4·1 we have

$$l_{\phi_{m,n}}(\eta, A_n) = l_{\phi_{m,n}}(\zeta, A_n). \tag{7·30}$$

Since F_n is of degree $2d-1$, we have

$$l_{\phi_{m+1,n}}\left(\zeta, \tilde{A}_n^1\right) \le (2d-1) \cdot l_{\phi_{m,n}}\left(\xi, A_n^1\right). \tag{7·31}$$

By Lemma 7·1 there is some $C_0 > 0$ independent of m and n such that

$$l_{\phi_{m,n}}(\zeta, A_n) < e^{2C_0} \cdot l_{\phi_{m+1,n}}(\zeta, A_n). \tag{7·32}$$

Since $A_n \subset \tilde{A}_n^1$, we have

$$l_{\phi_{m+1,n}}(\zeta, A_n) \le l_{\phi_{m+1,n}}\left(\zeta, \tilde{A}_n^1\right). \tag{7·33}$$

Now by tracking these inequalities according to the order (7·30, 7·32, 7·33, 7·31, 7·28), we have

$$l_{\phi_{m,n}}(\eta, A_n) < 2(2d-1)e^{2C_0} \cdot l_{\phi_{m,n}}(\gamma, A_n).$$

Since $1/l_{\phi_{m,n}}(\gamma, A_n) > La$, by (7·26) we get $1/l_{\phi_{m,n}}(\eta, A_n) > a$. By (i) it follows that $1/l_{\phi_{m,n}}(\eta, A_n) > La$ and thus η is homotopic in $\widehat{\mathbb{C}} \setminus A_n$ to some element $\beta \in \Sigma$. Since

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$\eta \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}$ by (7·29), we must have $\beta \in \Gamma$. Since both η and β do not intersect $\mathbb{T} \cup \cup_i \overline{D_i}$, and since $\mathbb{T}$ and each $\overline{D_i}$ contain points in A_n, the homotopy between η and β can be made in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup A_n \cup \cup_i \overline{D_i}) = \widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$. This proves that Γ is F_n-stable in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$. By symmetry Γ^* is also F_n-stable in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$. This proves (1).

The assertion (2) follows from Claim 3 and that $l_{\phi_{m,n}}(\gamma, A_n) < \kappa$.

Claim 5. Let Γ be a F_n-stable multi-curve in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$ described in Claim 4. Then Γ is a f-stable multi-curve in $\widehat{\mathbb{C}} \setminus (\overline{\Delta} \cup P_f \cup \cup_i \overline{D_i})$, and moreover, Γ induces the same linear transformation for F_n and f.

Proof. Let $\Gamma = \{\gamma_l : 1 \le l \le N\}$ with $1 \le N \le M$ and M being the constant in (7·21). By deforming γ_l in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$, we may assume that $\gamma_l \subset \widehat{\mathbb{C}} \setminus \{z : |z| \le 1 + r_1\}$ for all $1 \le l \le N$, where $r_1 > 0$ is the constant in Remark 3·2. Then Claim 5 follows obviously if the following three assertions hold. Let $\gamma_l \in \Gamma$ be an arbitrary element.

(i) If a component of $F_n^{-1}(\gamma_l)$ is non-peripheral in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$, then it is contained in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$.

(ii) $F_n^{-1}(\gamma_l) \cap (\widehat{\mathbb{C}} \setminus \overline{\Delta}) = f^{-1}(\gamma_l)$ and when restricted to each component of $f^{-1}(\gamma_l)$, F_n and f coincide with each other.

(iii) A component of $f^{-1}(\gamma_l)$ is non-peripheral in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$ if and only if it is non-peripheral in $\widehat{\mathbb{C}} \setminus (\overline{\Delta} \cup P_f \cup \cup_i \overline{D_i})$.

The first assertion follows since Γ is F_n-stable in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$ and all the elements in Γ are contained in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$. The second one follows from Remark 3·2.

To see the third one, let η be a component of $f^{-1}(\gamma_l)$. Suppose η is non-peripheral in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$. Then η is contained in $\widehat{\mathbb{C}} \setminus \overline{\Delta}$ and is homotopic to some element $\beta \in \Gamma$ in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$. Thus by (2) of Claim 4, η does not separate the point 1 and the points in $P_{F_n} \setminus (P_f \cup \overline{\Delta})$. This implies that the component of $\widehat{\mathbb{C}} \setminus \eta$, which does not contain $\mathbb{T}$, does not contain the points in $P_{F_n} \setminus (P_f \cup \overline{\Delta})$. So η must be non-peripheral in $\widehat{\mathbb{C}} \setminus (\overline{\Delta} \cup P_f \cup \cup_i \overline{D_i})$. On the other hand, suppose η is non-peripheral in $\widehat{\mathbb{C}} \setminus (\overline{\Delta} \cup P_f \cup \cup_i \overline{D_i})$. Since $P_f \subset P_{F_n}$, the component of $\widehat{\mathbb{C}}$ which does not contain $\mathbb{T}$ must contain at least two points in P_f and thus contains at least two points in P_{F_n}. Since the other component contains $\mathbb{T}$, it follows that η must be non-peripheral in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$.

