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Evanescent Schölte waves of arbitrary profile and direction

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Schölte waves, waves bound to the interface between a fluid and an elastic half-space, are, for many material combinations, *evanescent*; as they propagate, they are damped due to radiation. A representation of the general evanescent Schölte wave is here obtained in terms of a solution to the *membrane equation with complex speed*, linked, at each instant, to a complex-valued harmonic function in a half-space. This derivation generalises one obtained recently for (non-evanescent) Rayleigh, Stoneley and Schölte waves. An alternative description is also obtained, in which the time-evolution of the normal displacement of the interface satisfies a first-order, complex-valued, non-local evolution equation. Amongst some explicit solutions obtained are decaying solutions allied to a general solution to the Helmholtz equation, and a solution closely related to a Gaussian beam. In the plane–strain case, the general Schölte wave splits into two disturbances, one right-travelling and one left-travelling, each being described at *all* times in terms of a harmonic function in a half-plane, decaying with depth yet having arbitrary boundary values. This representation highlights the dual elliptic–hyperbolic nature typical of guided waves and gives a surprisingly compact representation for the two-dimensional case.

Key words: Schölte waves, evanescent, membrane equation, complex speed

1 Introduction

Rayleigh's [21] 1885 prediction that, along the surface of a linearly elastic half-space, sinusoidal waves of any wavelength may propagate at a unique speed, below that of both dilatational and shear waves within the half-space, was generalised by Stoneley [20] for waves at the plane interface between two dissimilar elastic media, and independently by Schölte [19] and Gogoladze [8] for waves at a fluid–solid interface. All such waves are non-dispersive. In fact, analogous waves exist at the surface of an anisotropic half-space or between two anisotropic half-spaces, though the propagation speed then depends upon direction but not wavelength (or frequency) of the wave. Such waves have been much studied and utilised (see e.g. [2–4, 7] and references therein) and the theory extended to include non-linear effects [16]. More recently, as a by-product of investigations into waves on plates and layered half-spaces [1, 14, 15], it was observed that cases involving isotropic (or transversely isotropic) half-spaces give rise to a substantially generalised theory [11,13]. Indeed, rotational invariance of material properties about the surface normal makes the wave speed independent of both wavelength and direction. This property is shared by the familiar *membrane equation*, $u_{tt} = c^2(u_{xx} + u_{yy})$, and it is a major result in [13] that Rayleigh, Stoneley and Schölte waves have explicit representations in terms of solutions to the membrane equation and, at each time, a linked solution to Laplace's equation in a half-space. The one restriction on that prediction for Stoneley and Schölte waves is that it applies only when the material combinations allow the wave speed to be real. Particularly for the Schölte waves, this requirement is known to be restrictive, so that, for example, at a steel–water interface, the 'wave speed' must have a small imaginary part (the wave must be accompanied by radiation away from the interface into the fluid – the *slower* medium). The aim of this paper is to analyse explicit representations for this case, specifically for evanescent Schölte waves. (It is anticipated that similar analysis applies, but with increased algebraic complexity, for Stoneley waves at the interface between two elastic materials in the case for which interface waves cause radiation into one of the half-spaces so that interface waves are evanescent.)

For evanescent Schölte waves, one representation is found here in terms of a complex-valued solution to the *membrane equation with complex speed*. Associated with this function at each time is a (complex-valued) solution to Laplace's equation in an abstract half-space. Then, at each instant, the displacements everywhere are given explicitly in terms of first derivatives of this 'potential'. This representation is a natural generalisation of the solution structure for the non-evanescent case [13], which has real propagation speed. This case shows the dual elliptic–hyperbolic nature of guided waves [6, 9, 10] – as is typical, the guided wave has structure defined through solution of a boundary value problem (a feature of elliptic systems) yet propagates according to the membrane equation (the wave equation in two dimensions, a hyperbolic equation). For Schölte waves, a standard analysis using complex exponentials shows, in Section 2, that both the non-evanescent and evanescent cases of spatially sinusoidal waves may be treated together, provided that the frequency (or, equivalently, the speed) is allowed to be complex. This allows time dependence which is a product of sinusoidal and exponential terms. The corresponding displacement field in the evanescent case is a linear combination of terms each having depth dependence which is a complex exponential decay. Superposing these evanescent waves of all wavelengths and directions then gives, in Section 3, a representation of a general disturbance in terms of *three copies of a single function*. This is closely similar to the representation in the non-evanescent case [13], where displacements are given in terms of three depth-scaled copies of a single scalar function, which is harmonic in an abstract half-space and has boundary values evolving in time as a solution of the membrane equation. Two copies apply within the elastic medium and one within the fluid.

The essential difference in the evanescent case is that the abstract 'depth coordinate' must become complex, while the surface values of the potential evolve according to the *membrane equation with complex speed*. However, a *general* solution to that equation requires initial conditions equivalent to four real functions defined over a plane. This is much more information than is required in the non-evanescent case. Hence, unsurprisingly, it is found that only a subset of solutions is relevant, in which the real and imaginary parts of the solution are interrelated at each time. It is shown in Section 4 that there are integral representations for initial conditions for the imaginary part in terms of those for the real part. Thus, within the initial conditions only two real functions may be independently specified over the plane interface. An alternative is to specify a single complex-valued function as the initial values of both the real and imaginary parts of the potential; for the initial values of the time derivative of that potential there is then no freedom – they are defined completely through an integral representation.

These features suggest that the potential evolves according to a first-order evolution rule. Indeed, this is so. It is shown, also in Section 4, that the whole solution may be described formally, at all times, in terms of a single complex-valued solution to Laplace's equation in a half-space, in which the coordinate normal to the surface is allowed to be complex (it is a *complex* linear combination of depth and time). This again highlights the elliptic–hyperbolic duality. In addition, it greatly helps, in Section 5, in the construction of explicit solutions for evanescent waves. Besides the expected unidirectional solutions that are either time-harmonic (hence spatially decaying) or spatially sinusoidal (hence, decaying with time), there are decaying solutions of fixed (complex) frequency corresponding to *any* solution to the Helmholtz equation (*cf.* Achenbach's analysis [1] for plate waves). Also, an exact and explicit solution analogous to a Gaussian beam is obtained (without any approximation using high-frequency asymptotics), similar in structure to a solution to the wave equation given by Kiselev [12]. This shows that beamlike solutions (with time dependence, which is not exactly sinusoidal) exist to the full set of equations and boundary conditions for an isotropic elastic solid adjoining a compressible fluid.

In Section 6, the special case of two-dimensional Schölte waves is revisited and it is shown that the evolution equation for normal displacements at the interface involves two real-valued functions and their Hilbert transforms, while interior displacements are represented in terms of the two associated pairs of conjugate harmonic functions in a half-plane. In fact, the general solution is a superposition of a right-propagating wave and a left-propagating wave, each having arbitrary initial form. However, the wave profile of each does not have permanent form, unlike in the non-evanescent case. It evolves with time (or, equivalently, distance). Moreover, its time-evolution is given by evaluating, at successively increasing depths, the harmonic function whose values on the Ox axis coincide with its initial waveform. This representation of a general two-dimensional, evanescent Schölte wave, in terms of two arbitrary harmonic functions is a further demonstration that interface waves have many features of elliptic equations as well as of hyperbolic waves.

2 Two-dimensional analysis for Schölte waves

At the interface $z = 0$ between a fluid occupying $z > 0$ and having density $\rho_{\rm f}$ and sound speed $c_{\rm f}$ and an isotropic elastic solid occupying $z < 0$ and having density ρ and Lamé constants λ and μ, and unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are taken along the axes Ox, Oy and Oz of cartesian coordinates (x, y, z). For waves travelling along Ox, interface waves have displacements $\mathbf{u} = u\mathbf{e}_1 + w\mathbf{e}_3 = u_1\mathbf{e}_1 + u_3\mathbf{e}_3$ (they are saggitally polarised, i.e. $u_2 \equiv \mathbf{u}\cdot\mathbf{e}_2 = 0$). Within $z > 0$, there is a displacement potential $\phi(x, z, t)$ such that

$$u = \phi_x, \quad w = \phi_z \qquad \text{with} \qquad \phi_{tt} = {c_{\rm f}}^2(\phi_{xx} + \phi_{zz}) \tag{2.1}$$

(subscripts x, y, z and t denote partial differentiation). The corresponding velocity components are $u_t = \phi_{xt}$ and $w_t = \phi_{zt}$ and the perturbation pressure is

$$\hat{p} = -\rho_{\rm f}\phi_{tt}.$$

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Within the elastic solid, the Navier equations are

$$(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2\mathbf{u} = \rho\mathbf{u}_{tt}, \tag{2.2}$$

while the traction components acting over $z = 0$ are (in indicial notation, with commas denoting partial differentiation with respect to $x \equiv x_1$, $y \equiv x_2$ or $x_3 \equiv z$)

$$\sigma_{i3} = \lambda(\nabla \cdot \mathbf{u})\delta_{i3} + \mu(u_{i,3} + u_{3,i}) \qquad (i = 1, 2, 3),$$

so that

$$\sigma_{13} = \mu(u_z + w_x), \qquad \sigma_{23} = 0, \qquad \sigma_{33} = \lambda u_x + (\lambda + 2\mu)w_z.$$

Thus, continuity of traction and (normal component of) displacement imposes the conditions (at $z = 0$)

$$u_z + w_x = 0, \qquad \lambda u_x + (\lambda + 2\mu)w_z = -\hat{p} = \rho_{\mathrm{f}}\phi_{tt}, \qquad w = \phi_z. \tag{2.3}$$

We now seek *generalised travelling wave solutions*

$$\mathbf{u} = A(k)\,\mathbf{V}(z,k)\,\mathrm{e}^{\mathrm{i}(kx-|k|ct)} + \text{c.c.} \tag{2.4}$$

with k real but c possibly complex, provided that $\operatorname{Im} c \leqslant 0$ (to ensure attenuation rather than amplification). The term c.c. denotes the complex conjugate. Observe that the case in which c is real corresponds to the non-evanescent Schölte waves, for which the membrane equation arises, as shown by Kiselev and Parker [13].

