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HANKEL MATRICES OVER RIGHT ORDERED AMENABLE GROUPS

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ABSTRACT. We extend to amenable right ordered groups the theorems of Nehari and Hartman on Hankel matrices.

INTRODUCTION

An infinite complex matrix $M = (m_{i,j})_{i,j\geq 1}$ is said to be a *Hankel matrix* if $m_{i,j}$ is given by some function of $i + j$. In other words the entries are required to be constant along every diagonal perpendicular to the main diagonal.

Extending a former result of Toepliz [13], Nehari gave in [8] a necessary and sufficient condition for a Hankel matrix to be bounded. The condition obtained by Nehari is that the first row of M (which clearly determines M itself) be given by the Fourier coefficients, with positive indices, of some measurable *bounded* periodic function f on the real line, i.e. $m_{1,j} = \hat{f}(j)$ for all $j \geq 1$.

One year after the publication of Nehari's paper Philip Hartman [7] found a characterization of compact of Hankel matrices. His result says that a Hankel matrix is compact if and only if its first row is given by the positive Fourier coefficients of a *continuous* periodic function.

Therefore, in a way, bounded Hankel matrices correspond to bounded functions while compact Hankel matrices correspond to continuous functions.

We refer the reader to S. Power's survey [11] and the references therein for more information on the classical theory of Hankel matrices and its interesting connections with other areas of analysis.

We propose, in this paper, to extend the characterizations above to the context of right ordered groups. We thus define Hankel matrices over a given right ordered group G and prove natural extensions of Nehari's theorem as well as Hartman's provided G is amenable. When G is the group of integers we recover Nehari's theorem and obtain a result closely related to Hartman's theorem.

This paper has two distinct parts the first of which is directed to a proof of the generalization of Nehari's theorem which is found in section III. The second part starts with section IV and is intended to construct a proof of the generalization of Hartman's theorem.

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Nehari's proof is based on the solution of certain functional equations which arise from a problem closely related to the Pick-Nevanlinna interpolation theory for bounded analytic functions, and are solved using a algorithm of Schur. Other techniques employed include the existence of radial limits for bounded analytic functions on the unit disc and the maximum principle for harmonic functions.

The harmonic analysis of right ordered groups does not yet include such powerful tools, so our techniques, as far as Part One is concerned, are forced in a different direction resembling somewhat the theory of nest algebras (see for example [1]).

In part two our techniques have a more classical flavor. Based on Pierre Eymard's thesis [5] we are able to extend to our context the notion of summability kernels which turns out to be a very useful tool. En passant we give a partial answer to a question posed by Arveson in [2, remarks 2.2.3 and 3.2.3] and prove a generalization of Sarason's theorem [12] on the closedness of $H^\infty + C$.

Our techniques break down completely without the assumption that our group be amenable. It is, nevertheless, an interesting project to study how much of all this still holds in the non-amenable case.

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PART ONE

1. **Lower triangular systems .** Suppose that $A = (a_{i,j})_{i,j\in Z}$ is an infinite matrix whose only known entries $a_{i,j}$ are those below the main diagonal (i.e. for $i - j \geq 0$); we propose to fill in the missing entries in such a way that A becomes the matrix of a bounded operator on $\ell_2(Z)$. Of course such a task is not always feasible, as for example, would be the case if there were submatrices of A, living in the known region, of arbitrarily large norm. Note that any such submatrix is in turn a submatrix of a "maximal known submatrix", i.e. a submatrix consisting of the entries $a_{i,j}$ for $i \geq n$ and $j \leq n$ for some n.

It is therefore reasonable to ask whether the existence of a uniform bound for the norms of all "maximal known submatrices" could make our task possible.

The goal of the present section is to answer this question. We do so in a much more general setting.

Let H_1 and H_2 be Hilbert spaces. By a flag between H_1 and H_2 we shall mean a triple (I, P, Q) where I is a linearly ordered index set, $P = \{p_i\}_{i\in I}$ and $Q = \{q_i\}_{i\in I}$ are increasing families of self-adjoint projections on H_1 and H_2 respectively. If the reader is used to the notion of nests he will recognize flags as natural generalizations of nests.