Claim 6. Let $A = 1/\log(\sqrt{2} + 1)$ and L be the constant defined by (7·26). Then there exists a $C_1 > 0$ such that for all $m \ge 0$ and all n large enough, if (i) and (ii) in Claim 4 hold for some $A \le a \le L^M A$, then

$$\sum_{\gamma \in \Gamma} 1/l_{\phi_{m+m_0,n}}(\gamma, A_n) < (2M)^{-1} \cdot \sum_{\gamma \in \Gamma} 1/l_{\phi_{m,n}}(\gamma, A_n) + C_1, \tag{7·34}$$

where Γ is the F_n-stable multi-curve guaranteed by Claim 4 and M is the constant defined by (7·21).

Proof. In Lemma 4·2 let $N_0 = m_0$ and define the set $\tilde{A}_n^{m_0}$. Then $A_n \subset \tilde{A}_n^{m_0}$. Let $p = |\tilde{A}_n^{m_0} \setminus A_n|$. Then $p \ge 1$ is independent of m and n. Let

$$K = \min\left\{\delta(m_0), \lambda(m_0), L^{-M} A^{-1}\right\} < \log\left(\sqrt{2} + 1\right),$$

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where $\delta(m_0)$ and $\lambda(m_0)$ are the constants guaranteed by Corollary 7·6 and Lemma 7·3 respectively. Let us prove Claim 6 by taking

$$C_1 = M\left(\frac{2}{\pi} + \frac{p+1}{K}\right).$$

Let $\Gamma = \{\gamma_i,\ 1 \le i \le N\}$ with $1 \le N \le M$. Recall that for each $\gamma_i \in \Gamma$, $l_{\phi_{m,n}}(\gamma_i, A_n) < \tau$, see (7·27). By Claim 1, there is a unique $(\phi_{m,n}, A_n^{m_0})$-geodesic γ_i' homotopic in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$ to γ_i. Let

$$\Gamma' = \left\{\gamma_i',\ 1 \le i \le N\right\}.$$

Note that $F_n^{m_0} : \widehat{\mathbb{C}} \setminus \tilde{A}_n^{m_0} \to \widehat{\mathbb{C}} \setminus A_n^{m_0}$ is a covering map by Lemma 4·2. Let $\gamma_{i,j,\alpha}'$ denote all the components of $F_n^{-m_0}(\gamma_j')$ which are homotopic to γ_i' in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \cup_i \overline{D_i})$. Then $\gamma_{i,j,\alpha}'$ are $(\phi_{m+m_0,n}, \tilde{A}_n^{m_0})$-geodesics. Let $d_{i,j,\alpha} \ge 1$ denote the covering degree of the map $F_n^{m_0} : \gamma_{i,j,\alpha}' \to \gamma_j'$. Now let us prove

$$\sum_i 1/l_{\phi_{m+m_0,n}}(\gamma_i, A_n) < \sum_{i,j,\alpha} 1/l_{\phi_{m+m_0,n}}\left(\gamma_{i,j,\alpha}', \tilde{A}_n^{m_0}\right) + C_1 \tag{7·35}$$

Note that $A_n \subset \tilde{A}_n^{m_0}$. In Lemma 8·3 let $X = \widehat{\mathbb{C}} \setminus \phi_{m+m_0}(A_n)$ and $P = \phi_{m+m_0}(\tilde{A}_n^{m_0} \setminus A_n)$. For each $\gamma_i \in \Gamma$, by the left-hand inequality in Lemma 8·3 we get

$$1/l_{\phi_{m+m_0,n}}(\gamma_i, A_n) < \sum_\eta 1/l_{\phi_{m+m_0,n}}\left(\eta, \tilde{A}_n^{m_0}\right) + \frac{2}{\pi} + \frac{p+1}{K}, \tag{7·36}$$