Details of the depth-dependence $\mathbf{V}(z,k)$ are found as follows:
In $z > 0$, take $\phi = b_+ A\,\mathrm{e}^{-\gamma_{\mathrm{f}}|k|z}\mathrm{e}^{\mathrm{i}(kx-|k|ct)} + \text{c.c.}$, so that

$$\mathbf{V} = b_+(\mathrm{i}k\mathbf{e}_1 - \gamma_{\mathrm{f}}|k|\mathbf{e}_3)\,\mathrm{e}^{-\gamma_{\mathrm{f}}|k|z}, \tag{2.5}$$

where $c^2 = (1-\gamma_{\mathrm{f}}^2)c_{\mathrm{f}}^2$, so that the 'attenuation factor' γ_{f} may be complex, with $\operatorname{Re}\gamma_{\mathrm{f}} > 0$.

In $z \leqslant 0$, the Navier equations allow special (separated) solutions

$$\mathbf{V} = (\mathrm{i}\gamma_1|k|\mathbf{e}_1 + k\mathbf{e}_3)\mathrm{e}^{\gamma_1|k|z} \equiv \mathbf{V}^{(1)}(z,k),$$
$$\mathbf{V} = (\mathrm{i}k\mathbf{e}_1 + \gamma_2|k|\mathbf{e}_3)\mathrm{e}^{\gamma_2|k|z} \equiv \mathbf{V}^{(2)}(z,k),$$

where

$$\gamma_1{}^2 = 1 - c^2/c_{\mathrm{S}}{}^2, \quad \gamma_2{}^2 = 1 - c^2/c_{\mathrm{L}}{}^2, \quad \text{with } \operatorname{Re}\gamma_1 > 0, \ \operatorname{Re}\gamma_2 > 0,$$

and $c_{\mathrm{S}} \equiv (\mu/\rho)^{1/2}$, $c_{\mathrm{L}} \equiv [(\lambda+2\mu)/\rho]^{1/2}$ are the shear and longitudinal speeds, respectively.

Taking the linear combination $\mathbf{V} = a_-\mathbf{V}^{(1)}(z,k) + b_-\mathbf{V}^{(2)}(z,k)$ leads to the expressions (in $z < 0$)

$$u_z + w_x = \mathrm{i}\big[a_-(1+\gamma_1{}^2)k^2\mathrm{e}^{\gamma_1|k|z} + 2b_-\gamma_2 k|k|\mathrm{e}^{\gamma_2|k|z}\big]\mathrm{e}^{\mathrm{i}(kx-c|k|t)} + \text{c.c.},$$
$$\lambda u_x + (\lambda+2\mu)w_z = \{a_- 2\mu\gamma_1 k|k|\mathrm{e}^{\gamma_1|k|z} + b_-[(\lambda+2\mu)\gamma_2{}^2 - \lambda]k^2\mathrm{e}^{\gamma_2|k|z}\}\mathrm{e}^{\mathrm{i}(kx-c|k|t)} + \text{c.c.},$$
$$w = \big[a_- k\mathrm{e}^{\gamma_1|k|z} + b_-\gamma_2|k|\mathrm{e}^{\gamma_2|k|z}\big]\mathrm{e}^{\mathrm{i}(kx-c|k|t)} + \text{c.c.}$$

Thus, the continuity conditions (2.3) reduce to the algebraic equations

$$\begin{aligned} 2\gamma_2|k|b_- + \left(1+\gamma_1{}^2\right)ka_- &= 0, \\ \rho_f c^2|k|b_+ + \mu\left(1+\gamma_1{}^2\right)|k|b_- + 2\mu\gamma_1 ka_- &= 0, \\ \gamma_f|k|b_+ + \gamma_2|k|b_- + ka_- &= 0, \end{aligned}$$

for which the compatibility condition becomes

$$\gamma_f\{4\gamma_1\gamma_2 - (1+\gamma_1{}^2)^2\} = \frac{\rho_f}{\rho}\gamma_2\left(1-\gamma_1{}^2\right)^2. \tag{2.6}$$

This is precisely the algebraic equation for c^2 governing the non-evanescent Schölte waves (i.e. c real). While equation (2.6), after being squared twice and having substitutions made for $\gamma_f{}^2$, $\gamma_1{}^2$ and $\gamma_2{}^2$, yields a polynomial equation for c^2 with real coefficients, this process introduces spurious roots. It is preferable to rewrite equation (2.6) and to seek $\xi \equiv c^2/c_S{}^2$, which satisfies

$$\sqrt{\beta^2-\xi}\,F(\xi) = \Delta\xi^2\sqrt{\sigma^2-\xi} \quad \text{with} \quad F(\xi) \equiv 4\sqrt{1-\xi}\,\sqrt{\sigma^2-\xi} - \sigma(2-\xi)^2, \tag{2.7}$$

where $\sigma \equiv c_L/c_S > 1$, $\beta \equiv c_f/c_S$ and $\Delta = \rho_f\beta/(\rho\sigma) = \rho_f c_f/(\rho c_L)$. Moreover, in equation (2.7) the real parts of $\sqrt{1-\xi} = \gamma_1$, $\sqrt{\sigma^2-\xi} = \sigma\gamma_2$ and $\sqrt{\beta^2-\xi} = \beta\gamma_f$ must be chosen as positive.

Real roots ξ, corresponding to the travelling (non-evanescent) Schölte waves, arise if equation (2.7) has a root in $0 < \xi < \min(1, \beta^2)$. However, equation $F(\xi) = 0$ corresponds to the secular equation $4\gamma_1\gamma_2 - (1+\gamma_1{}^2)^2 = 0$ of Rayleigh waves on a traction-free half-space $z < 0$ and so is known to have a single root $\xi = \xi_R \equiv c_R{}^2/c_S{}^2$ such that $0 < \xi_R < 1$. Moreover, it is readily shown that $F(0) = 0$ and $F'(\xi_R) < 0$. Thus, if $\beta^2 > \xi_R$, the function $\sqrt{\beta^2-\xi}\,F(\xi)$ is real and positive for $0 < \xi < \xi_R$ and real and negative for $\xi_R < \xi < \min(1, \beta^2)$. Since $\xi^2\sqrt{\sigma^2-\xi} > 0$ throughout $0 < \xi < \xi_R$, it can be shown that equation (2.7) has at least one root ξ in $(0, \xi_R)$ for all Δ (> 0). Moreover, for small Δ there is a root $\xi \approx \xi_R$. This defines a real speed $c < c_R$ and corresponding real, positive values for γ_1, γ_2 and γ_f.

The above case arises for $c_f > c_R$ so that a Rayleigh wave in $z < 0$ would travel more slowly than sound waves within $z > 0$. Otherwise, if $c_f < c_R$ so that $\beta^2 < \xi_R$, the function $\sqrt{\beta^2-\xi}\,F(\xi)$ is purely imaginary on $\xi \in (\beta^2, \xi_R)$. In general, solutions to equation (2.7) must be sought in $\operatorname{Im}\xi < 0$. For small Δ, with $\varepsilon \equiv \xi_R - \beta^2$ also small, writing $\xi = \xi_R - \varepsilon\bar{\xi}$ allows equation (2.7) to be approximated as

$$K\varepsilon^{3/2}\bar{\xi}\sqrt{\bar{\xi}-1} = -\Delta\xi_R^2\sqrt{\sigma^2-\xi_R}, \quad \text{where} \quad K \equiv -F'(\xi_R) > 0.$$

This implies that $\arg\bar{\xi} + \frac{1}{2}\arg(\bar{\xi}-1) = \pi$, for which a parametric representation of solutions is $\arg\bar{\xi} = q + \frac{1}{2}\pi$ and $\arg(\bar{\xi}-1) = \pi - 2q$, so leading to

$$\bar{\xi} = \frac{2\sin q}{1-4\sin^2 q}\mathrm{i}e^{\mathrm{i}q}$$

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with q related to ε through

$$\frac{2\sin q}{(1-4\sin^2 q)^{3/2}} = \frac{\Delta\xi_R^2\sqrt{\sigma^2-\xi_R}}{\varepsilon^{3/2}K} \qquad \text{for} \quad 0 < q < \frac{1}{6}\pi.$$

Note that as $\Delta/\varepsilon^{3/2}$ becomes small, $\arg\bar{\xi}$ decreases from $\frac{2}{3}\pi$ to $\frac{1}{2}\pi$ so that $\bar{\xi}$ becomes close to the positive imaginary axis, so keeping $\operatorname{Im}\xi < 0$, as required. In fact, for $\Delta \ll 1$ but with c_f significantly smaller than c_R so that $\xi_R - \beta^2 = O(1)$, the approximation to equation (2.7) simplifies considerably to

$$\xi \approx \xi_R - \frac{i\Delta\xi_R{}^2}{K}\sqrt{\frac{\sigma^2-\xi_R}{\xi_R-\beta^2}} \quad \text{or} \quad c^2 \approx c_R{}^2\left[1-\frac{i\Delta c_R{}^2}{Kc_S^2}\sqrt{\frac{c_L{}^2-c_R{}^2}{c_R{}^2-c_f{}^2}}\right].$$

Thus, evanescent Schölte waves with $\operatorname{Re} c > 0$ and $\operatorname{Im} c < 0$ certainly exist, for $c_f < c_R$ with Δ small. The temporal decay is due to radiation of energy into the fluid, which has sound speed below the Rayleigh speed of the solid.

