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CLASS NUMBERS OF CM ALGEBRAIC TORI, CM ABELIAN VARIETIES AND COMPONENTS OF UNITARY SHIMURA VARIETIES

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Abstract. We give a formula for the class number of an arbitrary complex mutliplication (CM) algebraic torus over $\mathbb{Q}$. This is proved based on results of Ono and Shyr. As applications, we give formulas for numbers of polarized CM abelian varieties, of connected components of unitary Shimura varieties and of certain polarized abelian varieties over finite fields. We also give a second proof of our main result.

§1. Introduction

An algebraic torus T over a number field k is a connected linear algebraic group over k such that $T\otimes_k \bar{k}$ isomorphic to $(\mathbb{G}_{\mathrm{m}})^d\otimes_k \bar{k}$ over the algebraic closure $\bar{k}$ of k for some integer $d\geq 1$. The class number, $h(T)$, of T is by definition, the cardinality of $T(k)\backslash T(\mathbb{A}_{k,f})/U_T$, where $\mathbb{A}_{k,f}$ is the finite adele ring of k and U_T is the maximal open compact subgroup of $T(\mathbb{A}_{k,f})$. As a natural generalization for the class number of a number field, Ono [18, 19] studied the class numbers of algebraic tori. Let K/k be a finite extension and let $R_{K/k}$ denote the Weil restriction of scalars form K to k, then we have the following exact sequence of tori defined over k

$$1\longrightarrow R^{(1)}_{K/k}(\mathbb{G}_{\mathrm{m},K})\longrightarrow R_{K/k}(\mathbb{G}_{\mathrm{m},K})\longrightarrow \mathbb{G}_{\mathrm{m},k}\longrightarrow 1,$$

where $R^{(1)}_{K/k}(\mathbb{G}_{\mathrm{m},K})$ is the kernel of the norm map $N: R_{K/k}(\mathbb{G}_{\mathrm{m},K})\longrightarrow \mathbb{G}_{\mathrm{m},k}$. It is easy to see that $h(R_{K/k}(\mathbb{G}_{\mathrm{m},K}))$ and $h(\mathbb{G}_{\mathrm{m},k})$ coincide with the class numbers h_K and h_k of K and k, respectively. In order to compute the class number $h(R^{(1)}_{K/k}(\mathbb{G}_{\mathrm{m},K}))$, Ono [21] introduced the arithmetic invariant

$$E(K/k):=\frac{h_K}{h_k\cdot h(R^{(1)}_{K/k}(\mathbb{G}_{\mathrm{m},K}))}$$

and expressed it in terms of certain cohomological invariants when K/k is Galois. In [10], Katayama proved a formula for $E(K/k)$ for any finite extension K/k. He also studied its dual arithmetic invariant $E'(K/k)$ and gave a similar formula. The latter gives a formula for the class number of the quotient torus $R_{K/k}(\mathbb{G}_{\mathrm{m},K})/\mathbb{G}_{\mathrm{m},k}$. The class numbers of general tori have been investigated by Shyr [23, Theorem 1], Morishita [16], González-Avilés [7, 8], and Tran [24].

Besides the class number $h(T)$ of an algebraic torus T, another important arithmetic invariant is the Tamagawa number $\tau(T)$. Roughly speaking, the Tamagawa number is the volume of a suitable fundamental domain. More precisely, for any connected semi-simple algebraic group G over k, one associates the group $G(\mathbb{A}_k)$ of adelic points on G, where $\mathbb{A}_k$

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is the adele ring of k. As $G(\mathbb{A}_k)$ is a unimodular locally compact group, it admits a unique Haar measure up to a scalar. Tamagawa defined a canonical Haar measure on $G(\mathbb{A}_k)$ now called the Tamagawa measure. The Tamagawa number $\tau(G)$ is then defined as the volume of the quotient space $G(k)\backslash G(\mathbb{A}_k)$ (or a fundamental domain of it) with respect to the Tamagawa measure. Similar to the case of class numbers, the calculation of the Tamagawa number is usually difficult. A celebrated conjecture of Weil states that any semi-simple simply connected algebraic group has Tamagawa number 1. The Weil conjecture has been proved in many cases by many people (Weil, Ono, Langlands, Lai, and others) and it is finally proved by Kottwitz [11].

For a more general linear algebraic group G, the quotient space $G(k)\backslash G(\mathbb{A}_k)$ may not have finite volume. This occurs precisely when G has nontrivial characters defined over k, that is also the case for tori. For this reason, the necessity of introducing convergence factors in a canonical way leads to the emergence of Artin L-functions. We shall recall the definition of the Tamagawa number $\tau(T)$ for any algebraic torus T. Then the famous analytic class number formula can be reformulated by the statement $\tau(\mathbb{G}_{\mathrm{m},k})=1$.

In this paper, we investigate the class numbers of CM tori. Let $K=\prod_{i=1}^r K_i$ be a CM algebra, where each K_i is a CM field. The subalgebra K^+ of elements in K fixed by the canonical involution is the product $K^+=\prod_{i=1}^r K_i^+$ of the maximal totally real subfield K_i^+ of K_i. Denote N_i as the norm map from K_i to K_i^+ and $N_{K/K^+}=\prod_{i=1}^r N_i : K\to K^+$ the norm map.

Now, we put $T^K=\prod_{i=1}^r T^{K_i}$ with $T^{K_i}=R_{K_i/\mathbb{Q}}(\mathbb{G}_{\mathrm{m},K_i})$, and $T^{K_i^+}=R_{K_i^+/\mathbb{Q}}(\mathbb{G}_{\mathrm{m},K_i})$. We denote

$$h_K=h(K):=\prod_{i=1}^r h(K_i),\quad h_{K^+}=h(K^+):=\prod_{i=1}^r h(K_i^+),\quad Q=Q_K:=\prod_{i=1}^r Q_i,$$

where $Q_i=Q_{K_i}:=[O_{K_i}^\times:\mu_{K_i}O_{K_i^+}^\times]$ is the Hasse unit index of the CM extension K_i/K_i^+ and μ_{K_i} is the torsion subgroup of $O_{K_i}^\times$. One has $h(T^K)=h(K)$ and $h(T^{K^+})=h(K^+)$. It is known that $Q_i\in\{1,2\}$. Finally, we let $t=\sum_{i=1}^r t_i$, where t_i is the number of primes in K_i^+ ramified in K_i. Then we have the following exact sequence of algebraic tori defined over $\mathbb{Q}$

$$0\longrightarrow T_1^K\longrightarrow T^K\xrightarrow{N_{K/K^+}} T^{K^+}\longrightarrow 0, \tag{1.1}$$

where $T_1^K:=\ker(N_{K/K^+})$, which is the product of norm one subtori $T_1^{K_i}:=\{x\in T^{K_i}\mid N_i(x)=1\}$. We regard $\mathbb{G}_{\mathrm{m}}$ as a $\mathbb{Q}$-subtorus of T^{K^+} via the diagonal embedding. Let $T^{K,\mathbb{Q}}$ denote the preimage of $\mathbb{G}_{\mathrm{m}}$ in T^K under the map N_{K/K^+}. We have the second exact sequence of algebraic tori over $\mathbb{Q}$ as follows. Here, for brevity, we write N for N_{K/K^+}.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_1^K & \xrightarrow{\iota} & T^K & \xrightarrow{N} & T^{K^+} & \longrightarrow & 0\\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & T_1^K & \xrightarrow{\iota} & T^{K,\mathbb{Q}} & \xrightarrow{N} & \mathbb{G}_m & \longrightarrow & 0. \end{array} \tag{1.2}$$

The purpose of this paper is concerned with the class number and the Tamagawa number of $T^{K,\mathbb{Q}}$. Let $T(\mathbb{Z}_p)$ denote the unique maximal open compact subgroup of $T(\mathbb{Q}_p)$.

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THEOREM 1.1. *Let $T^{K,\mathbb{Q}}$ denote the preimage of $\mathbb{G}_{\mathrm{m}}$ in T^K under the map N_{K/K^+} as in (1.2).*

(1) *We have*

$$\tau(T^{K,\mathbb{Q}}) = \frac{2^r}{[\mathbb{G}_{\mathrm{m}}(\mathbb{A}) : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{G}_{\mathrm{m}}(\mathbb{Q})]},$$

where r is the number of components of K.

(2) *We have*

$$\begin{aligned} h(T^{K,\mathbb{Q}}) &= \frac{\prod_{p\in S_{K/K^+}} e_{T,p}}{[\mathbb{G}_{\mathrm{m}}(\mathbb{A}) : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{G}_{\mathrm{m}}(\mathbb{Q})]} \cdot h(T_1^K) \\ &= \frac{\prod_{p\in S_{K/K^+}} e_{T,p}}{[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{Q}^\times]} \cdot \frac{h(K)}{h(K^+)} \cdot \frac{1}{2^{t-r}Q}, \end{aligned}$$

where $e_{T,p} := [\mathbb{Z}_p^\times : N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))]$ and S_{K/K^+} is the set of primes p such that there exists a place $v|p$ of K^+ ramified in K.

To make the formulas in Theorem 1.1 more explicit, one needs to calculate the indices $e_{T,p}$ and $[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{Q}^\times]$. We determine the index $e_{T,p}$ for all primes p; the description in the case where $p = 2$ requires local norm residue symbols. For the global index $[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{Q}^\times]$, we could only compute some special cases including the biquadratic fields and therefore obtain a clean formula for these CM fields. For example, if $K = \mathbb{Q}(\sqrt{p}, \sqrt{-1})$ with prime p, then

$$h(T^{K,\mathbb{Q}}) = \begin{cases} 1 & \text{if } p = 2; \\ h(-p) & \text{if } p \equiv 3 (\text{mod } 4); \\ h(-p)/2 & \text{if } p \equiv 1 (\text{mod } 4), \end{cases} \tag{1.3}$$

where $h(-p) := h(\mathbb{Q}(\sqrt{-p}))$. The global index $[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{Q}^\times]$ may serve another invariant which measures the complexity of CM fields and it requires further investigation. Nevertheless, the indices $e_{T,p}$ and $[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{Q}^\times]$ are all powers of 2 (in fact $e_{T,p} \in \{1, 2\}$ if $p \neq 2$). Then from Theorem 1.1, we deduce

$$h(T^{K,\mathbb{Q}}) = \frac{h_K}{h_{K^+}} \frac{2^e}{2^{t-r} \cdot Q},$$

where t and r are as in Theorem 1.1, e is an integer with $0 \le e \le e(K/K^+, \mathbb{Q})$, and $e(K/K^+, \mathbb{Q})$ is the invariant defined in (4.4). In particular, we conclude that $h(T^{K,\mathbb{Q}})$ is equal to h_K/h_{K^+} only up to 2-power.

It is well known that the double coset space $T^{K,\mathbb{Q}}(\mathbb{Q})\backslash T^{K,\mathbb{Q}}(\mathbb{A}_f)/T^{K,\mathbb{Q}}(\widehat{\mathbb{Z}})$ parameterizes CM abelian varieties with additional structures and conditions. Thus, Theorem 1.1 counts such CM abelian varieties and yields a upper bound for CM points of Siegel modular varieties. There are several investigations on CM points in the literature which have interesting applications and we mention a few for the reader's information. Ullmo and Yafaev [25] give a lower bound for Galois orbits of CM points in a Shimura variety. This plays an important role toward the proof of the André-Oort conjecture under the Generalized Riemann hypothesis. Under the same assumption, Daw [4] proves an upper bound of n-torsion of the class group of a CM torus, motivated from a conjecture of Zhang [32].

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On the other hand, one can also express the number of connected components of a complex unitary Shimura variety $\mathrm{Sh}_U(G,X)_{\mathbb{C}}$ as a class number of $T^{K,\mathbb{Q}}$ or T_1^K. Thus, our result also gives an explicit formula for $|\pi_0(\mathrm{Sh}_U(G,X)_{\mathbb{C}})|$. This information is especially useful when the Shimura variety $\mathrm{Sh}_U(G,X)$ (over the reflex field) has good reduction modulo p. Indeed, by the existence of a smooth toroidal compactification due to Lan [13], the geometric special fiber $Sh/\overline{\mathbb{F}}_p$ of $\mathrm{Sh}_U(G,X)$ has the same number of connected components of $\mathrm{Sh}_U(G,X)_{\mathbb{C}}$. In some special cases, one may be able to show that an stratum (e.g., Newton, EO, or leaves) in the special fiber is "as irreducible as possible," namely, the intersection with each connected component of $Sh/\overline{\mathbb{F}}_p$ is irreducible. In that case, the stratum then has the same number of irreducible components as those of $\mathrm{Sh}_U(G,X)_{\mathbb{C}}$.

In [2], Achter studies the geometry of the reduction modulo a prime p of the unitary Shimura variety associated to $GU(1,n-1)$, extending the work of Bültel and Wedhorn [3] (in fact Achter considers one variant of moduli spaces). Though the main result asserts the irreducibility of each nonsupersingular Newton stratum in the special fiber $Sh/\overline{\mathbb{F}}_p$, the proof actually shows the "relative irreducibility." That is, every nonsupersingular Newton stratum $\mathcal{W}$ in each connected component of $Sh/\overline{\mathbb{F}}_p$ is irreducible (and nonempty). Thus, $\mathcal{W}$ has $|\pi_0(Sh/\overline{\mathbb{F}}_p)|$ irreducible components and we give an explicit formula for the number of its irreducible components.

There is also a connection of class numbers of CM tori with the polarized abelian varieties over finite fields. Indeed, the set of polarized abelian varieties within a fixed isogeny class can be decomposed into certain orbits which are the analogue of genera of the lattices in a Hermitian space. When the common endomorphism algebra of these abelian varieties is commutative, each orbit is isomorphic to the double coset space associated to either $T^{K,\mathbb{Q}}$ or T_1^K (see Section 6). Marseglia [15] gives an algorithm to compute isomorphism classes of square-free polarized ordinary abelian varieties defined over a finite field. Achter et al. [1] also study principally polarized ordinary abelian varieties within an isogeny class over a finite field from a different approach. They utilize the Langlands–Kottwitz counting method and express the number of abelian varieties in terms of discriminants and a product of certain local density factors, reminiscent of the Smith–Minkowski Siegel formula (cf. [6, Section 10]).

This paper is organized as follows. Section 2 recalls the definition of the Tamagawa number of an algebraic torus. The proof of Theorem 1.1 is given in Section 3. In Section 4, we compute the local and global indices appearing in Theorem 1.1 and give an improvement and a second proof. We calculate the class number of the CM torus associated to any biquadratic CM field in Section 5. In the last section, we discuss applications of Theorem 1.1 to polarized CM abelian varieties, connected components of unitary Shimura varieties, and polarized abelian varieties with commutative endomorphism algebras over finite fields.

