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A NEW LOOK AT THE DIAMOND SEARCH MODEL: STOCHASTIC CYCLES AND EQUILIBRIUM SELECTION IN SEARCH EQUILIBRIUM

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We recast Diamond's search equilibrium model into that with a finite number of agents. The state of the model is described by a jump-Markov process, the transition rates of which are functions of the reservation cost, which are endogenously determined by value maximization by rational agents. The existence of stochastic fluctuations causes the fraction of the employed to move from one basin of attraction to the other with positive probabilities when the dynamics have multiple equilibria. Stochastic asymmetric cycles that arise are quite different from the cycles of the set of Diamond–Fudenberg nonlinear deterministic differential equations. By taking the number of agents to infinity, we get a limiting probability distribution over the stationary state equilibria. This provides a natural basis for equilibrium selection in models with multiple equilibria, which is new in the economic literature.

Keywords: Search Equilibrium, Jump-Marakov Processes, Fokker–Planck Equations, Asymmetric Cycles, Equilibrium Selection

1. INTRODUCTION

The search equilibrium of Diamond (1982) and its elaboration by Diamond and Fudenberg (1989) have been influential, as evidenced by frequent citations in the search literature. Their model has an infinite number of agents, and the fraction of the employed agents is the state variable. Let N denote the number of agents in the model for later references. By taking $N = \infty$, their dynamic analysis of the fraction is deterministic, as most macroeconomic models are, and totally abstracted from stochastic fluctuations of the fraction near equilibria. Any movements of the fraction are as parts of cycles of a set of nonlinear deterministic differential equations.

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As a modeling strategy, we can take $N = \infty$ or begin with models with $N < \infty$. We reexamine their model in the latter framework.[1] There are advantages and disadvantages in dealing with models with finite numbers of agents. With the conventional choice, dynamics are usually deterministic, and whatever results or the behavior in the models is also deterministic as most macroeconomic descriptions are. This is a great advantage when the formulation with $N = \infty$ is used. What are some of the disadvantages of this approach? One of the most serious ones is the loss of information about fluctuations, externalities, and so on. The cyclical variations exhibited by deterministic models are totally predictable, and are due to cycles of nonlinear deterministic dynamic equations. Despite the more complicated setups and calculations that we must face, the formulation with $N < \infty$ provides information on stochastic fluctuations that otherwise is not available. Another disadvantage of the formulation with $N = \infty$ is difficulty in resolving the equilibrium selection when models have several locally stable equilibria. We show that a natural selection criterion exists for models with $N < \infty$.

More specifically, we have two objectives in recasting the original model this way: One is to obtain information on fluctuations about the equilibria and to provide a simpler explanation than Diamond and Fudenberg did for cyclical behavior. The other is to provide a new and more natural basis for equilibrium selection for the model with multiple equilibria than those in the economic literature on equilibrium selection.

Dynamic behavior of our model is described by the backward Chapman–Kolmogorov equation, or what is called the master equation in the physics and ecology literature.[2] It describes how the probability for the fraction evolves over time. Then, this master equation is solved approximately to yield two equations: One is an ordinary differential equation for the average or expected value of the fraction of the employed. This macroeconomic equation embodies a new aggregation procedure as argued by Aoki (1996, Sect. 5.4; 1998). The other is a partial differential equation, known as the Fokker–Planck equation, for random deviations of the fraction about the mean. When we let the number of agents go to infinity, the equation for the mean reproduces the equation for the fraction derived by Diamond. The critical points of the ordinary differential equation and the endogenously determined reservation cost expression jointly yield information on the equilibria and asymmetrical cyclical behavior.

To derive these, we construct a jump-Markov process with transition rates that are coupled with the value maximization by rational agents. Put differently, unlike the use of jump-Markov processes in the probability literature, our jump-Markov processes have jump rates that depend on the reservation cost for accepting production opportunities, which is determined by comparing value functions for the alternative choices of becoming employed by accepting the production opportunity or remaining unemployed by rejecting the production prospect as being too costly. The model construction and transition rate specifications are discussed in Section 2.

On our second objective, we provide a quite different explanation for the occurrence of business cycles from that given by Diamond and Fudenberg—a stochastic asymmetrical business cycle. Fluctuations about aggregate dynamics occur in our analysis because microshocks intrinsic in our models do not vanish when the number of agents in the model is finite. With positive probabilities, net effects of arrivals of production and trading opportunities do not vanish, but accumulate to change the fraction of employed from one basin of attraction to the other.[3] In Section 3, we derive the aggregate (macroeconomic) dynamic equation, and in Section 4, we derive the Fokker–Planck equation for the fluctuations about the means.

Multiple equilibria in Diamond's model produce the problem of equilibrium selection. It often is argued that the expectation held by agents determines which equilibrium is to be realized. In our approach, we get the equilibrium transition rates of aggregate states, which generate a stationary (invariant) probability distribution over fractions of employed agents, and there are multiple basins of attraction associated with the stationary probability distribution. The probability that the economy stays in each basin of attraction can be calculated. As we bring the number of agents to infinity, our model converges to the deterministic model with multiple stationary states, but with some probability assigned to each stationary state. These probabilities can be used as a natural basis for the equilibrium selection criteria.[4] Section 5 derives the value functions of our model. In Section 6, we discuss examples with multiple equilibria.