It may be noted that even for the 'free space' case limit $\rho_f = 0$ which defines Rayleigh waves, the secular equation (2.6), or equivalently equation (2.7), possesses complex roots c^2, with correspondingly complex values for γ_1, γ_2 and γ_f. However, in the general case, only those roots with $|\operatorname{Im} c| \ll |\operatorname{Re} c|$ are of practical significance, since other roots correspond to modes that attenuate appreciably on the scale of a wavelength. For this reason, attention should be confined to cases with small, but negative, $\arg c$ and hence small $|\arg\xi|$. Analysis of uniqueness, or otherwise, of admissible solutions in a related problem for viscoelastic surface waves, in which the secular equation has complex coefficients, has been treated by Romeo [17].

The displacements within the generalised travelling wave (2.4) are found from

$$b_+ \;:\; b_- \;:\; a_- = \gamma_2(1-\gamma_1{}^2)k \;:\; -2\gamma_f\gamma_2|k| \;:\; \gamma_f(1+\gamma_1{}^2)k.$$

Choosing as normalisation the condition $\mathbf{e}_3 \cdot \mathbf{V}(0,k) = \frac{1}{2}$, so that $w = \operatorname{Re} A(k)e^{i(kx-|k|ct)}$, then gives

$$2\mathbf{V}(z,k) = \begin{cases} [-i(\operatorname{sgn} k)\mathbf{e}_1/\gamma_f + \mathbf{e}_3]e^{-\gamma_f|k|z} & \text{in } z \geqslant 0 \\ (1-\gamma_1{}^2)^{-1} i(\operatorname{sgn} k)[2\gamma_1 e^{\gamma_1|k|z} - \gamma_2{}^{-1}(1+\gamma_1{}^2)e^{\gamma_2|k|z}]\mathbf{e}_1 & \\ \quad +(1-\gamma_1{}^2)^{-1}[2e^{\gamma_1|k|z} - (1+\gamma_1{}^2)e^{\gamma_2|k|z}]\mathbf{e}_3 & \text{in } z \leqslant 0. \end{cases} \tag{2.8}$$

From (2.4) and (2.8) it is seen that, when c is written as $c = c_+ - ic_-$ with $c_- > 0$, the normal displacement $w(x,t)$ (and indeed each displacement component at each depth) propagates so that

$$\frac{\partial w}{\partial t} + c_+\frac{\partial w}{\partial x}\operatorname{sgn} k = -c_-|k|w.$$

This evolution equation has affinity with those derived recently for surface wave energy by Rousseau and Maugin [18], who interpret surface waves as quasi-particles. However, it applies only to waves with a single wave number. Since the decay in amplitude is

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inversely proportional to wavelength, no similar equation applies to waves propagating in all directions at all wave numbers, as developed in subsequent sections.

3 Superposition as a wave of general form and direction

Since the governing equations and interface conditions are invariant in form under rotations about the Oz axis, a generalisation to three dimensions of the complex-valued displacements (2.4) is given by first defining a (real) *surface wave vector* $\mathbf{k} \equiv k_1\mathbf{e}_1 + k_2\mathbf{e}_2$ and its associated unit vector $\hat{\mathbf{k}}$ as

$$\mathbf{u} = A(\mathbf{k})\mathbf{V}(z,\mathbf{k})\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-c|\mathbf{k}|t)} + A^*(\mathbf{k})\mathbf{V}^*(z,\mathbf{k})\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-c^*|\mathbf{k}|t)} \tag{3.1}$$

for any complex $A(\mathbf{k})$, where * denotes the complex conjugate and

$$\mathbf{V}(z,\mathbf{k}) \equiv \begin{cases} (2\gamma_\mathrm{f})^{-1}(\gamma_\mathrm{f}\mathbf{e}_3 - \mathrm{i}\hat{\mathbf{k}})\,\mathrm{e}^{-\gamma_\mathrm{f}|\mathbf{k}|z} & z \geqslant 0 \\ (1-\gamma_1{}^2)^{-1}(\mathbf{e}_3 + \mathrm{i}\gamma_1\hat{\mathbf{k}})\,\mathrm{e}^{\gamma_1|\mathbf{k}|z} & \\ \quad -[2\gamma_2(1-\gamma_1{}^2)]^{-1}(1+\gamma_1{}^2)(\gamma_2\mathbf{e}_3 + \mathrm{i}\hat{\mathbf{k}})\,\mathrm{e}^{\gamma_2|\mathbf{k}|z} & z \leqslant 0. \end{cases} \tag{3.2}$$

For any c chosen so that $\mathrm{Re}\,c > 0$ and $\mathrm{Im}\,c < 0$, equations (3.1) and (3.2) describe a real-valued, spatially sinusoidal wave travelling in the direction of $\hat{\mathbf{k}}$ and decaying with time. By selecting c as such a root of (2.6) (i.e. with $-\pi/2 < \arg c < 0$), we observe that $-c^*$ is also an allowable complex speed (since $\mathrm{Im} - c^* < 0$), but $\mathrm{Re} - c^* < 0$ so that the corresponding wave travels in the direction of $-\hat{\mathbf{k}}$. As

$$\gamma_1 \equiv \left(1 - c^2/c_\mathrm{S}{}^2\right)^{1/2}, \qquad \gamma_2 \equiv \left(1 - c^2/c_\mathrm{L}{}^2\right)^{1/2} \quad \text{and} \quad \gamma_\mathrm{f} \equiv \left(1 - c^2/c_\mathrm{f}{}^2\right)^{1/2}$$

are defined with $\mathrm{Re}\,\gamma_1 > 0$, $\mathrm{Re}\,\gamma_2 > 0$ and $\mathrm{Re}\,\gamma_\mathrm{f} > 0$, then

$$\gamma_1^* \equiv \left[1 - (-c^*)^2/c_\mathrm{S}{}^2\right]^{1/2}, \qquad \gamma_2^* \equiv \left[1 - (-c^*)^2/c_\mathrm{L}{}^2\right]^{1/2} \quad \text{and} \quad \gamma_\mathrm{f}^* \equiv \left[1 - (-c^*)^2/c_\mathrm{f}{}^2\right]^{1/2},$$

with $\mathrm{Re}\,\gamma_1^* > 0$, $\mathrm{Re}\,\gamma_2^* > 0$ and $\mathrm{Re}\,\gamma_\mathrm{f}^* > 0$.

Also, observe from equation (3.2) that displacements within the complex-valued attenuating wave with wave vector $\mathbf{k}$ but with complex speed $-c^*$ are given by

$$\mathbf{V}^*(z,-\mathbf{k})\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}+c^*|\mathbf{k}|t)}.$$

Hence, by writing the general superposition of attenuating travelling waves in two equivalent forms as

$$\begin{aligned} \mathbf{u} &= \int_{-\infty}^{\infty} \{A(\mathbf{k})\mathbf{V}(z,\mathbf{k})\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-c|\mathbf{k}|t)} + \bar{A}(\mathbf{k})\mathbf{V}^*(z,-\mathbf{k})\ \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}+c^*|\mathbf{k}|t)}\}\,\mathrm{d}k_1\,\mathrm{d}k_2 \\ &= \int_{-\infty}^{\infty} \{A(\mathbf{k})\mathbf{V}(z,\mathbf{k})\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-c|\mathbf{k}|t)} + \bar{A}(-\mathbf{k})\mathbf{V}^*(z,\mathbf{k})\ \mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-c^*|\mathbf{k}|t)}\}\,\mathrm{d}k_1\,\mathrm{d}k_2, \end{aligned} \tag{3.3}$$

we see that $\mathbf{u}$ is a real-valued, generalised, attenuating, interface-bound disturbance, provided that $\bar{A}(\mathbf{k})$ is chosen to satisfy

$$\bar{A}(\mathbf{k}) = A^*(-\mathbf{k}) \qquad (\text{i.e. } \bar{A}(-\mathbf{k}) = A^*(\mathbf{k})).$$

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Moreover, equations (3.2) and (3.3) may be written in terms of $P(\mathbf{k}) \equiv |\mathbf{k}|^{-1}A(\mathbf{k})$ as

$$\mathbf{u} = 2\,\mathrm{Re} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\mathbf{k})|\mathbf{k}|\mathbf{V}(z,\mathbf{k})\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-c|\mathbf{k}|t)}\,\mathrm{d}k_1\,\mathrm{d}k_2, \tag{3.4}$$

together with

$$|\mathbf{k}|\mathbf{V}(z,\mathbf{k}) = \begin{cases} -(2\gamma_{\mathrm{f}})^{-1}(\mathrm{i}\mathbf{k} - \gamma_{\mathrm{f}}|\mathbf{k}|\mathbf{e}_3)\,\mathrm{e}^{-\gamma_{\mathrm{f}}|\mathbf{k}|z}, & z \geqslant 0, \\ \left(1-\gamma_1{}^2\right)^{-1}(\mathrm{i}\gamma_1\mathbf{k} + |\mathbf{k}|\mathbf{e}_3)\;\mathrm{e}^{\gamma_1|\mathbf{k}|z} & \\ \quad -\frac{1+\gamma_1{}^2}{2\gamma_2(1-\gamma_1{}^2)}(\mathrm{i}\mathbf{k} + \gamma_2|\mathbf{k}|\mathbf{e}_3)\,\mathrm{e}^{\gamma_2|\mathbf{k}|z}, & z \leqslant 0. \end{cases} \tag{3.5}$$