Given a flag $F = (I, P, Q)$, an F-lower triangular system is by definition a family of bounded operators $\{T_i\}_{i\in I}$ in $B(H_1, H_2)$ such that

(a) $T_i = (1 - q_i)T_ip_i$,

(b) $(1 - q_i)T_i = T_jp_i$ for all $i \leq j$, and

(c) $\sup\{\|T_i\|, i \in I\} < \infty$.

An example is the family of "maximal known submatrices" as in our discussion above provided all boundedness conditions are satisfied.

The precise statement of our result is then

THEOREM 1. *Given a flag $F = (I, P, Q)$ between H_1 and H_2 and an F-lower triangular system $\{T_i\}_{i\in I}$, there exists a bounded operator $T \in B(H_1, H_2)$ such that*
(a) $T_i = (1 - q_i)Tp_i$ for all $i \in I$, and
(b) $\|T\| = \{\sup \|T_i\|, i \in I\}$.

PROOF. As a first step we shall prove the theorem for finite flags by induction on the cardinality $|I|$ of the index set.

If $|I| = 1$ then the operator $T = T_1$ clearly satisfies (a) and (b).

Assume now that $|I| = n > 1$ and let $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$.

Let $S = T_2 + q_2T_1$ which is a bounded operator. Then $S = (1 - q_1)Sp_2$ and by (a trivial modification of) Parrot's distance formula (Theorem 1 of [9]) we have

$$\mathrm{dist}(S, (q_2 - q_1)B(H_1, H_2)(p_2 - p_1)) = \max\{\|(1 - q_2)S\|, \|Sp_1\|\}.$$

Note that $(1 - q_2)S = T_2$ while $Sp_1 = T_1$. Therefore the distance form S to $(q_2 - q_1)B(H_1, H_2)(p_2 - p_1)$ is less than M where $M = \max\{\|T_i\|, i \in I\}$. So there exists an operator

$$X \in (q_2 - q_1)B(H_1, H_2)(p_2 - p_1)$$

such that $\|S - X\| \leq M$.

Consider now the flag $F' = (I', P', Q')$ where $P' = \{p_2, p_3, p_4, \ldots, p_n\}$ and $Q' = \{q_1, q_3, q_4, \ldots, q_n\}$, with I' as the obvious index set, and define an F'-lower triangular system as follows:

$$\{T'_j\}_{j\in I'} = \{S - X, T_3, T_4, \ldots, T_n\}.$$

One may easily verify that $\{T'_j\}_{j\in I'}$ is indeed an F'-lower triangular system. It is also clear that $\max\{\|T'_j\|, j \in I'\} \leq M$, and so the induction hypothesis provides us with an operator $T \in B(H_1, H_2)$ with $\|T\| \leq M$ such that

$$(1 - q_i)Tp_i = T_i \quad \text{for } i \geq 3,$$

and

$$(1 - q_1)Tp_2 = S - X.$$

The reader may now check that $(1 - q_1)Tp_1 = T_1$ and $(1 - q_1)Tp_2 = T_2$, completing the proof of the finite case.

For infinte flags we proceed as follows. Consider the directed set of all finite subsets J of I ordered by inclusion. For each such J let $F_J = (J, P|_J, Q|_J)$ denote the corresponding finite subflag and choose $T_J \in B(H_1, H_2)$ satisfying

$$(1 - q_j)T_Jp_j = T_j \quad \text{for all } j \in J$$

and

$$\|T_J\| \leq \max\{\|T_j\|, j \in J\}$$

(hence $\|T_J\| \leq M$), as constructed above. Viewing $\{T_J\}$ as a bounded net, let T be the weak limit of any weakly converging subnet. One may now verify that T satisfies the required properties. This completes the proof.

2. **Right ordered groups.** From now on we shall be concerned with C^*-algebras and W^*-algebras associated with discrete groups so we make a short pause to introduce some notation and recall some known facts.

The letter G will always denote a discrete group. Given G we shall denote by $\ell_2(G)$ the Hilbert space of complex square summable functions on G. For every g in G we shall denote by δ_g the function that vanishes everywhere on G except at the point g where its value is one. It is clear that the set of all δ_g's forms an orthonormal basis for $\ell_2(G)$.