§2. Tamagawa numbers of algebraic tori

Following [18], we recall the definition of Tamagawa number of an algebraic torus T over a number field k. Fix the natural Haar measure dx_v on k_v for each place v such that it has measure 1 on the ring of integers $\mathfrak{o}_v$ in the nonarchimedean case, measure 1 on $\mathbb{R}/\mathbb{Z}$ in the real place case, and measure 2 on $\mathbb{C}/\mathbb{Z}[i]$ in the complex place case. Let ω be a nonzero invariant differential form of T of highest degree defined over k. To each place v,

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one associates a Haar measure ω_v on $T(k_v)$. We say that the product of the Haar measures

$$\omega_{\mathbb{A}} = \prod_v \omega_v \tag{2.1}$$

converges absolutely if the product

$$\prod_{v \nmid \infty} \omega_v(T(\mathfrak{o}_v))$$

converges absolutely, where $T(\mathfrak{o}_v) \subset T(k_v)$ is the maximal open compact subgroup. In this case, one defines a Haar measure $\omega_{\mathbb{A}}$ on the locally compact topological group $T(\mathbb{A}_k)$. Since the space of invariant differential forms is a one-dimensional k-vector space, by the product formula, the Haar measure $\omega_{\mathbb{A}}$ does not depend on the choice of ω, which is called the canonical measure.

However, the measure (2.1) does not converge if T admits a nontrivial rational character. Thus, we must modify the local measures by suitable convergence factors λ_v for each v so that the product $\prod_v (\lambda_v \cdot \omega_v)$ is absolutely convergent on $T(\mathbb{A}_k)$. Such a collection $\lambda = \{\lambda_v\}$ is called a set of convergence factors for ω; the resulting measure is denoted by $\omega_{\mathbb{A},\lambda}$.

Suppose T splits over a Galois extension K/k with Galois group $\mathfrak{g}$. The group $\widehat{T} := \mathrm{Hom}_K(T, \mathbb{G}_{\mathrm{m}})$ of characters is a finite free $\mathbb{Z}$-module with a continuous action of $\mathfrak{g}$. Let $\chi_T : \mathfrak{g} \to \mathbb{C}$ be the character associated to the representation $\widehat{T} \otimes \mathbb{Q}$ of $\mathfrak{g}$.

Let χ_i, $1 \le i \le h$, be all the irreducible characters of $\mathfrak{g}$ and we denote χ_1 as the trivial character. Express $\chi_T = \sum_{i=1}^h a_i \chi_i$ as the sum of irreducible characters χ_i with non-negative integral coefficients. Note that a_1 is the rank of the group $(\widehat{T})^{\mathfrak{g}}$ of rational characters. The Artin L-function of χ_T with respect to the field extension K/k is equal to

$$L(s, \chi_T, K/k) = \zeta_k(s)^{a_1} \prod_{i=2}^h L(s, \chi_i, K/k)^{a_i}.$$

We define the number $\rho(T)$ to be the nonzero number $\lim_{s \to 1}(s-1)^{a_1} L(s, \chi_T, K/k)$, that is,

$$\rho(T) = (\mathrm{Res}_{s=1}\, \zeta_k(s))^{a_1} \prod_{i=2}^h L(1, \chi_i, K/k)^{a_i}.$$

On the other hand, note that there exists a finite set S of places of k such that $T \otimes k_v$ admits a smooth model over $\mathfrak{o}_v$ for each finite place v outside S. For such v, the reduction map $T(\mathfrak{o}_v) \to T(k(v))$ is surjective, where $k(v) \simeq \mathbb{F}_{q_v}$ is the residue field of $\mathfrak{o}_v$. Let $T^{(1)}(\mathfrak{o}_v)$ be the kernel of the reduction map. By [27, Theorem 2.2.5] and [18, (3.3.2)], we have

$$\begin{aligned} \int_{T(\mathfrak{o}_v)} \omega_v &= |T(k(v))| \times \int_{T^{(1)}(\mathfrak{o}_v)} \omega_v \\ &= \frac{|T(k(v))|}{q_v^d} = L_v(1, \chi_T, K/k)^{-1}, \end{aligned}$$

where d is the dimension of T and $L_v(s, \chi_T, K/k)$ is the local factor of the Artin L-function at v. We now choose the set of convergence factors $\{\lambda_v\}$ such that λ_v is equal to 1 if v is archimedean and is equal to $L_v(1, \chi_T, K/k)$ otherwise, and hence define a measure $\omega_{\mathbb{A},\lambda}$ on $T(\mathbb{A}_k)$.

Let $\xi_i, i=1,\ldots,a_1$, be a basis of $(\widehat{T})^{\mathfrak{g}}$. Define

$$\xi : T(\mathbb{A}_k) \to \mathbb{R}_+^{a_1}, \quad x \mapsto (||\xi_1(x)||, \ldots, ||\xi_{a_1}(x)||),$$

where $\mathbb{R}_+ := \{x > 0 \in \mathbb{R}\}$. Let $T(\mathbb{A}_k)^1$ denote the kernel of ξ; one has an isomorphism $T(\mathbb{A}_k)/T(\mathbb{A}_k)^1 \simeq \mathbb{R}_+^{a_1} \subset (\mathbb{R}^\times)^{a_1}$. Let $d^\times t := \prod_{i=1}^{a_1} dt_i/t_i$ be the canonical measure on $\mathbb{R}_+^{a_1}$. Let $\omega^1_{\mathbb{A},\lambda}$ be the unique Haar measure on $T(\mathbb{A}_k)^1$ such that $\omega_{\mathbb{A},\lambda} = \omega^1_{\mathbb{A},\lambda} \cdot d^\times t$, that is, for any measurable function F on $T(\mathbb{A}_k)$ one has

$$\int_{T(\mathbb{A}_k)/T(\mathbb{A}_k)^1} \int_{T(\mathbb{A}_k)^1} F(xt)\, \omega^1_{\mathbb{A},\lambda} \cdot d^\times t = \int_{T(\mathbb{A}_k)} F(x)\, \omega_{\mathbb{A},\lambda}.$$

By a well-known theorem of Borel and Harish-Chandra [22, Theorem 5.6], the quotient space $T(\mathbb{A}_k)^1/T(k)$ has finite volume with respect to a Haar measure. The Tamagawa number of T is then defined by

$$\tau(T) := \frac{|d_k|^{-\frac{\dim T}{2}} \cdot \int_{T(\mathbb{A}_k)^1/T(k)} \omega^1_{\mathbb{A},\lambda}}{\rho(T)}, \tag{2.2}$$

where d_k is the discriminant of the field k.

§3. Proof of Theorem 1.1

3.1 *q*-Symbols and relative class numbers

Suppose $\alpha : G \to G'$ is a homomorphism of abelian groups such that $\ker \alpha$ and coker α are finite. Following Tate, the *q-symbol of* α is defined by

$$q(\alpha) := |\text{coker } \alpha| / |\ker \alpha|.$$

It is easy to see whenever both G and G' are finite, one has $q(\alpha) = |G'|/|G|$. Let $\Gamma = \Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denote the Galois group of $\mathbb{Q}$. For any isogeny $\lambda : T \to T'$ of algebraic tori defined over $\mathbb{Q}$, we have the following induced maps:

$$\begin{aligned}
&\hat{\lambda} : \hat{T}' \to \hat{T}, \quad \hat{\lambda}^\Gamma : (\hat{T}')^\Gamma \to (\hat{T})^\Gamma,\\
&\lambda_p^c = \lambda_{\mathbb{Z}_p} : T(\mathbb{Z}_p) \to T'(\mathbb{Z}_p),\\
&\lambda_\infty : T(\mathbb{R}) \to T'(\mathbb{R}),\\
&\lambda_{\mathbb{Z}} : T(\mathbb{Z}) \to T'(\mathbb{Z}).
\end{aligned}$$

Thus, we have the corresponding q-symbols. Note that $T(\mathbb{Z}) = T(\mathbb{Q}) \cap [T(\mathbb{R}) \times \prod_{p<\infty} T(\mathbb{Z}_p)]$.

Shyr [23] showed that these q-symbols play a role in the connection between the ratios of Tamagawa numbers and class numbers of T and T' as follows.

THEOREM 3.1. *Let $\lambda : T \to T'$ be an isogeny of algebraic tori defined over $\mathbb{Q}$. Then*

$$\frac{h(T)}{h(T')} = \frac{\tau(T)}{\tau(T')} \cdot \frac{q(\lambda_\infty)}{q(\lambda_{\mathbb{Z}})q(\hat{\lambda}^\Gamma)} \cdot \prod_{p<\infty} q(\lambda_p^c).$$

Proof. See [23, Theorem 2]. ∎

For any exact sequence

$$(E) \qquad 0 \longrightarrow T' \xrightarrow{\ \iota\ } T \xrightarrow{\ N\ } T'' \longrightarrow 0$$

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of algebraic tori defined over $\mathbb{Q}$, we associate a number to the exact sequence (E) by [19, Section 4]

$$\tau(E) := \tau(T'') \cdot \tau(T')/\tau(T). \tag{3.1}$$

THEOREM 3.2. *Let* $\mu : T''(\mathbb{Q})/N(T(\mathbb{Q})) \to T''(\mathbb{A})/N(T(\mathbb{A}))$ *and* $\hat{\iota}^\Gamma : \hat{T}^\Gamma \to \widehat{T'}^\Gamma$ *be the maps derived from the exact sequence* (E). *Then the subgroups* $\operatorname{coker}\mu$ *and* $\ker\mu$ *are finite, and we have*

$$\tau(E) = q(\hat{\iota}^\Gamma)q(\mu) = |\operatorname{coker}\ \hat{\iota}^\Gamma| \cdot \frac{[T''(\mathbb{A}) : N(T(\mathbb{A})) \cdot T''(\mathbb{Q})]}{[N(T(\mathbb{A})) \cap T''(\mathbb{Q}) : N(T(\mathbb{Q}))]}.$$

Proof. See [19, Section 4.3 and Theorem 4.2.1]. ∎

Now we let

$$T = T^{K,\mathbb{Q}}, \quad T' = T_1^K, \quad \text{and} \quad T'' = \mathbb{G}_{\mathrm{m}}.$$

Since $x^2N(x)^{-1}$ is of norm 1, the map λ defined by

$$\lambda : T \to T' \times T'', \quad x \mapsto (x^2N(x)^{-1}, N(x)) \tag{3.2}$$

is an isogeny. Applying Theorem 3.1 to this λ and Theorem 3.2 to the exact sequence (1.2) together, we have

$$\begin{aligned} \frac{h(T)}{h(T' \times T'')} &= \tau(E)^{-1} \cdot \frac{q(\lambda_\infty)}{q(\lambda_\mathbb{Z})q(\hat{\lambda}^\Gamma)} \prod_p q(\lambda_p^c) \\ &= |\operatorname{coker}\ \hat{\iota}^\Gamma|^{-1} \cdot \left(\frac{[T''(\mathbb{A}) : N(T(\mathbb{A})) \cdot T''(\mathbb{Q})]}{[N(T(\mathbb{A})) \cap T''(\mathbb{Q}) : N(T(\mathbb{Q}))]} \right)^{-1} \\ &\quad \cdot \frac{q(\lambda_\infty)}{q(\lambda_\mathbb{Z})q(\hat{\lambda}^\Gamma)} \cdot \prod_{p<\infty} q(\lambda_p^c). \end{aligned} \tag{3.3}$$

As $h(\mathbb{G}_{\mathrm{m}}) = 1$, we obtain

$$\begin{aligned} \frac{h(T^{K,\mathbb{Q}})}{h(T_1^K)} &= |\operatorname{coker}\ \hat{\iota}^\Gamma|^{-1} \cdot \frac{[N(T^{K,\mathbb{Q}}(\mathbb{A})) \cap \mathbb{G}_{\mathrm{m}}(\mathbb{Q}) : N(T^{K,\mathbb{Q}}(\mathbb{Q}))]}{[\mathbb{G}_{\mathrm{m}}(\mathbb{A}) : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{G}_{\mathrm{m}}(\mathbb{Q})]} \\ &\quad \cdot \frac{q(\lambda_\infty)}{q(\lambda_\mathbb{Z})q(\hat{\lambda}^\Gamma)} \cdot \prod_{p<\infty} q(\lambda_p^c). \end{aligned} \tag{3.4}$$

We shall determine each term in (3.4).

3.2 Calculation of cokernel

LEMMA 3.3. *The cardinality of* $\operatorname{coker}\hat{\iota}^\Gamma$ *is* 1.

Proof. Taking the character groups of (1.2), we have

$$0 \longrightarrow (\widehat{\mathbb{G}_{\mathrm{m}}})^\Gamma \xrightarrow{\hat{N}^\Gamma} (\widehat{T^{K,\mathbb{Q}}})^\Gamma \xrightarrow{\hat{\iota}^\Gamma} (\widehat{T_1^K})^\Gamma \longrightarrow \operatorname{coker}\hat{\iota}^\Gamma \longrightarrow 0. \tag{3.5}$$

Thus, it suffices to show $(\widehat{T_1^K})^\Gamma = 0$. Again from (1.2), we have

$$(*) \quad 0 \longrightarrow (\widehat{T^{K^+}})^\Gamma \xrightarrow{\hat{N}} (\widehat{T^K})^\Gamma \longrightarrow (\widehat{T_1^K})^\Gamma.$$

Recall that $T^K = \prod_{i=1}^r T^{K_i}$ with $T^{K_i} = R_{K_i/\mathbb{Q}}\mathbb{G}_{m,K_i}$. Let $\Gamma_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Also, note that

$$\widehat{T^{K_i}} = \mathbb{Z}[\Gamma_{\mathbb{Q}}/\Gamma_{K_i}], \qquad \widehat{T^{K_i^+}} = \mathbb{Z}[\Gamma_{\mathbb{Q}}/\Gamma_{K_i^+}],$$
$$(\widehat{T^{K_i}})^\Gamma = \mathbb{Z}[\sum_{\sigma\in\Gamma_{\mathbb{Q}}/\Gamma_{K_i}} \sigma], \qquad (\widehat{T^{K_i^+}})^\Gamma = \mathbb{Z}[\sum_{\sigma\in\Gamma_{\mathbb{Q}}/\Gamma_{K_i^+}} \sigma].$$

The norm map N_i sends x to $x\bar{x}$, where $x \in K_i$ and $\bar{x}$ is the complex conjugate of x. Therefore, $\widehat{N_i}(\chi_i) = \chi_i + \bar{\chi}_i$ for $\chi_i \in \widehat{T^{K_i^+}}$. This shows $(\widehat{T^{K_i^+}})^\Gamma \xrightarrow{\sim} (\widehat{T^{K_i}})^\Gamma$. Note that the left exact sequence $(*) \otimes_{\mathbb{Z}}\mathbb{Q}$ is also right exact. Thus, $(\widehat{T_1^{K_i}})^\Gamma \otimes \mathbb{Q} = 0$ and $(\widehat{T_1^{K_i}})^\Gamma$ is a torsion $\mathbb{Z}$-module. It follows that $(\widehat{T_1^{K_i}})^\Gamma = 0$, because it is a submodule of a finite free $\mathbb{Z}$-module $(\widehat{T_1^{K_i}})$. It follows that $|\mathrm{coker}\,\hat{\iota}^\Gamma| = 1$. ∎

We remark that Lemma 3.3 also follows from $H^1(\Gamma, \widehat{\mathbb{G}}_m) = \mathrm{Hom}(\Gamma, \mathbb{Z}) = 1$. The proof will be also used in Lemma 3.8.