Just as the non-evanescent case (c real) allows $\mathbf{u}$ to be written [13] compactly in terms of copies of a single scalar function $\Phi(x,y,Z\,;t)$ (with Z being an abstract variable formally independent from the spatial coordinate z), representations (3.4) and (3.5) allow a representation in terms of a complex-valued potential $\chi(x,y,Z\,;t)$. First, define $\chi(x,y,Z\,;t)$ as

$$\chi(x,y,Z\,;t) \equiv 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\mathbf{k})\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-c|\mathbf{k}|t)}\mathrm{e}^{-|\mathbf{k}|Z}\,\mathrm{d}k_1\,\mathrm{d}k_2, \tag{3.6}$$

which is readily seen to satisfy

$$\chi_{xx} + \chi_{yy} + \chi_{ZZ} = 0, \tag{3.7}$$

so that at *each* instant t the function $\chi(x,y,Z\,;t)$ is a complex-valued harmonic function of x, y and Z. Moreover, its evolution with time t is simply given by

$$\chi_t = \mathrm{i}c\chi_Z. \tag{3.8}$$

Indeed, by defining $\zeta = Z + \mathrm{i}ct$, it is found that

$$\chi(x,y,Z\,;t) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\mathbf{k})\,\mathrm{e}^{(\mathrm{i}\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|\zeta)}\mathrm{d}k_1\,\mathrm{d}k_2 \equiv \Theta(x,y,\zeta), \tag{3.9}$$

so that Θ satisfies

$$\Theta_{xx} + \Theta_{yy} + \Theta_{\zeta\zeta} = 0. \tag{3.10}$$

The connection with $\mathbf{u}$ is seen by observing that, in the fluid-filled region $z > 0$,

$$\begin{aligned} \mathbf{u} &= -\mathrm{Re}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\mathbf{k})\gamma_{\mathrm{f}}{}^{-1}(\mathrm{i}\mathbf{k} - \gamma_{\mathrm{f}}|\mathbf{k}|\mathbf{e}_3)\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-c|\mathbf{k}|t)}\mathrm{e}^{-\gamma_{\mathrm{f}}|\mathbf{k}|z}\mathrm{d}k_1\,\mathrm{d}k_2 \\ &= -\mathrm{Re}\,\gamma_{\mathrm{f}}{}^{-1}\nabla\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\mathbf{k})\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-c|\mathbf{k}|t)}\mathrm{e}^{-\gamma_{\mathrm{f}}|\mathbf{k}|z}\mathrm{d}k_1\,\mathrm{d}k_2 \\ &= -\mathrm{Re}\,\gamma_{\mathrm{f}}{}^{-1}\nabla\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\mathbf{k})\,\mathrm{e}^{[\mathrm{i}\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|(\gamma_{\mathrm{f}}z+\mathrm{i}ct)]}\mathrm{d}k_1\,\mathrm{d}k_2 \\ &= -\mathrm{Re}\,(2\gamma_{\mathrm{f}})^{-1}\nabla\Theta(x,y,\gamma_{\mathrm{f}}z+\mathrm{i}ct). \end{aligned} \tag{3.11}$$

A similar representation exists in the half-space $z < 0$, in the form

$$\mathbf{u} = \operatorname{Re}\left(\frac{\gamma_1}{1-\gamma_1{}^2}\nabla + \frac{\mathbf{e}_3}{\gamma_1}\frac{\partial}{\partial z}\right)\Theta(x, y, \gamma_1|z| + \mathrm{i}ct)$$
$$- \operatorname{Re}\frac{1+\gamma_1{}^2}{2\gamma_2(1-\gamma_1{}^2)}\nabla\Theta(x, y, \gamma_2|z| + \mathrm{i}ct). \tag{3.12}$$

In particular, the normal displacement $\bar{w}(x, y, t)$ at the interface $z = 0$ is given by

$$\bar{w}(x, y, t) \equiv w(x, y, 0; t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{A(\mathbf{k})\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-c|\mathbf{k}|t)} + A^*(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-c^*|\mathbf{k}|t)}\}\,\mathrm{d}k_1\,\mathrm{d}k_2$$
$$= 2\operatorname{Re}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A(\mathbf{k})\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-c|\mathbf{k}|t)}\,\mathrm{d}k_1\,\mathrm{d}k_2$$
$$= -\operatorname{Re}\chi_Z(x, y, 0; t) = -\operatorname{Re}\Theta_\zeta(x, y, \mathrm{i}ct). \tag{3.13}$$

Hence, the general displacement field carried along with the evolving normal displacement $\bar{w}(x, y, t)$ is expressed at typical z through (3.11) and (3.12) in terms of three depth-scaled and time-shifted versions of the complex-valued harmonic function $\chi(x, y, Z; t) = \Theta(x, y, \zeta)$.

Moreover, differentiation of (3.8) shows that

$$\chi_{tt} = -c^2\chi_{ZZ},$$

so that substitution into (3.7) then yields

$$\chi_{tt} = c^2(\chi_{xx} + \chi_{yy}). \tag{3.14}$$

Thus, at *each* Z, the complex potential χ satisfies the *membrane equation with complex speed*. So also does the function $W(x, y, Z; t) \equiv -\chi_Z(x, y, Z; t) = -\Theta_\zeta(x, y, \zeta)$ and, in particular, its boundary value $\bar{W}(x, y, t) \equiv W(x, y, 0; t) = \Theta_\zeta(x, y, \mathrm{i}ct)$. Specifically this gives

$$\bar{W}_{tt} = c^2(\bar{W}_{xx} + \bar{W}_{yy}), \tag{3.15}$$

so that the normal displacement $\bar{w}(x, y, t) = \operatorname{Re}\bar{W}(x, y, t)$ is the real part of a function satisfying the *membrane equation with complex speed* (3.15). It is readily seen that, whenever c is real, equation (3.15) reduces to the membrane equation, and the description reduces to that of non-evanescent Schölte waves, as described in [13].

4 Evolution of the interface data

While $\bar{w}(x, y, t)$ has been shown in Section 3 to evolve as the real part of a solution to (3.15), in order to identify a solution to that equation it is necessary to specify initial conditions for both of the complex-valued functions $\bar{W}$ and $\bar{W}_t$. However, these are not independent, since $W(x, y, Z; t)$ satisfies Laplace's equation

$$W_{xx} + W_{yy} + W_{ZZ} = 0, \tag{4.1}$$

for each t and, moreover, decays as $x^2+y^2+Z^2 \to \infty$ in $Z>0$. It has a Fourier representation

$$W(x,y,Z;t) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\mathbf{k})\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-c|\mathbf{k}|t)}\mathrm{e}^{-|\mathbf{k}|Z}\,\mathrm{d}k_1\,\mathrm{d}k_2,$$

from which it is readily found (*cf.* (3.8)) that

$$W_t = \mathrm{i}cW_Z. \tag{4.2}$$

However, by applying Green's identity to the pair of functions $W(x',y',Z';t)$ and

$$G_+ \equiv \frac{1}{[(x-x')^2+(y-y')^2+(Z-Z')^2]^{1/2}} + \frac{1}{[(x-x')^2+(y-y')^2+(Z+Z')^2]^{1/2}},$$

in the half-space $Z'>0$, excluding a small sphere centered at $(x',y',Z')=(x,y,Z)$, yields a representation for $W(x,y,Z;t)$ in terms of its normal derivative as

$$W(x,y,Z;t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{W_Z(x',y',0;t)}{[(x-x')^2+(y-y')^2+Z^2]^{1/2}}\,\mathrm{d}x'\,\mathrm{d}y'. \tag{4.3}$$

Setting $Z=0$ and using equation (4.2) then shows that $W_t(x,y,0;0)$ defines $W(x,y,0;0)$. More generally, W and W_t are interrelated at each value of t.