For every g in G we denote by L_g (resp. R_g) the unitary operator on $\ell_2(G)$ given by $L_g(\delta_h) = \delta_{gh}$ (resp. $R_g(\delta_h) = \delta_{hg}$).

The von Neumann algebra generated by $\{L_g : g \in G\}$ (resp. $\{R_g : g \in G\}$) is called the left (resp. right) von Neumann algebra of G and is denoted by $W^*(G)$ (resp. $W^*(G)_r$). It is a well known fact that the commutant of $W^*(G)$ is $W^*(G)_r$ and vice versa.

The C^*-algebra generated by $\{L_g : g \in G\}$ is called the reduced C^*-algebra of G and is denoted by $C^*_{\text{red}}(G)$.

The linear functional τ defined on $W^*(G)$ by

$$\tau(a) = \langle a(\delta_e), \delta_e \rangle, \quad a \in W^*(G)$$

is a non-degenerate normalized trace. The Fourier transform $\hat{a}$ of an element a in $W^*(G)$ is defined to be the complex-valued function on G given by

$$\hat{a}(g) = \tau(L_{g^{-1}}a) = \langle a(\delta_e), \delta_g \rangle, \qquad g \in G.$$

If S is any subset of G we set

$$W(S) = \{a \in W^*(G) : \hat{a}(g) = 0 \text{ for all } g \in G \setminus S\}$$

and $\tilde{W}(S)$ equal to the σ-weakly closed linear span of $\{L_s : s \in S\}$.

Again, if S is a subset of G we set

$$C(S) = \{a \in C^*_{\text{red}} : \hat{a}(g) = 0 \text{ for all } g \in G \setminus S\}$$

and $\tilde{C}(S)$ equal to the norm-closed linear span of $\{L_s : s \in S\}$. The linear span of $\{L_s : s \in S\}$ is denoted by $\mathbb{C}(S)$.

We shall say that G is a right ordered group if G comes with an order relation $\leq$ satisfying

(i) for every g, h in G, either $g \leq h$ or $h \leq g$, and

(ii) for every g, s, t in G if $s \leq t$ then $sg \leq tg$.

Given a right ordered group G we set

$$\begin{aligned} G^+ &= \{g \in G : g \geq e\}, \\ G^- &= \{g \in G : g \leq e\}, \\ G_0^+ &= \{g \in G : g > e\} \quad \text{and} \\ G_0^- &= \{g \in G : g < e\}. \end{aligned}$$

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We also let H^+, H^-, H_0^+, and H_0^- be the closed subspaces of $\ell_2(G)$ spanned by the $\delta_g' s$ with g in G^+, G^-, G_0^+, and G_0^- respectively, and p^+, p^-, p_0^+, and p_0^- denote the orthogonal projections onto these subspaces.

3. **Bounded Hankel matrices .** Let G be a right ordered group.

DEFINITION 2 . A Hankel matrix over G is a complex matrix

$$M = (m_{t,s})_{t \in G^+, s \in G_0^-}$$

satisfying $m_{t,s} = \alpha(ts^{-1})$ for some complex function α defined on G_0^+, which we shall call the defining function of M. A Hankel operator is by definition a bounded operator from H_0^- to H^+ whose matrix with respect to the canonical basis of H_0^- and H^+ is a Hankel matrix. A Hankel matrix is said to be *bounded* if it is the matrix of some Hankel operator.

As an example note that if a is an element of $W^*(G)$ then the operator $h(a)$ defined by

$$h(a) = p^+ a|_{H_0^-}$$

is a Hankel operator (as the reader may easily verify). The corresponding Hankel matrix is the matrix $(m_{t,s})_{t \in G^+, s \in G_0^-}$ given by

$$m_{t,s} = \langle p^+ a|_{H_0^-} (\delta_s), \delta_t \rangle = \langle a(\delta_s), \delta_t \rangle = \hat{a}(ts^{-1}) \rangle.$$

It is clear that a is in $W(G^-)$ if, only if, $h(a) = 0$.