where the sum is taken over all $(\phi_{m+m_0}, \tilde{A}_n^{m_0})$-geodesics η which are homotopic to γ_i in $\widehat{\mathbb{C}} \setminus A_n$ and satisfy $l_{\phi_{m+m_0,n}}(\eta, \tilde{A}_n^{m_0}) < K$. For any such η, since $F_n^{m_0} : \widehat{\mathbb{C}} \setminus \tilde{A}_n^{m_0} \to \widehat{\mathbb{C}} \setminus A_n^{m_0}$ is a covering map, it follows from Lemma 8·2 that $F_n^{m_0}(\eta)$ is a $(\phi_{m,n}, A_n^{m_0})$-geodesic with

$$l_{\phi_{m,n}}\left(F_n^{m_0}(\eta), A_n^{m_0}\right) \le l_{\phi_{m+m_0,n}}\left(\eta, \tilde{A}_n^{m_0}\right) < K.$$

By Corollary 7·6 and Lemma 7·3 $F_n^{m_0}(\eta)$ does not intersect $\mathbb{T} \cup \cup_i \overline{D_i}$, hence η does not intersect $\mathbb{T} \cup \cup_i \overline{D_i}$ either. Since η is homotopic to $\gamma_i \in \Gamma$ in $\widehat{\mathbb{C}} \setminus A_n$ and $\gamma_i \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}$, we must have $\eta \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}$. Since $(A_n^{m_0} \setminus A_n) \subset (\mathbb{T} \cup \cup_i \overline{D_i})$ and $F_n^{m_0}(\eta)$ does not intersect $\mathbb{T} \cup \cup_i \overline{D_i}$, and since each D_i contains two points a_i and b_i in A_n, $F_n^{m_0}(\eta)$ must be non-peripheral in $\widehat{\mathbb{C}} \setminus A_n$. Since $A_n \subset A_n^{m_0}$, we have $l_{\phi_{m,n}}(F_n^{m_0}(\eta), A_n) \le l_{\phi_{m,n}}(F_n^{m_0}(\eta), A_n^{m_0}) < K \le 1/a$. Thus $1/l_{\phi_{m,n}}(F_n^{m_0}(\eta), A_n) > a$. By (i) of Claim 4, we must have $1/l_{\phi_{m,n}}(F_n^{m_0}(\eta), A_n) > La$. Thus either $F_n^{m_0}(\eta) \subset \Delta$ and is homotopic in $\widehat{\mathbb{C}} \setminus A_n$ to some element in Γ^*, or $F_n^{m_0}(\eta) \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}$ and is homotopic in $\widehat{\mathbb{C}} \setminus A_n$ to some element in Γ. Since $F_n^{m_0}(\eta)$ and all the elements in Γ and Γ^* do not intersect $\mathbb{T} \cup \cup_i \overline{D_i}$, and since $\mathbb{T}$ and each D_i contain points in A_n, in both the cases, the homotopy can be made in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \overline{D_i})$. The first case can not happen. Because otherwise, since Γ^* is F_n-stable and η is a non-peripheral component of $F_n^{-m_0}(F_n^{m_0}(\eta))$, η would be homotopic in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \overline{D_i})$ to some element in Γ^*. But this is impossible since $\eta \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}$ and all the elements in Γ^* are contained in Δ. So $F_n^{m_0}(\eta) \subset \widehat{\mathbb{C}} \setminus \overline{\Delta}$ and is homotopic in $\widehat{\mathbb{C}} \setminus (\mathbb{T} \cup P_{F_n} \cup \overline{D_i})$ to some element in Γ. This, together with the definition

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of Γ', implies that $F_n^{m_0}(\eta) \in \Gamma'$. Assume that $F_n^{m_0}(\eta) = \gamma'_j$. Then η must be one of the $\gamma'_{i,j,\alpha}$. From (7·36) we get

$$1/l_{\phi_{m+m_0,n}}(\gamma_i, A_n) < \sum_{j,\alpha} 1/l_{\phi_{m+m_0,n}}\left(\gamma_{i,j,\alpha}, \tilde{A}_n^{m_0}\right) + \frac{2}{\pi} + \frac{p+1}{K}.$$

From $N \leq M$, (7·35) follows by taking the sum of the above inequalities for $1 \leq i \leq N$. By Claim 5. Γ induces the same linear transformation for F_n and f. Thus by (2·1, and (7·22) we have

$$\max_j \sum_{i,\alpha} \frac{1}{d_{i,j,\alpha}} = \max_j \sum_i b_{i,j} < (2M)^{-1}. \tag{7·37}$$