Observe that, for all t, displacements are represented, through equations (3.11) and (3.12), in terms of the complex-valued potential $\chi(x,y,Z;t)$. Since, at each time t, χ satisfies equations (3.7) and (3.8) thoughout $Z>0$ and decays as $x^2+y^2+Z^2\to\infty$ in $Z>0$, Green's identity may be applied to $\chi(x',y',Z';t)$ and

$$G_- \equiv \frac{1}{[(x-x')^2+(y-y')^2+(Z-Z')^2]^{1/2}} - \frac{1}{[(x-x')^2+(y-y')^2+(Z+Z')^2]^{1/2}}$$

in the half-space $Z'>0$, again excluding a small sphere centered at $(x',y',Z')=(x,y,Z)$, to yield

$$\chi(x,y,Z;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{Z\chi(x',y',0;t)}{[(x-x')^2+(y-y')^2+Z^2]^{3/2}}\,\mathrm{d}x'\,\mathrm{d}y'. \tag{4.4}$$

A similar equation applies to W in the form

$$W(x,y,Z;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{ZW(x',y',0;t)}{[(x-x')^2+(y-y')^2+Z^2]^{3/2}}\,\mathrm{d}x'\,\mathrm{d}y'. \tag{4.5}$$

Then, by differentiating equations (4.4) and (4.5) and specialising to $Z=0$, it is found that

$$\chi_Z(x,y,0;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\chi(x',y',0;t)}{[(x-x')^2+(y-y')^2]^{3/2}}\,\mathrm{d}x'\,\mathrm{d}y' \equiv \mathsf{L}\,\chi(x,y,0;t),$$
$$W_Z(x,y,0;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{W(x',y',0;t)}{[(x-x')^2+(y-y')^2]^{3/2}}\,\mathrm{d}x'\,\mathrm{d}y' \equiv \mathsf{L}\,\bar{W}. \tag{4.6}$$

Hence, one form of the evolution equation is, from (3.8) and (4.2),

$$\chi_t(x,y,0;t) = \mathrm{i}c\mathsf{L}\,\chi(x,y,0;t) \quad \text{and, similarly,} \qquad \bar{W}_t = \mathrm{i}c\mathsf{L}\,\bar{W}, \tag{4.7}$$

where the integral operator L is the 'Dirichlet-to-Neumann' map for Laplace's equation (3.7) in $Z \geqslant 0$ subject to $|\chi| \to 0$ as $x^2 + y^2 + Z^2 \to \infty$. The inverse M of L is found by using the representation for χ throughout $Z > 0$ (analogous to (4.3))

$$\chi(x, y, Z; t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\chi_Z(x', y', 0; t)}{[(x - x')^2 + (y - y')^2 + Z^2]^{1/2}} \, \mathrm{d}x' \, \mathrm{d}y'.$$

Specialisation to $Z = 0$ then gives the operator M (the 'Neumann-to-Dirichlet' map) as

$$\chi(x, y, 0; t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\chi_Z(x', y', 0; t)}{[(x - x')^2 + (y - y')^2]^{1/2}} \, \mathrm{d}x' \, \mathrm{d}y' \equiv \mathsf{M}\,\chi. \tag{4.8}$$

Note that L and M must satisfy

$$\mathsf{LM} = \mathsf{ML} = \mathsf{I}, \tag{4.9}$$

where I is the identity operator.

Now, splitting $\chi(x, y, 0; t) \equiv \bar{\chi}(x, y, t)$ into its real and imaginary parts as $\bar{\chi}(x, y, t) = \tilde{\chi}(x, y, t) + \mathrm{i}\hat{\chi}(x, y, t)$ gives

$$\tilde{\chi}_t = -c_+ \mathsf{L}\,\hat{\chi} + c_- \mathsf{L}\,\tilde{\chi} \quad \text{and} \quad \hat{\chi}_t = c_+ \mathsf{L}\,\tilde{\chi} + c_- \mathsf{L}\,\hat{\chi}.$$

Upon rearrangement, using the operator M, these show that at *each* instant t the imaginary parts of χ and χ_t may be expressed in terms of the real parts, using the pair of maps L and M, as

$$\hat{\chi}(x, y, t) = (-\mathsf{M}\,\tilde{\chi}_t + c_- \tilde{\chi})/c_+ \quad \text{and} \quad \hat{\chi}_t(x, y, t) = (|c|^2 \mathsf{L}\,\tilde{\chi} - c_- \tilde{\chi}_t)/c_+. \tag{4.10}$$

Equations (4.10) allow *initial conditions* for the membrane equation with complex speed (3.14) (evaluated at $Z = 0$) to be expressed in terms of just *two* real-valued functions $f(x, y) \equiv \tilde{\chi}(x, y, 0)$ and $g(x, y) \equiv \tilde{\chi}_t(x, y, 0)$ as

$$\bar{\chi}(x, y, 0) = \tilde{\chi}(x, y, 0) + \mathrm{i}\hat{\chi}(x, y, 0) = \frac{1}{c_+}[c^* f(x, y) - \mathrm{i}\mathsf{M}\, g(x, y)]; \tag{4.11}$$

$$\bar{\chi}_t(x, y, 0) = \tilde{\chi}_t(x, y, 0) + \mathrm{i}\hat{\chi}_t(x, y, 0) = \frac{1}{c_+}[cg(x, y) + \mathrm{i}|c|^2 \mathsf{L}\, f(x, y)]. \tag{4.12}$$

Alternatively, the initial conditions may be expressed, using $\bar{\chi}_t = \mathrm{i}c\mathsf{L}\bar{\chi}$, in terms of a *single* complex-valued function $F(x, y) \equiv f(x, y) + \mathrm{i}\hat{f}(x, y)$ as

$$\chi(x, y, 0; 0) = F(x, y, \qquad \chi_t(x, y, 0; 0) = \mathrm{i}c\,\mathsf{L}\,F, \tag{4.13}$$

or, equivalently as the *real* conditions

$$\tilde{\chi}(x, y, 0) = f(x, y), \quad \hat{\chi}(x, y, 0) = \hat{f}(x, y),$$
$$\tilde{\chi}_t(x, y, 0) = c_- \mathsf{L} f - c_+ \mathsf{L}\hat{f} \ (= g), \quad \hat{\chi}_t(x, y, 0) = c_+ \mathsf{L} f + c_- \mathsf{L}\hat{f}.$$

It should not be surprising that only *one* complex-valued function $F(x, y)$ (or *two* real-valued functions $f(x, y)$ and $g(x, y)$) may be *independently* specified, since equation (3.8) is a first-order evolution equation, while equation (4.4) determines $\chi(x,y,Z;t)$ at *all* $Z>0$ in terms of its boundary behaviour $\bar{\chi}(x,y,t)=\chi(x,y,0;t)$ at that same instant.

4.1 A first-order, non-local equation

In the non-evanescent case described in [13], where $c_-=0$, expressions (4.10) reduce to $\hat{\chi}=-c^{-1}\mathsf{M}\tilde{\chi}_t$ and $\hat{\chi}_t=c\mathsf{L}\tilde{\chi}$, so explaining why it was unnecessary there to introduce a second function $\hat{w}$ satisfying the membrane equation (with real c) – it is no more than a Dirichlet-to-Neumann map of $\tilde{w}$ and so does not allow additional initial conditions to be specified.

In the evanescent case, solutions to equation (3.15) subject to initial conditions in either of the forms (4.11) or (4.13) should lie in the subspace for which $\bar{W}_t-\mathrm{i}c\mathsf{L}\bar{W}=0$ and, equivalently, $c\bar{W}-\mathrm{i}\mathsf{M}\bar{W}_t=0$ for all subsequent times. This may be checked by defining the quantities $\tilde{s}\equiv\tilde{w}_t$, $\hat{s}\equiv\hat{w}_t$ and $\bar{S}\equiv\tilde{s}+\mathrm{i}\hat{s}$ so that equation (3.15) becomes equivalent to

$$\bar{S}_t=c^2{\nabla_2}^2\bar{W} \qquad \text{with} \quad \bar{W}_t=\bar{S}, \tag{4.14}$$

where ${\nabla_2}^2=\partial^2/\partial x^2+\partial^2/\partial y^2$ is the two-dimensional Laplacian. It is readily checked that

$${\nabla_2}^2(\mathsf{M}\bar{S})=\mathsf{L}\bar{S},$$

so that the use of identities (4.6) and (4.8) then yields

$$\frac{\partial}{\partial t}[\bar{S}-\mathrm{i}c\mathsf{L}\bar{W}]=c^2{\nabla_2}^2\bar{W}-\mathrm{i}c\mathsf{L}\bar{S}=-\mathrm{i}c\left[\mathrm{i}c{\nabla_2}^2\bar{W}+{\nabla_2}^2(\mathsf{M}\bar{S})\right]=-\mathrm{i}c{\nabla_2}^2\{\mathsf{M}(\bar{S}-\mathrm{i}c\mathsf{L}\bar{W})\}=0.$$

Hence, whenever the constraint $\bar{S}=\mathrm{i}c\mathsf{L}\bar{W}$ (or, equivalently, $c\bar{W}=\mathrm{i}\mathsf{M}\bar{S}$) is applied at $t=0$ for all (x,y), it is found that

$$\bar{S}=\mathrm{i}c\mathsf{L}\bar{W} \qquad \text{and} \qquad c\bar{W}=\mathrm{i}\mathsf{M}\bar{S} \qquad \text{for } all\ (x,y,t). \tag{4.15}$$

Thus, the evolution of the interface values $\bar{W}(x,y,t)$ is governed by the single first-order, non-local evolution equation

$$\bar{W}_t=\mathrm{i}c\,\mathsf{L}\bar{W}=\frac{\mathrm{i}c}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{W}(x',y',t)}{[(x-x')^2+(y-y')^2]^{3/2}}\,\mathrm{d}x'\,\mathrm{d}y'. \tag{4.16}$$

However, in a numerical integration scheme, evaluation of an integral at each time step is best avoided by integration of the system (4.14), with accuracy controlled through intermittent projection into the subspaces $\bar{S}=\mathrm{i}c\mathsf{L}\bar{W}$, $c\bar{W}=\mathrm{i}\mathsf{M}\bar{S}$.