3.3 Calculation of indices of rational points

Recall that for any commutative $\mathbb{Q}$-algebra R, the groups of R-points of T^K and $T^{K,\mathbb{Q}}$ are

$$T^K(R) = (K\otimes R)^\times \quad \text{and} \quad T^{K,\mathbb{Q}}(R) = \left\{a \in (K\otimes R)^\times : N(a) \in R^\times\right\},$$

respectively. For $v \in V_K$, the union of the sets V_{K_i} of places of K_i^+ for $1 \le i \le r$, we put $K_v = (K_i)_v$ if $v \in V_{K_i}$. For any prime p, let S_p be the set of places of K^+ lying over p. We have

$$T^{K,\mathbb{Q}}(\mathbb{Q}_p) = \{((x_v)_v, x_p) \in \prod_{v\in S_p} K_v^\times \times \mathbb{Q}_p^\times \mid N(x_v) = x_p \ \forall v\}, \tag{3.6}$$

and

$$T^{K,\mathbb{Q}}(\mathbb{Z}_p) = \{((x_v)_v, x_p) \in \prod_{v\in S_p} O_{K_v}^\times \times \mathbb{Z}_p^\times \mid N(x_v) = x_p \ \forall v\}. \tag{3.7}$$

Note that x_p is uniquely determined by $(x_v)_v$ and we may also represent an element x in $T^{K,\mathbb{Q}}(\mathbb{Q}_p)$ by $(x_v)_v$.

LEMMA 3.4. *We have $N(T^{K,\mathbb{Q}}(\mathbb{Q}_p)) \cap \mathbb{Z}_p^\times = N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))$ for every prime number p.*

Proof. Clearly, $N(T^{K,\mathbb{Q}}(\mathbb{Q}_p)) \cap \mathbb{Z}_p^\times \supset N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))$. We must prove the other inclusion $N(T^{K,\mathbb{Q}}(\mathbb{Q}_p)) \cap \mathbb{Z}_p^\times \subset N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))$. Suppose $x = (x_v) \in T^{K,\mathbb{Q}}(\mathbb{Q}_p) \cap N^{-1}(\mathbb{Z}_p^\times)$. We will find an element $x' = (x'_v) \in T^{K,\mathbb{Q}}(\mathbb{Z}_p)$ such that $N(x) = N(x')$.

Suppose v is inert or ramified in K and let w be the unique place of K over v. Then $x_v \in K_v = K_w$ and $N(x_v) \in \mathbb{Z}_p^\times$. We have

$$\mathrm{ord}_w N(x_v) = \mathrm{ord}_w(x_v) + \mathrm{ord}_w \bar{x}_v = 2\,\mathrm{ord}_w x_v = 0$$

and hence $x_v \in O_{K_w}^\times$.

Suppose $v = w\bar{w}$ splits in K. Then $x_v = (x_w, x_{\bar{w}}) \in K_v = K_w \times K_{\bar{w}} = K_v^+ \times K_v^+$ and $N(x_v) = x_w x_{\bar{w}} \in \mathbb{Z}_p^\times$. We have $\mathrm{ord}_v x_w = -\mathrm{ord}_v x_{\bar{w}} = a_v$ for some $a_v \in \mathbb{Z}$. Put $x'_v := (\varpi_v^{-a_v} x_w, \varpi_v^{a_v} x_{\bar{w}})$, where ϖ_v is a uniformizer of K_v^+. Clearly, $x'_v \in O_{K_v}^\times$ and $N(x_v) = N(x'_v)$.

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Now suppose $y \in N(T^{K,\mathbb{Q}}(\mathbb{Q}_p)) \cap \mathbb{Z}_p^\times$ and $N(x) = y$ for some $x \in T^{K,\mathbb{Q}}(\mathbb{Q}_p)$. Set $x' := (x'_v)$ with $x'_v = x_v$ if v is inert or ramified in K, and x'_v as above if v splits in K. Then $y = N(x) = N(x') \in N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))$. This proves $N(T^{K,\mathbb{Q}}(\mathbb{Q}_p)) \cap \mathbb{Z}_p^\times \subset N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))$. ∎

Lemma 3.5. *We have* $[N(T^{K,\mathbb{Q}}(\mathbb{A})) \cap \mathbb{G}_{\mathrm{m}}(\mathbb{Q}) : N(T^{K,\mathbb{Q}}(\mathbb{Q}))] = 1$ *and*

$$\tau(E) = [\mathbb{G}_{\mathrm{m}}(\mathbb{A}) : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{G}_{\mathrm{m}}(\mathbb{Q})].$$

Proof. Since $T^{K,\mathbb{Q}}(\mathbb{A}) = \{x \in \mathbb{A}_K^\times \mid N(x) \in \mathbb{A}^\times\}$, we have $N(T^{K,\mathbb{Q}}(\mathbb{A})) = N(\mathbb{A}_K^\times) \cap \mathbb{A}^\times$. Applying the norm theorem [19, Theorem 6.1.1], we have $N(\mathbb{A}_K^\times) \cap (K^+)^\times = N(K^\times)$. Hence

$$N(T^{K,\mathbb{Q}}(\mathbb{A})) \cap \mathbb{Q}^\times = N(\mathbb{A}_K^\times) \cap \mathbb{Q}^\times = N(K^\times) \cap \mathbb{Q}^\times = N(T^{K,\mathbb{Q}}(\mathbb{Q})).$$

This proves the first statement. The second statement then follows from Lemma 3.3 and Theorem 3.2. ∎

3.4 Calculation of q-symbols

We are going to evaluate each q-symbol in (3.4). Recall the isogeny in (3.2), namely

$$\lambda : T^{K,\mathbb{Q}} \to T_1^K \times \mathbb{G}_{\mathrm{m}}, \ x \mapsto (x^2 N(x)^{-1}, N(x)). \tag{3.8}$$

Note that $\ker \lambda = \{x \in T^{K,\mathbb{Q}} \mid N(x) = 1, \ x^2 = 1\} = \{x \in T_1^K \mid x^2 = 1\}$. Hence, we have $\ker \lambda = \ker \mathrm{Sq}_{T_1^K}$, where $\mathrm{Sq}_{T_1^K} : T_1^K \to T_1^K, \ x \mapsto x^2$ is the squared map.

Lemma 3.6. *Suppose* $d = [K^+ : \mathbb{Q}]$. *The q-symbol of* λ_∞ *is equal to* 2^{-d+1}.

Proof. Since $K = \prod_{i=1}^r K_i$ is a CM algebra, we have $T^K(\mathbb{R}) = (K \otimes_{\mathbb{Q}} \mathbb{R})^\times = (\mathbb{C}^d)^\times$. According to (3.8), we have

$$\begin{aligned} T^{K,\mathbb{Q}}(\mathbb{R}) &= \{(x_i) \in (\mathbb{C}^\times)^d \mid N(x_i) = N(x_j) \in \mathbb{R}, \ \forall i, \ j\}, \\ T_1^K(\mathbb{R}) &= \{(x_i) \in (\mathbb{C}^\times)^d \mid x_i \bar{x}_i = 1 \text{ for all } i\} = (S^1)^d, \\ \ker \lambda_\infty &= \ker \mathrm{Sq}_{T_1^K(\mathbb{R})} = \{\pm 1\}^d, \end{aligned}$$

and hence the exact sequence

$$0 \longrightarrow \{\pm 1\}^d \longrightarrow T^{K,\mathbb{Q}}(\mathbb{R}) \xrightarrow{\lambda_\infty} (S^1)^d \times \mathbb{R}^\times.$$

Since $N(\mathbb{C}^\times) = \mathbb{R}_+^\times$ is connected, the image of λ_∞ is $(S^1)^d \times \mathbb{R}_+^\times$ and $|\mathrm{coker}\, \lambda_\infty| = |\mathbb{R}^\times / \mathbb{R}_+^\times| = 2$. Therefore, $q(\lambda_\infty) = 2/|\{\pm 1\}^d| = 2^{-d+1}$. ∎

Lemma 3.7. *The q-symbol of* $\lambda_{\mathbb{Z}}$ *is equal to* 2.

Proof. It is clear that $N(x) = 1$ for $x \in T^{K,\mathbb{Q}}(\mathbb{Z})$. Note that any element $x \in O_K^\times$ with $x\bar{x} = 1$ is a root of unity. Then

$$T^{K,\mathbb{Q}}(\mathbb{Z}) = T_1^K(\mathbb{Z}) = \prod_{i=1}^r \mu_{K_i} =: \mu_K,$$

where μ_{K_i} is the group of roots of unity in K_i. Since $T^{K,\mathbb{Q}}(\mathbb{Z})$ and $(T_1^K \times \mathbb{G}_{\mathrm{m}})(\mathbb{Z})$ are finite, we have

$$q(\lambda_{\mathbb{Z}}) = \frac{|(T_1^K \times \mathbb{G}_m)(\mathbb{Z})|}{|T^{K,\mathbb{Q}}(\mathbb{Z})|} = \frac{|\mu_K \times \{\pm 1\}|}{|\mu_K|} = 2.$$ ∎

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LEMMA 3.8. *The q-symbol of* $\hat{\lambda}^\Gamma$ *is equal to* 1.

Proof. In the proof of Lemma 3.3, we have showed that $(\widehat{T_1^K})^\Gamma = 0$. Therefore, the map $\hat{\lambda}: (\widehat{\mathbb{G}_\mathrm{m}})^\Gamma \times (\widehat{T_1^K})^\Gamma \to (\widehat{T^{K,\mathbb{Q}}})^\Gamma$ is just given by $\hat{N}: (\widehat{\mathbb{G}_\mathrm{m}})^\Gamma \to (\widehat{T^{K,\mathbb{Q}}})^\Gamma$. The map $\hat{N}$ is in fact an isomorphism from (3.5). Therefore, $q(\hat{\lambda}^\Gamma) = 1$. ∎

LEMMA 3.9. *Let* $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ *be group homomorphisms of abelian groups with finite* $\ker \beta\alpha$ *and* $\operatorname{coker} \beta\alpha$. *Then the cardinality of* $\operatorname{coker}\beta$ *is equal to*

$$\begin{aligned}[C : \mathrm{im}\beta] &= [C : \mathrm{im}\beta\alpha][\mathrm{im}\beta : \mathrm{im}\beta\alpha]^{-1} \\ &= [C : \mathrm{im}\beta\alpha][B : \mathrm{im}\alpha]^{-1} \cdot [\ker\beta : \ker\beta\alpha/\ker\alpha]. \end{aligned} \tag{3.9}$$

Proof. The first equality is obvious and the second equality follows by applying the snake lemma to the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\beta\alpha/\ker\alpha & \longrightarrow & A/\ker\alpha & \longrightarrow & A/\ker\beta\alpha & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker\beta & \longrightarrow & B & \longrightarrow & B/\ker\beta & \longrightarrow & 0. \end{array}$$

∎

LEMMA 3.10. *We have*

$$q(\lambda_p^c) = \begin{cases} e_{T,p} & \text{if } p \neq 2; \\ e_{T,2} \cdot 2^d & \text{if } p = 2, \end{cases}$$

where $e_{T,p} = [\mathbb{Z}_p^\times : N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))]$ *and* $d = [K^+ : \mathbb{Q}]$.

Proof. By (3.7), every element $x \in T^{K,\mathbb{Q}}(\mathbb{Z}_p)$ is of the form $((x_v)_{v\in S_p}, x_p)$ with $N(x_v) = x_p$ for all $v \in S_p$. Consider the homomorphisms

$$(T_1^K \times \mathbb{G}_\mathrm{m})(\mathbb{Z}_p) \xrightarrow{m} T^{K,\mathbb{Q}}(\mathbb{Z}_p) \xrightarrow{\lambda_p^c} (T_1^K \times \mathbb{G}_\mathrm{m})(\mathbb{Z}_p),$$

where $((y_v)_{v\in S_p}, y_p) \overset{m}{\longmapsto} ((y_v y_p)_{v\in S_p}, y_p^2)$ and recall that $((x_v)_v, x_p) \overset{\lambda_p^c}{\longmapsto} ((x_v^2)_v x_p^{-1}, x_p)$. It is easy to see that the composition $\lambda_p^c \circ m$ is the squared map $\mathrm{Sq} : (y, y') \mapsto (y^2, (y')^2)$. Therefore, by Lemma 3.9, we have

$$q(\lambda_p^c) = \frac{|\operatorname{coker}\lambda_p^c|}{|\ker\lambda_p^c|} = |\operatorname{coker}\mathrm{Sq}| \cdot |\operatorname{coker} m|^{-1} \cdot [\ker\mathrm{Sq} : \ker m]^{-1}.$$

First, $|\operatorname{coker}\mathrm{Sq}| = [T_1^K(\mathbb{Z}_p) : T_1^K(\mathbb{Z}_p)^2] \cdot [\mathbb{Z}_p^\times : (\mathbb{Z}_p^\times)^2]$. Suppose $y = ((y_v)_v, y_p) \in \ker\mathrm{Sq}$. Then $y_p = \pm 1$, and $y_v^2 = 1$ $\forall v$. If v is inert or ramified in K, then $y_v = \pm 1$. If v splits in K, then $y_v = (y_w, y_{\bar{w}})$, and $y_w = \pm 1$, $y_{\bar{w}} = \pm 1$. Since $N(y_v) = 1$, $y_v = (1,1)$ or $(-1,-1)$, that is, $y_v = \pm 1$. We conclude that $\ker\mathrm{Sq} = \{\pm 1\}^{S_p} \times \{\pm 1\}$.

On the other hand, since $\ker N \subset \mathrm{im}m$, we have $\operatorname{coker} m \simeq N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))/N(\mathrm{im}m)$. Recall that $\mathrm{im}m = \{((x_v x_p)_{v\in S_p}, x_p^2)\}$, one has $N(\mathrm{im}m) = \{x_p^2 \mid x_p \in \mathbb{Z}_p^\times\}$. Therefore, $|\operatorname{coker} m| = [N(T^{K,\mathbb{Q}}(\mathbb{Z}_p)) : (\mathbb{Z}_p^\times)^2]$. Clearly, $\ker m = \{\pm 1\}$.

Now,

$$\begin{aligned} q(\lambda_p^c) &= [T_1^K(\mathbb{Z}_p) : T_1^K(\mathbb{Z}_p)^2] \cdot [\mathbb{Z}_p^\times : (\mathbb{Z}_p^\times)^2] \cdot [N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))) : (\mathbb{Z}_p^\times)^2]^{-1} \cdot 2^{-|S_p|} \\ &= [T_1^K(\mathbb{Z}_p) : T_1^K(\mathbb{Z}_p)^2] \cdot [\mathbb{Z}_p^\times : N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))] \cdot 2^{-|S_p|} \\ &= [T_1^K(\mathbb{Z}_p) : T_1^K(\mathbb{Z}_p)^2] \cdot 2^{-|S_p|} \cdot e_{T,p}. \end{aligned} \tag{3.10}$$

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The lemma then follows from Lemma 3.12. ∎

For proving Lemma 3.12, we recall the structure theorem of p-adic local units.

PROPOSITION 3.11. *Let $k/\mathbb{Q}_p$ be a finite extension of degree d with ring of integers O_k and residue field $\mathbb{F}_q$. Then*

$$k^\times \simeq \mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} \oplus \mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}_p^d$$

and

$$O_k^\times \simeq \mathbb{Z}/(q-1)\mathbb{Z} \oplus \mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}_p^d,$$

where $p^a = |\mu_{p^\infty}(O_k)|$.