As we shall now see every Hankel operator arises in this way when G is amenable.

THEOREM 3. *Let G be an amenable right ordered group and let*

$$M = (m_{t,s})_{t \in G^+, s \in G_0^-}$$

be a Hankel matrix over G. Then M is bounded if and only if M is $h(a)$ for some a in $W^*(G)$. In this case one can choose a with $\|a\| = \|M\|$.

PROOF. The "if" part being trivial we move on to the "only if" part.

The proof will be based on Section I so let us identify the flag which will be relevant for our purposes.

For every g in G we denote by H_g the closed linear span of $\{\delta_s : s < g\}$ and write p_g for the orthogonal projection onto H_g.

If P is the family of all such $p_g' s$ then the triple $F = (G, P, P)$ is clearly a flag between H and itself.

Given that M represents a bounded operator, let b be such an operator. Obviously, $\|b\| = \|M\|$. We shall assume that b is defined on the whole of $\ell_2(G)$ by putting $b|_{H^+} = 0$.

For every g in G set $b_g = R_g b R_{g^{-1}}$. With a little effort the reader may verify that the family $\{b_g\}_{g \in G}$ is an F-lower triangular system with bound equal to $\|M\|$. We may therefore use Theorem 1 to conclude that there exists a bounded operator a on $\ell_2(G)$ such that $\|a\| \leq \|M\|$ and for all g in G one has $b_g = (1 - p_g) a p_g$.

Let S denote the set of all such operators. One may show that S is convex and weakly compact and that $R_{g^{-1}}SR_g = S$ for all g in G. That is, S is invariant under the action of G by conjugation by the operators R_g.

Since G is amenable it follows by [6] that there exists a fixed point, i.e. there exists some a in S such that $R_{g^{-1}}aR_g = a$ for all g in G. This says that a commutes with the right von Neumann algebra of G. Therefore a is in $W^*(G)$ and we have for $t \geq e$, $s < e$

$$\begin{aligned} m_{t,s} &= \langle b(\delta_s), \delta_t \rangle \\ &= \langle b_e(\delta_s), \delta_t \rangle \\ &= \langle (1 - p_e)ap_e(\delta_s), \delta_t \rangle \\ &= \langle a(\delta_s), \delta_t \rangle \\ &= \hat{a}(ts^{-1}). \end{aligned}$$

Finally, note that

$$\|a\| \geq \|(1 - p_e)ap_e\| = \|b_e\| = \|b\| = \|M\|$$

and so

$$\|a\| = \|M\|.$$

COROLLARY 4. *Let G be an amenable right ordered group. Then for every a in $W^*(G)$ we have*

$$\operatorname{dist}(a, W(G^-)) = \|h(a)\|.$$

PROOF. If b is in $W(G^-)$ we have $h(b) = 0$ and so

$$\|a - b\| \geq \|h(a - b)\| = \|h(a)\|,$$

so that $\operatorname{dist}(a, W(G^-)) \geq \|h(a)\|$. To prove the converse inequality note that by Theorem 3 there exists b in $W^*(G)$ with $h(b) = h(a)$ and $\|b\| = \|h(a)\|$. So we have $h(a-b) = 0$, whence $a - b$ is in $W(G^-)$ and thus

$$\operatorname{dist}(a, W(G^-)) \leq \|a - (a - b)\| = \|b\| = \|h(a)\|,$$

completing the proof.

PART TWO

4. **Summability kernels .** In order to extend the notion of summability kernels to non-commutative groups we must make use of the work of P. Eymard [5] which we now briefly describe, for the convenience of the reader and also because, in our context of discrete groups, we can make some minor improvements on it. These improvements will turn out to be important in the sequel. See [5] for details.

As before let G be a discrete group. The full C^*-algebra of G, denoted by $C^*(G)$, is by definition the C^*-algebra generated by the range of the universal representation U of G.