4.2 An alternative evolution equation

Recall that $W(x,y,Z;t)=-\chi_Z(x,y,Z;t)$, so that there exist two companion harmonic functions $P(x,y,Z;t)\equiv\chi_x$ and $Q(x,y,Z;t)\equiv\chi_y$ (each being complex-valued), which satisfy (at each and every instant)

$$P_Z=\chi_{xZ}=-W_x, \qquad Q_Z=\chi_{yZ}=-W_y, \qquad P_x+Q_y=-\chi_{ZZ}=W_Z. \tag{4.17}$$

Then, differentiation of equation (4.4) gives

$$\begin{aligned} P(x,y,Z;t) = \chi_x &= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(x-x')W(x',y',0;t)}{[(x-x')^2+(y-y')^2+Z^2]^{3/2}}\,\mathrm{d}x'\,\mathrm{d}y' \equiv \mathsf{M}_1\bar{W}(x,y,t),\\ Q(x,y,Z;t) = \chi_y &= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(y-y')W(x',y',0;t)}{[(x-x')^2+(y-y')^2+Z^2]^{3/2}}\,\mathrm{d}x'\,\mathrm{d}y' \equiv \mathsf{M}_2\bar{W}(x,y,t), \end{aligned} \tag{4.18}$$

which shows how P and Q are related to $\bar{W}(x,y,t)$ through the operators M_1 and M_2. In particular, after setting $Z=0$ to define operators $\bar{\mathsf{M}}_1$ and $\bar{\mathsf{M}}_2$, these specialise to

$$\bar{P}(x,y,t) \equiv P(x,y,0;t) = \bar{\mathsf{M}}_1\bar{W}(x,y,t), \quad \bar{Q}(x,y,t) \equiv Q(x,y,0;t) = \bar{\mathsf{M}}_2\bar{W}(x,y,t), \tag{4.19}$$

while, since an alternative form for (4.3) is obtained through the use of Green's identity as

$$\begin{aligned} \bar{W}(x,y,t) &= -\chi_Z(x,y,0;t)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(x-x')\chi_x(x',y',0;t)+(y-y')\chi_y(x',y',0;t)}{[(x-x')^2+(y-y')^2]^{3/2}}\,\mathrm{d}x'\mathrm{d}y', \end{aligned}$$

then

$$\bar{W}(x,y,t) = -\bar{\mathsf{M}}_1\bar{P}(x,y,t) - \bar{\mathsf{M}}_2\bar{Q}(x,y,t). \tag{4.20}$$

Observe that (4.19) and (4.20) imply the identity relation $\bar{\mathsf{M}}_1^2+\bar{\mathsf{M}}_2^2+\mathsf{I}=0$. Also, use of $W_t = \mathrm{i}cW_Z$ gives the non-local evolution equation for $\bar{W} \equiv \tilde{w}+\mathrm{i}\hat{w}$ as

$$\bar{W}_t = \mathrm{i}c[(\mathsf{M}_1\bar{W})_x + (\mathsf{M}_2\bar{W})_y], \tag{4.21}$$

which is an alternative version of the non-local equation (4.16).

From equations (4.17) and (3.8) it is found that

$$\bar{P}_t = -\mathrm{i}c\bar{W}_x, \quad \bar{Q}_t = -\mathrm{i}c\bar{W}_y, \quad \bar{W}_t = \mathrm{i}c(\bar{P}_x+\bar{Q}_y)$$

and that every solution has the representation

$$\bar{P}(x,y,t) = \bar{\chi}_x(x,y,t), \;\; \bar{Q}(x,y,t) = \bar{\chi}_y(x,y,t). \tag{4.22}$$

Hence, the required solutions evolve according to

$$\bar{W}_t = \mathrm{i}c{\nabla_2}^2\bar{\chi} \qquad \text{together with} \qquad \bar{\chi}_t = -\mathrm{i}c\bar{W} = \mathrm{i}c(\bar{M}_1\bar{\chi}_x + \bar{M}_2\bar{\chi}_y). \tag{4.23}$$

This version shows (like (4.13)) that initial conditions $\bar{\chi}(x,y,0) = F(x,y)$ are sufficient to define the general wave-like solution. Alternatively, for (4.21) initial conditions $\bar{W}(x,y,0) = \tilde{w}(x,y,0)+\mathrm{i}\tilde{w}(x,y,0)$ are sufficient.

The relationship of either of the above initial conditions to general initial conditions in the two media requires further elucidation. Of course, interface waves form only a small part of the excitation due to general initial conditions. Even for surface waves on an isotropic half-space, efficient techniques for projecting this component out of the general excitation are still being sought (see e.g. Touhei [22]).

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5 Some explicit solutions for $W(x, y, Z\,; t)$

Equations (3.11) and (3.12) give expressions for displacements $\mathbf{u}(x, y, z, t)$ in terms of a single complex-valued scalar function $\chi(x, y, Z\,; t) = \Theta(x, y, Z + \mathrm{i}ct)$. Here, some explicit forms for $W(x, y, Z\,; t)$ satisfying equations (4.1) and (4.2) are obtained, each allowing χ to be determined from $\chi_Z = -W(x, y, Z\,; t)$.

5.1 Spatially sinusoidal disturbances

By seeking $W(x, y, Z\,; t)$ in the form $W = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}V(Z, t)$, it is found from $V_t = \mathrm{i}cV_Z$ and $V_{ZZ} = -\nabla_2{}^2 V = \mathbf{k}\cdot\mathbf{k}V$ so that $V \propto \mathrm{e}^{-|\mathbf{k}|(Z+\mathrm{i}ct)}$. Hence, for each $a =$ constant,

$$W = a \exp \mathrm{i}(\mathbf{k}\cdot\mathbf{x} - c_+|\mathbf{k}|t)\, \mathrm{e}^{-|\mathbf{k}|Z}\, \mathrm{e}^{-|\mathbf{k}|c_- t},$$

describes waves travelling at speed c_+, which decay with $c_- t$ as well as with the depth Z.

Clearly, the analogous solution for χ is $\chi = W(x, y, Z\,; t)/|\mathbf{k}|$ and has the same functional form as $W(x, y, Z\,; t)$.

5.2 An omni-directional generalisation

It is readily seen that $W = U(x, y)\, \mathrm{e}^{-|\mathbf{k}|(Z+\mathrm{i}ct)}$ satisfies both (4.1) and (4.2) whenever (complex-valued) U satisfies the two-dimensional Helmholtz (reduced membrane) equation

$$\nabla_2{}^2 U + |\mathbf{k}|^2 U = U_{xx} + U_{yy} + |\mathbf{k}|^2 U = 0. \tag{5.1}$$

Corresponding solutions for χ also satisfy $\nabla_2{}^2\chi + |\mathbf{k}|^2\chi = 0$, while decaying with both Z and t. They describe oscillatory patterns decaying with depth and decaying exponentially with time.

5.3 Time-harmonic leaky waves

Expressions $W = \mathrm{e}^{-\mathrm{i}\omega t}V(x, y, Z)$ satisfy equation (4.2) whenever $V = \mathrm{e}^{-\omega Z/c}U(x, y)$, with U a solution to

$$\nabla_2{}^2 U + \frac{\omega^2}{c^2} U = 0, \tag{5.2}$$

which is the two-dimensional Helmholtz equation with complex parameter.

Special solutions travelling in the direction of $\mathbf{n} = \hat{\mathbf{k}}$ have

$$U \propto \exp \mathrm{i}(\omega/c)\mathbf{n}\cdot\mathbf{x}.$$

The substitutions $c = c_+ - \mathrm{i}c_- \equiv c_+(1 - \mathrm{i}\beta)$ and $\mathbf{k} = \omega c_+\mathbf{n}/|c|^2 = \omega\mathbf{n}/[c_+(1+\beta^2)]$ convert this to

$$U \propto \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{e}^{-\beta\mathbf{k}\cdot\mathbf{x}},$$

so showing that $2\pi\beta$ is the radiative damping factor per wavelength. The corresponding expression for W may be written as

$$W(x, y, Z\,; t) \propto \mathrm{e}^{-|\mathbf{k}|Z}\mathrm{e}^{-\beta\mathbf{k}\cdot\mathbf{x}} \exp \mathrm{i}\{\mathbf{k}\cdot\mathbf{x} - (1+\beta^2)c_+|\mathbf{k}|t - \beta|\mathbf{k}|Z\}. \tag{5.3}$$

Again, it is readily confirmed that solutions for χ may have the same functional form

$$\chi = b\mathrm{e}^{-|\mathbf{k}|Z}\,\mathrm{e}^{-\beta\mathbf{k}\cdot\mathbf{x}}\exp\mathrm{i}\{\mathbf{k}\cdot\mathbf{x}-(1+\beta^2)c_+|\mathbf{k}|t-\beta|\mathbf{k}|Z\} \tag{5.4}$$

for *all* vectors $\mathbf{k} \equiv k_1\mathbf{e}_1 + k_2\mathbf{e}_2$. Using this within equation (3.9) then gives $\Theta(x, y, \zeta) = b\exp[(1+\mathrm{i}\beta)(\mathrm{i}\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|\zeta)]$, so that expression (3.11) gives, within the fluid,

$$\begin{aligned}\mathbf{u} = -\,\mathrm{Re}\frac{b(1+\mathrm{i}\beta)}{2\gamma_\mathrm{f}}[\mathrm{i}\mathbf{k}-\gamma_\mathrm{f}|\mathbf{k}|\mathbf{e}_3]\exp-[\beta\mathbf{k}\cdot\mathbf{x}+({\gamma_\mathrm{f}}^+-\beta{\gamma_\mathrm{f}}^-)|\mathbf{k}|z]\\ \times\exp\mathrm{i}\{\mathbf{k}\cdot\mathbf{x}-\omega t-(\beta{\gamma_\mathrm{f}}^++{\gamma_\mathrm{f}}^-)|\mathbf{k}|z\},\end{aligned} \tag{5.5}$$

where c and γ have been split into their real and imaginary parts as $c = c_+ + \mathrm{i}c_-$ and $\gamma_\mathrm{f} = {\gamma_\mathrm{f}}^+ + \mathrm{i}{\gamma_\mathrm{f}}^-$ and where $\omega = c_+(1+\beta^2)|\mathbf{k}|$. Only for $\beta \ll 1$ do expressions (5.3) and (5.4) describe weakly evanescent (leaky) waves.