We thus have

$$\begin{aligned}\sum_{i,j,\alpha} 1/l_{\phi_{m+m_0,n}}\left(\gamma'_{i,j,\alpha}, \tilde{A}_n^{m_0}\right) &= \sum_j \left(\sum_{i,\alpha} \frac{1}{d_{i,j,\alpha}}\right) 1/l_{\phi_{m,n}}\left(\gamma'_j, A_n^{m_0}\right) \\ &< (2M)^{-1} \cdot \sum_j 1/l_{\phi_{m,n}}\left(\gamma'_j, A_n^{m_0}\right) \\ &< (2M)^{-1} \cdot \sum_j 1/l_{\phi_{m,n}}\left(\gamma'_j, A_n\right)\end{aligned} \tag{7·38}$$

The last inequality holds since $A_n \subset A_n^{m_0}$ and thus $l_{\phi_{m,n}}(\gamma'_j, A_n) \leq l_{\phi_{m,n}}(\gamma'_j, A_n^{m_0})$. (7·34) then follows from (7·35), (7·38) and that $l_{\phi_{m,n}}(\gamma'_j, A_n) = l_{\phi_{m,n}}(\gamma_j, A_n)$.

Now let us complete the proof in the Step 5. Let $N^* \geq 1$ be an integer such that all the assertions proved until now hold for all $m \geq 0$ and all $n \geq N^*$, see Remark 4·2. Take an arbitrary $n \geq N^*$ and let it be fixed. For $m \geq 0$, let

$$x_m = \max_\gamma\{1/l_{\phi_{m,n}}(\gamma, A_n)\},$$

where the max is taken over all non-peripheral curves γ in $\widehat{\mathbb{C}} \setminus A_n$. Since $\phi_{0,n} = \mathrm{id}$ and since the spherical distance between any two distinct points in A_n has a positive lower bound independent of n (see Proposition 4·1), it follows that $l_{\phi_{0,n}}(\gamma, A_n)$ has a positive lower bound independent of γ and n. So we have a constant $0 < c_0 < \infty$ independent of n so that $x_0 \leq c_0$. From Lemma 7·1 it follows that $x_{m+1}/x_m \leq e^{2C_0}$ holds for all $m \geq 0$ with $0 < C_0 < \infty$ being the constant in Lemma 7·1.

Let $M_0 = \max\{2C_1, A \cdot e^{2m_0C_0} \cdot L^{M+1}\}$ with $C_0 > 0$ being the constant in Lemma 7·1, $L > 1$ being the constant given by (7·26) and $C_1 > 0$ being the constant in Claim 6 and $A = 1/\log(\sqrt{2}+1)$.

Now suppose $x_m \geq M_0$ for some $m \geq 0$. Then there is a $(\phi_{m,n}, A_n)$-geodesic η so that $x_m = 1/l_{\phi_{m,n}}(\eta, A_n) \geq A \cdot e^{2m_0C_0} \cdot L^{M+1}$. Since there are at most M $(\phi_{m,n}, A_n)$-geodesics with length less than $\log(\sqrt{2}+1)$, there is some $1 \leq i \leq M+1$ such that there is no $(\phi_{m,n}, A_n)$-geodesic γ satisfying $L^{i-1}A < 1/l_{\phi_{m,n}}(\gamma, A_n) \leq L^i A$. In Claim 4 let $a = L^{i-1}A \leq L^M A$. Then the two conditions in Claim 4 are satisfied. Let Γ be the F_n-stable multi-curve guaranteed by Claim 4.

Suppose $x_{m+m_0} = 1/l_{\phi_{m+m_0,n}}(\xi, A_n)$ for some $(\phi_{m+m_0,n}, A_n)$-geodesic ξ. There are two cases.

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In the first case, $1/l_{\phi_{m,n}}(\xi, A_n) \le La$. By Lemma 7·1 we have

$$x_{m+m_0} = 1/l_{\phi_{m+m_0,n}}(\xi, A_n) \le e^{2m_0C_0}/l_{\phi_{m,n}}(\xi, A_n) \le A \cdot e^{2m_0C_0} \cdot L^{M+1} \le x_m.$$

In the second case, $1/l_{\phi_{m,n}}(\xi, A_n) > La$. By the definition of Γ, it follows that ξ is homotopic in $\widehat{\mathbb{C}} \setminus A_n$ to some $\gamma \in \Gamma$. By Claim 6 we have

$$\begin{aligned} x_{m+m_0} &\le \sum_{\gamma\in\Gamma} 1/l_{\phi_{m+m_0,n}}(\gamma, A_n) < (2M)^{-1} \cdot \sum_{\gamma\in\Gamma} 1/l_{\phi_{m,n}}(\gamma, A_n) + C_1 \\ &\le \frac{1}{2}x_m + C_1 \le x_m. \end{aligned}$$

The last inequality holds because $x_m \ge M_0 \ge 2C_1$ by the definition of M_0.