According to [5] the Banach space dual of $C^*(G)$ may be identified with the subspace $B(G)$ of $\ell_\infty(G)$ spanned by the positive definite functions on G. The identification is as follows: given ψ in $C^*(G)^*$, the corresponding element in $\ell_\infty(G)$ is the function

$$g \in G \mapsto \psi(U_g) \in \mathbb{C}.$$

In turns out that $B(G)$ is a subalgebra of $\ell_\infty(G)$ under pointwise multiplication. Moreover it is an involutive Banach algebra with the norm inherited from $C^*(G)^*$ and involution defined by

$$f^*(g) = \overline{f(g^{-1})}, \quad g \in G$$

for all f in $B(G)$.

The subset of $B(G)$ comprising the positive definite functions is a closed convex generating cone which is denoted $B_+(G)$. Eymard calls $B(G)$ the Fourier-Stieltjes algebra of G.

The pre-dual $W^*(G)_*$ of $W^*(G)$ may be identified with a closed self-adjoint ideal $A(G)$ of $B(G)$ called the Fourier algebra of G. The identification maps each ϕ in $W^*(G)_*$ to the function

$$g \in G \mapsto \phi(L_g) \in \mathbb{C}.$$

The intersection of $B_+(G)$ with $A(G)$ is denoted by $A_+(G)$.

Keeping the above identifications in mind we shall refer to $C^*(G)^*$ and $W^*(G)_*$ as $B(G)$ and $A(G)$ respectively (and vice versa) since no confusion will arise. This will of course lead to an identification between the notations $\phi(L_g)$ and $\phi(g)$.

For every g in G let χ_g be the characteristic function of the singleton $\{g\}$. Each χ_g belongs to $A(G)$. Actually $A(G)$ contains the space $K(G)$ of finitely supported functions on G as a dense subspace. Given ψ in $B(G)$ and a in $W^*(G)$ the map

$$\phi \in A(G) \mapsto (\phi\psi)(a) \in \mathbb{C}$$

(note that the right hand side makes sense because $A(G)$ is an ideal) is clearly a bounded linear funcitonal on $A(G)$ with norm at most $\|\psi\|\|a\|$. But since $W^*(G)$ is the dual of $A(G)$ there must exist a uniquely defined element in $W^*(G)$ which we denote $\psi \times a$ such that

$$(\phi\psi)(a) = \phi(\psi \times a)$$

for all ϕ in $A(G)$. This gives $W^*(G)$ a $B(G)$-module structure.

As an example note that if ϕ is in $B(G)$ and g is a group element one has

$$\psi \times L_g = \psi(L_g)L_g$$

since for all ϕ in $A(G)$ we have

$$\phi(\psi \times L_g) = (\phi\psi)(L_g) = \phi(L_g)\psi(L_g) = \phi(\psi(L_g)L_g).$$

PROPOSITION 5. *Given a discrete group G the following statements hold:*

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(i) For all ψ in $B(G)$ the map

$$a \in W^*(G) \mapsto \psi \times a \in W^*(G)$$

is σ-weakly continuous.
(ii) For all ψ in $B(G)$, a in $W^*(G)$ and g in G

$$(\widehat{\psi \times a})(g) = \psi(g)\hat{a}(g).$$

(iii) $C^*_{\mathbf{red}}(G)$ is a $B(G)$-submodule of $W^*(G)$. In other words

$$B(G) \times C^*_{\mathbf{red}}(G) \subset C^*_{\mathbf{red}}(G).$$

(iv) $A(G) \times W^*(G) \subset C^*_{\mathbf{red}}(G)$.

PROOF. Since the σ-weak topology on $W^*(G)$ coincides with the weak topology arising from the duality between $A(G)$ and $W^*(G)$ all we must check in order to prove (i) is that for every ϕ in $A(G)$ the map

$$a \in W^*(G) \mapsto \phi(\psi \times a) \in \mathbb{C}$$

is σ-weakly continuous. But note that $\phi(\psi \times a) = (\phi\psi)(a)$ and $\phi\psi \in A(G)$ because $A(G)$ is an ideal. Since $A(G)$ corresponds exactly to the σ-weakly continuous linear functionals on $W^*(G)$ we have proven (i).

