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Analytical and numerical treatment of a complex model for Hele-Shaw moving boundary value problems with kinetic undercooling regularization

M. REISSIG[1], S. V. ROGOSIN[2] with an appendix by F. HÜBNER[1]

[1] *Faculty of Mathematics and Computer Science TU Bergakademie Freiberg, Bernhard von Cotta Str. 2, 09596 Freiberg, Germany*
(*email*: reissig@mathe.tu-freiberg.de)
[2] *Faculty of Mathematics and Mechanics, Belarusian State University, F.Skaryny av. 4, 220050 Minsk, Belarus*
(*email*: rogosin@mmf.bsu.unibel.by)

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In this paper, a complex variable model for Hele-Shaw moving boundary problems with kinetic undercooling regularization is studied. This model consists of an abstract Cauchy–Kovalevsky problem and of a Riemann–Hilbert–Poincaré problem for holomorphic functions. The local existence of holomorphic solutions is shown. Some numerical results for this model complete the paper.

1 Introduction

In Vindogradou *et al.* [1] and Richardson [2], a mathematical model was derived for describing Hele-Shaw flows with a free boundary produced by injection/suction of fluid into/from a narrow channel supposing constant atmospheric pressure on the moving boundary. This model can be represented in the following form, where $f = f(z,t)$ is the Riemann mapping function from the unit disk G_1 onto the region occupied by fluid at the time t, and $Q(t)$ is the rate of injection/suction:

Let the functions $f_0 = f_0(z)$ and $Q = Q(t)$ be given. We suppose that f_0 is holomorphic and univalent in a neighbourhood of the unit disk $G_1, f_0(0) = 0$, and Q is continuous in a right-sided neighbourhood of $t = 0$. Find a function $f = f(z,t)$, holomorphic and univalent as a function of z in a neighbourhood of G_1, continuously differentiably with respect to t in a right-sided neighbourhood of $t = 0$, satisfying

$$\Re\left(\frac{1}{z}\frac{\partial f}{\partial t}(z,t)\overline{\frac{\partial f}{\partial z}(z,t)}\right) = Q(t), \quad z \in \partial G_1, \quad t \in [0,T); \tag{1.1}$$

$$f(z,0) = f_0(z), \quad z \in G_1; \tag{1.2}$$

$$f(0,t) = 0, \quad t \in [0,T). \tag{1.3}$$

Under the additional assumption $f'_z(0,t) > 0$ one gets with the results of Vindogradou *et al.* [1] the local existence and uniqueness of solutions in the case $Q(t) \equiv 1$, where the solution depends analytically upon t.

In Gustaffson [3], a more elementary proof of local existence and uniqueness of solutions for the above model is given in the case that f_0 is a polynomial or a rational function. The solution remains polynomial (rational) with respect to z if the initial function f_0 is. Later, in Reissig & von Wolfersdorf [4], a generalization of the above model was interpreted as an abstract Cauchy–Kovalevsky problem and solved by using a theorem of Nirenberg–Nishida [5].

The first exact solutions for Hele-Shaw flows driven by a single sink at the origin were constructed by Polubarinova-Kochina [6] and Galin [7]. They have finite-time blow-up and cusp formation at the free boundary before the moving boundary reaches the sink (e.g. see Howison & Hohlov [8]). Consequently, the above model is globally ill-posed in this respect. The physical meaning of this situation is that the velocity of points on the moving boundary tends to infinity at the cusp. There are different approaches to the regularization of the suction problem by incorporating extra terms at the free boundary conditions to penalize large curvatures or large normal derivatives (see Tanveer [9] and Hohlov *et al.* [10]). The most common corrections are to include a Gibbs–Thomson term proportional to the curvature or a kinetic undercooling term proportional to the normal velocity at the moving boundary.

This paper is concerned with the kinetic undercooling regularization. For Hele-Shaw flows this regularization first appeared in the doctoral thesis [11] (see § 4 for a typical configuration and also the review in Saffman [12]). A local linear stability analysis shows that this regularization successfully penalises the short wavelength growth which is usually associated with blow-up [13]. In Hohlov & Howison [14], the following problem (P_α) was derived for a complex model which describes Hele-Shaw flows with kinetic undercooling regularization.

Problem (P_α) *Let the two functions $f_0 = f_0(z)$ and $Q = Q(t)$ be given. We suppose that $f_0 = f_0(z)$ is holomorphic and univalent in a neighbourhood of the unit disk $G_1, f_0(0) = 0$, and $Q(t)$ is continuous in a right-sided neighbourhood of $t = 0$. Find two functions $f = f(z,t)$ and $w_{reg} = w_{reg}(z,t)$ such that f is holomorphic and univalent as a function of z in a neighbourhood of G_1, continuously differentiably with respect to t in a right-sided neighbourhood of $t = 0$, and w_{reg} is holomorphic in z in a neighbourhood of G_1, continuous with respect to t in a right-sided neighbourhood of $t = 0$, and both functions satisfy for all $z \in \partial G_1,\ \ t \in [0,T)$,*

$$\Re\left(\frac{1}{z}\frac{\partial f}{\partial t}(z,t)\overline{\frac{\partial f}{\partial z}(z,t)}\right) = Q(t) + \Re\left(zw_{reg}(z,t)\right), \tag{1.4}$$

$$f(z,0) = f_0(z), \tag{1.5}$$

$$f(0,t) = 0, \tag{1.6}$$

$$\Im\left(zw_{reg}(z,t)\right) = \alpha\partial_\theta\left(\left|\frac{\partial f}{\partial z}\right|^{-1}\left(Q(t) + \Re\left(zw_{reg}(z,t)\right)\right)\right), \tag{1.7}$$

where $\alpha > 0, z = re^{i\theta}$, and $Q(t) < 0$ in the case of suction.

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If $\alpha = 0$, then $\Im\left(zw_{reg}(z,t)\right) = 0, z \in \partial G_1$. Consequently, $\Re\left(zw_{reg}(z,t)\right) = const, z \in \partial G_1$, that is $w_{reg}(z,t) \equiv 0, z \in \overline{G}_1$. This means our starting model (1.1)–(1.3) coincides with (P_0).

Remark 1 Let us make some comments about the conditions (1.4) and (1.7). If one is interested in the kinetic undercooling regularization, then one has to take into account the kinematic boundary condition $V_n = U_n$ and the dynamic boundary condition $p \sim V_n$ at the moving boundary $\Gamma(t)$ in the physical plane. Here p denotes the pressure, V_n the normal component of velocity of the fluid and U_n the normal component of the velocity of $\Gamma(t)$. Using $f = f(z,t)$ and the complex potential $\chi = \chi(f(z,t),t)$ both conditions can be written in the mathematical plane as

$$\Re\left(\frac{1}{z}\frac{\partial f}{\partial t}\overline{\frac{\partial f}{\partial z}}\right) = \Re\left(z\frac{\partial \chi}{\partial z}\right),$$

$$\Re\chi = -\alpha\left|\frac{\partial f}{\partial z}\right|^{-1}\Re\left(\frac{1}{z}\frac{\partial f}{\partial t}\overline{\frac{\partial f}{\partial z}}\right), \quad \alpha > 0.$$

The ansatz for the complex potential $\chi = Q(t)\log z + \chi_{reg}$ and for the conjugate $\overline{w} := \partial_z\chi = Q(t)/z + w_{reg}$ of the complex velocity w gives immediately the kinematic boundary condition (1.4). After differentiation of the second condition with respect to θ some calculations lead to the dynamic boundary condition (1.7).

Remark 2 We mention that there exists another mathematical model for Hele-Shaw flows supposing constant atmospheric pressure [15, 16] and kinetic undercooling regularization [17], respectively. This model is based on a real-valued vector-function representation of the moving boundary. Local existence results were proved elsewhere [18, 17]. Recently, local existence results were obtained for Hele-Shaw models incorporating both kinds of regularizations together by using real methods [19, 20, 21].