Observe that unless $\arg\gamma_\mathrm{f} < \arg c = -\tan^{-1}\beta$ so that $\beta{\gamma_\mathrm{f}}^+ + {\gamma_\mathrm{f}}^- < 0$, the loci of constant phase at each $z > 0$ *precede* those at the interface $z = 0$. This behaviour would not be expected of solutions to hyperbolic problems. It is a manifestation of the dual elliptic–hyperbolic nature of surface- and interface-guided waves.

5.4 A Gaussian beam

In equation (3.10), write $\xi \equiv x + \mathrm{i}\zeta$, $\eta \equiv x - \mathrm{i}\zeta$ and $\Theta(x, y, \zeta) \equiv \Phi(\xi, \eta, y)$, so giving

$$\Phi_{yy} + 4\Phi_{\xi\eta} = 0. \tag{5.6}$$

Then, seek solutions of the form

$$\Phi(\xi, \eta, y) = \mathrm{e}^{\mathrm{i}a\xi}\phi(\eta, y),$$

where a is a complex constant, so that equation (5.6) yields

$$\phi_{yy} + 4\mathrm{i}a\phi_\eta = 0. \tag{5.7}$$

The fundamental similarity solution to (5.7) (*cf.* the heat equation) is

$$\phi(\eta, y) = (\eta+\eta_0)^{-1/2}\,\mathrm{e}^{\mathrm{i}aq}, \quad \text{where} \quad q \equiv \frac{(y+y_0)^2}{\eta+\eta_0}.$$

Thus, for any complex a, a solution to equation (3.9) is

$$\Theta(x, y, \zeta) = \frac{\mathrm{e}^{\mathrm{i}a(x+\mathrm{i}\zeta)}}{(x-\mathrm{i}\zeta+\eta_0)^{1/2}}\exp\mathrm{i}a\frac{(y+y_0)^2}{x-\mathrm{i}\zeta+\eta_0}. \tag{5.8}$$

Here, y_0 and η_0, like a, are arbitrary complex constants. Since

$$x + \mathrm{i}\zeta = x - c_+t + \mathrm{i}(Z + c_-t) \;\text{ and }\; x - \mathrm{i}\zeta + \eta_0 = x + c_+t + \eta_0^+ - \mathrm{i}(Z + c_-t - \eta_0^-),$$

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the choice $y_0 = 0$, with a and η_0 real, gives

$$\chi(x,y,Z,t) = \frac{e^{-a(Z+c_-t)}e^{ia(x-c_+t)}}{[x-c_+t+\eta_0-i(Z+c_-t)]^{1/2}}\exp\frac{-ay^2}{Z+c_-t+i(x+c_+t+\eta_0)}$$
$$\equiv \Theta(x,y,Z+ict), \tag{5.9}$$

from which both $W(x,y,Z,t) = -\chi_Z$ and $\mathbf{u}(x,y,z,t)$ are readily derived through the use of equations (3.11) and (3.12).

Equation (5.9) has the form of a Gaussian beam symmetric in the plane $y = 0$. However, like expressions (72)–(75) of Kiselev [12] for solutions to the wave equation in three dimensions, these are *exact solutions* to the original set of partial differential equations, rather than to a paraxial approximation arising through the use of high-frequency asymptotics. Observe that, associated with the basic phase variable $a(x - c_+t)$, there is time-decay e^{-ac_-t} (*evanescence*) as well as decay with $|z|$. Indeed, within the fluid, where $Z = \gamma_f z$, the first factor is

$$e^{-a(Z+c_-t)} = e^{-a(c_-t+\gamma_f{}^+z)}e^{-ia\gamma_f{}^-z}.$$

Also, the 'beam width' Δ_f described by

$$\Delta_f{}^2 = [(\gamma_f{}^+z+c_-t)^2+(x+c_+t+\gamma_f{}^-z+\eta_0)^2]/a$$

depends not only on the propagation coordinate x but also on time t and depth z. There are also, of course, corrections to the phase due to the arguments of the two denominators in equation (5.8).

5.5 Fundamental 'source' solution

Equation (3.10) has, by analogy with Laplace's equation, a fundamental solution

$$\Theta = H(x,y,\zeta+\zeta_0) \equiv [x^2+y^2+(\zeta+\zeta_0)^2]^{-1/2},$$

which, in $z > 0$ where $\zeta = \gamma_f z + ict$, gives the fluid displacements as

$$\mathbf{u} = \mathrm{Re}\frac{1}{2\gamma_f}\frac{x\mathbf{e}_1+y\mathbf{e}_2+(\gamma_f z+ict+\zeta_0)\mathbf{e}_3}{[x^2+y^2+(\gamma_f z+ict+\zeta_0)^2]^{3/2}}.$$

The choice of complex constant $\zeta_0 = \gamma_f z_0$, with z_0 real, gives (in $z > 0$)

$$\mathbf{u} = \mathrm{Re}\frac{1}{2\gamma_f}\frac{x\mathbf{e}_1+y\mathbf{e}_2+[\gamma_f(z+z_0)+ict]\mathbf{e}_3}{\{x^2+y^2+[\gamma_f(z+z_0)+ict]^2\}^{3/2}}. \tag{5.10}$$

Since the denominator is $[x^2+y^2+\{\gamma_f{}^+(z+z_0)+c_-t+i[c_+t+\gamma_f^-(z+z_0)]\}^2]^{3/2}$, this describes an axisymmetric disturbance first converging upon, and later diverging from, $(x,y,z) = (0,0,-z_0)$. Since this is associated with two similar disturbances in the solid, expression (5.10) appears to have little physical applicability.

6 The plane–strain case

In the two-dimensional (plane–strain) case, evanescent Schölte waves may be regarded as a superposition of right- and left-travelling disturbances. Although each has a compact representation in terms of conjugate harmonic functions, neither may be regarded as a wave of permanent form as in the non-evanescent case. The representation generalising that of Kiselev and Parker [13] and of Chadwick [5] becomes more subtle.

In equations (3.9)–(3.12), taking $\Theta_y \equiv 0$ gives

$$\Theta = \Theta(x,\zeta) = \chi(x,Z\,;t) \quad \text{with } \zeta = Z + \mathrm{i}ct = Z + c_- t + \mathrm{i}c_+ t,$$

where $Z \equiv \gamma z$ and $\gamma \equiv \gamma^+ + \mathrm{i}\gamma^-$ takes, in turn, one of the values γ_f, $-\gamma_1$ and $-\gamma_2$. Thus, equation (3.9) specialises to

$$\chi(x,Z\,;t) = 2\int_{-\infty}^{\infty} P(k)\mathrm{e}^{\mathrm{i}kx-|k|(Z+c_-t+\mathrm{i}c_+t)}\mathrm{d}k = 2\int_{-\infty}^{\infty} P(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-|k|\zeta}\,\mathrm{d}k. \tag{6.1}$$

Similarly, the function $W(x,Z\,;t) \equiv -\chi_Z$ specialises to

$$W = 2\int_{-\infty}^{\infty} A(k)\,\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-|k|\zeta}\,\mathrm{d}k,$$

so that, at $z = 0$,

$$\tilde{w} + \mathrm{i}\hat{w} \equiv \bar{W}(x,t) = W(x,0;t) = 2\int_{-\infty}^{\infty} A(k)\,\mathrm{e}^{\mathrm{i}[kx-|k|(c_+-\mathrm{i}c_-)t]}\mathrm{d}k. \tag{6.2}$$

Now define, in the upper half $Y > 0$ of a real (x, Y) plane, the two complex-valued functions

$$\Upsilon^{\pm}(x,Y) = 2\int_{0}^{\infty} A(\pm k)\,\mathrm{e}^{\pm\mathrm{i}kx}\mathrm{e}^{-kY}\,\mathrm{d}k, \tag{6.3}$$

and also write

$$\Upsilon(x,Y) \equiv \Upsilon^{+}(x,Y) + \Upsilon^{-}(x,Y) = 2\int_{-\infty}^{\infty} A(k)\,\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-|k|Y}\,\mathrm{d}k, \tag{6.4}$$

in which Υ^+ is the contribution due to $k > 0$, while Υ^- is the contribution due to $k < 0$. When the functions $\Upsilon^{\pm}$ are split into their real and imaginary parts $\xi^{\pm}$ and $\eta^{\pm}$ through

$$\begin{aligned} \xi^{\pm}(x,Y) &= \int_{0}^{\infty} [A(\pm k)\mathrm{e}^{\pm\mathrm{i}kx} + A^*(\pm k)\mathrm{e}^{\mp\mathrm{i}kx}]\mathrm{e}^{-kY}\,\mathrm{d}k, \\ \eta^{\pm}(x,Y) &= \int_{0}^{\infty} \mathrm{i}[A^*(\pm k)\mathrm{e}^{\mp\mathrm{i}kx} - A(\pm k)\mathrm{e}^{\pm\mathrm{i}kx}]\mathrm{e}^{-kY}\,\mathrm{d}k, \end{aligned} \tag{6.5}$$

it is readily seen that each of the functions $\xi^{\pm}(x,Y)$ and $\eta^{\pm}(x,Y)$ is harmonic in the upper half $Y > 0$ of the real (x, Y) plane, with

$$\xi^+_x = \eta^+_Y, \quad \xi^+_Y = -\eta^+_x; \qquad \xi^-_x = -\eta^-_Y, \quad \xi^-_Y = \eta^-_x. \tag{6.6}$$

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Indeed, since $\xi^+ + i\eta^+ = \Upsilon^+(x, Y)$ and $\xi^- - i\eta^- = \Upsilon^{-*}(x, Y)$ each are analytic functions of $x + iY$, expressions (6.6) are merely the Cauchy–Riemann equations stating that (ξ^+, η^+) and $(\xi^-, -\eta^-)$ each are pairs of conjugate harmonic functions throughout $Y > 0$. Moreover, the functions decay as $Y \to \infty$.