Proof. See [17, Proposition 5.7, p. 140]. ∎

LEMMA 3.12. *We have*

$$[T_1^K(\mathbb{Z}_p) : T_1^K(\mathbb{Z}_p)^2] = \begin{cases} 2^{|S_p|} & \text{if } p \neq 2; \\ 2^{|S_p|+d} & \text{if } p = 2, \end{cases}$$

and

$$q(\mathrm{Sq}_{T_1^K(\mathbb{Z}_p)}) = \begin{cases} 1 & \text{if } p \neq 2; \\ 2^d & \text{if } p = 2, \end{cases}$$

where $d = [K^+ : \mathbb{Q}]$ and S_p is the set of places of K^+ lying over p.

Proof. Note that $T_1^K(\mathbb{Z}_p) = \prod_{v \in S_p} O_{K_v}^{(1)}$, where $O_{K_v}^{(1)}$ consists of norm one elements in $O_{K_v}^\times$. We need to calculate $[O_{K_v}^{(1)} : (O_{K_v}^{(1)})^2]$ for $v \in S_p$.

Let $d_v = [(K^+)_v : \mathbb{Q}_p]$ and consider the exact sequence

$$1 \to O_{K_v}^{(1)} \to O_{K_v}^\times \to N(O_{K_v}^\times) \to 1.$$

Note that $\mathrm{rank}_{\mathbb{Z}_p} O_{K_v}^\times = 2d_v$. Since $[O_{K_v^+}^\times : N(O_{K_v}^\times)]$ is finite, we have $\mathrm{rank}_{\mathbb{Z}_p} N(O_{K_v}^\times) = \mathrm{rank}_{\mathbb{Z}_p} O_{K_v^+}^\times = d_v$ and hence $\mathrm{rank}_{\mathbb{Z}_p} O_{K_v}^{(1)} = d_v$.

By Proposition 3.11, we have $O_{K_v}^{(1)} \simeq A \oplus B \oplus \mathbb{Z}_p^{d_v}$, where A is a finite cyclic group of prime-to-p order and B is a finite cyclic group of p-power order.

Suppose p is odd. Then $O_{K_v}^{(1)}/(O_{K_v}^{(1)})^2 \simeq A/2A$. Since $O_{K_v}^{(1)}$ contains -1, we have $A/2A = \mathbb{Z}/2\mathbb{Z}$. Thus, $[O_{K_v}^{(1)} : (O_{K_v}^{(1)})^2] = 2$ and $[T_1^K(\mathbb{Z}_p) : T_1^K(\mathbb{Z}_p)^2] = \prod_{v \in S_p} [O_{K_v}^{(1)} : (O_{K_v}^{(1)})^2] = 2^{|S_p|}$.

Suppose p is even. The group $O_{K_v}^{(1)}$ contains -1; Therefore, $B/2B = \mathbb{Z}/2\mathbb{Z}$. We have $O_{K_v}^{(1)}/(O_{K_v}^{(1)})^2 \simeq \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{d_v}$. Therefore, $[O_{K_v}^{(1)} : (O_{K_v}^{(1)})^2] = 2^{1+d_v}$ and $[T_1^K(\mathbb{Z}_p) : T_1^K(\mathbb{Z}_p)^2] = \prod_{v \in S_p} [O_{K_v}^{(1)} : (O_{K_v}^{(1)})^2] = 2^{|S_p|+d}$ as $\sum_{v \in S_p} d_v = d$. This proves the first result.

The second result follows from the first result and $|\ker \mathrm{Sq}_{T_1^K(\mathbb{Z}_p)}| = |\{\pm 1\}^{S_p}| = 2^{|S_p|}$ (see the proof of Lemma 3.10). ∎

3.5 Proof of Theorem 1.1

(1) By [20, Remark, p. 128], we have $\tau(\mathbb{G}_{\mathrm{m}})=1$, and $\tau(T_1^{K_i})=2$ for each $i$and we conclude

$$\tau(T^{K,\mathbb{Q}})=\frac{2^r}{[\mathbb{G}_{\mathrm{m}}(\mathbb{A}):N(T^{K,\mathbb{Q}}(\mathbb{A}))\cdot\mathbb{G}_{\mathrm{m}}(\mathbb{Q})]}$$

from (3.1), Theorem 3.2 and Lemma 3.5.

(2) By (3.4) and Lemmas 3.3, 3.5, 3.6, 3.7, 3.8, and 3.10, we obtain

$$\begin{aligned}h(T^{K,\mathbb{Q}})&=h(T_1^K)\cdot\frac{1}{[\mathbb{G}_{\mathrm{m}}(\mathbb{A}):N(T^{K,\mathbb{Q}}(\mathbb{A}))\cdot\mathbb{G}_{\mathrm{m}}(\mathbb{Q})]}\cdot\frac{2^{-d+1}}{2\cdot 1}\cdot\prod_{p\in S_{K/K^+}}e_{T,p}\cdot 2^d.\\&=\frac{h(T_1^K)\cdot\prod_{p\in S_{K/K^+}}e_{T,p}}{[\mathbb{G}_{\mathrm{m}}(\mathbb{A}):N(T^{K,\mathbb{Q}}(\mathbb{A}))\cdot\mathbb{G}_{\mathrm{m}}(\mathbb{Q})]}.\end{aligned}$$

It is known (see [23, (16), p. 375]) that

$$h(T_1^{K_i})=\frac{h_{K_i}}{h_{K_i^+}}\frac{1}{Q_i\cdot 2^{t_i-1}},$$

where Q_i is the Hasse unit index of K_i/K_i^+, and t_i is the number of primes of K_i^+ ramified in K_i. Thus,

$$h(T_1^K)=\prod_{i=1}^r h(T_1^{K_i})=\prod_{i=1}^r\frac{h_{K_i}}{h_{K_i^+}}\frac{1}{Q_i\cdot 2^{t_i-1}}=\frac{h_K}{h_{K^+}}\frac{1}{Q\cdot 2^{t-r}}.$$

This completes the proof of the theorem.

§4. Local and global indices

4.1 Local indices

Keep the notation of the previous section. Denote by f_v, the inertia degree of a finite place v of K^+. Let $N_v:=N(O_{K_v}^\times)$ and $H_v:=\mathbb{Z}_p^\times\cap N_v$, if $v|p$. We define an integer $e(K/K^+,\mathbb{Q},p)$, where p is a prime, as follows. For $p\neq 2$, let

$$e(K/K^+,\mathbb{Q},p):=\begin{cases}1 & \text{if there exists } v\in S_p \text{ ramified in } K \text{ such that } f_v \text{ is odd;}\\ 0 & \text{otherwise.}\end{cases}\tag{4.1}$$

Suppose $p=2$. For any place $v|2$ of K^+, let $\Phi_v:\mathbb{Z}_2^\times\to O_{K_v^+}^\times/(O_{K_v^+}^\times)^2$ be the natural map. Consider the condition

$$\Phi_v(\mathbb{Z}_2^\times)\not\subset\overline{N}_v,\tag{4.2}$$

where $\overline{N}_v$ is the image of N_v in $O_{K_v^+}^\times/(O_{K_v^+}^\times)^2$. Note that $[\mathbb{Z}_2^\times:H_v]=2$ if and only if (4.2) holds (see an explanation in the proof of Proposition 4.1(4)). Define

$$e(K/K^+,\mathbb{Q},2):=\begin{cases}0 & \text{if } \Phi_v(\mathbb{Z}_2^\times)\subset\overline{N}_v \text{ for all } v|2;\\ 2 & \text{if there exist two places } v_1,v_2|2 \text{ of } K^+ \text{ satisfying (4.2) and}\\ & H_{v_1}\neq H_{v_2};\\ 1 & \text{otherwise.}\end{cases}\tag{4.3}$$

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Define

$$e(K/K^+,\mathbb{Q}) := \sum_p e(K/K^+,\mathbb{Q},p). \tag{4.4}$$

Note that $e(K/K^+,\mathbb{Q},p)=0$ if $p\notin S_{K/K^+}$. Recall that $e_{T,p}=[\mathbb{Z}_p^\times : N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))]$. We shall evaluate $e_{T,p}$ and interpret $e_{T,p}$ by using the invariant $e(K/K^+,\mathbb{Q})$ in the following.

PROPOSITION 4.1. (1) *We have* $e_{T,p}=[\mathbb{Z}_p^\times : \cap_{v\in S_p} H_v]$.

(2) *We have* $e_{T,p}|2$ *if* $p\neq 2$, *and* $e_{T,2}|4$.

(3) *Suppose* $p\neq 2$. *Then* $e_{T,p}=2$ *if and only if there exists a place* $v\in S_p$ *such that* v *is ramified in* K *and* f_v *is odd.*

(4) *Suppose* $p=2$. *Then* $2|e_{T,p}$ *if and only if there exists* $v\in S_p$ *satisfying (4.2). Moreover,* $e_{T,p}=4$ *if and only if there exist two places* $v_1,v_2\in S_p$ *satisfying (4.2) and* $H_{v_1}\neq H_{v_2}$.

Proof. (1) It follows immediately from the description of $T^{K,\mathbb{Q}}(\mathbb{Z}_p)$ that $N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))=\cap_{v\in S_p} H_v$.

(2) Consider the inclusion $\mathbb{Z}_p^\times/H_v \hookrightarrow O_{K_v^+}^\times/N(O_{K_v}^\times)$. The latter group has order dividing 2 by the local norm index theorem. Thus, $H_v\subset\mathbb{Z}_p^\times$ is of index 1 or 2. In particular H_v contains $(\mathbb{Z}_p^\times)^2$. It follows that the intersection $\cap H_v$ contains $(\mathbb{Z}_p^\times)^2$, and that $e_{T,p}$ divides $[\mathbb{Z}_p^\times:(\mathbb{Z}_p^\times)^2]$, which is 4 or 2 according as $p=2$ or not.

(3) As $p\neq 2$, we have $e_{T,p}=2$ if and only if there exists $v\in S_p$ such that $[\mathbb{Z}_p^\times:H_v]=2$. We must show that $[\mathbb{Z}_p^\times:H_v]=2$ if and only if v is ramified in K and f_v is odd. If v is unramified in K, then $N_v:=N(O_{K_v}^\times)$ is equal to $O_{K_v^+}^\times$ and $[\mathbb{Z}_p^\times:H_v]=1$. Thus, $[\mathbb{Z}_p^\times:H_v]=2$ only when v is ramified, and we assume this now. Note that $N_v\subset O_{K_v^+}^\times$ is the unique subgroup of index 2 and it contains the principal unit subgroup $1+\pi_v O_{K_v^+}$. Therefore, its image $\overline{N}_v$ in the residue field $\kappa^\times=\mathbb{F}_{q_v}^\times$ is equal to $(\mathbb{F}_{q_v}^\times)^2$. The inclusion $\mathbb{Z}_p^\times\hookrightarrow O_{K_v^+}^\times$ induces the inclusion $\mathbb{F}_p^\times\hookrightarrow\mathbb{F}_{q_v}^\times$ for which the image $\overline{H}_v$ of H_v is equal to $\mathbb{F}_p^\times\cap\overline{N}_v$. Since $\overline{N}_v$ is the unique subgroup of $\mathbb{F}_{q_v}^\times$ of index 2, we have

$$[\mathbb{Z}_p^\times:H_v]=[\mathbb{F}_p^\times:\overline{H}_v]=[\mathbb{F}_p^\times:\mathbb{F}_p^\times\cap(\mathbb{F}_{q_v}^\times)^2].$$

Since $\mathbb{F}_{q_v}^\times$ is cyclic, $\mathbb{F}_p^\times\subset(\mathbb{F}_{q_v}^\times)^2$ if and only if $p-1|(q_v-1)/2$. The latter is equivalent to $2|(p^{f_v-1}+\cdots+1)$ or that f_v is even. Thus, $[\mathbb{Z}_p^\times:H_v]=2$ if and only if f_v is odd.

(4) Since N_v contains $(O_{K_v^+}^\times)^2$, we have $[\mathbb{Z}_p^\times:H_v]=[\Phi_v(\mathbb{Z}_p^\times):\Phi_v(\mathbb{Z}_p^\times)\cap\overline{N}_v]$, which is equal to 2 if and only if $\Phi_v(\mathbb{Z}_p^\times)\not\subset\overline{N}_v$. This proves the first statement. Clearly, $e_{T,p}=4$ if and only if there exist two places $v_1,v_2\in S_p$ such that $[\mathbb{Z}_p^\times:H_{v_1}]=[\mathbb{Z}_p^\times:H_{v_2}]=2$ and $H_{v_1}\neq H_{v_2}$. Then the second statement follows from what we have just proved. ∎

COROLLARY 4.2. *We have* $e_{T,p}=2^{e(K/K^+,\mathbb{Q},p)}$ *for all primes* p.

By Proposition 4.1, one has $e_{T,2}|4$. We now give an example of $T=T^{K,\mathbb{Q}}$ such that $e_{T,2}=4$. Let E and E' be two imaginary quadratic field such that 2 is ramified in both E and E'. We also assume that $N_{E_2/\mathbb{Q}_2}(O_{E_2}^\times)\neq N_{E_2'/\mathbb{Q}_2}(O_{E_2'}^\times)$. Put $K:=E\times E'$, the product of E and E' (not the composite), and then $K^+=\mathbb{Q}\times\mathbb{Q}$. Let v_1 and v_2 be the two places of K^+ over 2. We have $H_{v_1}=N_{E_2/\mathbb{Q}_2}(O_{E_2}^\times)$ and $H_{v_2}=N_{E_2'/\mathbb{Q}_2}(O_{E_2'}^\times)$. Then we see that $[\mathbb{Z}_2^\times:H_{v_1}]=[\mathbb{Z}_2^\times:H_{v_2}]=2$, and that by Proposition 4.1(1), $e_{T,2}=[\mathbb{Z}_2^\times:H_{v_1}\cap H_{v_2}]=4$.

For $f\in\mathbb{N}$ and $q=p^f$, denote by $\mathbb{Q}_q$ the unique unramified extension of $\mathbb{Q}_p$ of degree f and $\mathbb{Z}_q$ the ring of integers.

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LEMMA 4.3. *Let $f \in \mathbb{N}$ and $q = 2^f$.*

(1) $\mathbb{Z}_q^\times/(\mathbb{Z}_q^\times)^2 \simeq (1+2\mathbb{Z}_q)/(1+2\mathbb{Z}_q)^2 \simeq (\mathbb{Z}/2\mathbb{Z})^{f+1}$.