We show in Section 7 that the larger the basin of attraction for a stationary state, or the smaller the fluctuation around the stationary state, the more likely it is that such a stationary state well be selected as the equilibrium.

2. MODEL

2.1. Setup

There is a large but a finite number, N, of agents,[5] who are in one of two possible states, employed and unemployed. Of the N agents in the model, n of them are employed, and $N-n$ are unemployed. The state of the collection of the N agents is n or, equivalently, the fraction $e = n/N$. Each of the $N-n$ unemployed persons independently encounters a production opportunity that appears at the rate of $a\Delta t$ in a small time interval Δt. If the opportunity is accepted, it yields the unit output and at the cost c, where c is a nonnegative random number with a known distribution function G. There is a reservation or threshold cost $c^*(n)$, to be determined endogenously later, above which the opportunity is rejected as being too costly. When the opportunity is accepted, the person's status changes from being unemployed to being employed. Each of n employed persons independently encounters a trading opportunity at the rate $b(n/N)$ per unit time. When an employed person encounters a trading opportunity, he forms a pair with another randomly selected employed person, and the pair trade and each of the pair consumes the output of the partner to receive instantaneous utility v and their status changes from being employed to being unemployed. See Diamond (1982) for some explanations for these assumptions.

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Let $W_e(n, t)$ be the present discounted value of lifetime utility of an employed person, and let $W_u(n, t)$ be that of an unemployed person when the state is n. Because n is a random variable in this paper, we take the expectation of these random value functions after we derive the stationary distribution of n. We drop t from the argument of the value functions because dynamic programming involves an infinite horizon and the problem is time-homogeneous.

The value functions are evaluated in Section 5 after we discuss the dynamics for the mean of the fraction and a Fokker–Planck equation for the fluctuations about the mean in Sections 3 and 4. Section 6 provides the special case of the model that creates two locally stable equilibria. With this example, we discuss the problems of first passage times between two locally stable equilibria and equilibrium selection in Section 7. We conclude in Section 8.

2.2. Transition Rates

We model the problem as a jump-Markov process. Thus, the model is completely specified by the transition rates that describe movements of agents over a small interval of time.

To an unemployed agent, production opportunities arrive at the rate a as a Poisson process. Each production opportunity if undertaken yields a unit of output with cost c. Only production with cost c^* or less will be undertaken. The transition rate from n to $n+1$ is given by $(N-n)aG(c^*)$, where c^* is the "reservation" cost in the sense that only the production with cost $c \le c^*$ is undertaken. Because this reservation cost is a choice variable and depends on n/N, we write it as $c^*(n/N)$, or as $c^*(n)$ for short, in the following. This transition rate is thus endogenous, unlike most rates in the probability textbooks. The determination of the reservation cost is discussed in Section 5.

For an employed agent, trading opportunities arrive as a Poisson process at the rate $\beta(n/N)$. His probability for being one of the random pair is $1 - C_{n-1,2}/C_{n,2} = 2/n$. We define the arrival rate of trading opportunity for an agent to be $b(n/N) := (2/n)\beta(n/N)$. While an employed agent waits for a trading partner, the probability is $[C_{n-1,2}/C_{n,2}]\beta = [(n-2)/n]\beta$ that a pair involving other employed agents trade, thus decreasing n to $n-2$. In aggregate, then, the transition rate from state n to $n-2$ is given by $(n/2)b(n/N)$.

3. AGGREGATE DYNAMICS: DYNAMICS FOR THE MEAN OF THE FRACTION

Denote the probability distribution that n persons are being employed at time t by $P_n(t)$. The master equation [see Aoki (1996, Sect. 5.1)] is

$$dP_n(t)/dt = r_{n-1}P_{n-1}(t) + l_{n+2}P_{n+2}(t) - (r_n + l_n)P_n(t),$$

with obvious boundary conditions imposed at $n = 0$ and N, and near these values as shown in Section 5.

From our previous discussion,

$$r_n = (N-n)aG[c^*(n/N)] = N(1-e)aG[c^*(n/N)],$$

and

$$l_n = \frac{n}{2} b\left(\frac{n}{N}\right).$$

Because this equation cannot be solved exactly, we proceed as in Aoki (1996, p. 123) to derive an approximate solution. To do so, we change variables as

$$\frac{n}{N} = \phi + \frac{\xi}{\sqrt{N}}. \tag{1}$$

The variable ϕ is the expected fraction of employed and ξ represents random fluctuations about the mean. The idea of this change in variables is that we presume that there is a peak of $P_n(t)$ at $n = \phi N$ of order N, while its width will be of order $\sqrt{N}$.[6]

This presumption is correct if distribution of ξ turns out to be that with mean zero and some variance. It is shown in Section 4 that it is so.