We hope that this paper will be of interest for the following reasons:

- As far as the authors know the local existence and uniqueness of solutions for model (P_α) is still an open problem. The authors propose an approach basing on complex methods to solve this problem.
- The papers by Deckelnick & Elliot [19] and Fahuai [20] prove local existence results in spaces with finite smoothness. In Prokert [21], moreover, local existence results are proved in spaces of smooth functions. In this paper, we are working in the framework of holomorphic functions. Our approach allows to give a precise description of properties of the solution, describes a minimal life-span of the solution by including a conical weight and a scale of Banach spaces in the definition of the basic function space.
- If we have a local existence result for the complex model, then tools from function theory can be used to get more information about properties of the solution than in the real model case.
- The model (P_α) is very convenient (maybe in contrast to the model from Hohlov & Reissig [17]) for numerical treatment (see § 4).
- It may be possible that the approach of this paper can be applied to nonlinear kinetic undercooling regularizations of the form $p \sim (\partial p/\partial n)^\beta$, $\beta > 1$. Instead of (1.7) we have in this case to study a nonlinear Riemann–Hilbert–Poincaré problem.

2 On the structure of the problem (P_α)

Let us discuss the structure of (1.4)–(1.7), and derive an equivalent problem to (P_α) to which we will apply complex-analytic methods. Under the additional assumption

$$\Im\left(\frac{1}{z}\frac{\partial f}{\partial t}(z,t)\left(\frac{\partial f}{\partial z}\right)^{-1}(z,t)\right)(0,t)=0, \tag{2.1}$$

one can rewrite (1.4) by using Schwarz's integral formula as follows:

$$\frac{\partial f}{\partial t}(z,t)-z\frac{\partial f}{\partial z}(z,t)\frac{1}{2\pi i}\int_{|\zeta|=1}\left|\frac{\partial f}{\partial \zeta}\right|^{-2}\left(Q(t)+\Re(\zeta w_{reg})\right)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}=0.$$

Differentiating this equation with respect to z and setting $\omega(z,t):=zw_{reg}(z,t)$, $\phi(z,t):=\left(\frac{\partial f}{\partial z}\right)^{-1}(z,t)$ gives the following equivalent form of equation (1.4) (cf. Reissig & von Wolfersdorf [4] for the model (1.1)–(1.3), (2.1)):

$$\frac{\partial \phi}{\partial t}(z,t)-z\frac{\partial \phi}{\partial z}(z,t)\mathbf{T}_t(\phi,\omega)+\phi(z,t)\frac{\partial}{\partial z}\left(z\mathbf{T}_t(\phi,\omega)\right)=0,$$

$$\phi_0(z):=\phi(z,0)=\left(\frac{\partial f_0}{\partial z}\right)^{-1},$$

where

$$\mathbf{T}_t(\phi,\omega):=\frac{1}{2\pi i}\int_{|\zeta|=1}|\phi|^2\left(Q(t)+\Re\omega\right)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}. \tag{2.2}$$

Using the above notation, equation (1.7) can be written in form of a Riemann–Hilbert–Poincaré problem for $\omega=\omega(z,t)$:

$$\Im\left(\omega(z,t)\right)=\alpha\partial_\theta\Big(|\phi(z,t)|\big(Q(t)+\Re\left(\omega(z,t)\right)\big)\Big),\qquad \omega(0,t)=0.$$

In the following we study the next problem, which is equivalent to problem (P_α) (cf. [4] for the model (1.1)–(1.3), (2.1)):

Problem (Q_α) *Let the two functions $\phi_0=\phi_0(z)$ and $Q=Q(t)$ be given. We suppose that ϕ_0 is holomorphic and non-vanishing in a neighbourhood of the unit disk G_1, and Q is continuous in a right-sided neighbourhood of $t=0$. Find two functions $\phi=\phi(z,t)$ and $\omega=\omega(z,t)$: ϕ is holomorphic as a function of z and non-vanishing in a neighbourhood of G_1, continuously differentiably with respect to t in a right-sided neighbourhood of $t=0$, and ω is holomorphic in z in a neighbourhood of G_1, continuous with respect to t in a right-sided neighbourhood of $t=0$, both functions satisfy*

- *the abstract Cauchy–Kovalevsky problem (or Cauchy problem for the abstract evolution equation)*

$$\frac{\partial \phi}{\partial t}(z,t)-z\frac{\partial \phi}{\partial z}(z,t)\mathbf{T}_t(\phi,\omega)+\phi(z,t)\frac{\partial}{\partial z}\left(z\mathbf{T}_t(\phi,\omega)\right)=0, \tag{2.3}$$

$$\phi(z,0)=\left(\frac{\partial f_0}{\partial z}\right)^{-1}, \tag{2.4}$$

in a cylinder $\{(z,t)\in G_1\times[0,T)\}$;

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- *the Riemann–Hilbert–Poincaré problem with respect to* ω

$$\Im(\omega(z,t)) = \alpha\partial_\theta\Big(|\phi(z,t)|\big(Q(t) + \Re(\omega(z,t))\big)\Big) \quad on \quad \partial G_1 \times [0,T), \tag{2.5}$$

under the additional constraint

$$\omega(0,t) = 0, \tag{2.6}$$

where $\mathbf{T}_t(\phi,\omega)$ *is defined by* (2.2), $\alpha > 0$, *and* $z = re^{i\theta}$.

The goal of our paper is to prove a local existence result (in time) for (2.3)–(2.6), which gives immediately the next statement for our starting problem (P_α):

There exists an interval of time $[0,b)$, *in general small, such that the problem* (P_α) *has a uniquely determined solution* $\big(f(z,t), w_{reg}(z,t)\big)$. *The first component* f (*the second component* w_{reg}) *of the solution depends continuously differentiably* (*continuously*) *upon* t, *respectively.*

3 Mathematical treatment of problem (Q_α)

3.1 A Banach space of holomorphic functions

To study the problem (Q_α), we introduce a function space for $\phi = \phi(z,t)$ and $\omega = \omega(z,t)$. To define this space we need constants r_0 and r_1 with $1 < r_0 < r_1$, a positive constant b, and a parameter $s \in (0,1)$. By $\mathscr{H}(G_{(s)})$ we denote the space of functions which are holomorphic in $G_{(s)}$, where $G_{(s)} := \{z \in \mathbb{C} : |z| < r_0 + s(r_1 - r_0)\}$. Then we define the space

$$\mathbf{B} := \Bigg\{ g = g(z,t) \in \bigcup_{0<s<1} \mathscr{C}\big([0, b(1-s)), \mathscr{H}(G_{(s)}) \cap \mathscr{C}^{1,\lambda}(\overline{G}_{(s)})\big) :$$

$$\|g\|_{\mathbf{B}} = \max \Bigg\{ \sup_{s\in(0,1),h<b(1-s)} \max_{t\in[0,h]} \|g(\cdot,t)\|_{\mathscr{C}^\lambda(\overline{G}_{(s)})};$$

$$\sup_{s\in(0,1),t<b(1-s)} \left\| \frac{\partial g}{\partial z}(\cdot,t) \right\|_{\mathscr{C}^\lambda(\overline{G}_{(s)})} \left(1 - \frac{t}{b(1-s)}\right)^{\frac{1}{2}} \Bigg\} < \infty \Bigg\}.$$

Lemma 1 *The function space* **B** *is a Banach space.*

Lemma 2 *The function space* **B** *is an algebra with*

$$\|g \cdot h\|_{\mathbf{B}} \leqslant 2\|g\|_{\mathbf{B}}\|h\|_{\mathbf{B}} \quad \textit{for all} \quad g,h \in \mathbf{B}.$$

3.2 The abstract Cauchy–Kovalevsky problem

To study the evolution problem (2.3), (2.4) we want to understand how the operators ∂_z, $\mathbf{T}_t(\phi,\omega)$ act on **B**, $\mathbf{B}\times\mathbf{B}$, respectively. It is clear that ∂_z is an unbounded operator on **B**. If we use the operator $(\mathbf{J}\psi)(\cdot,t) := \int_0^t \psi(\cdot,\tau)d\tau$, then it turns out that $\mathbf{A} := \mathbf{J}\circ\partial_z$ is a continuous operator on **B** (see Tutschke [22]).