(2) *We have $1+8\mathbb{Z}_q \subset (1+2\mathbb{Z}_q)^2 \subset 1+4\mathbb{Z}_q$. Under the isomorphism $(1+4\mathbb{Z}_q)/(1+8\mathbb{Z}_q) \simeq \mathbb{F}_q([1+4a] \mapsto \bar{a})$, the subgroup $(1+2\mathbb{Z}_q)^2/(1+8\mathbb{Z}_q)$ corresponds to the image $\varphi(\mathbb{F}_q)$ of the Artin–Schreier map $\varphi(x) = x^2 - x : \mathbb{F}_q \to \mathbb{F}_q$.*

(3) *We have*

$$\mathbb{Z}_2^\times \cap (1+2\mathbb{Z}_q)^2 = \begin{cases} 1+4\mathbb{Z}_2 & \text{if } f \text{ is even;} \\ 1+8\mathbb{Z}_2 & \text{otherwise.} \end{cases}$$

Proof. (1) The Teichmüller lifting ω gives a splitting of the exact sequence $1 \to (1+2\mathbb{Z}_q) \to \mathbb{Z}_q^\times \to \mathbb{F}_q^\times \to 1$. Thus, $\mathbb{Z}_q^\times = \mathbb{F}_q^\times \times (1+2\mathbb{Z}_q)$ and hence $(\mathbb{Z}_q^\times)^2 = \mathbb{F}_q^\times \times (1+2\mathbb{Z}_q)^2$ because $q-1$ is odd. This proves the first isomorphism. The second isomorphism follows from $1+2\mathbb{Z}_q \simeq \{\pm 1\} \times \mathbb{Z}_2^f$; see Proposition 3.11.

(2) For $a \in \mathbb{Z}_q$, we have $(1+2a)^2 = 1+4(a^2+a)$. Therefore, $(1+2\mathbb{Z}_q)^2 \subset 1+4\mathbb{Z}_q$. On the other hand, for any $b \in \mathbb{Z}_q$, the equation $T^2+T = 2b$ has a solution in $\mathbb{Z}_q$ by Hensel's lemma. This proves $1+8\mathbb{Z}_q \subset (1+2\mathbb{Z}_q)^2$. The image of $(1+2\mathbb{Z}_q)^2/(1+8\mathbb{Z}_q)$ in $\mathbb{F}_q$ consists of elements $\bar{a}^2+\bar{a} = \varphi(\bar{a})$ for all $\bar{a} \in \mathbb{F}_q$.

(3) It is clear that $1+8\mathbb{Z}_2 \subset \mathbb{Z}_2^\times \cap (1+2\mathbb{Z}_q)^2 \subset 1+4\mathbb{Z}_2$. Note that $1+4\mathbb{Z}_2 \subset (1+2\mathbb{Z}_q)^2$ if and only if $5 \in (1+2\mathbb{Z}_q)^2$. The latter is also equivalent to that the equation $1 = t^2 - t$ is solvable in $\mathbb{F}_q$ by (2). Since $\mathbb{F}_4 = \mathbb{F}_2[t]/(t^2+t+1)$, the previous condition is the same as $\mathbb{F}_4 \subset \mathbb{F}_q$, or equivalently, $2|f$. ∎

LEMMA 4.4. *Let the notation be as in Lemma 4.3.*

(1) *If f is even, then there are $2(2^f-1)$ (respectively 2^{f+1}) ramified quadratic field extensions $K/\mathbb{Q}_q$ such that $\mathbb{Z}_2^\times \subset N(O_K^\times)$ (resp. $\mathbb{Z}_2^\times \not\subset N(O_K^\times)$).*

(2) *If f is odd, then there are 2^f-2 (respectively $2^{f+2}-2^f$) ramified quadratic field extensions $K/\mathbb{Q}_q$ such that $\mathbb{Z}_2^\times \subset N(O_K^\times)$ (respectively $\mathbb{Z}_2^\times \not\subset N(O_K^\times)$).*

Proof. (1) Since $\mathbb{Q}_q^\times/(\mathbb{Q}_q^\times)^2 \simeq (\mathbb{Z}/2\mathbb{Z})^{f+2}$, there are $2^{f+2}-1$ subgroups $\widetilde{N} \subset \mathbb{Q}_q^\times$ of index 2. By the local class field theory, there are $2^{f+2}-1$ quadratic extensions $K/\mathbb{Q}_q$, and $2^{f+2}-2$ of them are ramified. On the other hand, since $\mathbb{Z}_q^\times/(\mathbb{Z}_q^\times)^2 \simeq (\mathbb{Z}/2\mathbb{Z})^{f+1}$, there are $2^{f+1}-1$ subgroups $N \subset \mathbb{Z}_q^\times$ of index 2. It is not hard to see that for each N there are exactly two ramified extensions $K/\mathbb{Q}_q$ such that $N = N(O_K^\times)$. Suppose $2|f$, then $\Phi(\mathbb{Z}_2^\times)$ is a one-dimensional subspace in $\mathbb{Z}_q^\times/(\mathbb{Z}_q^\times)^2 = (\mathbb{Z}/2\mathbb{Z})^{f+1}$ by Lemma 4.3(3). Therefore, there are 2^f-1 subspaces $\overline{N}$ of co-dimension one containing $\Phi(\mathbb{Z}_2^\times)$, and 2^f subspaces $\overline{N}$ of co-dimension one not containing $\Phi(\mathbb{Z}_2^\times)$.

(2) Suppose that f is odd. By Lemma 4.3(3), $\Phi(\mathbb{Z}_2^\times)$ is a two-dimensional subspace in $\mathbb{Z}_q^\times/(\mathbb{Z}_q^\times)^2 = (\mathbb{Z}/2\mathbb{Z})^{f+1}$. There are $2^{f-1}-1$ subspaces $\overline{N}$ of co-dimension one containing $\Phi(\mathbb{Z}_2^\times)$, and the other $2^{f+1}-2^{f-1}$ subspaces $\overline{N}$ not containing $\Phi(\mathbb{Z}_2^\times)$. This proves the lemma. ∎

LEMMA 4.5. *Let $F/\mathbb{Q}_2$ be a finite extension of $\mathbb{Q}_2$, and let L/F be a quadratic extension of F. Then $\mathbb{Z}_2^\times \subset N_{L/F}(O_L^\times)$ if and only if the norm residue symbols $(-1, L/F) = 1$ and $(5, L/F) = 1$.*

Proof. This follows directly from the basic fact that $\mathbb{Z}_2^\times = \{\pm 1\} \times \overline{\langle 5 \rangle}$, where $\overline{\langle 5 \rangle}$ is the closure of the cyclic subgroup $\langle 5 \rangle$ in $\mathbb{Z}_2^\times$. □

4.2 Global indices

We obtain the following partial results for the global index $[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{Q}^\times]$.

LEMMA 4.6. (1) *We have*

$$[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{Q}^\times] = [\widehat{\mathbb{Z}}^\times : \widehat{\mathbb{Z}}^\times \cap N(T^{K,\mathbb{Q}}(\mathbb{A}_f)) \cdot \mathbb{Q}_+],$$

where $\mathbb{Q}_+ := \mathbb{Q}^\times \cap \mathbb{R}_+$.

(2) *The index* $[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{Q}^\times]$ *divides* $\prod_{p \in S_{K/K^+}} e_{T,p}$.

Proof. (1) We have $\mathbb{A}^\times = \mathbb{R}^\times \times \mathbb{A}_f^\times$ and $N(T^{K,\mathbb{Q}}(\mathbb{A})) = \mathbb{R}_+ \times N(T^{K,\mathbb{Q}}(\mathbb{A}_f))$. We use $\mathbb{Q}^\times$ to reduce $\mathbb{R}^\times$ to $\mathbb{R}_+$. Thus,

$$\begin{aligned} \frac{\mathbb{A}^\times}{[N(T^{K,\mathbb{Q}}(\mathbb{A}))] \cdot \mathbb{Q}^\times} &\simeq \frac{\mathbb{R}^\times \times \mathbb{A}_f^\times}{[\mathbb{R}_+ \times N(T^{K,\mathbb{Q}}(\mathbb{A}_f))] \cdot \mathbb{Q}^\times} \\ &\simeq \frac{\mathbb{R}_+ \times \mathbb{A}_f^\times}{[\mathbb{R}_+ \times N(T^{K,\mathbb{Q}}(\mathbb{A}_f))] \cdot \mathbb{Q}_+} \\ &\simeq \frac{\mathbb{Q}_+ \cdot \widehat{\mathbb{Z}}^\times}{N(T^{K,\mathbb{Q}}(\mathbb{A}_f)) \cdot \mathbb{Q}_+} \\ &\simeq \frac{\widehat{\mathbb{Z}}^\times}{\widehat{\mathbb{Z}}^\times \cap N(T^{K,\mathbb{Q}}(\mathbb{A}_f)) \cdot \mathbb{Q}_+}. \end{aligned} \tag{4.5}$$

(2) Since $\widehat{\mathbb{Z}}^\times \cap N(T^{K,\mathbb{Q}}(\mathbb{A}_f)) \cdot \mathbb{Q}_+ \supset \widehat{\mathbb{Z}}^\times \cap N(T^{K,\mathbb{Q}}(\mathbb{A}_f))$, the group

$$\mathbb{A}^\times / \left(N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{Q}^\times\right) \simeq \widehat{\mathbb{Z}}^\times / \left(\widehat{\mathbb{Z}}^\times \cap N(T^{K,\mathbb{Q}}(\mathbb{A}_f)) \cdot \mathbb{Q}_+\right)$$

is a quotient of $\widehat{\mathbb{Z}}^\times / \left(\widehat{\mathbb{Z}}^\times \cap N(T^{K,\mathbb{Q}}(\mathbb{A}_f))\right)$. On the other hand, we have

$$\widehat{\mathbb{Z}}^\times \cap N(T^{K,\mathbb{Q}}(\mathbb{A}_f)) = \prod_p [\mathbb{Z}_p^\times \cap N(T^{K,\mathbb{Q}}(\mathbb{Q}_p))] = \prod_p N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))$$

by Lemma 3.4. Therefore, $[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{Q}^\times]$ divides

$$\prod_p [\mathbb{Z}_p^\times : N(T^{K,\mathbb{Q}}(\mathbb{Z}_p))] = \prod_p e_{T,p} = \prod_{p \in S_{K/K^+}} e_{T,p}.$$

This proves the lemma. □

LEMMA 4.7. *Suppose K is a CM field which contains two distinct imaginary quadratic fields E_1 and E_2. Then* $[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{Q}^\times] = 1$.

Proof. By the global norm index theorem, $[\mathbb{A}^\times : N(\mathbb{A}_{E_i}^\times) \cdot \mathbb{Q}^\times] = 2$ for $i = 1, 2$. Since $E_1 \neq E_2$, the subgroup $N(\mathbb{A}_{E_1}^\times) \cdot N(\mathbb{A}_{E_2}^\times) \cdot \mathbb{Q}^\times$ of $\mathbb{A}^\times$ generated by $N(\mathbb{A}_{E_i}^\times) \cdot \mathbb{Q}^\times$ $(i = 1, 2)$ strictly contains $N(\mathbb{A}_{E_1}^\times) \cdot \mathbb{Q}^\times$. Thus, $[\mathbb{A}^\times : N(\mathbb{A}_{E_1}^\times) \cdot N(\mathbb{A}_{E_2}^\times) \cdot \mathbb{Q}^\times] = 1$. On the other hand, the subgroup $N(T^{K,\mathbb{Q}}(\mathbb{A}))$ contains $N(\mathbb{A}_{E_i}^\times)$ for $i = 1, 2$. Therefore, $[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{Q}^\times] = 1$. □

4.3 Consequences and a second proof

Using our computation of the local index $e_{T,p}$ and $[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A}))\cdot\mathbb{Q}^\times]$, we obtain the following improvement of Theorem 1.1.

THEOREM 4.8. *Let the notation be as in Theorem 1.1. We have*

$$h(T^{K,\mathbb{Q}}) = \frac{h_K}{h_{K^+}}\frac{2^e}{2^{t-r}\cdot Q_K},$$

where e is an integer with $0 \le e \le e(K/K^+,\mathbb{Q})$, where $e(K/K^+,\mathbb{Q})$ is the invariant defined in (4.4).

Proof. This follows from Theorem 1.1, Corollary 4.2, and Lemma 4.6(2). ∎

Since the Hasse unit index Q_K is a power of 2, we obtain the following result from Theorem 4.8.

PROPOSITION 4.9. *The class number $h(T^{K,\mathbb{Q}})$ is equal to h_K/h_{K^+} up to a power of 2.*

A second proof of Theorem 1.1. Put $T := T^{K,\mathbb{Q}}$ and $T' := T^K_1$. We first show that the sequence

$$1 \to T'(\mathbb{A}_f)/[T'(\mathbb{Q})T'(\widehat{\mathbb{Z}})] \to T(\mathbb{A}_f)/[T(\mathbb{Q})T(\widehat{\mathbb{Z}})] \xrightarrow{N} N(T(\mathbb{A}_f))/N[T(\mathbb{Q})T(\widehat{\mathbb{Z}})] \to 1. \tag{4.6}$$

is exact. The kernel of N is

$$T'(\mathbb{A}_f)T(\mathbb{Q})T(\widehat{\mathbb{Z}})/[T(\mathbb{Q})\cdot T(\widehat{\mathbb{Z}})] \simeq T'(\mathbb{A}_f)/[T'(\mathbb{A}_f)\cap T(\mathbb{Q})\cdot T(\widehat{\mathbb{Z}})].$$

If $t = qu \in T'(\mathbb{A}_f)\cap T(\mathbb{Q})\cdot T(\widehat{\mathbb{Z}})$ with $q \in T(\mathbb{Q})$ and $u \in T(\widehat{\mathbb{Z}})$, Then

$$N(q) = N(u)^{-1} \in T(\mathbb{Q})\cap T(\widehat{\mathbb{Z}}) = T(\mathbb{Z}) = T'(\mathbb{Z}).$$

So $q \in T'(\mathbb{Q})$ and $u \in T'(\widehat{\mathbb{Z}})$ and $T'(\mathbb{A}_f)\cap T(\mathbb{Q})\cdot T(\widehat{\mathbb{Z}}) = T'(\mathbb{Q})\cdot T'(\widehat{\mathbb{Z}})$. This proves the exactness of (4.6).

We now prove that

$$N(T(\mathbb{A}_f))/N[T(\mathbb{Q})\cdot T(\widehat{\mathbb{Z}})] \simeq N(T(\mathbb{A}_f))\cdot\mathbb{Q}_+/[N(T(\widehat{\mathbb{Z}}))\cdot\mathbb{Q}_+]. \tag{4.7}$$

Suppose $t = qu \in N(T(\mathbb{A}_f))\cap N(T(\widehat{\mathbb{Z}}))\cdot\mathbb{Q}_+$ with $q \in \mathbb{Q}_+^\times$ and $u \in N(T(\widehat{\mathbb{Z}}))$. Then q is a local norm everywhere. Thus, there is an element $x \in K^\times$ such that $N(x) = q$ by the Hasse principle. By the definition, the element x lies in $T(\mathbb{Q})$ and hence $q \in N(T(\mathbb{Q}))$. This verifies (4.7).

Note that

$$[N(T(\mathbb{A}_f))\cdot\mathbb{Q}_+ : N(T(\widehat{\mathbb{Z}}))\cdot\mathbb{Q}_+] = [\mathbb{A}_f^\times : N(T(\widehat{\mathbb{Z}}))\cdot\mathbb{Q}_+]\cdot[\mathbb{A}_f^\times : N(T(\mathbb{A}_f))\cdot\mathbb{Q}_+]^{-1},$$

By (4.5), $[\mathbb{A}_f^\times : N(T(\mathbb{A}_f))\cdot\mathbb{Q}_+] = [\mathbb{A}^\times : N(T(\mathbb{A}))\cdot\mathbb{Q}^\times]$. It is also easy to see $[\mathbb{A}_f^\times : N(T(\widehat{\mathbb{Z}}))\cdot\mathbb{Q}_+] = \prod_p[\mathbb{Z}_p^\times : N(T(\mathbb{Z}_p))][\mathbb{A}_f^\times : N(T(\widehat{\mathbb{Z}}))\cdot\mathbb{Q}_+^\times]$. Then Theorem 1.1 follows from (4.6) and (4.7).