To prove (ii) recall that for g in G, χ_g is in $A(G)$. Viewing χ_g as an element of $W^*(G)_*$, it is clear that $\chi_g(a) = \hat{a}(g)$ for all a in $W^*(G)$. We then have

$$(\widehat{\psi \times a})(g) = \chi_g(\psi \times a) = (\chi_g\psi)(a) = (\psi(g)\chi_g)(a) = \psi(g)\hat{a}(g).$$

In order to prove (iii) take ψ in $B(G)$ and a in $\mathbb{C}(G)$. Then a is of the form

$$a = \sum_{i=1}^{n} \lambda_i L_{g_i} \quad \lambda_i \in \mathbb{C},\ g_i \in G$$

and we have

$$\psi \times a = \sum_{i=1}^{n} \lambda_i \psi \times L_{g_i} = \sum_{i=1}^{n} \lambda_i \psi(L_{g_i}) L_{g_i}$$

which clearly belongs to $\mathbb{C}(G)$. That is, $B(G) \times \mathbb{C}(G)$ is contained in $\mathbb{C}(G)$. But since $C^*_{\mathbf{red}}(G)$ is the norm closure of $\mathbb{C}(G)$ the result follows from continuity.

To prove (iv) pick ϕ in $K(G)$ and a in $W^*(G)$. Then by (ii)

$$(\widehat{\phi \times a})(g) = \phi(g)\hat{a}(g)$$

so that $\phi \times a$ has finitely many non-zero Fourier coefficients, whence

$$\phi \times a = \sum_{g \in G} (\widehat{\phi \times a})(g) L_g,$$

so that $\phi \times a$ is in $\mathbb{C}(G)$. This says that $K(G) \times W^*(G)$ is contained in $\mathbb{C}(G)$. The result now follows from the fact that $K(G)$ is dense in $A(G)$.

Since most of our results are only proved for amenable groups we will, from now on, concentrate our attention on groups having this property.

If G is amenable then the trivial representation of G extends to a one dimensional representation T of $C^*_{\mathbf{red}}(G)$ (which in this case is isomorphic to $C^*(G)$; cf. [10, 7.3.9]) such that $T(L_g) = 1$ for all g in G.

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LEMMA 6. *Let G be amenable. Then for every ϕ in $A(G)$ and a in $W^*(G)$ we have $\phi(a) = T(\phi \times a)$.*

PROOF. First assume that $a = L_g$ for some g in G. Then

$$T(\phi \times a) = T(\phi \times L_g) = T(\phi(L_g)L_g) = \phi(L_g) = \phi(a).$$

So by linearity and continuity the lemma holds for all a in $C^*_{\text{red}}(G)$. Now suppose that a is in $W^*(G)$ and $\phi = \phi_1\phi_2$ for some ϕ_1, ϕ_2 in $A(G)$. Then $\phi_2 \times a$ belongs to $C^*_{\text{red}}(G)$ and we have by what was said above

$$T(\phi \times a) = T(\phi_1 \times (\phi_2 \times a)) = \phi_1(\phi_2 \times a) = (\phi_1\phi_2)(a) = \phi(a).$$

It is now enough to prove that

$$A(G)^2 = \text{span}\{\phi_1\phi_2 : \phi_1, \phi_2 \in A(G)\}$$

is dense in $A(G)$. But this becomes clear once we note that $K(G)^2 = K(G)$.

DEFINITION 7. A *summability kernel* for a group G is a net $\{\phi_i\}$ of elements of $A(G)$ such that

(i) for all a in $C^*_{\text{red}}(G)$ the net $\{\phi_i \times a\}$ is a norm convergent to a,

(ii) for all a in $W^*(G)$ the net $\{\phi_i \times a\}$ is a σ-weakly convergent to a.