In this change of variables, note that $(n+1)/N = \phi + N^{-1/2}(\xi + 1/\sqrt{N})$, and so on. For example, ξ changes by $2/\sqrt{N}$ in l_{n+2}. Thus, the function $P_n(t)$ transforms into a function $\Pi(\xi, t)$ of ξ according to

$$P_n(t) = P_{N\phi+\sqrt{N}\xi}(t) := \Pi(\xi, t).$$

The left-hand side of the master equation is now rewritten in terms of Π by noting that

$$\frac{dP_n(t)}{dt} = \frac{\partial \Pi}{\partial t} + \frac{\partial \Pi}{\partial \xi}\frac{d\xi}{dt},$$

where $d\xi/dt$ is obtained by differentiating (1) with respect to t with n fixed:

$$\frac{d\xi}{dt} = -\sqrt{N}\frac{d\phi}{dt}.$$

The right-hand side of the master equation can be expanded in terms of orders $\sqrt{N}$, N^0, $N^{-1/2}$, and so on. It is given as (see Appendix for the derivation)

$$-\Phi(\phi)\frac{\partial \Pi}{\partial \xi}\sqrt{N} - \left\{ \Phi'(\phi)\Pi + \Phi'(\phi)\xi\frac{\partial \Pi}{\partial \xi} - \left[\frac{\Phi(\phi) + 3\phi b(\phi)}{2}\right]\frac{\partial^2 \Pi}{\partial \xi^2} \right\} N^0 + O(N^{-1/2}),$$

where

$$\Phi(\phi) := (1-\phi)aG[c(\phi)] - \phi b(\phi).$$

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We match the left-hand side of the order $\sqrt{N}$ with the terms of the same order on the right-hand side and derive the aggregate dynamic equation for ϕ as

$$\frac{d\phi}{dt} = \Phi(\phi) = (1-\phi)aG(c^*) - \phi b(\phi). \tag{2}$$

This is in agreement with the dynamic equation for e in Diamond (1982, Eq. 1).

4. DYNAMICS FOR THE FLUCTUATIONS

The rest of terms are for determining the distribution of ξ. By collecting terms of order $O(N^0)$ in the Taylor-series expansion, this equation is seen to be given by

$$\frac{\partial \Pi}{\partial t} = A(\phi)\Pi + A(\phi)\xi\frac{\partial \Pi}{\partial \xi} + C(\phi)\frac{\partial^2 \Pi}{\partial \xi^2} + O(N^{-1/2}), \tag{3}$$

with

$$A(\phi) = -\Phi'(\phi)$$

and

$$C(\phi) = (1/2)(1-\phi)aG(c^*) + \phi b(\phi).$$

This is a type of Fokker–Planck equation that can be solved as discussed by Aoki (1996, Sect. 5.13), for example. As we discuss shortly, the local equilibria of the dynamics are the zeros of the function Φ. Its derivative Φ' is negative at those local equilibria that are locally asymptotically stable; that is, at those locally stable equilibria, $A(\phi)$ is positive. Note that the coefficient $C(\phi)$ is $2\phi b(\phi)$ at the critical points, that is, when $\Phi(\phi) = 0$.

The stationary distribution of ξ is derived by solving equation (3) with $\partial\Pi/\partial t = 0$. The equilibrium distribution $\Pi^e(\xi)$ is given by

$$\Pi^e(\xi) = \frac{1}{\sqrt{2\pi C(\phi)/A(\phi)}} \exp\left[-\frac{A(\phi)}{C(\phi)}\frac{\xi^2}{2}\right].$$

We have thus shown that the stationary distribution for ξ is normally distributed with mean zero and variance $C(\phi)/A(\phi)$. Its variance is given by

$$\mathrm{Var}(\xi) = \frac{C(\phi)}{A(\phi)} := \sigma^2(\phi).$$

Recall the change of variable (1). From this, we can readily see that $e := n/N$ has the following density function around the locally stable critical point ϕ:

$$\frac{1}{\sqrt{2\pi\sigma^2(\phi)/N}} \exp\left[-\frac{(e-\phi)^2}{2\sigma^2(\phi)/N}\right]. \tag{4}$$

With two or more locally stable equilibria, the probability mass around each of the critical points may overlap and assign positive probability to the neighboring

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critical points. This is one sufficient condition for fluctuations to spill over to the neighboring basins of attraction. Even if this does not happen, we show later that expected first passage times from one basin to the neighboring ones are finite; that is, cycles are possible.

5. VALUE FUNCTIONS

5.1. Determination of Reservation Cost

Denote the discount rate by r. Value functions depend on the fraction n/N rather than on n directly. For shorter notation, however, we denote them as $W_e(n)$ and $W_u(n)$ for the employed and unemployed when the number of the employed is n. For an employed agent, we obtain the relation for the value functions as

$$\begin{aligned} rW_e(n) &= b\left(\frac{n}{N}\right)[v + W_u(n-2) - W_e(n)] \\ &\quad + (N-n)aG[c^*(n)][W_e(n+1) - W_e(n)] \\ &\quad + \frac{n-2}{2}b\left(\frac{n}{N}\right)[W_e(n-2) - W_e(n)] \end{aligned} \tag{5}$$

for n between 3 and $N-1$, and for an unemployed agent,[7]

$$\begin{aligned} rW_u(n) &= a\int_0^{c^*(n)} [W_e(n+1) - W_u(n) - z]\,dG(z) \\ &\quad + (N-n-1)aG[c^*(n)][W_u(n+1) - W_u(n)] \\ &\quad + \frac{n}{2}b\left(\frac{n}{N}\right)[W_u(n-2) - W_u(n)], \end{aligned} \tag{6}$$

for $n = 2, 3, \ldots, N-1$. The boundary relations are

$$rW_e(N) = b(1)[v + W_u(N-2) - W_e(N)] + \frac{N-2}{2}b(1)[W_e(N-2) - W_e(N)] \tag{7}$$

and

$$rW_e(n) = (N-n)aG^*(n)[W_e(n+1) - W_e(n)], \tag{8}$$

for $n = 1, 2$, where $G^*(n) := G[c^*(n)]$. Finally,

$$rW_u(n) = aG^* + (N-n-1)aG^*(n)[W_u(n+1) - W_u(n)] \tag{9}$$

for $n = 0, 1$.