QUESTION 4.10. (1) Let $N : \widetilde{T} \to T$ be a homomorphism of algebraic tori over $\mathbb{Q}$ such that $T' := \ker N$ is again an algebraic torus. Then by Lang's theorem, the map $N : \widetilde{T}(\mathbb{A}) \to T(\mathbb{A})$ is open and then $[T(\mathbb{A}) : N(\widetilde{T}(\mathbb{A}))\cdot T(\mathbb{Q})]$ is finite. What is the index $[T(\mathbb{A}) : N(\widetilde{T}(\mathbb{A}))\cdot T(\mathbb{Q})]$? When $\widetilde{T} = T^K$, $T = T^k$ and N is the norm map, where K/k is

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a finite extension of number fields, then $[T(\mathbb{A}) : N(\widetilde{T}(\mathbb{A})) \cdot T(\mathbb{Q})] = [\mathbb{A}_k^\times : k^\times N(\mathbb{A}_K^\times)]$ is nothing but the global norm index and it is equal to the degree $[K_0 : k]$ of the maximal abelian subextension K_0 of k in K [14, IX, Section 5, p. 193]. The global norm index theorem requires deep analytic results. It is also expected that one may equally need deep analytic and arithmetic results for computing $[T(\mathbb{A}) : N(\widetilde{T}(\mathbb{A})) \cdot T(\mathbb{Q})]$.

(2) Suppose $\lambda : T \to T'$ is an isogeny of tori over $\mathbb{Q}$ of degree d. Is it true that for any prime $\ell \nmid d$, the ℓ-primary parts of $h(T)$ and $h(T')$ are the same? This is inspired by Proposition 4.9.

§5. Examples

5.1 Imaginary quadratic fields

Suppose K is an imaginary quadratic field. Then $K^+ = \mathbb{Q}$ and $T^{K,\mathbb{Q}} = T^K$. Thus, we have $h(T^{K,\mathbb{Q}}) = h(T^K) = h_K$ without any computation. On the other hand, we use Theorem 1.1 to compute $h(T^{K,\mathbb{Q}})$. It is easy to compute that $Q = 1$, $e_{T,p} = 2$ for each $p \in S_{K/K^+}$ and we have

$$[\mathbb{G}_{\mathrm{m}}(\mathbb{A}) : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{G}_{\mathrm{m}}(\mathbb{Q})] = [\mathbb{A}^\times : N_{K/\mathbb{Q}}(\mathbb{A}_K^\times) \cdot \mathbb{Q}^\times] = 2$$

by the global norm index theorem. Thus, we have

$$\tau(T^{K,\mathbb{Q}}) = 1 \quad \text{and} \quad h(T^{K,\mathbb{Q}})/h(T_1^K) = 2^{t-1},$$

where t is the number of rational primes ramified in K. This also gives the result $h(T^{K,\mathbb{Q}}) = h_K$.

5.2 Biquadratic CM fields

Let K be a biquadratic CM field and F the unique real quadratic subfield. Write $F = \mathbb{Q}(\sqrt{d})$, where $d > 0$ is the unique square-free positive integer determined by F, and $K = EF$, where $E = \mathbb{Q}(\sqrt{-j})$ for a square-free positive integer j. Finite places of F, E, and K will be denoted by v, u, and w, respectively. Recall that $S_{K/F}$ denotes the set of primes p such that there exists a place $v|p$ of F which is ramified in K.

LEMMA 5.1. *Let $K = EF = \mathbb{Q}(\sqrt{d}, \sqrt{-j})$ be a biquadratic CM field over $\mathbb{Q}$. Assume that none of primes of $\mathbb{Q}$ is totally ramified in K.*

(1) *A prime p lies in $S_{K/F}$ if and only if p is ramified in E and is unramified in F.*
(2) *If p is ramified in E and splits in F, then $e_{T,p} = 2$.*
(3) *If p is ramified in E and is inert in F, then $e_{T,p} = 1$.*

Proof. (1) Suppose a prime p is unramified in E. Then every place $v|p$ of F remains unramified in K (see Lang [14, Chapter II, Section 4, Proposition 8(ii)]) and hence $p \notin S_{K/F}$.

Suppose a prime p is both ramified in E and in F. Then the unique place $v|p$ must be unramified in K, because if v is ramified in K then p is totally ramified in K which contradicts to our assumption. Thus, p lies in $S_{K/F}$ if and only if it is ramified in E and is unramified in F.

(2) Let v_1, v_2 be the places of F over p. One has $F_{v_i} = \mathbb{Q}_p$, $K_p = E_u \times E_u$ and $H_{v_i} = N(O_{E_u}^\times)$ for $i = 1, 2$. Thus, $e_{T,p} = [\mathbb{Z}_p^\times : N(O_{E_u}^\times)] = 2$.

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(3) Suppose first that $p \neq 2$. We have $F_v = \mathbb{Q}_{p^2}$ with inertia degree $f = 2$. Then $e_{T,p} = 1$ follows from Proposition 4.1(3). Now assume $p = 2$. By Lemma 4.3(3), one has $5 \in N(O_{K_w}^\times)$ because $5 \in 1 + 4\mathbb{Z}_2 \subset (1 + 2\mathbb{Z}_4)^2 \subset N(O_{K_w}^\times)$. By Lemma 5.2, we also have $-1 \in N(O_{K_w}^\times)$. Thus, by Lemma 4.5, $\mathbb{Z}_2^\times \subset N(O_{K_w}^\times)$ and we obtain $e_{T,2} = 1$. This proves the lemma. ∎

LEMMA 5.2. *Let $\mathsf{E}/\mathbb{Q}_2$ be a ramified quadratic extension of $\mathbb{Q}_2$, and let $\mathsf{L} = \mathsf{E}\cdot\mathbb{Q}_4$ be the composite of E and $\mathbb{Q}_4$. Then $-1 \in N_{\mathsf{L}/\mathbb{Q}_4}(O_{\mathsf{L}}^\times)$.*

Proof. Since $\mathsf{E}/\mathbb{Q}_2$ is ramified, we can write $\mathsf{E} = \mathbb{Q}_2(\sqrt{d_{\mathsf{E}}})$ and $O_{\mathsf{E}} = \mathbb{Z}_2[\sqrt{d_{\mathsf{E}}}]$ for some $d_{\mathsf{E}} \in \{3, 7, 2, 6, 10, 14\}$. Put $j := -d_{\mathsf{E}}$ and then $j \bmod 8 \in \{1, 5, 2, 6\}$.

Note that $\mathbb{Q}_4 = \mathbb{Q}_2[t]$ with $t^2 + t + 1 = 0$. For each element $a \in \mathbb{Q}_4$, write $a = a_0 + a_1 t$ with $a_0,\ a_1 \in \mathbb{Q}_2$. For $x \in O_{\mathsf{L}}$, we write $x = a + b\sqrt{-j}$ where $a, b \in \mathbb{Z}_4$ and $a = a_0 + a_1 t$, $b = b_0 + b_1 t$. Then

$$\begin{aligned} N(x) &= a^2 + jb^2 = (a_0^2 + 2a_1 a_0 t + a_1^2 t^2) + j(b_0^2 + 2b_0 b_1 t + b_1^2 t^2) \\ &= [a_0^2 + jb_0^2 - (a_1^2 + jb_1^2)] + [2a_0 a_1 + 2jb_0 b_1 - (a_1^2 + jb_1^2)]t. \end{aligned}$$

Hence for $x = a + b\sqrt{-j}$ satisfying $N(x) \in \mathbb{Z}_2$, the element x must satisfy the condition

$$2a_0 a_1 + 2jb_0 b_1 - (a_1^2 + jb_1^2) = 0. \tag{5.1}$$

Observe $1 + 8\mathbb{Z}_4 \subset (1 + 2\mathbb{Z}_4)^2 \subset N(O_{\mathsf{L}}^\times)$. If $-1 \in N(O_{\mathsf{L}}^\times)/(1 + 8\mathbb{Z}_4) \subset (\mathbb{Z}_4/8\mathbb{Z}_4)^\times$, then $-1 \in N(O_{\mathsf{L}}^\times)$. We may solve the equation $N(x) \equiv -1 \pmod 8$, and regard $a, b \in \mathbb{Z}_4/8\mathbb{Z}_4$. For simplicity, let $j \in \{1,\ 5,\ 2,\ 6\}$.

Case j is even: Since j is even, by (5.1), we have $2|a_1$ and write $a_1 = 2c_1$. Consider the case $2|b_1$. Then we have $2a_0 a_1 - a_1^2 = 4a_0 c_1 - 4c_1^2 = 0 \pmod 8$. Hence $(a_0,\ c_1) \equiv (1,\ 1)$, $(0,\ 0)$, or $(1,\ 0) \pmod 2$.

We have $N(x) = [a_0^2 + jb_0^2 - (a_1^2 + jb_1^2)] = a_0^2 + jb_0^2 - 4c_1^2$. For $j = 2$, take $(a_0,\ b_0,\ c_1) \equiv (1,\ 1,\ 1) \pmod 2$; for $j = 6$, take $(a_0,\ b_0,\ c_1) \equiv (1,\ 1,\ 0) \pmod 2$. Then $N(x) = -1$.

Case j is odd: Consider the case $2 \nmid a_1$ and $2 \nmid b_1$. Since j is odd, the condition (5.1) is equivalent to

$$a_0 a_1 + b_0 b_1 - (1 + j)/2 \equiv 0 \pmod 4. \tag{5.2}$$

By (5.2), we require $a_0 - b_0 \equiv 1 \pmod 2$. Suppose $2|a_0$ and $2 \nmid b_0$, and write $a_0 = 2c_0$. Then $N(x) = -1$ gives the equation $a_0^2 + jb_0^2 - (a_1 + jb_1^2) = 4c_0^2 + j - 1 - j = -1 \pmod 8$. Thus, $c_0 = 2d_0$ for some $d_0 \in \mathbb{Z}_4/8\mathbb{Z}_4$. Moreover, substituting $a_0 = 4d_0$ into (5.2), we have the condition $b_0 b_1 - (1 + j)/2 \equiv 0 \pmod 4$. Since $j \in \{1, 5\}$, there exists b_1 satisfying this condition. Conclusively, if we take $(c_0,\ b_0,\ a_1,\ b_1) \equiv (0,\ 1,\ 1,\ 1) \pmod 2$ and $b_0 b_1 = (1 + j)/2$, then $N(x) = -1$. ∎

Let ζ_n denote a primitive nth root of unity.

LEMMA 5.3. *Let L be a totally ramified biquadratic field extension of $\mathbb{Q}_p$. Then*

(1) *$p = 2$ and $\mathsf{L} \simeq \mathbb{Q}(\zeta_8) \otimes \mathbb{Q}_2 = \mathbb{Q}_2[t]$ with relation $t^4 + 1 = 0$;*
(2) *for any quadratic subextension E of L over $\mathbb{Q}_2$, one has $N_{\mathsf{L}/\mathsf{E}}(O_{\mathsf{L}}^\times) \supset \mathbb{Z}_2^\times$.*

Proof. (1) By local class field theory [14], $\mathbb{Z}_p^\times / N_{\mathsf{L}/\mathbb{Q}_p}(O_{\mathsf{L}}^\times) \simeq I_p = \mathrm{Gal}(\mathsf{L}/\mathbb{Q}_p) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where I_p is the inertia group of p. As $\mathbb{Z}_p^\times$ is pro-cyclic for odd prime p, this

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is possible only when $p=2$. Clearly, 2 is totally ramified in $\mathbb{Q}(\zeta_8)$. Thus, it suffices to show that $\mathbb{Q}_2$ has only one totally ramified biquadratic extension, that is, there is only one subgroup $H \subset \mathbb{Z}_2^\times$ satisfying $\mathbb{Z}_2^\times/H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by the existence theorem. Now $\mathbb{Z}_2^\times \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_2$ and one easily checks that $H = \{0\} \times 2\mathbb{Z}_2$ is the unique subgroup satisfying $\mathbb{Z}_p^\times/H \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This proves (1).

(2) Put $\mathsf{E}_1 = \mathbb{Q}_2(\sqrt{2}) = \mathbb{Q}_2[t-t^3]$, $\mathsf{E}_2 = \mathbb{Q}_2(\sqrt{-2}) = \mathbb{Q}_2[t+t^3]$ and $\mathsf{E}_3 = \mathbb{Q}_2(\sqrt{-1}) = \mathbb{Q}_2[t^2]$. The Galois group $\mathrm{Gal}(\mathsf{L}/\mathbb{Q}_2) = \{1,\sigma_1,\sigma_2,\sigma_3\}$, where $\sigma_1(t) = t^{-1}$, $\sigma_2(t) = t^3$ and $\sigma_3(t) = t^5 = -t$. Then E_i is the fixed subfield of the element σ_i for each $i = 1,2,3$. We choose a uniformizer π_i of E_i as $t-t^3$, $t+t^3$, and t^2-1 for $i=1,2,3$, respectively. Thus, we have $(\mathsf{E},\pi) = (\mathsf{E}_i,\pi_i)$ for some i. Let $x = a+bt+ct^2+dt^3 \in O_{\mathsf{L}}$ with $a,b,c,d \in \mathbb{Z}_2$. It is well-known that every element in $1+4\pi O_{\mathsf{E}}$ is a square, and hence $N_{\mathsf{L}/\mathsf{E}}(O_{\mathsf{L}}^\times) \supset (O_{\mathsf{E}}^\times)^2 \supset 1+4\pi O_{\mathsf{E}}$. To show $N_{\mathsf{L}/\mathsf{E}}(O_{\mathsf{L}}^\times) \supset \mathbb{Z}_2^\times$, it suffices to show that the group $N_{\mathsf{L}/\mathsf{E}}(O_{\mathsf{L}}^\times)$ mod $4\pi O_{\mathsf{E}}$ contains $1,3,5,7$ mod 8.For $i=1$, we compute

$$N_{\mathsf{L}/\mathsf{E}_1}(x) = (a^2+b^2+c^2+d^2) + (ab-ad+bc+cd)(t-t^3).$$

Put $(a,b,c,d) = (1,1,0,1)$, one has $N(x) \bmod 4\pi_1$ is equal to $3 \bmod 8$. Put $(a,b,c,d) = (1,0,2,0)$, one has $N(x) \bmod 4\pi_1$ is equal to $7 \bmod 8$. Thus, $N_{\mathsf{L}/\mathsf{E}_1}(O_{\mathsf{L}}^\times) \supset \mathbb{Z}_2^\times$,

For $i=2$, we compute

$$N_{\mathsf{L}/\mathsf{E}_2}(x) = (a^2-b^2+c^2-d^2) + (ab+ad-bc+cd)(t+t^3).$$

Put $(a,b,c,d) = (2,0,1,0)$, one has $N(x) \bmod 4\pi_2$ is equal to $5 \bmod 8$. Put $(a,b,c,d) = (0,1,1,1)$, one has $N(x) \bmod 4\pi_2$ is equal to $7 \bmod 8$. Thus, $N_{\mathsf{L}/\mathsf{E}_2}(O_{\mathsf{L}}^\times) \supset \mathbb{Z}_2^\times$,

For $i=3$, we compute

$$N_{\mathsf{L}/\mathsf{E}_3}(x) = (a^2-b^2-c^2+d^2+2bd+2ac) + (2ac-b^2+d^2)(t^2-1).$$

Put $(a,b,c,d) = (1,1,0,1)$, one has $N(x) \bmod 4\pi_3$ is equal to $3 \bmod 8$. Put $(a,b,c,d) = (2,0,1,0)$, one has $N(x) \bmod 4\pi_3$ is equal to $7 \bmod 8$.Thus, $N_{\mathsf{L}/\mathsf{E}_3}(O_{\mathsf{L}}^\times) \supset \mathbb{Z}_2^\times$, ∎

COROLLARY 5.4. *Let $K = EF$ be a biquadratic CM field. If p is a prime totally ramified in K, then $p=2$ and $e_{T,2} = 1$.*

PROPOSITION 5.5. *Let F be a real quadratic field and E an imaginary quadratic field, and let $K = EF$. Then*

$$h(T^{K,\mathbb{Q}}) = \frac{h_K}{h_F} \cdot \frac{2^s}{2^{t-1} \cdot Q}, \tag{5.3}$$

where t is the number of places of F ramified in K, s is the number of primes p that are ramified in E and split in F, and $Q = Q_K$.