Lemma 3 *Let $\phi \in \mathbf{B}$. Then the operator*

$$\mathbf{A} : \phi \mapsto \int_0^t \partial_z \phi(\cdot, \tau) d\tau, \qquad t \in [0, b(1-s)), \quad s \in (0,1),$$

is a continuous operator mapping $\mathbf{B}$ *into itself, and satisfying the estimate*

$$\|\mathbf{A}\phi\|_{\mathbf{B}} \leqslant Cb\|\phi\|_{\mathbf{B}}, \tag{3.1}$$

where $C = C(r_0, r_1)$.[1]

Proof The main ideas of this proof are taken from Tutschke [22]. The idea is to consider the differential operator ∂_z in the scale of Banach spaces $\{\mathscr{H}(G_{(s)}) \cap \mathscr{C}^\lambda(\overline{G}_{(s)})\}_{0<s<1}$. The operator of integration with respect to time acts as a regularizing one; this means it compensates the unboundedness of ∂_z.[2] □

To study the nonlinear operator $\mathbf{T}_t(\phi, \omega)$ for given $\phi, \omega \in \mathbf{B}$ we introduce another function space (Banach space)

$$\mathbf{B}_a := \Big\{ g = g(z,t) \in \bigcup_{0<s<1} \mathscr{C}\left([0, b(1-s)), \mathscr{H}(A_{(s)}) \cap \mathscr{C}^{1,\lambda}(\overline{A}_{(s)})\right) :$$
$$\|g\|_{\mathbf{B}_a} = \max\Big\{ \sup_{s\in(0,1), h<b(1-s)} \max_{t\in[0,h]} \|g(\cdot,t)\|_{\mathscr{C}^\lambda(\overline{A}_{(s)})};$$
$$\sup_{s\in(0,1), t<b(1-s)} \left\| \frac{\partial g}{\partial z}(\cdot, t) \right\|_{\mathscr{C}^\lambda(\overline{A}_{(s)})} \left(1 - \frac{t}{b(1-s)}\right)^{\frac{1}{2}} \Big\} < \infty \Big\},$$

where

$$A_{(s)} = \left\{ z \in \mathbb{C} : \frac{1}{r_0 + (r_1 - r_0)s} < |z| < r_0 + (r_1 - r_0)s \right\}, \ 0 < s < 1.$$

The following lemma is evident.

Lemma 4 *If* ϕ *belongs to* $\mathbf{B}$, *then the function* $\tilde{\phi} = \tilde{\phi}(z,t) := \overline{\phi(\frac{1}{\bar{z}}, t)}$ *as well as the product* $\phi\tilde{\phi}$ *belongs to* $\mathbf{B}_a$. *Besides,*

$$\|\tilde{\phi}\|_{\mathbf{B}_a} \leqslant C\|\phi\|_{\mathbf{B}}.$$

Now we have all tools for the consideration of $\mathbf{T}_t(\phi, \omega)$ on $\mathbf{B}$.

Lemma 5 *The nonlinear operator* $\mathbf{T}_t(\phi, \omega)$ *is a continuous operator mapping* $\mathbf{B} \times \mathbf{B}$ *into* $\mathbf{B}$. *Moreover, there exists a constant* $C = C(\lambda, Q, r_0, r_1)$ *such that*

$$\|\mathbf{T}_t(\phi, \omega)\|_{\mathbf{B}} \leqslant C\|\phi\|^2_{\mathbf{B}}(1 + \|\omega\|_{\mathbf{B}}).$$

[1] Up to the end of the paper we use C as a universal constant.
[2] The goal of this paper is to prove local existence in time. So, we have no need to minimize the constant Cb in (3.1).

Proof Let the given functions ϕ and ω belong to $\mathbf{B}$. Then

$$\begin{aligned}\mathbf{T}_t(\phi,\omega)(z,t) &:= \frac{1}{2\pi i}\int_{|\zeta|=1} |\phi|^2 \left(Q(t)+\Re\omega\right)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} \\ &= \frac{1}{2\pi i}\int_{|\zeta|=1} \phi(\zeta,t)\overline{\phi\left(\frac{1}{\overline{\zeta}},t\right)}\left(Q(t)+\frac{\omega(\zeta,t)+\overline{\omega(\frac{1}{\overline{\zeta}},t)}}{2}\right)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}.\end{aligned}$$

Due to Lemma 4 the functions $\tilde{\phi}=\tilde{\phi}(z,t):=\overline{\phi(\frac{1}{\overline{z}},t)}$ and $\tilde{\omega}=\tilde{\omega}(z,t):=\overline{\omega(\frac{1}{\overline{z}},t)}$ belong to $\mathbf{B}_a$ (but not to $\mathbf{B}$). This is the motivation for us to consider the Banach space $\mathbf{B}_a$. For our purpose it is necessary to use the domains $A_{(s)}, s\in(0,1)$, because the product $\phi\tilde{\phi}$ is defined only on these sets. Due to the definition of the functions $\tilde{\phi},\tilde{\omega}$ and their properties in $A_{(s)}$, $0<s<1$, we have

$$\begin{aligned}\mathbf{T}_t(\phi,\omega)(z,t) &= \frac{1}{2\pi i}\int_{|\zeta|=1}\phi(\zeta,t)\tilde{\phi}(\zeta,t)\left(Q(t)+\frac{\omega(\zeta,t)+\tilde{\omega}(\zeta,t)}{2}\right)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} \\ &= \frac{1}{2\pi i}\int_{\partial G_{(s)}}\phi(\zeta,t)\tilde{\phi}(\zeta,t)\left(Q(t)+\frac{\omega(\zeta,t)+\tilde{\omega}(\zeta,t)}{2}\right)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}\end{aligned}$$

for all $s\in(0,1), t\in[0,b(1-s))$ and all $z\in G_{(s)}$. The last formula gives us immediately $\mathbf{T}_t(\phi,\omega)\in\mathscr{C}\left([0,b(1-s)),\mathscr{H}(G_{(s)})\cap\mathscr{C}^{1,\lambda}(\overline{G}_{(s)})\right)$ for all $s\in(0,1)$. Using Hölder estimates for the Schwarz's integral we have with a constant $C=C(\lambda,Q,r_0,r_1)$ the estimate

$$\|\mathbf{T}_t(\phi,\omega)(\cdot,t)\|_{\mathscr{C}^{\lambda}(\overline{G}_{(s)})}\leqslant C\|\phi\|_{\mathbf{B}}\|\tilde{\phi}\|_{\mathbf{B}_a}(1+\tfrac{1}{2}(\|\omega\|_{\mathbf{B}}+\|\tilde{\omega}\|_{\mathbf{B}_a})).$$

It follows from this inequality together with Lemma 4

$$\begin{aligned}&\|\mathbf{T}_t(\phi,\omega)(\cdot,t)\|_{\mathscr{C}^{\lambda}(\overline{G}_{(s)})}\leqslant C\|\phi\|^2_{\mathbf{B}}(1+\|\omega\|_{\mathbf{B}}),\\ &\sup_{s\in(0,1),h<b(1-s)}\ \max_{t\in[0,h]}\|\mathbf{T}_t(\phi,\omega)(\cdot,t)\|_{\mathscr{C}^{\lambda}(\overline{G}_{(s)})}\leqslant C\|\phi\|^2_{\mathbf{B}}(1+\|\omega\|_{\mathbf{B}}).\end{aligned}\tag{3.2}$$

The estimation of $\partial_z\mathbf{T}_t(\phi,\omega)$ can be reduced to estimate corresponding Cauchy-type integrals and Schwarz-type integrals in Hölder spaces, where the functions $\phi,\tilde{\phi},\omega,\tilde{\omega}$ appear in the densities of these integrals. Taking account that these densities contain at most one first derivative of these functions we get

$$\sup_{s\in(0,1),t\in[0,b(1-s))}\|\partial_z\mathbf{T}_t(\phi,\omega)(\cdot,t)\|_{\mathscr{C}^{\lambda}(\overline{G}_{(s)})}\left(1-\frac{t}{b(1-s)}\right)^{\frac{1}{2}}\leqslant C\|\phi\|^2_{\mathbf{B}}(1+\|\omega\|_{\mathbf{B}}).\tag{3.3}$$