In this paper, we examine the optimal search rule for $c^*(n)$:

$$c^*(n) = \max\{0, W_e(n+1) - W_u(n)\} \tag{10}$$

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for $n = 0, 1, 2, \ldots, N-1$. This is the rule used by Diamond and is obtained by maximizing the integral term in equation (6).[8] The set of equations (5)–(10) determines the optimal values of $W_e(n)$, $W_u(n)$, and $c^*(n)$. With this optimal cutoff level, we can determine the transition rates given in preceding section.

5.2. Expected Value Functions

We next take the expected values of these value functions with respect to the stationary distributions of n. By changing variables as done in (1), we have shown that the stationary distribution for ξ is normally distributed with mean zero and variance that is a function of ϕ but is independent of N. This is a posteriori justification for the change of variables that we have performed.

Rather than obtaining optimal sequences of the cutoff levels by solving the set of equations displayed above, we first derive the expressions for the expected values of the value functions around ϕ and then derive the expression for the cutoff levels as functions of ϕ up to terms $O(1/N)$.

We change variables as indicated earlier, and define

$$V_u(\phi + \xi/\sqrt{N}) := W_u(n),$$

$$V_e(\phi + \xi/\sqrt{N}) := W_e(n),$$

and

$$c(\phi + \xi/\sqrt{N}) := c^*(n).$$

Performing the Taylor-series expansion of the preceding three equations around ϕ and noting that $E\xi = 0$, and $E\xi^2 = \sigma^2$, we find that, after dropping terms of the order $1/N$ or less, the expected value functions and cutoff level become

$$rV_u(\phi) = aG[c(\phi)][V_e(\phi) - V_u(\phi)] - a\hat{c} + \Phi(\phi)V_u'(\phi), \tag{11}$$

$$rV_e(\phi) = b(\phi)[v + V_u(\phi) - V_e(\phi)] + \Phi(\phi)V_e'(\phi), \tag{12}$$

and

$$c(\phi) = V_e(\phi) - V_u(\phi), \tag{13}$$

where

$$\hat{c} = \int_{\underline{c}}^{c(\phi)} z\, dG(z).$$

Details of the algebra are in the Appendix. Equations (11) and (12) correspond to equations (4a) and (4b) of Diamond and Fudenberg (1989).

Making use of the fact that $\Phi(\phi_e) = 0$, where ϕ_e denotes locally stable equilibrium points, (11) and (12) yield equilibrium value functions

$$V_e(\phi_e) = \frac{r + aG[c(\phi)]b(\phi_e)v - ab(\phi_e)\hat{c}}{r[r + b(\phi_e) + aG(\phi)]}$$

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and

$$V_u(\phi_e) = \frac{b(\phi_e)aG[c(\phi)]v - a[r + b(\phi_e)]\hat{c}}{r\{r + b(\phi_e) + aG[c(\phi)]\}},$$

where $\hat{c}$ is evaluated at ϕ_e.

By subtracting (11) from (12), the expected cutoff level is given implicitly by

$$rc(\phi) = b(\phi)[v - c(\phi)] - a\{c(\phi)G[c(\phi)] - \hat{c}\} + \Phi(\phi)[V_e'(\phi) - V_u'(\phi)],$$

where the last term is recognized as $dc(\phi)/d\tau = (dc^*/d\phi)(d\phi/d\tau)$.

Thus, the parallel developments or the correspondence with the case of infinite number of agents holds. We can actually see that this choice of $c(\phi)$ is optimal by differentiating the expected value functions with respect to $c(\phi)$, noting that $b(\phi)$ is exogenously specified and its derivative with respect to $c(\phi)$ is zero. Solving for the derivatives of the expected value functions with respect to $c(\phi)$, we see that they are both zero. This is the first-order condition for optimality. The second-order condition may be shown to hold by taking derivatives once more.

6. MULTIPLE EQUILIBRIA AND CYCLES: AN EXAMPLE WITH DISCRETE COST DISTRIBUTION

Our method of solving the master equation requires that ϕ be at the interior of $[0, 1]$. Thus, to see the stochastic dynamics of the model with multiple equilibria, we need to construct a model with multiple locally stable equilibria with each of them in the interior of $[0, 1]$. In this section, we provide an example with two locally stable interior equilibria. We use this example for the discussion in the following sections.

Let $b(\phi) = \phi$. Also, suppose that the costs are distributed in a discrete manner. More specifically, suppose that there are two possible costs: $c_1 = 0$ and $c_2 \geq 0$, that is, the distribution function $G(c)$ is a step function; $G(c_1) = p > 0$; $G(c_2) = 1$. We also assume that $v > c_2$. The right-hand side of the dynamics for the aggregate equation (2) is either $\Phi_1(\phi) := [(1 - \phi)ap - \phi^2]$ or $\Phi_2(\phi) := [(1 - \phi)a - \phi^2]$, depending on the range of the argument ϕ.