Proof. By Lemma 5.1 and Corollary 5.4, $\prod_{p \in S_{K/F}} e_{T,p} = 2^s$. Since K contains two distinct imaginary quadratic fields, by Lemma 4.7, we have $[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A})) \cdot \mathbb{Q}^\times] = 1$. Thus, the formula (5.3) follows from Theorem 1.1. ∎

Note that we may rewrite (5.3) as

$$h(T^{K,\mathbb{Q}}) = \frac{h_K}{h_F} \cdot \frac{1}{2^{|S_{K/F}|-1} \cdot Q_K}. \tag{5.4}$$

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Indeed, suppose we let m be the number of primes p that are ramified in E and inert in F. Then $t=m+2s+\delta$ and $|S_{K/F}|=m+s+\delta$, where $\delta=1$ if 2 is totally ramified in K and $\delta=0$ otherwise. So $t-s=|S_{K/F}|$.

If $K\neq\mathbb{Q}(\sqrt{2},\sqrt{-1})$, then by Herglotz [9] (cf. [28, Section 2.10])

$$h_K=Q_K\cdot h_F\cdot h_E\cdot h_{E'}/2, \tag{5.5}$$

where $E'\subset K$ is the other imaginary quadratic field. By (5.4) and (5.5), we have

$$\begin{aligned} h(T^{K,\mathbb{Q}}) &= \frac{h_E\cdot h_{E'}\cdot Q_K}{2}\frac{1}{2^{|S_{K/F}|-1}Q_K} \\ &= \frac{h_E\cdot h_{E'}}{2^{|S_{K/F}|}}, \quad \text{if } K\neq\mathbb{Q}(\sqrt{2},\sqrt{-1}). \end{aligned} \tag{5.6}$$

For $K=\mathbb{Q}(\sqrt{2},\sqrt{-1})=\mathbb{Q}(\zeta_8)$, it is known that $h(\mathbb{Q}(\zeta_8))=1$, and $Q_{\mathbb{Q}(\zeta_8)}=1$ as 8 is a prime power [26, Corollary 4.13, p. 39]. Moreover, $S_{K/F}=\{2\}$ and $e_{T,2}=1$ (Corollary 5.4). Thus,

$$h(T^{K,\mathbb{Q}})=1, \quad \text{for } K=\mathbb{Q}(\sqrt{2},\sqrt{-1}). \tag{5.7}$$

We specialize to the case where $K=K_j=FE=\mathbb{Q}(\sqrt{p},\sqrt{-j})$, where $F=\mathbb{Q}(\sqrt{p})$, $E=\mathbb{Q}(\sqrt{-j})$, p is a prime and $j\in\{1,2,3\}$. Note that we have $\mathbb{Q}(\sqrt{2},\sqrt{-2})=\mathbb{Q}(\sqrt{2},\sqrt{-1})=\mathbb{Q}(\zeta_8)$ and $\mathbb{Q}(\sqrt{3},\sqrt{-3})=\mathbb{Q}(\sqrt{3},\sqrt{-1})=\mathbb{Q}(\zeta_{12})$. We may assume that $p\neq 2$ if $j=2$ and $p\neq 3$ if $j=3$.

The set $S_{K/F}$ is given as follows:

(i) $S_{K_1/F}=\{2\}$ if $p\equiv 1(\text{mod } 4)$, and $S_{K_1/F}=\emptyset$ otherwise;
(ii) $S_{K_2/F}=\{2\}$ if $p\equiv 1(\text{mod } 4)$, and $S_{K_2/F}=\emptyset$ otherwise;
(iii) $S_{K_3/F}=\{3\}$ always (recall $p\neq 3$).

Thus,

$$|S_{K/F}|=\begin{cases}0 & \text{if} j\in\{1,2\} \text{ and } p\not\equiv 1 \pmod 4;\\ 1 & \text{otherwise.}\end{cases} \tag{5.8}$$

By (5.6) and (5.8), if $K\neq\mathbb{Q}(\sqrt{2},\sqrt{-1})$, we have

$$h(T^{K,\mathbb{Q}})=\begin{cases}h(\mathbb{Q}(\sqrt{-jp})) & \text{if } j\in\{1,2\} \text{ and } p\not\equiv 1 \pmod 4;\\ h(\mathbb{Q}(\sqrt{-jp}))/2 & \text{otherwise.}\end{cases} \tag{5.9}$$

REMARK 5.6. Observe from formula (5.6) that for computing the class number $h(T^{K,\mathbb{Q}})$ or $h(T_1^K)$, one needs not to calculate the Hasse unit index Q_K. For the case where $F=\mathbb{Q}(\sqrt{p})$ with prime p, one has $Q_K=2$ if and only if $p\equiv 3(\text{mod } 4)$, and either $K=\mathbb{Q}(\sqrt{p},\sqrt{-1})$ or $K=\mathbb{Q}(\sqrt{p},\sqrt{-2})$; see [28, Proposition 2.7].

§6. Polarized CM abelian varieties and unitary Shimura varieties

6.1 CM points

Let (K,O_K,V,ψ,Λ,h) be a PEL-datum, where

- $K=\prod_{i=1}^r K_i$ be a product of CM fields K_i with canonical involution $^-$;
- O_K the maximal order of K;

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- V is a free K-module of rank one;
- $\psi : V \times V \to \mathbb{Q}$ be a nondegenerate alternating pairing such that

$$\psi(ax, y) = \psi(x, \bar{a}y), \quad \forall a \in K,\ x, y \in V;$$

- Λ be an O_K-lattice with $\psi(\Lambda, \Lambda) \subset \mathbb{Z}$;
- $h : \mathbb{C} \to \mathrm{End}_{K_{\mathbb{R}}}(V_{\mathbb{R}})$ be an $\mathbb{R}$-algebra homomorphism such that

$$\psi(h(z)x, y) = \psi(x, h(\bar{z})y), \quad \text{for } z \in \mathbb{C},\ x, y \in V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R},$$

and that the pairing $(x, y) := \psi(h(i)x, y)$ is symmetric and positive definite.

Let $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$ be the decomposition into $\mathbb{C}$-subspaces such that $h(z)$ acts by z (respectively $\bar{z}$) on $V^{-1,0}$ (respectively $V^{0,-1}$). Let $T = T^{K,\mathbb{Q}}$ and $U \subset T(\mathbb{A}_f)$ be an open compact subgroup. Put $g = \frac{1}{2}\dim_{\mathbb{Q}} V$. Let $M_{(\Lambda,\psi),U}$ be the set of isomorphism classes of tuples $(A, \lambda, \iota, \bar{\eta})_{\mathbb{C}}$, where

- A is a complex abelian variety of dimension g;
- $\iota : O_K \to \mathrm{End}(A)$ is a ring monomorphism;
- $\lambda : A \to A^t$ is an O_K-linear polarization, that is, it satisfies $\lambda\iota(\bar{b}) = \iota(b)\lambda$ for all $b \in O_K$;
- $\bar{\eta}$ is an U-orbit of $O_K \otimes \widehat{\mathbb{Z}}$-linear isomorphisms

$$\eta : V \otimes \widehat{\mathbb{Z}} \to T(A) := \prod_{\ell} T_\ell(A)$$

preserving the pairing up to a scalar in $\widehat{\mathbb{Z}}^\times$, where $T_\ell(A)$ is the ℓ-adic Tate module of A such that

(a) $\det(b; V^{-1,0}) = \det(b; \mathrm{Lie}(A))$ for all $b \in O_K$;
(b) there exists a K-linear isomorphism

$$(V, \psi) \simeq (H_1(A, \mathbb{Q}), \langle , \rangle_\lambda) \tag{6.1}$$

that preserves the pairings up to a scalar in $\mathbb{Q}^\times$, where $\langle , \rangle_\lambda$ is the pairing induced by the polarization λ.

Two members $(A_1, \lambda_1, \iota_1, \bar{\eta}_1)$ and $(A_2, \lambda_2, \iota_2, \bar{\eta}_2)$ are said to be isomorphic if there exists an O_K-linear isomorphism $\varphi : A_1 \to A_2$ such that $\varphi^*\lambda_2 = \lambda_1$ and $\varphi_*\bar{\eta}_1 = \bar{\eta}_2$.

LEMMA 6.1. *Let T be an algebraic torus over $\mathbb{Q}$, $U \subset T(\mathbb{A}_f)$ an open compact subgroup, and $U_\infty \subset T(\mathbb{R})$ an open subgroup. Then*

$$\begin{aligned}[T(\mathbb{A}) : T(\mathbb{Q})U_\infty U] &= \frac{[U_T : U]}{[T(\mathbb{Z})_\infty : T(\mathbb{Z})_\infty \cap U]} \cdot [T(\mathbb{A}) : T(\mathbb{Q})U_\infty U_T] \\ &= \frac{[U_T : U]}{[T(\mathbb{Z})_\infty : T(\mathbb{Z})_\infty \cap U]} \cdot [T(\mathbb{R}) : T(\mathbb{Z})U_\infty] \cdot h(T), \end{aligned} \tag{6.2}$$

where $U_T := T(\widehat{\mathbb{Z}})$ is the maximal open compact subgroup of $T(\mathbb{A}_f)$ and $T(\mathbb{Z})_\infty = T(\mathbb{Z}) \cap U_\infty$.

Proof. Let $T(\mathbb{Q})_\infty = T(\mathbb{Q}) \cap U_\infty$. One has $[T(\mathbb{A}) : T(\mathbb{Q})U_\infty U] = [T(\mathbb{A}_f) : T(\mathbb{Q})_\infty U]$. Now consider the exact sequence

$$0 \longrightarrow \frac{U_T}{U \cdot T(\mathbb{Q})_\infty \cap U_T} \simeq \frac{U_T \cdot G(\mathbb{Q})_\infty}{U \cdot T(\mathbb{Q})_\infty} \longrightarrow \frac{T(\mathbb{A}_f)}{U \cdot T(\mathbb{Q})_\infty} \longrightarrow \frac{T(\mathbb{A}_f)}{U_T \cdot T(\mathbb{Q})_\infty} \longrightarrow 1. \tag{6.3}$$

It is easy to verify $U\cdot T(\mathbb{Q})_\infty \cap U_T = U\cdot T(\mathbb{Z})_\infty$. Using the exact sequence (6.3) and the following one

$$0 \longrightarrow \frac{T(\mathbb{Z})_\infty}{T(\mathbb{Z})_\infty \cap U} \longrightarrow \frac{U_T}{U} \longrightarrow \frac{U_T}{U\cdot T(\mathbb{Z})_\infty} \longrightarrow 1, \tag{6.4}$$

we obtain the first equation of (6.2).

Now we prove the second equality. Consider the exact sequence

$$0 \longrightarrow \frac{T(\mathbb{R})}{U_T\cdot T(\mathbb{Q})\cdot U_\infty \cap T(\mathbb{R})} \longrightarrow \frac{T(\mathbb{A})}{U_T\cdot T(\mathbb{Q})\cdot U_\infty} \longrightarrow \frac{T(\mathbb{A})}{U_T\cdot T(\mathbb{Q})\cdot T(\mathbb{R})} \longrightarrow 1, \tag{6.5}$$

where the inclusion $T(\mathbb{Q}) \hookrightarrow T(\mathbb{A})$ is given by the diagonal map, the map $T(\mathbb{R}) \hookrightarrow T(\mathbb{A}) = T(\mathbb{R}) \times T(\mathbb{A}_f)$ sends $t_\infty \mapsto (t_\infty, 1)$, and the intersection $U_T\cdot T(\mathbb{Q})\cdot U_\infty \cap T(\mathbb{R})$ is taken in $T(\mathbb{A})$. Suppose $utu_\infty = (tu_\infty, tu)$ is an element in $U_T\cdot T(\mathbb{Q})\cdot U_\infty \cap T(\mathbb{R})$. Then $tu = 1$ and $t = u^{-1} \in T(\mathbb{Q}) \cap U_T = T(\mathbb{Z})$. Thus, $U_T\cdot T(\mathbb{Q})\cdot U_\infty \cap T(\mathbb{R}) = T(\mathbb{Z})\cdot U_\infty \subset T(\mathbb{R})$ and we have $[T(\mathbb{A}) : T(\mathbb{Q})U_\infty U_T] = [T(\mathbb{R}) : T(\mathbb{Z})U_\infty]\cdot h(T)$. ∎

By [5, 4.11], the set $M_{(\Lambda,\psi),U}$ is isomorphic to the Shimura set $\mathrm{Sh}_U(T,h) \simeq T(\mathbb{Q})\backslash T(\mathbb{A}_f)/U$. By Lemma 6.1, we have

$$|M_{(\Lambda,\psi),U}| = \frac{[T(\widehat{\mathbb{Z}}) : U]}{[T(\mathbb{Z}) : T(\mathbb{Z}) \cap U]}\cdot h(T) = \frac{[T(\widehat{\mathbb{Z}}) : U]}{[\mu_K : \mu_K \cap U]}\cdot h(T), \tag{6.6}$$

where $\mu_K = \prod_{i=1}^r \mu_{K_i}$ and $T = T^{K,\mathbb{Q}}$. Using Theorem 1.1, we obtain the following result.

Proposition 6.2. *We have*

$$|M_{(\Lambda,\psi),U}| = \frac{[T(\widehat{\mathbb{Z}}) : U]}{[\mu_K : \mu_K \cap U]}\cdot \frac{h_K}{h_{K^+}}\cdot \frac{1}{Q\cdot 2^{t-r}}\cdot \frac{\prod_{p\in S_{K/K^+}} e_{T,p}}{[\mathbb{A}^\times : N(T(\mathbb{A}))\cdot \mathbb{Q}^\times]}, \tag{6.7}$$

where $r, t, Q, S_{K/K^+}$ and $e_{T,p}$ are as in Theorem 1.1.