The inequalities (3.2) and (3.3) yield

$$\|\mathbf{T}_t(\phi,\omega)\|_{\mathbf{B}}\leqslant C\|\phi\|^2_{\mathbf{B}}(1+\|\omega\|_{\mathbf{B}}),$$

i.e. $\mathbf{T}_t(\phi,\omega)$ belongs to $\mathbf{B}$. Following the same approach implies, moreover,

$$\|\mathbf{T}_t(\phi_1,\omega)-\mathbf{T}_t(\phi_2,\omega)\|_{\mathbf{B}}\leqslant C\max\{\|\phi_1\|_{\mathbf{B}},\|\phi_2\|_{\mathbf{B}}\}(1+\|\omega\|_{\mathbf{B}})\|\phi_1-\phi_2\|_{\mathbf{B}},\tag{3.4}$$

$$\|\mathbf{T}_t(\phi,\omega_1)-\mathbf{T}_t(\phi,\omega_2)\|_{\mathbf{B}}\leqslant C\|\phi\|^2_{\mathbf{B}}\|\omega_1-\omega_2\|_{\mathbf{B}}.\tag{3.5}$$

Thus, $\mathbf{T}_t$ depends continuously upon ϕ and ω. All the statements of our lemma are proved. □

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Corollary 1 *For each* $\omega \in \mathbf{B}$ *there exists a constant* $b = b(\omega)$ *such that the abstract Cauchy–Kovalevsky problem* (2.3)–(2.4) *has a uniquely determined solution* $\phi \in \mathbf{B}$.

Proof The problem (2.3)–(2.4) can be rewritten in the following operator form:

$$\mathbf{I}(\phi) = \mathbf{I}(\phi_0) + \mathbf{A} \circ \mathbf{M}(\phi, \mathbf{M}(z, \mathbf{T}_t(\phi, \omega))) - 2\mathbf{J} \circ \mathbf{M}(\phi, \partial_z \circ \mathbf{M}(z, \mathbf{T}_t(\phi, \omega))), \tag{3.6}$$

where $\mathbf{M}(\phi, \omega) := \phi \cdot \omega$ is the operator of multiplication, $\mathbf{J}\psi := \int_0^t \psi(\cdot, \tau) d\tau$ is the operator of integration, $\mathbf{I}$ is the identity operator, and $\circ$ denotes the superposition operator. Lemmas 2, 3 and 5 imply immediately

$$\|\mathbf{A} \circ \mathbf{M}(\phi, \mathbf{M}(z, \mathbf{T}_t(\phi, \omega)))\|_{\mathbf{B}} \leqslant Cb\|\phi\|_{\mathbf{B}}^3(1 + \|\omega\|_{\mathbf{B}}), \tag{3.7}$$

$$\|2\mathbf{J} \circ \mathbf{M}(\phi, \partial_z \circ \mathbf{M}(z, \mathbf{T}_t(\phi, \omega)))\|_{\mathbf{B}} \leqslant Cb\|\phi\|_{\mathbf{B}}^3(1 + \|\omega\|_{\mathbf{B}}), \tag{3.8}$$

respectively, where in the last two formulas the constants C are independent of ϕ and ω.

The assumptions on ϕ and ω from problem (Q_α) guarantee the existence of r_0, r_1 such that $\phi_0 \in \mathbf{B}$. Let us fix a constant $R > 0$. Then a suitable small b makes sure together with (3.6), (3.7), and (3.8) that the right-hand side of (3.6) maps $\{\phi \in \mathbf{B} : \|\phi - \phi_0\|_{\mathbf{B}} \leqslant R\}$ into itself. By (3.1) this mapping is a contractive one if we eventually choose a smaller b. Banach's fixed point theorem leads to a uniquely determined solution $\phi \in \mathbf{B}$ of (2.3)–(2.4). This completes the proof. □

3.3 Riemann–Hilbert–Poincaré problem

3.3.1 *Analytic continuation of the solution*

Here we discuss the holomorphy of the second component $w_{reg} = w_{reg}(z, t)$ of solutions of problem (P_α) (or equivalently $\omega = \omega(z, t)$ of problem (Q_α)). As we have already shown, the solution $\phi = \phi(z, t)$ of (2.3)–(2.4) admits an analytic continuation into a larger domain if we suppose a corresponding property for ω. Without loss of generality we can suppose that this domain is a disk $G_d = \{z : |z| < d\}, d > 1$, and it is a common domain of holomorphy for all functions $\phi = \phi(\cdot, t)$, $t \in [0, T_d)$. In problem (2.5)–(2.6) the variable t is only a parameter. For this reason let us omit it till the end of this section (and use for sake of brevity the notation $\phi(z), \omega(z), Q$ etc.).

Let ϕ be a given holomorphic and non-vanishing function in G_d. Our goal is to prove that ω connected with ϕ by

$$\Im(\omega(z)) = \alpha\partial_\theta(|\phi(z)|(Q + \Re(\omega(z)))) \;\; on \;\; \partial G_1, \tag{3.9}$$

$$\omega(0) = 0, \tag{3.10}$$

is also holomorphic in G_d. Therefore, we need the following auxiliary result:

Lemma 6 *Let the non-vanishing function* ϕ *belong to* $\mathscr{H}(G_1) \cap \mathscr{C}^{1,\lambda}(\overline{G}_1)$. *Then the boundary value problem* (3.9)–(3.10) *possesses a uniquely determined solution* $\omega(z) \in \mathscr{H}(G_1) \cap \mathscr{C}^{1,\lambda}(\overline{G}_1)$.

Proof Using $\Im(\omega(0)) = 0$ we can rewrite the boundary condition (3.9) in the equivalent form

$$-\mathbf{H}(\Re\omega)(z) = \alpha Q \partial_\theta |\phi(z)| + \alpha\partial_\theta(|\phi(z)|\Re(\omega(z))), \; z = re^{i\theta} \in \partial G_1, \tag{3.11}$$

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where $\mathbf{H}$ is the Hilbert transform for the unit circle,

$$\mathbf{H}g(e^{i\eta}) := \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\sigma})\cot\frac{\eta-\sigma}{2}d\sigma, \qquad \eta\in[0,2\pi).$$

Using $\Re(\omega(0)) = 0$ and applying $\mathbf{H}$ once more we get

$$\Re(\omega(z)) = \alpha Q\partial_\theta \mathbf{H}(|\phi(z)|) + \alpha\partial_\theta(\mathbf{H}(|\phi(z)|\Re(\omega(z)))). \tag{3.12}$$

Let us introduce a new function $U = U(z,\bar{z})$, which is harmonic in the unit disk, and which satisfies the boundary relation

$$U(z,\bar{z}) = |\phi(z)|\Re(\omega(z)), \qquad z\in\partial G_1. \tag{3.13}$$

Using the Cauchy–Riemann equations on the unit circle we get finally, that (3.9)–(3.10) can be rewritten as the third-kind boundary problem for U,

$$\partial_r U + \frac{1}{\alpha|\phi(z)|}U = Q\mathbf{H}\left(\partial_\theta|\phi|\right)(z), \qquad z\in\partial G_1.$$

The coefficient $(\alpha|\phi(z)|)^{-1}$, and the right-hand side of this boundary condition belong to $\mathscr{C}^\lambda(\partial G_1)$. It is known that this problem is uniquely solvable (e.g. see Ladyzhenskaya & Uraltseva [23]) in the space $\mathscr{C}^{1,\lambda}\left(\overline{G}_1\right)$. By Poisson's formula and (3.13) we determine the unique harmonic function $\Re(\omega)$. The formula (3.12) implies $\int_0^{2\pi}\Re(\omega(e^{i\theta}))d\theta = 0$, i.e. $\Re(\omega(0)) = 0$. The corresponding unique solution ω of problem (3.9)–(3.10) is obtained by Schwarz's formula under the additional constraint $\Im(\omega(0)) = 0$. Using the assumption for ϕ and the regularity of U gives immediately the statement of the lemma. □

Let the function ϕ be holomorphic and non-vanishing in the disk G_d, $d > 1$.[3] One can introduce the following holomorphic function:

$$h^+(z) := (\phi(z))^{\frac{1}{2}}, \qquad z\in G_d,$$

choosing any branch of the root. We can define another holomorphic function by

$$h^-(z) := \overline{h^+\left(\frac{1}{\bar{z}}\right)} \quad \text{for} \quad \left\{z : |z| > \frac{1}{d}\right\}.$$

Both of these functions $h^+(z)$, and $h^-(z)$ as well as their product $m(z) := h^+(z)h^-(z)$ are holomorphic in the annulus $A = \left\{z : \frac{1}{d} < |z| < d\right\}$. Besides we have on the unit circle

$$|\phi(z)| = h^+(z)h^-(z), \qquad z\in\partial G_1.$$

In the same manner we can introduce

$$\omega^-(z) := \overline{\omega^+\left(\frac{1}{\bar{z}}\right)} \quad \text{for} \quad \{z : |z| > 1\},$$

where $\omega^+(z) := \omega(z)$ in the unit disk G_1. With this notation the boundary condition (3.9) can be rewritten on ∂G_1 in the equivalent form

$$\omega^+(z) - \omega^-(z) = 2i\alpha Q\partial_\theta m(z) + i\alpha\partial_\theta(m(z)(\omega^+(z)+\omega^-(z))),$$

[3] We will use further the same notations for functions given in the unit disk as for their analytic continuations into larger domains.