There are thus two critical points. They are the roots of $\Phi_i(\phi) = 0$, $i = 1, 2$, and are given by

$$\phi_1 = [\sqrt{a^2p^2 + 4ap} - ap]/2$$

and

$$\phi_2 = [\sqrt{a^2 + 4a} - a]/2 := \kappa.$$

We see that $\Phi_i'(\phi_i)$ are negative for $i = 1, 2$; that is, the critical points are locally stable.

From the optimality condition, $c_1^* = c(\phi_1)$ is determined by

$$rc_1^* = \phi_1(v - c_1^*) - apc_1^*$$

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or

$$c_1^* = \frac{\phi_1 v}{r + \phi_1 + ap},$$

if $0 < c_1^* < c_2$.

The second value $c_2^* = c(\phi_2)$ is determined by

$$rc_2^* = \phi_2\left(v - c_2^*\right) - ac_2^* + ac_2(1 - p)$$

or

$$c_2^* = \frac{\phi_2 v + ac_2(1 - p)}{r + \phi_2 + a},$$

if $c_2 < c_2^*$. It is clear that $\Phi_1(\phi_1) = 0$ and $\Phi_2(\phi_2) = 0$ are valid only if $0 \le c_1^* < c_2 < c_2^*$. The last inequalities are represented by the conditions

$$\phi_1 < \frac{(r + ap)c_2}{v - c_2} < \phi_2,$$

which can be rearranged as

$$\begin{aligned} &ap[1 - c_2(r + ap)(v - c_2)](v - c_2)^2 < (v - c_2)^2 c_2^2 \\ &\quad < a[1 - (r + ap)(v - c_2)](v - c_2)^2. \end{aligned} \qquad \textbf{(14)}$$

The two basins of attraction are separated at

$$\psi = \frac{c_2(r + ap)}{(v - c_2)};$$

that is, the value of Φ undergoes a discontinuous change at this value:

$$\Phi_1(\psi-) < 0$$

and

$$\Phi_2(\psi+) > 0.$$

See Figure 1. The conditions to ensure that $0 \le \phi_1 < \psi < \phi_2 \le 1$ are the same as (14). Thus, a small p and a not-too-large c_2 will suffice to satisfy these conditions. We can also construct the examples of three or more discrete costs. However, the conditions on parameters are more complicated to state.

If there is a large positive disturbance near ϕ_1 that makes the variable ϕ cross the boundary at ψ, then the derivative is positive and the disturbance is amplified and ϕ is attracted to ϕ_2. Conversely, a large negative disturbance near ϕ_2 will cause the state variable to be attracted to ϕ_1. This is what we mentioned at the end of Section 4.

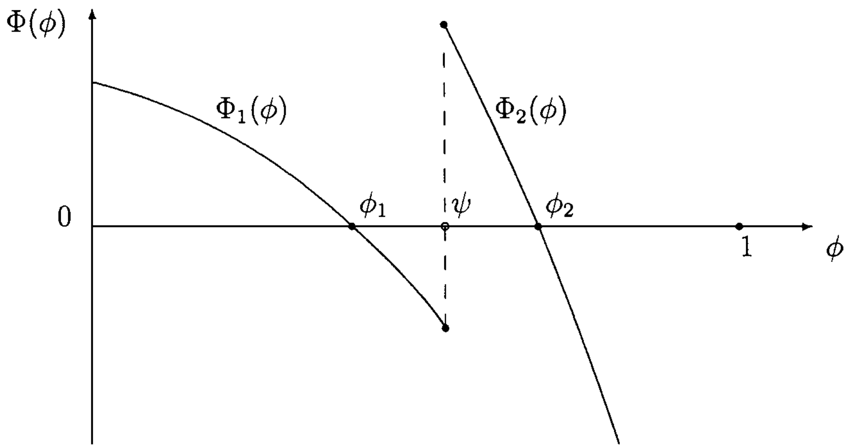

FIGURE 1. Example of the aggregate dynamic equation, equation (2).

7. MEAN FIRST PASSAGE TIME AND EQUILIBRIUM SELECTION

This section shows how we approximately evaluate the mean transition times from one basin of attraction to the other, and calculate the equilibrium probabilities that the employed fraction of population stays in each of the two basins of attraction. Moreover, we show that our analysis provides the basis for equilibrium selection in the deterministic version of the model. Because the states off the equilibria approach them exponentially fast, only the events of disturbances that are large enough to bring the states from one basin of attraction to another will force the model to move from one equilibrium to another. These are rare events in the large deviation theory, and can be analyzed as such.[9] In this paper we merely provide an approximate analysis of the next subsection.

7.1. Approximate Analysis

First we recognize that we need to calculate only the event from one of the equilibrium states to the boundary between two basins of attraction, ψ, which is introduced in the example above. The reason is the same one used by van Kampen (1992) as quoted by Aoki (1996, p. 151). The time needed for ϕ to reach its equilibrium value, ϕ_1, or ϕ_2, depending on the initial value, is much shorter than the time needed to go from one basin of attraction to the other.

A quick way to see this is to solve the deviational equation for ϕ.[10] To be definite, suppose that ϕ is in the domain of attraction to ϕ_1 and let $x := \phi - \phi_1$. Then, it is governed by

$$\frac{dx}{d\tau} = \Phi_1'(\phi_1)x = -A(\phi_1)x,$$

with the initial condition $x(0) = \phi(0) - \phi_1$.