6.2 Connected components of unitary Shimura varieties

In this subsection we consider a PEL-datum $(K, O_K, V, \psi, \Lambda, h)$, where

- K is a CM field with canonical involution $^-$;
- V is a free K-module of rank $n > 1$;
- O_K, ψ, h are as in Section 6.1.

Let $G = GU_K(V,\psi)$ be the group of unitary similitudes of (V,ψ). The kernel of the multiplier homomorphism $c : G \to \mathbb{G}_{\mathrm{m}}$ is the unitary group $U_K(V,\psi)$ associated to (V,ψ). Let X be the $G(\mathbb{R})$-conjugacy class of h, and $U \subset G(\mathbb{A}_f)$ an open compact subgroup. The complex Shimura variety associated to the PEL datum is defined by

$$Sh_U(G,X)_\mathbb{C} := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/U. \tag{6.8}$$

As in Section 6.1, we define $M_{(\Lambda,\psi),U}$ as the moduli space of complex abelian varieties $(A,\lambda,\iota,\bar{\eta})$ with additional structures satisfying the conditions (a) and (b). By [5, 4.11], one has $Sh_U(G,X)_\mathbb{C} \simeq M_{(\Lambda,\psi),U}$; this provides the modular interpretation of the Shimura variety $Sh_U(G,X)_\mathbb{C}$. We are interested in the number of the connected components of the moduli space $M_{(\Lambda,\psi),U}$, or equivalently, those of the Shimura variety $\mathrm{Sh}_U(G,X)_\mathbb{C}$.

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Let X^+ be the connected component of X that contains the base point h, and let $G(\mathbb{R})_+ := \mathrm{Stab}_{G(\mathbb{R})} X^+$ be the stabilizer of X^+ in $G(\mathbb{R})$. We have

$$\pi_0(\mathrm{Sh}_U(G,X)_\mathbb{C}) \simeq G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/U \simeq G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})_+U, \tag{6.9}$$

where $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+$. Let G^{der} be the derived group of G, and let $D := G/G^{\mathrm{der}}$ be the quotient torus. Denote by $\nu : G \to D$ the natural homomorphism. Note that the derived group $G^{\mathrm{der}} = SU_K(V,\psi)$ is semi-simple and simply connected.

Theorem 6.3. *Assume that $G^{\mathrm{der}}(\mathbb{R})$ is not compact. Then the complex Shimura variety $\mathrm{Sh}_U(G,X)_\mathbb{C}$ has*

$$\frac{[D(\widehat{\mathbb{Z}}) : \nu(U)]}{[\mu_K : \mu_K \cap \nu(U)]} \cdot \frac{h_K}{h_{K^+}} \cdot \frac{1}{2^{t-1}Q_K} \cdot \begin{cases} 1 & \text{if } n \text{ is even;} \\ \frac{\prod_{p \in S_{K/K^+}} e_{T,p}}{[\mathbb{A}^\times : N(T^{K,\mathbb{Q}}(\mathbb{A}) \cdot \mathbb{Q}^\times]} & \text{if } n \text{ is odd,} \end{cases} \tag{6.10}$$

connected components, where t, Q_K and $e_{T,p}$ are as in Theorem 1.1.

Proof. Using the strong approximation argument and Kneser's theorem (namely, $H^1(\mathbb{Q}_p, G^{\mathrm{der}}) = 1$ for all primes p), the morphism ν induces a bijection

$$\nu : G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/U \xrightarrow{\sim} \nu(G(\mathbb{Q})_+)\backslash D(\mathbb{A}_f)/\nu(U) \tag{6.11}$$

(see [30, Lemma 2.2]). By [12, Section 7, pp. 393 – 394], one has

$$D \simeq \begin{cases} T^{K,1} \times \mathbb{G}_\mathrm{m} & \text{if } n \text{ is even;} \\ T^{K,\mathbb{Q}} & \text{if } n \text{ is odd.} \end{cases} \tag{6.12}$$

Using the Hasse principle, one shows that $\nu(G(\mathbb{Q})_+) = D(\mathbb{Q}) \cap \nu(G(\mathbb{R})_+)$. One directly checks

$$\nu(G(\mathbb{R})_+) \simeq \begin{cases} T^{K,1}(\mathbb{R}) \times \mathbb{R}_+ & \text{if } n \text{ is even;} \\ T^{K,\mathbb{Q}}(\mathbb{R}) & \text{if } n \text{ is odd.} \end{cases} \tag{6.13}$$

As a result, the intersection $D(\mathbb{Z})_\infty := D(\mathbb{Z}) \cap \nu(G(\mathbb{R})_+)$ is equal to μ_K for all n. Applying Lemma 6.1, (6.9), (6.11), and the formula for $h(D)$ using Theorem 1.1, we obtain the result. ∎

6.3 Polarized abelian varieties over finite fields

In this subsection, we formulate two counting problems for polarized abelian varieties over finite fields in an isogeny class and compute their cardinality using the class number formula of CM tori. Let k be a finite field.

Definition 6.4. Let $\underline{A}_1 = (A_1, \lambda_1)$ and $\underline{A}_2 = (A_2, \lambda_2)$ be two polarized abelian varieties over k.

(1) They ($\underline{A}_1$ and $\underline{A}_2$) are *isomorphic*, denoted $\underline{A}_1 \simeq \underline{A}_2$, if there exists an isomorphism $\alpha : A_1 \xrightarrow{\sim} A_2$ such that $\alpha^*\lambda_2 = \lambda_1$. Similarly, their polarized ℓ-divisible groups $\underline{A}_1[\ell^\infty]$ and $\underline{A}_2[\ell^\infty]$ are said to be *isomorphic*, denoted $\underline{A}_1[\ell^\infty] \simeq \underline{A}_2[\ell^\infty]$, if there exists an isomorphism $\alpha_\ell : A_1[\ell^\infty] \xrightarrow{\sim} A_2[\ell^\infty]$ such that $\alpha_\ell^*\lambda_2 = \lambda_1$.

(2) They are said to be *in the same isogeny class* if there exists a quasi-isogeny $\alpha : A_1 \to A_2$ (i.e., a multiple of α by an integer is an isogeny) such that $\alpha^*\lambda_2 = \lambda_1$. Denote by

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Isog(A_1, λ_1) the set of isomorphism classes of (A_2, λ_2) lying in the same isogeny class of (A_1, λ_1).

(3) They are said to be *similar*, denoted $\underline{A}_1 \sim \underline{A}_2$, if there exists an isomorphism $\alpha : A_1 \to A_2$ such that $\alpha^* \lambda_2 = q\lambda_1$ for some $q \in \mathbb{Q}_{>0}$. Similarly, their polarized ℓ-divisible groups $\underline{A}_1[\ell^\infty]$ and $\underline{A}_2[\ell^\infty]$ are said to be *similar*, denoted $\underline{A}_1[\ell^\infty] \sim \underline{A}_2[\ell^\infty]$, if there exists an isomorphism $\alpha_\ell : A_1[\ell^\infty] \xrightarrow{\sim} A_2[\ell^\infty]$ such that $\alpha_\ell^* \lambda_2 = q\lambda_1$, for some $q \in \mathbb{Q}_\ell^\times$.

(4) They are said to be *isomorphic locally everywhere*, if

$$\underline{A}_1[\ell^\infty] \simeq \underline{A}_2[\ell^\infty] \quad \text{over } k$$

for all primes ℓ including the prime char k.

(5) They are said to be *similar locally everywhere*, if

$$\underline{A}_1[\ell^\infty] \sim \underline{A}_2[\ell^\infty] \quad \text{over } k$$

for all primes ℓ (also including the prime char k).

Now we start with a polarized abelian variety (A_0, λ_0) over k. Assume that the endomorphism algebra $\mathrm{End}^0(A_0)$ is commutative. So $\mathrm{End}^0(A_0) = K$ for some CM algebra K, and $R := \mathrm{End}(A_0) \subset K$ is a CM order. Let $T := T^{K,\mathbb{Q}}$ and $T^1 := T_1^K$. On the other hand, consider

$$\Lambda(A_0, \lambda_0) := \left\{ (A, \lambda) \in \mathrm{Isog}(A_0, \lambda_0) \,\middle|\, \begin{array}{l} (A, \lambda) \text{ is locally isomorphic} \\ \text{to } (A_0, \lambda_0) \text{ everywhere} \end{array} \right\} \tag{6.14}$$

and

$$I(A_0, \lambda_0) := \left\{ \begin{array}{l} \text{similitude classes of} \\ (A, \lambda) \in \mathrm{Isog}(A_0, \lambda_0) \end{array} \,\middle|\, \begin{array}{l} (A, \lambda) \text{ is locally similar} \\ \text{to } (A_0, \lambda_0) \text{ everywhere} \end{array} \right\}. \tag{6.15}$$

Proposition 6.5. *Let $U := T(\mathbb{A}_f) \cap \widehat{R}^\times$ and $U^1 := T^1(\mathbb{A}_f) \cap \widehat{R}^\times$. We have*

$$|\Lambda(A_0, \lambda_0)| = \frac{[T^1(\widehat{\mathbb{Z}}) : U^1]}{[\mu_K : \mu_K \cap U^1]} \cdot \frac{h_K}{h_{K^+}} \cdot \frac{1}{2^{t-r} Q_K}, \tag{6.16}$$

and

$$|I(A_0, \lambda_0)| = \frac{[T(\widehat{\mathbb{Z}}) : U]}{[\mu_K : \mu_K \cap U]} \cdot \frac{h_K}{h_{K^+}} \cdot \frac{1}{2^{t-r} Q_K} \cdot \frac{\prod_{p \in S_{K/K^+}} e_{T,p}}{[\mathbb{A}^\times : N(T(\mathbb{A})) \cdot \mathbb{Q}^\times]}. \tag{6.17}$$

Proof. The main point is the following natural bijections

$$\Lambda(A_0, \lambda_0) \simeq T^1(\mathbb{Q}) \backslash T^1(\mathbb{A}_f) / U^1, \quad \text{and} \quad I(A_0, \lambda_0) \simeq T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / U. \tag{6.18}$$

See [29, Theorem 5.8 and Section 5.4]; also see [31, Theorem 2.2]. Then formulas (6.16) and (6.17) follow from (6.18), Theorem 1.1, and Lemma 6.1. ∎

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References

[1] J. D. Achter, *Irreducibility of Newton strata in GU$(1,n-1)$Shimura varieties*, Proc. Amer. Math. Soc. Ser. B**1** (2014), 79–88.

[2] J. Achter, S. A. Altug, J. Gordon, W.-W. Li, and T. Rüd, *Counting abelian varieties over finite fields via Frobenius densities*, preprint, 2019, arXiv:1905.11603.

[3] O. Bültel, and T. Wedhorn, *Congruence relations for Shimura varieties associated to some unitary groups*, J. Inst. Math. Jussieu **5** (2006), 229–261.

[4] C. Daw, *On torsion of class groups of CM tori*, Mathematika **58** (2012), 305–318.

[5] P. Deligne, *Travaux de Shimura*. In *Séminaire Bourbaki*, 23ème année (1970/71), Exp. No. 389. Lecture Notes in Math. **244**, Springer, Berlin, 1971.

[6] W. T. Gan, and J.-K. Yu, *Group schemes and local densities*, Duke Math. J. **105** (2000), 497–524.

[7] C. D. González-Avilés, *Chevalley's ambiguous class number formula for an arbitrary torus*, Math. Res. Lett. **15** (2008), 1149–1165.

[8] C. D. González-Avilés, *On Néron-Raynaud class groups of tori and the capitulation problem*, J. Reine Angew. Math. **648** (2010), 149–182.

[9] G. Herglotz, *Über einen Dirichletschen Satz*, Math. Z. **12** (1922), 255–261.

[10] S.-I. Katayama, *Isogenous tori and the class number formulae*, J. Math. Kyoto Univ. **31** (1991), 679–694.

[11] R. E. Kottwitz, *Tamagawa numbers*, Ann. Math **127** (1988), 629–646.

[12] R. E. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), 373–444.

[13] K.-W. Lan, *Arithmetic Compactifications of PEL-type Shimura Varieties,* London Mathematical Society Monographs Series, **36**, Princeton University Press, Princeton, NJ, 2013.

[14] S. Lang, *Algebraic Number Theory,* Graduate Texts in Mathematics, **110**, 2nd ed ., Springer, New York, NY, 1994.

[15] S. Marseglia, *Computing square-free polarized abelian varieties over finite fields*, preprint, 2018, arXiv:1805.10223. To appear in Mathemtics of Computation.

[16] M. Morishita, *OnS-class number relations of algebraic tori in Galois extensions of global fields*, Nagoya Math. J. **124** (1991), 133–144.

[17] J. Neukirch, *Algebraic Number Theory,* Grundlehren der Mathematischen Wissenschaften, **322**, Springer, Berlin, Germany, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher.

[18] T. Ono, *Arithmetic of algebraic tori*, Ann. Math **74** (1961), 101–139.

[19] T. Ono, *On the Tamagawa number of algebraic tori*, Ann. Math **78** (1963), 47–73.

[20] T. Ono, *On Tamagawa numbers*. In Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), American Mathematical Society, Providence, RI, 1966, 122–132.

[21] T. Ono, *On some class number relations for Galois extensions*, Nagoya Math. J. **107** (1987), 121–133.

[22] V. Platonov, and A. Rapinchuk, *Algebraic Groups and Number Theory,* Pure and Applied Mathematics, **139**, Academic Press, Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[23] J. M. Shyr, *On some class number relations of algebraic tori*, Michigan Math. J. **24** (1977), 365–377.

[24] M.-H. Tran, *A formula for the$S$-class number of an algebraic torus*, J. Number Theory **181** (2017), 218–239.

[25] E. Ullmo, and A. Yafaev, *Nombre de classes des tores de multiplication complexe et bornes inférieures pour les orbites galoisiennes de points spéciaux*, Bull. Soc. Math. France **143** (2015), 197–228.

[26] L. C. Washington, *Introduction to Cyclotomic Fields,* Graduate Texts in Mathematics, **83**, 2nd ed ., Springer, New York, NY, 1997.

[27] A. Weil, *Adèles et groupes algébriques*. In Séminaire Bourbaki, vol. **5**, Soc. Math. France, Paris, 1995, 249–257.

[28] J. Xue, T.-C. Yang, and C.-F. Yu, *Numerical invariants of totally imaginary quadratic$\mathbb{Z}\left[\sqrt{p}\right]$-orders*, Taiwanese J. Math **20** (2016), 723–741.

[29] J. Xue, and C.-F. Yu, *On counting certain abelian varieties over finite fields*, preprints, 2018, to appear in Acta Math. Sin. (Engl. Ser.).

[30] C.-F. Yu, *Connected components of certain complex Shimura varieties*. In preparation.

[31] C.-F. Yu, *Simple mass formulas on Shimura varieties of PEL-type*, Forum Math. **22** (2010): 565–582,

[32] S.-W. Zhang, *Equidistribution of CM-points on quaternion Shimura varieties*, Int. Math. Res. Not. **59** (2005), 3657–3689.

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