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or

$$\alpha z m(z) d_z \omega^+(z) + (1 + \alpha z d_z m(z))\omega^+(z) + \alpha Q z d_z m(z) = -\alpha z m(z) d_z \omega^-(z) + (1 - \alpha z d_z m(z))\omega^-(z) - \alpha Q z d_z m(z). \quad (3.14)$$

Let us note that all the terms on the left-hand side of (3.14) are in fact holomorphic on the internal annulus $A^i := \left\{z : \frac{1}{d} < |z| < 1\right\}$ and Hölder-continuous up to ∂G_1, but those at the right-hand side of (3.14) are holomorphic on the external annulus $A^e := \{z : 1 < |z| < d\}$ and Hölder-continuous up to ∂G_1. Therefore, due to the theorem on analytic continuation the left- (right-) hand side of (3.14) is the restriction of a holomorphic function (say $F = F(z)$) defined on the annulus A to A^i (A^e). This means that (3.14) is equivalent to the system of differential equations

$$\left.\begin{aligned} \alpha z m(z) d_z \omega^+ + (1 + \alpha z d_z m(z))\omega^+ + \alpha Q z d_z m(z) = F(z),\ z \in A^i, \\ -\alpha z m(z) d_z \omega^- + (1 - \alpha z d_z m(z))\omega^- - \alpha Q z d_z m(z) = F(z),\ z \in A^e. \end{aligned}\right\} \quad (3.15)$$

By Lemma 6 the function $\omega^+(z)$ is uniquely determined in $\overline{G}_1$. We have due to Schwarz's Reflection Principle that

$$F(z) = \overline{F\left(\frac{1}{\bar{z}}\right)}, \quad \text{for} \quad z \in A^e; \quad (3.16)$$

taking into consideration that the values of $F(z)$ on ∂G_1 are real, this means that (3.16) holds on ∂G_1. Consequently, we can use the differential equation

$$\alpha z m(z) d_z \omega^+ + (1 + \alpha z d_z m(z))\omega^+ + \alpha Q z d_z m(z) = F(z), \quad z \in A, \quad (3.17)$$

for the continuation of ω^+ from $A^i \cup \partial G_1$ into A^e. Now we are in position to formulate the next statement. We omit the proof.

Lemma 7 *Let the function $\omega^i = \omega^i(z)$ be a single-valued holomorphic function in an open annulus A^i whose boundary consists of two disjoint circles Γ_i, Γ_0 ($\Gamma_i \subset int\ \Gamma_0$). Let ω^i satisfy on A^i the first-order differential equation*

$$d_z \omega + a(z)\omega = b(z), \quad (3.18)$$

where $a, b \in \mathscr{H}(A^i)$.

Let us additionally suppose that a, b admit analytic continuations into an annulus A^e whose boundary consists of two disjoint circles Γ_0, Γ_e ($\Gamma_0 \subset int\ \Gamma_e$). Then ω^i admits an analytic continuation ω into A^e, too.

Corollary 2 *If in* (2.5) *the function $\phi = \phi(z,t)$ admits an analytic continuation to a larger disk G_d, $d > 1$, for certain $t \in [0, T)$, then the corresponding solution $\omega = \omega(z,t)$ of* (2.5)–(2.6) *admits an analytic continuation into G_d for the same t, too.*

3.3.2 *An existence result*

We suppose that $\phi_0 = \phi_0(z)$ is non-vanishing in a neighbourhood of the unit disk $\overline{G}_1$. Thus there exists $r_1 > 1$ such that

$$\rho_0 := \inf_{z\in G_{r_1}} |\phi_0(z)| = \inf_{z\in G_{r_1}} \left| \left(\frac{\partial f_0}{\partial z}(z) \right)^{-1} \right| > 0. \tag{3.19}$$

Using Corollary 2 the solution $\omega_0 = \omega_0(z)$ of the problem (3.9)–(3.10) with $\phi := \phi_0$ is holomorphic in $G_{(1)} = G_{r_1}$, too. The next lemma describes properties of continuations of solutions for (3.9)–(3.10). Let us introduce the function space

$$\mathbf{B}_1 = \{\phi \in \mathscr{H}\left(G_{(1)}\right) : \|\phi\|_1 := \sup_{s\in(0,1)} \|\phi\|_{\mathscr{C}^{1,\lambda}(\overline{G}_{(s)})} < \infty\}.$$

Moreover, let $B_1(\phi_0, \rho) \subset \mathbf{B}_1$ be the ball around ϕ_0 with radius ρ.

Lemma 8 *If* $\phi \in B_1(\phi_0, \rho), \rho < \rho_0$, *then the continuations* $\omega = \omega(z)$ *of solutions for* (3.9)–(3.10) *belong to* $\mathbf{B}_1$ *and satisfy*

$$\|\omega\|_1 \leqslant C, \quad \textit{where} \quad C = C(\lambda, Q, r_0, r_1, \rho, \rho_0).$$

Proof By the aid of Lemma 6 and following its proof, we obtain, with the same notation,

$$\|U(z)\|_{\mathscr{C}^{1,\lambda}(\overline{G}_1)} \leqslant C\|\phi(z)\|_{\mathscr{C}^{1,\lambda}(\overline{G}_1)}.$$

Using (3.19) and the assumption $\rho < \rho_0$ we have the estimate

$$\||\phi(z)|^{-1}\|_{\mathscr{C}^{1,\lambda}(\overline{G}_1)} \leqslant C.$$[4]

Taking into account (3.7), (3.8) and the properties of the Hilbert transform, we get

$$\|\omega\|_{\mathscr{C}^{1,\lambda}(\overline{G}_1)} \leqslant C. \tag{3.20}$$

From Corollary 2 we get the existence of ω in $G_{(1)}$. The continuation of ω from $\overline{G}_1$ to $G_{(1)}$ is defined in the annulus $A = \{z \in \mathbb{C} : \frac{1}{r_1} < |z| < r_1\}$ as the solution of the differential equation

$$d_z\omega^+ + \frac{1 + \alpha z d_z m(z)}{\alpha z m(z)}\omega^+ = \frac{F(z)}{\alpha z m(z)} - \frac{Q d_z m(z)}{m(z)}. \tag{3.21}$$

Remember that $m(z) = (\phi(z))^{\frac{1}{2}}(\overline{\phi(\frac{1}{\overline{z}})})^{\frac{1}{2}}, z \in A$, and $F(z)$ is defined in $A^i = \{z \in \mathbb{C} : \frac{1}{r_1} < |z| < 1\}$ by (3.15) and in $A^e = \{z \in \mathbb{C} : 1 < |z| < r_1\}$ by (3.16).

Taking points $z_1, \cdots, z_n$, n sufficiently large, we choose an equidistant partition of ∂G_1 and overlapping sectors $S_1, \cdots, S_n$ defined as follows:

$$S_k := \left\{ z \in \mathbb{C} : \frac{1}{r_1} < |z| < r_1, -\frac{\pi}{n} - \delta < \arg z - \delta_k < \frac{\pi}{n} + \delta \right\},$$

where $\delta_k = \arg z_k$, δ is a sufficiently small positive number. It is enough to get an estimate for $\|\omega\|_{\mathscr{C}^{1,\lambda}(\overline{G}_{(s)}\cap S_k)}$ uniformly for all $s \in (0, 1)$, and $\phi \in B_1(\phi_0, \rho)$. All these estimates together yield an estimate for $\|\omega\|_{\mathscr{C}^{1,\lambda}(\overline{G}_{(s)})}$.