Now by Lemma 8·4 we have

$$x_m \le D = \max\left\{b_0^{m_0-1}c_0, b_0^{m_0}M_0\right\}, \ \forall m \ge 0.$$

Since $n \ge N^*$ is arbitrary and b_0, c_0, m_0, M_0 are independent of n, the proof of the Step 5 is completed by taking $\delta = D^{-1}$. Lemma 4·3 has been proved.

8. *Appendix*

For the convenience of the reader we state three lemmas which are repeatedly used in Section 7. For the proofs, see [**3**].

LEMMA 8·1. [**3**, corollary 6·6] *Let X be a hyperbolic Riemann surface and γ_1, γ_2 be two simple closed geodesics with length $< \log(\sqrt{2}+1)$. Then either $\gamma_1 = \gamma_2$ or $\gamma_1 \cap \gamma_2 = \emptyset$.*

LEMMA 8·2. [**3**, corollary 6·7] *Let X be a hyperbolic Riemann surface. Let γ be a geodesic in X which intersects itself transversally at least once. Then $l_X(\gamma) > 2\log(\sqrt{2}+1)$. In particular, if $f : X \to Y$ is a holomorphic covering map between two hyperbolic Riemann surfaces, and if $\gamma \subset X$ is a simple closed geodesic with $l_X(\gamma) < 2\log(\sqrt{2}+1)$, then $f(\gamma)$ must be a simple closed geodesic in Y such that*

$$l_X(\gamma) = d_0 \cdot l_Y(f(\gamma))$$

with $d_0 \ge 1$ being the covering degree of $f : \gamma \to f(\gamma)$.

LEMMA 8·3. [**3**, theorem 7·1] *Let X be a hyperbolic Riemann surface, $P \subset X$ a finite set with $|P| = p > 0$. Choose $K < \log(\sqrt{2}+1)$. Let $X' = X - P$. Let γ be a simple closed geodesic in X and $\{\gamma_1', \ldots, \gamma_s'\}$ be the simple closed geodesic in X' which is homotopic to γ in X with length less than K. Let $l = l_X(\gamma)$ and $l_i' = l_{X'}(\gamma_i)$. Then $s \le p+1$ and*

$$\frac{1}{l} - \frac{2}{\pi} - \frac{p+1}{K} < \sum_{1\le i\le s} \frac{1}{l_i'} < \frac{1}{l} + \frac{2(p+1)}{\pi}.$$

In particular, there is a constant $C(p) > 1$ depending only on p such that if $l < C(p)^{-1}$, then there is some γ_i' with $1 \le i \le s$ so that

$$l' < C(p) \cdot l.$$

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Note. The last assertion of Lemma 8·3 follows by taking $K = (1/2)\log(\sqrt{2}+1)$ and

$$C(p) = \max\left\{2(p+1),\, 2\left(\frac{p+1}{K} + \frac{2}{\pi}\right)\right\}.$$

LEMMA 8·4. ([7, Lemma 7·6]) *Let $b_0 > 1$, c_0, $M_0 > 0$ and integer $m_0 \geq 1$ be given. Then for any sequence $\{x_m\}_{m\geq 0}$ of positive numbers satisfying:*

(i) $x_0 \leq c_0$;
(ii) $x_{m+1}/x_m \leq b_0$;
(iii) *if $x_m \geq M_0$, then $x_{m+m_0} \leq x_m$.*

Then

$$x_m \leq \max\{b_0^{m_0-1}c_0,\, b_0^{m_0}M_0\},\ \forall m \geq 0.$$

The following is a technical lemma on homotopy. We use it in Section 5. Its proof is rather elementary and we leave it to the reader.

LEMMA 8·5. *Let U_i, $1 \leq i \leq l$, be Jordan domains with $\overline{U_i} \subset \Delta$ and $\overline{U_i} \cap \overline{U_{i'}} = \emptyset$ for $1 \leq i \neq i' \leq l$. Let $Q = \{q_1, \ldots, q_m\} \subset \Delta$ be a set consisting of finitely many points. Suppose $Q \cap \cup_i \overline{U_i} = \emptyset$. Let $X = \Delta \setminus (Q \cup \cup_i \overline{U_i})$. Then there exists a $\tau > 0$ such that for any homeomorphism $h : \overline{X} \to \overline{X}$, if $\mathrm{dist}(h, \mathrm{id}) < \tau$ and $h|\partial X = \mathrm{id}$, then h is homotopic to id rel ∂X.*

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