THEOREM 8. *If G is amenable then there exists a summability kernel $\{\phi_i\}$ such that each ϕ_i is in $A_+(G) \cap K(G)$ and $\|\phi_i\| \leq 1$.*

PROOF. According to [10, 7.3.8] there exists a net $\{f_i\}$ in the unit sphere of $\ell_2(G)$ such that $f_i * f_i^*$ (convolution product) converges pointwise to 1. We may clearly assume that $f_i \in K(G)$. Define

$$\phi_i = f_i * f_i^*$$

so that ϕ_i belongs to $A_+(G) \cap K(G)$ and

$$\|\phi_i\| \leq \|f_i\|^2 \leq 1.$$

Observe that for all g in G, $\phi_i \times L_g = \phi_i(L_g)L_g$ converges to L_g. Thus, by linearity, $\phi_i \times a$ converges to a for al a in $\mathbb{C}(G)$ and since $\{\phi_i\}$ is uniformly bounded, $\phi_i \times a$ converges to a for all a in $C^*_{\text{red}}(G)$.

To prove (7.ii) we first claim that for all b in $C^*_{\text{red}}(G)$,

$$\lim_i \phi_i(b) = T(b).$$

This is clear if $b = L_g$ for some g in G, so also for all b in $\mathbb{C}(G)$. In the general case the claim follows from the uniform boundedness of $\{\phi_i\}$.

Now let a be in $W^*(G)$. To prove that $\{\phi_i \times a\}$ converges σ-weakly to a it is enough to show that for all ϕ in $A(G)$ one has

$$\lim_i \phi(\phi_i \times a) = \phi(a).$$

But

$$\phi(\phi_i \times a) = (\phi\phi_i)(a) = (\phi_i\phi)(a) = \phi_i(\phi \times a),$$

which, by the claim above, converges to $T(\phi \times a)$ since $\phi \times a$ is in $C^*_{\text{red}}(G)$. Therefore

$$\lim_i \phi(\phi_i \times a) = T(\phi \times a) = \phi(a).$$

As a first application of the results of this section we can give a partial answer to a question posed by Arveson in [2, remarks 2.2.3 and 3.2.3].

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THEOREM 9. *Let G be amenable. Given any subset S of G (in particular G^+ in case G is a right ordered group, which is the case of interest in* [2]*) we have $W(S) = \tilde{W}(S)$ and $C(S) = \tilde{C}(S)$.*

PROOF. . It is clear that $\tilde{W}(S)$ is contained in $W(S)$ and $\tilde{C}(S)$ is contained in $C(S)$. To prove the reverse inclusions let $\{\phi_i\}$ be a summability kernel for G with $\phi_i \in K(G)$.

Given a in $W(S)$, for all i we have that the non-zero Fourier coefficients of $\phi_i \times a$ correspond to a finite subset of S by Proposition 5 (ii). Thus $\phi_i \times a$ is in $\mathbb{C}(S)$, hence in $\tilde{W}(S)$. Since a is the σ-weak limit of $\phi_i \times a$ it follows that a belongs to $\tilde{W}(S)$. The inclusion $C(S) \subset \tilde{C}(S)$ is proved in a similar way.

We should note that after the present paper was first submitted for publication, we have been able to find a complete answer to Arveson's question in the finite case(see [4]). In particular, the equality $W(G^+) = \tilde{W}(G^+)$ holds for all right ordered groups regardless of amenability.

5. **Approximately finite Hankel matrices.** Our goal in this section is to extend Hartman's theorem concerning compact Hankel matrices. Hartman proves in [7] that a Hankel matrix is compact if and only if its first row is given by the Fourier coefficients, with positive indices, of some continuous periodic function.

Over some right ordered groups (as is the case of the discrete real line) there is no non-zero compact Hankel matrix. The reason being that there may not exist finite ordered intervals except for the degenerate ones. Because of this we must work with an alternative notion of "compactness" for Hankel matrices which we now describe.

DEFINITION 10. Let M be a Hankel matrix over the right ordered group G and let α be its defining function. We shall say taht M is finite if the support of α is a finite subset of G. We shall say that M is approximately finite if M is in the operator norm closure of the space of finite Hankel matrices.

It is clear that finite and hence also approximately finite Hankel matrices are bounded.

Our first step towards a "right ordered" form of Hartman's theorem is a generalization of a result of Sarason [12] on the closedness of $H^\infty + C$. Thanks to the existence of summability kernels over amenable groups we are able to give an "ipsis litteris" translation of Zalcman's proof [14] of Sarason's theorem.