[4] In this section the constant C is independent of $\phi \in B_1(\phi_0, \rho)$ and of $s \in (0, 1)$.

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The solution of (3.21) can be represented in S_k in the form

$$\omega(z) := J_1(z) + J_2(z) := \omega(z_k)\exp\left\{-\int_{z_k}^{z}\frac{1+\alpha\zeta d_\zeta m(\zeta)}{\alpha\zeta m(\zeta)}d\zeta\right\}$$
$$+\int_{z_k}^{z}\left(\frac{F(\zeta)}{\alpha\zeta m(\zeta)}-\frac{Qd_\zeta m(\zeta)}{m(\zeta)}\right)\exp\left\{\int_{z}^{\zeta}\frac{1+\alpha\xi d_\xi m(\xi)}{\alpha\xi m(\xi)}d\xi\right\}d\zeta.$$

For $J_1(z)$ we have

$$J_1(z) = \omega(z_k)\frac{m(z_k)}{m(z)}\exp\left\{-\int_{z_k}^{z}\frac{d\zeta}{\alpha\zeta m(\zeta)}\right\}.$$

The function m^{-1} belongs to $\mathscr{C}^{1,\lambda}(G_{(1)}\cap S_k)$, where $\|m^{-1}\|_{\mathscr{C}^{1,\lambda}(\overline{G}_{(s)}\cap S_k)} \leqslant C$. Here we use $|m(z)| \geqslant \rho_0 - \rho > 0$ for all $\phi \in B_1(\phi_0, \rho)$. Consequently,

$$\|J_1(z)\|_{\mathscr{C}^{1,\lambda}(\overline{G}_{(s)}\cap S_k)} \leqslant C. \tag{3.22}$$

The discussion of $J_2(z)$ brings no new difficulties besides the consideration of the integral $\int_{z_k}^{z}\frac{F(\zeta)}{\alpha\zeta m(\zeta)}d\zeta$. From (3.15) and (3.20) we get

$$\|F\|_{\mathscr{C}^{\lambda}(G_1\cap S_k)} \leqslant C \quad \text{for all} \quad \phi \in B_1(\phi_0, \rho).$$

Hence, by (3.16) we have

$$\|F\|_{\mathscr{C}^{\lambda}(\overline{G}_{(s)}\cap S_k)} \leqslant C.$$

But this immediately gives that $J_2(z) \in \mathscr{C}^{1,\lambda}(G_{(1)}\cap S_k)$ with

$$\|J_2(z)\|_{\mathscr{C}^{1,\lambda}(\overline{G}_{(s)}\cap S_k)} \leqslant C. \tag{3.23}$$

The inequalities (3.22), and (3.23) imply $\|\omega\|_{\mathscr{C}^{1,\lambda}(G_{(1)}\cap S_k)} \leqslant C$, that is $\omega \in \mathbf{B}_1$ with $\|\omega\|_1 \leqslant C$ for all $\phi \in B_1(\phi_0, \rho)$. This completes the proof. □

Corollary 3 *If $\phi \in B_1(\phi_0, \rho)$, then $\omega \in B_1(\omega_0, \eta)$ with a constant η depending on ρ. If ρ tends to 0, then η tends to 0, too.*

Proof Let ω, ω_0 be the solutions of (3.9)–(3.10) for ϕ, ϕ_0, respectively. Then $v := \omega - \omega_0$ solves the following Riemann–Hilbert–Poincaré Problem on ∂G_1 :

$$\Im(v(z)) = \alpha\partial_\theta(|\phi(z)|\Re(v(z))) + \alpha\partial_\theta((|\phi(z)| - |\phi_0(z)|)(Q + \Re(\omega_0(z)))),$$
$$v(0) = 0.$$

The statement follows from the fact that $v \equiv 0$ is a solution of

$$\Im(v(z)) = \alpha\partial_\theta(|\phi(z)|\Re(v(z))),\ v(0) = 0 \text{ on } \partial G_1,$$

and $\alpha\partial_\theta\big((|\phi(z)| - |\phi_0(z)|)\big(Q + \Re(\omega_0(z))\big)\big)$ can be considered as a small perturbation. To complete the proof we follow the same approach as in the proof of Lemma 8. □

Remark 3 The statements of Lemma 8 and Corollary 3 remain true if we replace $\mathbf{B}_1$ by $\mathbf{B}_{s_0}, s_0 \in (0,1)$, where

$$\mathbf{B}_{s_0} := \{\phi \in \mathscr{H}(G_{(s_0)}) : \|\phi\|_{s_0} := \sup_{s\in(0,s_0)} \|\phi\|_{\mathscr{C}^{1,\lambda}(\overline{G}_{(s)})} < \infty\}.$$

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3.4 Main result

The results of the previous sections serve as preparations to prove our main result.

Theorem 1 (Main Theorem) *There exists an in general small interval of time* $[0, b)$ *such that the problem* (Q_α) *has a uniquely determined solution* (ϕ, ω). *The first component* $\phi = \phi(z, t)$ *has no zeros on* $\overline{G}_{r_0} \times [0, b)$ *and belongs to the space* $\mathscr{C}^1\left([0, b), \mathscr{H}(G_{r_0}) \cap \mathscr{C}^{1,\lambda}(\overline{G}_{r_0})\right)$. *The second component* $\omega = \omega(z, t)$ *belongs to the space* $\mathscr{C}\left([0, b), \mathscr{H}(G_{r_0}) \cap \mathscr{C}^{1,\lambda}(\overline{G}_{r_0})\right)$. *The constant* r_0 *is taken as in the definition of* **B**.

Corollary 4 *There exists an in general small interval of time* $[0, b)$ *such that the problem* (P_α) *has a uniquely determined solution* (f, w_{reg}). *The first component* $f = f(z, t)$ *is univalent with respect to* z *on* G_{r_0} *for* $t \in [0, b)$ *and belongs to* $\mathscr{C}^1\left([0, b), \mathscr{H}(G_{r_0}) \cap \mathscr{C}^{2,\lambda}(\overline{G}_{r_0})\right)$. *The second component* $w_{reg} = w_{reg}(z, t)$ *belongs to* $\mathscr{C}\left([0, b), \mathscr{H}(G_{r_0}) \cap \mathscr{C}^{1,\lambda}(\overline{G}_{r_0})\right)$. *The constant* r_0 *is taken as in the definition of* **B**.

This corollary is a direct consequence of the theorem. Univalence of the function f follows from the properties of initial function f_0, the choice of $\rho < \rho_0$ (see (3.19)) and the chosen norm of **B** which is stronger than the sup-norm.

Proof of Theorem 1 Let us consider the problem (Q_α), that is (2.3)–(2.6). Let ω_0 be the solution of (2.5)–(2.6) for $\phi = \phi_0(z)$. One can find constants r_0, r_1 such that ϕ_0 and ω_0 belong to **B**. We will prove the existence of solutions $\phi = \phi(z, t)$ and $\omega = \omega(z, t)$ belonging to $M_R(\phi_0) := \{\phi : \|\phi - \phi_0\|_{\mathbf{B}} \leqslant R\}$ and $M_K(\omega_0) := \{\omega : \|\omega - \omega_0\|_{\mathbf{B}} \leqslant K\}$, respectively, where the constants R and K will be chosen later.

Step 1. Abstract Cauchy–Kovalevsky problem

Using Corollary 1, to each $\omega \in M_K(\omega_0)$ corresponds a constant $b = b(\omega)$ such that (2.3)–(2.4) has a uniquely determined solution $\phi \in \mathbf{B}$. We can choose the constant b in such a way that $\phi \in M_R(\phi_0)$ uniformly for all $\omega \in M_K(\omega_0)$. Let us define the operator

$$\mathbf{P}_1 : \omega \in M_K(\omega_0) \mapsto \phi = \phi(\omega) \in M_R(\phi_0),$$

which maps $\omega \in M_K(\omega_0)$ to the uniquely determined solution $\phi = \phi(\omega)$ of (2.3)–(2.4). Taking into consideration (3.4), (3.5), the operator $\mathbf{P}_1$ depends continuously on ω. The inequalities (3.4), (3.5) and (3.7), (3.8) yield the existence of a constant $C_1 = C_1(R, K)$ independent of b such that

$$\|\mathbf{P}_1(\omega_1) - \mathbf{P}_1(\omega_2)\|_{\mathbf{B}} \leqslant C_1 b \|\omega_1 - \omega_2\|_{\mathbf{B}}. \tag{3.24}$$

Consequently, this inequality remains valid if we choose a smaller b.