Now, equation (6.2) shows that $\bar{W}(x,t) = \Upsilon^+(x - c_+t, c_-t) + \Upsilon^-(x + c_+t, c_-t)$, which may be regarded as the superposition of right- and left-travelling disturbances. Moreover, splitting into real and imaginary parts shows that

$$\begin{aligned} \tilde{w}(x,t) &= \xi^+(x - c_+t, c_-t) + \xi^-(x + c_+t, c_-t), \\ \hat{w}(x,t) &= \eta^+(x - c_+t, c_-t) + \eta^-(x + c_+t, c_-t), \end{aligned} \tag{6.7}$$

so that the normal displacement at the interface is given in terms of the two harmonic functions $\xi^\pm(x \mp c_+t, Y)$, which correspond to right- and left-travelling disturbances. The accompanying function $\hat{w}(x,t)$ is given in terms of their harmonic conjugates. However, during propagation the evolution and decay of the surface disturbance are simply given by evaluating the two harmonic functions $\xi^\pm(x \mp c_+t, Y)$ at 'depth' $Y = c_-t$. This result brings into prominence the elliptic nature of the evanescent Schölte waves, since the surface displacement is related to its initial value just through the depth-dependence of the two harmonic functions $\xi^\pm_Y(x, Y)$.

Representation (6.7) suggests the splitting of $\bar{W}$, $\tilde{w}$ and $\hat{w}$ into right- and left-travelling parts, defined by

$$\tilde{w}^\pm(x,t) = \xi^\pm(x \mp c_+t, c_-t), \quad \hat{w}^\pm(x,t) = \eta^\pm(x \mp c_+t, c_-t), \quad \bar{W}^\pm(x,t) = \tilde{w}^\pm(x,t) + i\hat{w}^\pm(x,t).$$

Then, differentiation shows that

$$\bar{W}_t = \bar{W}^+_t + \bar{W}^-_t = -c\bar{W}^+_x + c\bar{W}^-_x,$$

which leads to the two uncoupled pairs of equations

$$\tilde{w}^+_t + c_+\tilde{w}^+_x = -c_-\hat{w}^+_x, \quad \hat{w}^+_t + c_+\hat{w}^+_x = c_-\tilde{w}^+_x$$

and

$$\tilde{w}^-_t - c_+\tilde{w}^-_x = c_-\hat{w}^-_x, \quad \hat{w}^-_t - c_+\hat{w}^-_x = -c_-\tilde{w}^-_x.$$

Each pair may be readily transformed into one of the sets of the Cauchy–Riemann equations in (6.6), so confirming the representations (6.7) for both $\tilde{w}$ and $\hat{w}$. However, two useful observations follow.

Since all the harmonic functions decay as $Y \to +\infty$, their values at $Y = 0$ (and indeed at *each* constant value of Y) are related as the Hilbert transforms, namely

$$\xi^\pm(x, Y) = \pm \mathsf{H}\,\eta^\pm(x, Y), \quad \eta^\pm(x, Y) = \mp \mathsf{H}\,\xi^\pm(x, Y), \tag{6.8}$$

where H is defined through the principal-value integral

$$\mathsf{H}\,\phi(x, Y) \equiv \frac{-1}{\pi} \;⨍_{-\infty}^{\infty} \frac{\phi(\bar{x}, Y)}{x - \bar{x}}\, d\bar{x}. \tag{6.9}$$

Thus, the evolution equations may be replaced by the pair of non-local equations

$$\tilde{w}_t^{\pm} + c_+\tilde{w}_x^{\pm} = \mp(\mathsf{H}\,\tilde{w}^{\pm})_x \tag{6.10}$$

for the left- and right-travelling parts of the normal displacement at the interface.

Observe that, since the equation $\bar{W}_{tt} = c^2\bar{W}_{xx}$ obtained from (3.15) may be factorised formally as

$$\left(\frac{\partial}{\partial x} - c\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + c\frac{\partial}{\partial t}\right)\bar{W} = 0,$$

it is hardly surprising that its general solution $\bar{W}(x,t)$ may be written as

$$\bar{W} = \bar{W}^+(x,t) + \bar{W}^-(x,t) = \Upsilon^+(x - c_+t, c_-t) + \Upsilon^{-*}(x + c_+t, c_-t). \tag{6.11}$$

Moreover, the two functions $\Upsilon^+(x,Y)$ and $\Upsilon^-(x,Y)$ defined by (6.3) are analytic functions of $x+\mathrm{i}Y$ and $x-\mathrm{i}Y$, respectively. Each is determined throughout $Y \geqslant 0$ by the boundary data $\tilde{w}^{\pm}(x,0) = \xi^{\pm}(x,0) = \mathsf{Re}\,\Upsilon^{\pm}(x,0)$. Consequently, the displacements in both $z < 0$ and $z > 0$ may be represented at all later times, using the two analytic functions $\Upsilon^{\pm}(x,Y)$, in terms of the two real functions $\tilde{w}^{\pm}(x,0)$.

In the fluid-filled region ($z > 0$), equations (3.11) give the displacements

$$\mathbf{u} = -\mathsf{Re}\,\frac{1}{\gamma_\mathrm{f}}[\nabla\Upsilon^+(x - c_+t - \gamma_\mathrm{f}^-z, \gamma_\mathrm{f}^+z + c_-t) + \nabla\Upsilon^-(x + c_+t + \gamma_\mathrm{f}^-z, \gamma_\mathrm{f}^+z + c_-t)], \tag{6.12}$$

while, in the elastic medium ($z < 0$), equation (3.12) gives

$$\begin{aligned}\mathbf{u} =\ &\mathsf{Re}\left(\frac{\gamma_1}{1-{\gamma_1}^2}\nabla + \frac{\mathbf{e}_3}{\gamma_1}\frac{\partial}{\partial z}\right)[\Upsilon^+(x - c_+t + \gamma_1^-z, -\gamma_1^+z + c_-t)\\ &+ \Upsilon^-(x + c_+t - \gamma_1^-z, -\gamma_1^+z + c_-t)]\\ &- \mathsf{Re}\frac{1+{\gamma_1}^2}{2\gamma_2(1-{\gamma_1}^2)}\nabla[\Upsilon^+(x - c_+t + \gamma_2^-z, -\gamma_2^+z + c_-t)\\ &+ \Upsilon^-(x + c_+t - \gamma_2^-z, -\gamma_2^+z + c_-t)].\end{aligned} \tag{6.13}$$

Thus, all displacements are represented in terms of three copies of the two pairs ξ_x^+, ξ_Y^+ and ξ_x^-, ξ_Y^- of conjugate harmonic functions. However, at each time t, only the portions $Y \geqslant c_-t$ of the (x,Y) plane are relevant.

In particular, since $\Upsilon_x^{\pm} = \xi_x^{\pm} + \mathrm{i}\eta_x^{\pm} = \mp\mathrm{i}(\xi_Y^{\pm} + \mathrm{i}\eta_Y^{\pm}) = \mp\mathrm{i}\Upsilon_Y^{\pm}$, expression (6.12) yields throughout the fluid-filled region $z > 0$

$$\begin{aligned}w &= -\mathsf{Re}\frac{1}{\gamma_\mathrm{f}}(\gamma_\mathrm{f}^+\Upsilon_Y^+ - \gamma_\mathrm{f}^-\Upsilon_x^+ + \gamma_\mathrm{f}^+\Upsilon_Y^- + \gamma_\mathrm{f}^-\Upsilon_x^-)\\ &= -\xi_Y^+(x - c_+t - \gamma_\mathrm{f}^-z, \gamma_\mathrm{f}^+z + c_-t) - \xi_Y^-(x + c_+t + \gamma_\mathrm{f}^-z, \gamma_\mathrm{f}^+z + c_-t).\end{aligned} \tag{6.14}$$

7 Conclusion

This paper describes how the recently derived representation of omni-directional Rayleigh, Stoneley and Schölte–Gogoladze waves [13] in terms of a solution of the membrane equation and Laplace's equation in a half-space having the solution of the membrane equation as boundary data may itself be generalised to the evanescent case. Taking Schölte waves at the interface between a fluid and an isotropic elastic half-space as an illustrative case, a representation in terms of a solution of the membrane equation with complex speed is obtained. It is demonstrated that though this solution is complex-valued, initial conditions for the imaginary part cannot be specified independently of those for the real part. Furthermore, alternative descriptions in terms of a first-order, non-local evolution equation for a complex-valued potential exist. In the two-dimensional case, for which the membrane equation reduces to the wave equation with complex speed, there also exists a representation in terms of right-and left-travelling disturbances, each described in terms of an analytic function in a half-plane. This, like the earlier representations, is a telling illustration of the fact that surface and interfacial waves have both an elliptic and a hyperbolic nature.

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