THEOREM 11. *Let G be an amenable group and let S be a subset of G. Then*
(i) $C^*_{\mathbf{red}}(G) + W(S)$ is closed,
(ii) for all a in $C^*_{\mathbf{red}}(G)$, $\mathrm{dist}(a, W(S)) = \mathrm{dist}(a, C(S))$.

PROOF. We prove (ii) first. Let $\{\phi_i\}$ be a finitely supported summability kernel for G, each ϕ_i having norm one as in Thoerem 8. Given a in $C^*_{\mathbf{red}}(G)$ and b in $W(S)$ we have for all i

$$\|a - \phi_i \times b\| \leq \|a - \phi_i \times a\| + \|\phi_i \times a - \phi_i \times b\| \leq \|a - \phi_i \times a\| + \|a - b\|.$$

Since $\|a - \phi_i \times a\| \longrightarrow 0$ and since each $\phi_i \times b$ is in $C(S)$ we conclude that $\mathrm{dist}(a, C(S)) \leq \mathrm{dist}(a, W(S))$. The reverse inequality is trivial.

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To prove (i) note that from (ii) the natural map

$$C^*_{\text{red}}(G)/\,C(s) \longmapsto W^*(G)/\,W(S)$$

is isometric and so its range is closed. Therefore $C^*_{\text{red}}(G)+W(S)$, being the inverse image of that range under the quotient map

$$W^*(G) \longmapsto W^*(G)/\,W(S),$$

is also closed.

We can now prove the main result of this section.

THEOREM 12. *Let G be an amenable right ordered group. A necessary and sufficient condition for a given Hankel matrix M to be approximately finite is that M be $h(a)$ for some a in $C^*_{\text{red}}(G)$. In this case, for every $\epsilon > 0$, one can find a in $C^*_{\text{red}}(G)$ with $M = h(a)$ and $\|a\| \leq \|M\| + \epsilon$.*

PROOF. Since every a in $C^*_{\text{red}}(G)$ is the norm limit of elements in $\mathbb{C}(G)$, it is clear that $h(a)$ is approximately finite.

Conversely let M be an approximately finite Hankel matrix. Given $\varepsilon > 0$ one can find a sequence $\{M_i\}$ of finite Hankel matrices such that

$$M = \sum_{i=1}^{\infty} M_i,$$
$$\|M_1\| \leq \|M\| + \varepsilon/4 \text{ and}$$
$$\|M_i\| \leq 2^{-i-1}\varepsilon$$

for all $i \geq 2$. Since M_i is finite for every i we may obviously choose a_i in $\mathbb{C}(G)$ such that $M_i = h(a_i)$.

We have by Corollary 4 and Theorem 11,

$$\text{dist}(a_i, C(G^-)) = \text{dist}(a_i, W(G^-)) = \|h(a_i)\| = \|M_i\|.$$

So for every i we may take b_i in $C(G^-)$ such that

$$\|\, a_i - b_i\| \leq \|M_i\| + 2^{-i-1}\varepsilon.$$

The series $\sum_{i=1}^{\infty} a_i - b_i$ is therefore absolutely summable. Denote its sum by a. We have

$$\|a\| \leq \sum_{i=1}^{\infty} \|a_i - b_i\| \leq \sum_{i=1}^{\infty} \|M_i\| + 2^{-i-1}\varepsilon \leq \|M\| + \varepsilon.$$

Moreover,

$$h(a) = \sum_{i=1}^{\infty} h(a_i - b_i) = \sum_{i=1}^{\infty} h(a_i) = \sum_{i=1}^{\infty} M_i = M.$$

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THEOREM 13. *Let a be in* $W^*(G)$. *Then a is in* $C^*_{\text{red}}(G) + W(G^-)$ *if and only if* $h(a)$ *is approximately finite.*

PROOF. Suppose that $h(a)$ is approximately finite. Then there exists b in $C^*_{\text{red}}(G)$ such that $h(b) = h(a)$. It follows that $h(a - b) = 0$ and hence $a - b$ is in $W(G^-)$. Writing $a = b + (a - b)$ we see that a is in $C^*_{\text{red}}(G) + W(G^-)$. The "only if" part is clear.

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