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The solution is $x(\tau) = x(0)\exp[-A(\phi_1)\tau]$. Recalling that $\exp(-4.5) = 0.01$, we find that it takes about $\tau = 4.5/A(\phi_1)$ to reduce the distance from ϕ_1 to about 0.01 of the original value. In the case where $a = 1$ and $p = 0.2$, we have $\phi_1 = 0.358$ and $A(\phi_1) = 0.92$. Thus, it takes about 4.9 or 5 time units to reach the equilibrium point. As we show later in Example 2, the mean first passage time for this example is of the order of 10^3 when $N = 100$. Therefore, we are justified in assuming that ϕ is initially at one of the equilibrium points when we calculate the mean first passage time. The procedure is as outlined by Aoki (1998, Appendix). We set up a two-state Markov chain because there are two locally stable equilibria in the example. Let π_1 and $\pi_2 = 1 - \pi_1$ be the probability that the employed fraction are in basins of attraction for ϕ_1 and ϕ_2, respectively. These probabilities evolve according to the differential equation

$$\frac{d\pi_1}{d\tau} = W_{2,1}\pi_2 - W_{1,2}\pi_1,$$

where $W_{1,2}$ is the transition rate from ϕ_1 to the boundary of the two basins of attraction (i.e., ψ) and $W_{2,1}$ is that from ϕ_2 to ψ. In the stationary state $d\pi_1/d\tau = 0$, we have

$$\pi_1 = \frac{1}{W_{1,2}/W_{2,1} + 1}. \tag{15}$$

The mean first passage time is given by

$$\tau_{1,2} = \frac{1}{W_{1,2}}.$$

See Aoki (1996, p. 152).

To calculate $W_{1,2}$ we use the probability that

$$W_{1,2} = \Pr[\xi \geq \xi_c],$$

with

$$\phi_1 + \frac{\xi_c}{\sqrt{N}} = \psi.$$

Analogously, $W_{2,1}$ is approximated by the probability that ξ is smaller than $\sqrt{N}(\psi - \phi_2)$ or, equivalently, it is larger than $\sqrt{N}(\phi_2 - \psi)$.

The following examples suggest that there are positive probabilities assigned for the system to move from one basin of attraction to the other and vice versa when the number of agents is finite.

Example 1

Let $a = 1$, $r = 0.1$, $p = 0.3$, and $c_2/v = 0.55$. Then, the boundary of the two basins is located at $\psi = 0.489$.

The variance of ξ in Basin 1 is $\Sigma = \phi_1^2/A(\phi_1)$. The standard deviation $sd_1 = 0.374$, and $sd_2 = 0.413$. Thus,

$$W_{1,2} = \Pr[\xi \geq \sqrt{N}(\psi - \phi_1)]$$

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and

$$W_{2,1} = \Pr[\xi \geq \sqrt{N}(\phi_2 - \psi)].$$

Suppose that $N = 50$. Then, $W_{1,2} = 0.00665$, and $W_{2,1} = 0.01375$.

We thus estimate the mean first passage times as $\tau_{1,2} = 150.4$, $\tau_{2,1} = 72.7$, $\pi_1 = 0.67$, and $\pi_2 = 0.33$. The ratio of the mean first passage times is 1.9.

If the number of agents is $N = 100$, then $W_{1,2} = 0.00025$, $W_{21} = 0.0009$, $\tau_{1,2} = 4{,}000$, $\tau_{2,1} = 1{,}100$, and $\pi_1 = 0.78$.

Example 2

Let $a = 1$, $r = p = 0.2$, and $c_2/v = 0.55$. The boundary value in this case is given by $\psi = 0.49$.

With $N = 50$, $W_{1,2} = 0.0062$, $W_{2,1} = 0.0119$, $\tau_{1,2} = 161.3$, $\tau_{2,1} = 84.03$, $\pi_1 = 0.657$.

With $N = 100$, the mean first passage times increase to $\tau_{1,2} = 5{,}000$ and $\tau_{2,1} = 1{,}000$ with $\pi_1 = 0.83$.

7.2. Equilibrium Selection

As we increase the number of agents in the economy to infinity, our model converges to that of Diamond. This can be seen by the fact that variance of the employed fraction of agents converges to zero as N is taken to infinity, as indicated by the density function (4). This suggests that the stationary (invariant) distribution over the fraction of employed agents in the economy appears as spikes, with probability masses of π_1 and π_2 assigned for employed fractions $e = \phi_1$ and $e = \phi_2$, respectively.

These probability masses for each locally stable critical point provide the criteria for equilibrium selection for the model of multiple equilibria with an infinite number of agents. One can easily check that our special case given in Section 6 yields exactly the same stationary fractions of employed agents ϕ_1 and ϕ_2 in Diamond's model if we set the same matching function b and cost distribution function G.