Concerning the choice of b, let us choose $R > 0$. From (3.7), (3.8) we obtain

$$\|\phi - \phi_0\|_{\mathbf{B}} \leqslant Cb(\|\phi_0\|_{\mathbf{B}} + R)^3(1 + K).$$

Hence,

$$b \leqslant \frac{R}{C\left(\|\phi_0\|_{\mathbf{B}} + R\right)^3(1 + K)} \tag{3.25}$$

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guarantees, that $\phi = \phi(\omega) \in M_R(\phi_0)$ for all $\omega \in M_K(\omega_0)$. Using (3.19) a sufficiently small choice of R gives additionally that $\phi \in M_R(\phi_0)$ has no zeros in $\bigcup_{0<s<1} G_{(s)} \times [0, b(1-s))$.

Step 2. Riemann–Hilbert–Poincaré problem
Due to Corollary 2, we can define a mapping

$$\mathbf{P}_2 : \phi \in M_R(\phi_0) \mapsto \tilde{\omega},$$

where $\tilde{\omega}$ is the uniquely determined solution of (2.5)–(2.6). Due to the results from § 3.3 this mapping takes values in $\mathbf{B}$. Repeating the proof of Lemma 8, taking into account Corollary 3, leads to the next statement:

Lemma 9 *If*

$$\max \left\{ \sup_{s\in(0,1),h<b(1-s)} \max_{t\in[0,h]} \|\phi(\cdot,t)\|_{\mathscr{C}^\lambda(\overline{G}_{(s)})}; \quad \sup_{s\in(0,1),h<b(1-s)} \|\frac{\partial \phi}{\partial z}(\cdot,t)\|_{\mathscr{C}^\lambda(\overline{G}_{(s)})} C_{t,s} \right\} < \infty,$$

then the same is true for $\tilde{\omega} = \tilde{\omega}(z,t)$.

Proof It is clear because the derivatives $\partial_z J_1(z)$, $\partial_z J_2(z)$ and the left-hand side of the differential equation (3.17) depend linearly on the derivatives $\partial_z \phi$ and $\partial_z \tilde{\phi}$. □

Using $C_{t,s} = \left(1 - \frac{t}{b(1-s)}\right)^{\frac{1}{2}}$ gives that $\mathbf{P}_2$ maps $\phi \in M_R(\phi_0)$ to $\tilde{\omega} = \mathbf{P}_2(\phi) \in \mathbf{B}$. By Corollary 3 the function $\tilde{\omega}$ belongs to $M_K(\omega_0)$ for all $\phi \in M_R(\phi_0)$. The proof of this corollary implies the existence of a constant $C_2 = C_2(R,K)$ independent of b such that

$$\|\mathbf{P}_2(\phi_1) - \mathbf{P}_2(\phi_2)\|_{\mathbf{B}} \leqslant C_2 b \|\phi_1 - \phi_2\|_{\mathbf{B}}. \tag{3.26}$$

The inequalities (3.24), (3.26) give for the mapping

$$\mathbf{P}_2 \circ \mathbf{P}_1 : \omega \in M_K(\omega_0) \mapsto \tilde{\omega} \in M_{\tilde{K}}(\omega_0)$$

the estimate

$$\|\tilde{\omega}_1 - \tilde{\omega}_2\|_{\mathbf{B}} = \|\mathbf{P}_2 \circ \mathbf{P}_1(\phi_1) - \mathbf{P}_2 \circ \mathbf{P}_1(\phi_2)\|_{\mathbf{B}} \leqslant C_2 C_1 b \|\omega_1 - \omega_2\|_{\mathbf{B}}.$$

Discussion of the choice of K, R *and* b
If $\tilde{K} > K$, then we choose $K := \tilde{K}$, otherwise it is unchanged. The constant R is determined by Corollary 3 in such a way, that $\phi \in M_R(\phi_0)$ is mapped by $\mathbf{P}_2$ into $M_K(\omega_0)$. Then (3.25) and $b < (C_1 C_2)^{-1}$ (the constants C_1, C_2 are independent of b) ensure that $\mathbf{P}_2 \circ \mathbf{P}_1$ is a contractive mapping on $M_K(\omega_0)$. Consequently, there exists a uniquely determined fixed point $\omega_{fix} \in M_K(\omega_0)$. This fixed point and $\phi := \mathbf{P}_1(\omega_{fix})$ form the unique solution $(\phi, \omega) := (\mathbf{P}_1(\omega_{fix}), \omega_{fix})$ of (Q_α). All statements of the theorem are proved. □

4 Conclusion

In this paper the local existence (in time) is proved for a complex variable Hele-Shaw model with a linear kinetic undercooling condition. Different authors propose to include into the model a nonlinear kinetic undercooling condition of the form $p \sim (\partial p/\partial n)^\beta$, $\beta > 1$. In forthcoming paper we will study such a model. The question if the suction problem is globally well-posed remained open but the numerical calculations from the appendix strongly suggest that linear kinetic undercooling regularisation leads to global well-posedness.

Appendix A Numerical results

This section is an extract from the student research report of F. Hübner [24] to summarize and to compare numerical results for the Hele-Shaw flow with and without kinetic undercooling regularization. The considerations of the previous chapters led to a local existence result in time for the model (P_α). This result motivates numerical experiments shown below. All the examples force the supposition that kinetic undercooling regularization probably avoids singularities and the blow-up time of the classical model could be exceeded. The classical Hele-Shaw problem (that means without any regularization)

$$\begin{aligned} \Re\left[z\frac{\partial f}{\partial z}\overline{\frac{\partial f}{\partial t}}\right] &= Q, && z \in \partial G_1,\ t \in [0,T]; \\ f(z,0) &= f_0(z), && z \in \partial G_1; \\ f(0,t) &= 0, && t \in [0,T); \end{aligned} \tag{A 1}$$

was handled numerically with the method of Hohlov, Ibragimov & Zhiltsova [25]. Supposing $f = u + iv$, the governing equations are

$$\begin{aligned} u_t &= Q\left(-u_\theta \mathbf{H}\left[\frac{1}{u_\theta^2+v_\theta^2}\right] + v_\theta\left(\frac{1}{u_\theta^2+v_\theta^2}\right)\right), \\ v_t &= Q\left(-v_\theta \mathbf{H}\left[\frac{1}{u_\theta^2+u_\theta^2}\right] - u_\theta\left(\frac{1}{u_\theta^2+v_\theta^2}\right)\right). \end{aligned} \tag{A 2}$$

With the finite-difference procedure, different kinds of singularities, which are known from analytical investigations, can be reproduced more or less precisely, where the singularity appears at the exact time T_*.

The initial functions $f_0(z) = R_0 z$, $R_0 > 0$, are the only ones for which all the fluid can be extracted (Figure 1). All other polynomial initial functions develop singularities before the boundary reaches the sink (for a proof, see Hohlov & Howison [26], Figure 2). For rational initial functions at least three situations are possible:

- the moving boundary reaches the sink with cusp formation (Figure 3);
- the moving boundary reaches the sink without cusp formation (smooth margin) (Figure 4);
- the moving boundary does not reach the sink but forms a cusp like in the case of polynomial initial functions (Figure 5).

The model (P_α), $\alpha > 0$, can be written in the following form, where we use $Q(t) \equiv -1$:

$$\begin{aligned} \frac{\partial f}{\partial t} &= z\frac{\partial f}{\partial z}\left\{\left|\frac{\partial f}{\partial z}\right|^{-2}(-1+\Re[z\,w_{reg}]) - i\mathbf{H}\left[\left|\frac{\partial f}{\partial z}\right|^{-2}(-1+\Re[z\,w_{reg}])\right]\right\}, \\ \Im[z\,w_{reg}] &= \alpha\frac{\partial}{\partial\theta}\mathbf{H}\left[\left|\frac{\partial f}{\partial z}\right|^{-1}(-1+\Re[z\,w_{reg}])\right]. \end{aligned} \tag{A 3}$$

The Hilbert operator $\mathbf{H}$ is handled with the Fast Fourier Transform (FFT) and for the

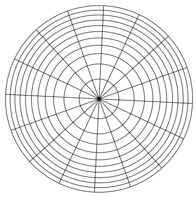

FIGURE 1.
$f_0(z) = R_0\, z$;
$T_* = \frac{\pi}{-q} R_0^2$.

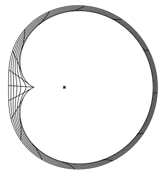

FIGURE 2. The classical example of Polubarinova–Kochina with
$f_0(z) = z + \frac{1}{4}\, z^2$;
$T_* = 0.566$.

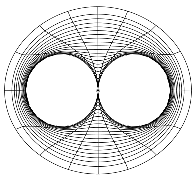

FIGURE 3.
$f_0(z) = z\,\frac{3-a_0^2-2a_0z^2}{3\sqrt{a_0}(1-a_0z^2)}$,
$a_0 = \sqrt{84} - 9$;
$T_* = 4\pi$.

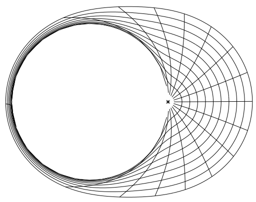

FIGURE 4.
$f_0(z) = z\,\frac{4-a_0z}{2(z-a_0)}$,
$a_0 = \sqrt{5/2}$;
$T_* = \pi$.

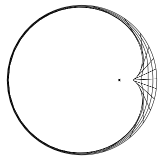

FIGURE 5.
$f_0(z) =$
$\left(\frac{b_0z-1}{z-1}\right)\left(z\,\frac{1-a_0^2b_0^2}{2a_0b_0^2} + a_0b_0\right) - a_0$,
$a_0 = 0.5, \quad b_0 = -2$;
$T_* = 0.349$.

derivatives the usual difference schemes are used. The calculations suggest that the kinetic undercooling regularization avoids the cusp formation that occurs in the examples above.

For the initial functions $f_0(z) = R_0 z$, $R_0 > 0$, the second equation of (A 3) can be written as

$$\Im[z\, w_{reg}] = \alpha\, R_0^{-1} \frac{\partial}{\partial\theta} \mathbf{H}\left[\Re[z\, w_{reg}]\right]$$

on the unit circle and it follows $\Im[z\, w_{reg}] = 0$ on ∂G_1, $w_{reg} \equiv 0$ on G_1, respectively. This means, a circle remains a circle for all time.

A comparison of the areas occupied by the remained fluid shows that more fluid can be extracted (λ is the ratio of the initial area to the remainder):

Figure 10 shows the results for the classical Polubarinova–Kochina example with different undercooling parameters α.

Remark 4 During the numerical calculations one has to take into consideration different numerical effects as, for example, crowding (see trace lines in the figures).

Remark 5 Contrary to the evaluation of the Hilbert operator with the Wittich matrix in Hohlov & Howison [26], here a FFT-algorithm is used with a $n \log n$ complexity instead of n^2.[5]

[5] Readers interested in the numerical programme to simulate Hele-Shaw flows with and without kinetic undercooling regularization are welcome to contact one of the following: {huebner,reissig}@mathe.tu-freiberg.de

Without any regularization at blow-up time $t = T_*$

With kinetic undercooling regularization at time $t = T_0$

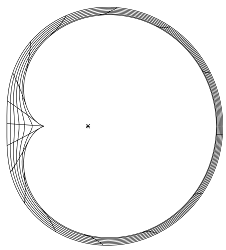

FIGURE 6a. $\alpha = 0$; $f_0(z) = z + \frac{1}{4}z^2$; $T_* = 0.566$; $\lambda = 0.86$.

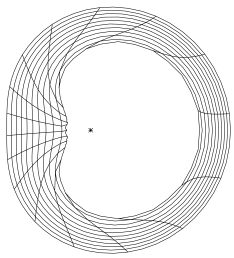

FIGURE 6b. $\alpha = 1$; $f_0(z) = z + \frac{1}{4}z^2$; $T_0 = 1.51$; $\lambda = 0.44$.

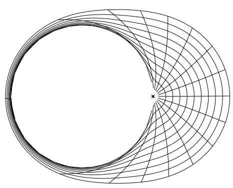

FIGURE 7a. $\alpha = 0$; $f_0(z) = z\,\frac{4-a_0 z}{2(z-a_0)}$, $a_0 = \sqrt{5/2}$; $T_* = \pi$; $\lambda = 0.4943$.

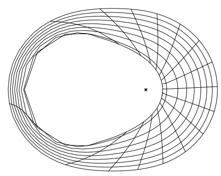

FIGURE 7b. $\alpha = 1$; $f_0(z) = z\,\frac{4-a_0 z}{2(z-a_0)}$, $a_0 = \sqrt{5/2}$; $T_0 = 3.65$; $\lambda = 0.348$.

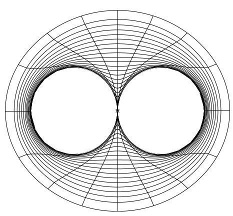

FIGURE 8a. $\alpha = 0$; $f_0(z) = z\,\frac{3-a_0^2-2a_0 z^2}{3\sqrt{a_0}(1-a_0 z^2)}$, $a_0 = \sqrt{84} - 9$; $T_* = 4\pi$; $\lambda = 0.237$.

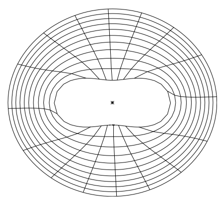

FIGURE 8b. $\alpha = 1$; $f_0(z) = z\,\frac{3-a_0^2-2a_0 z^2}{3\sqrt{a_0}(1-a_0 z^2)}$, $a_0 = \sqrt{84} - 9$; $T_0 = 15.8$; $\lambda = 0.154$.

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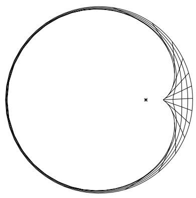

FIGURE 9a. $\alpha = 0$;
$f_0(z) = \left(\frac{b_0 z - 1}{z-1}\right)\left(z\frac{1-a_0^2 b_0^2}{2a_0 b_0^2} + a_0 b_0\right) - a_0$,
$a_0 = 0.5, \quad b_0 = -2$;
$T_* = 0.349$;
$\lambda = 0.88$.

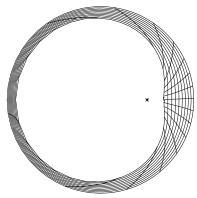

FIGURE 9b. $\alpha = 1$;
$f_0(z) = \left(\frac{b_0 z - 1}{z-1}\right)\left(z\frac{1-a_0^2 b_0^2}{2a_0 b_0^2} + a_0 b_0\right) - a_0$,
$a_0 = 0.5, \quad b_0 = -2$;
$T_0 = 0.89$;
$\lambda = 0.716$.

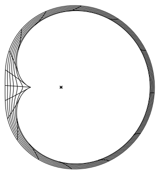

FIGURE 10a. $\alpha = \mathbf{0}$;
$T_* = 0.566$;
$\lambda = 0.866$.

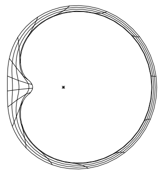

FIGURE 10b. $\alpha = \mathbf{0.01}$;
$T_0 = 0.65$;
$\lambda = 0.81$.

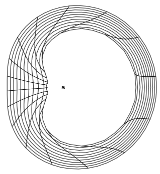

FIGURE 10c. $\alpha = \mathbf{1}$;
$T_0 = 1.93$;
$\lambda = 0.44$.

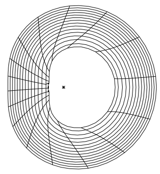

FIGURE 10d. $\alpha = \mathbf{10^6}$;
$T_0 = 2.74$;
$\lambda = 0.22$.

Acknowledgments

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