What we are left to do is to calculate π_1 when N is taken to infinity. As suggested earlier, we have

$$W_{1,2} = \Pr[\xi > \sqrt{N}(\psi - \phi_1)]$$
$$= \int_{\sqrt{N}(\psi-\phi_1)}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(\phi_1)}} \exp\left[-\frac{\xi^2}{2\sigma^2(\phi_1)}\right] d\xi$$

and

$$W_{2,1} = \Pr[\xi < \sqrt{N}(\psi - \phi_2)]$$
$$= \int_{\sqrt{N}(\phi_2-\psi)}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(\phi_2)}} \exp\left[-\frac{\xi^2}{2\sigma^2(\phi_2)}\right] d\xi.$$

It is easy to see that both $W_{1,2}$ and $W_{2,1}$ approach zero as N is brought to infinity. Hence, we can approximate $\lim_{N\to\infty} W_{1,2}/W_{2,1}$ by $\lim_{N\to\infty}(dW_{1,2}/dN)/(dW_{2,1}/N)$. This is given by

$$\lim_{N\to\infty}\frac{W_{1,2}}{W_{2,1}} = \lim_{N\to\infty}\frac{dW_{1,2}/dN}{dW_{2,1}/dN}$$
$$= \lim_{N\to\infty}\exp\left\{\left[\frac{(\phi_2-\psi)^2}{2\sigma^2(\phi_2)} - \frac{(\psi-\phi_1)^2}{2\sigma^2(\phi_1)}\right]N\right\}\frac{(\psi-\phi_1)\sigma(\phi_2)}{(\phi_2-\psi)\sigma(\phi_1)}. \quad \textbf{(16)}$$

From this, it is straightforward to see that, as N approaches infinity, $W_{1,2}/W_{2,1}$ approaches 0 if and only if $(\psi-\phi_1)/\sigma(\phi_1) > (\phi_2-\psi)/\sigma(\phi_2)$; $W_{1,2}/W_{2,1}$ approaches infinity if and only if $(\psi-\phi_1)/\sigma(\phi_1) < (\phi_2-\psi)/\sigma(\phi_2)$; and $W_{1,2}/W_{2,1}$ approaches 1 if and only if $(\psi-\phi_1)/\sigma(\phi_1) = (\phi_2-\psi)/\sigma(\phi_2)$.

The result is summarized as follows:

$$\lim_{N\to\infty}\pi_1 = \begin{cases} 1 & \text{if } (\psi-\phi_1)/\sigma(\phi_1) > (\phi_2-\psi)/\sigma(\phi_2), \\ 1/2 & \text{if } (\psi-\phi_1)/\sigma(\phi_1) = (\phi_2-\psi)/\sigma(\phi_2), \text{ and} \\ 0 & \text{if } (\psi-\phi_1)/\sigma(\phi_1) < (\phi_2-\psi)/\sigma(\phi_2). \end{cases}$$

The larger the distance between the critical point and the boundary of the basins of attraction, or the smaller the variance of fluctuation around the critical point, the more likely it is that this critical point will be selected as an equilibrium in a model with an infinite number of agents.

8. CONCLUSION

We have reexamined the Diamond search model for the case of a finite number of agents to show potential advantages of not assuming that there are an infinite number of agents in the model from the beginning.

Using the Diamond model as a vehicle of illustration, this paper has dealt with three important issues in modeling agent interactions. One is the use of $N < \infty$ (a finitely model, that is, the number of agents in the model is finite), the second is to couple rational choices with the jump-Markov processes with endogenous transition rates to describe dynamic discrete-choice situations, and the third is the equilibrium selection procedure. With a finite number of agents, the fraction of employed agents is a random variable that fluctuates about its mean value. We have shown that the fluctuations are of the form $\xi/\sqrt{N}$, where N is the total number of agents, and the distribution function for ξ is Gaussian with mean and finite variance, which is a function of the expected fraction of the employed agents. We have illustrated by a simple example that the model can have several locally stable equilibria and that the fraction of the employed agents may fluctuate between the pair of equilibria. We have shown that this leads to a simpler explanation of asymmetrical cycles, among others.

Moreover, if the number of agents are taken to infinity, we acquire the probability distribution for multiple stationary state equilibria of the deterministic model.

This provides the basis for equilibrium selection for the model with multiple equilibria.

NOTES

1. This example was suggested by Orszak (1997) as one of his comments on Aoki (1996).

2. See Aoki (1996, Sect. 5.1) or van Kampen (1992, p. 97) for the source of this name.

3. The idea of microshocks creating aggregate risk is pointed out by Jovanovic (1987). His main point is that, in the nonlinear systems, microshocks intrinsic to the model do not vanish. Kirman (1993) discusses a mechanism of stochastic cycles. The focus of his model is on herding effects. See Aoki (1998, p. 436) for comparison of our method with that of Kirman. Furthermore, our model in this paper has optimizing agents. In such a model, fluctuations among two basins of attractions are still possible.

4. There is a vast literature on equilibrium selection in evolutionary game theory [see Kandori (1997)]. However, most of their approaches introduce exogenous mutations to generate stochastic perturbation in the dynamics. Moreover, their analyses are done in simple games with multiple equilibria. How it can be applied to equilibrium selection in "economic models" with multiple equilibria is not clear.

5. We take N to be a large fixed number. It is straightforward to let N be random.

6. That this is the correct order is indicated by the fact that the coefficients of the Fokker–Planck equation for ξ, to be derived later, are independent of N. See Kelly (1979), for example.

7. The probability intensity for the transition from state n to state $n+1$ is $(N-n)aG^*$, where G^* is shorthand for $G(c^*)$, of which $(N-n-1)aG^*$ is the intensity for other unemployed agents to become employed while he remains unemployed. The intensity for him to become employed is aG^*.

8. The optimum rule is obtained by maximizing the integral expression in $W_u(n)$ with respect to $c^*(n)$, recognizing that $c^*(n)$ in the second term is due to the choice by others, and appealing to the exchangeability of agents.

9. See Aoki (1996, Sect. 3.6.2) for a quick look, or Dembo and Zeitouni (1993) for a more rigorous discussion.

10. In the example of the multiple equilibria with discrete and finite support for the distribution function G, one can actually calculate the time to reach a small neighborhood of the equilibrium point by direct integration of the dynamics in the phase plane. This yields results similar to those obtained by deviational analysis.

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APPENDIX

TAYLOR-SERIES EXPANSION OF THE MASTER EQUATION

The master equation is given by

$$\frac{dP_n(t)}{dt} = (N-n+1)aG(c^{n-1})P_{n-1}(t) + \frac{n+2}{2}b\left(\frac{n+2}{N}\right)P_{n+2}(t)$$
$$-\left[(N-n)aG(c^n) + \frac{n}{2}b\left(\frac{n}{N}\right)\right]P_n(t);$$

c^n can be considered as a function for n/N. Hence we can write $c^n = c(n/N)$. Moreover, we have $n/N = \phi + \xi/\sqrt{N}$. Also, setting $P_n(t) = \Pi(\xi, t)$ and through Taylor-series expansions of functions $b(\cdot)$, $c(\cdot)$, and $\Pi(\xi, t)$, we can rewrite the master equation as

$$\frac{d\Pi(\xi,t)}{dt}\left(=\frac{\partial \Pi}{\partial t} - \frac{\partial \Pi}{\partial \xi}\frac{d\phi}{dt}\sqrt{N}\right)$$
$$= aN\left(1-\phi-\frac{\xi}{\sqrt{N}}+\frac{1}{N}\right)\left[G(c)+G'(c)c'\left(\frac{\xi}{\sqrt{N}}-\frac{1}{N}\right)\right]$$
$$\times\left[\Pi - \frac{\partial \Pi}{\partial \xi}\frac{1}{\sqrt{N}} + \frac{1}{2}\frac{\partial^2 \Pi}{\partial \xi^2}\frac{1}{N}\right]$$
$$+\frac{N}{2}\left(\phi+\frac{\xi}{\sqrt{N}}+\frac{2}{N}\right)\left[b(\phi)+b'(\phi)\left(\frac{\xi}{\sqrt{N}}+\frac{2}{N}\right)\right]\left[\Pi + \frac{\partial \Pi}{\partial \xi}\frac{2}{\sqrt{N}} + \frac{\partial^2 \Pi}{\partial \xi^2}\frac{2}{N}\right]$$
$$-aN\left(1-\phi-\frac{\xi}{\sqrt{N}}\right)\left(G(c)+G'(c)c'\frac{\xi}{\sqrt{N}}\right)\Pi$$
$$-\frac{N}{2}\left(\phi+\frac{\xi}{\sqrt{N}}\right)\left[b(\phi)+b'(\phi)\frac{\xi}{\sqrt{N}}\right]\Pi + O(N^{-1})$$
$$= aG(c)\Pi + NaG'c'\left(1-\phi-\frac{\xi}{\sqrt{N}}+\frac{1}{N}\right)\left(\frac{\xi}{\sqrt{N}}-\frac{1}{N}\right)\Pi$$
$$-aNG'c'\left(1-\phi-\frac{\xi}{\sqrt{N}}\right)\frac{\xi}{\sqrt{N}}\Pi$$
$$-aNG\left(1-\phi-\frac{\xi}{\sqrt{N}}+\frac{1}{N}\right)\left(\frac{\partial \Pi}{\partial \xi}\frac{1}{\sqrt{N}} - \frac{1}{2}\frac{\partial^2 \Pi}{\partial \xi^2}\frac{1}{N}\right)$$
$$-aNG'c'\left(1-\phi-\frac{\xi}{\sqrt{N}}+\frac{1}{N}\right)\left(\frac{\xi}{\sqrt{N}}-\frac{1}{N}\right)\left(\frac{\partial \Pi}{\partial \xi}\frac{1}{\sqrt{N}} - \frac{1}{2}\frac{\partial^2 \Pi}{\partial \xi^2}\frac{1}{N}\right)$$
$$+b(\phi)\Pi + \frac{Nb'}{2}\left(\phi+\frac{\xi}{\sqrt{N}}+\frac{2}{N}\right)\left(\frac{\xi}{\sqrt{N}}+\frac{2}{N}\right)\Pi$$

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$$- \frac{Nb'}{2}\left(\phi + \frac{\xi}{\sqrt{N}}\right)\frac{\xi}{\sqrt{N}}\Pi + \frac{Nb}{2}\left(\phi + \frac{\xi}{\sqrt{N}} + \frac{2}{N}\right)\left(\frac{\partial \Pi}{\partial \xi}\frac{2}{\sqrt{N}} + \frac{\partial^2 \Pi}{\partial \xi^2}\frac{2}{N}\right)$$

$$+ \frac{Nb'}{2}\left(\phi + \frac{\xi}{\sqrt{N}} + \frac{2}{N}\right)\left(\frac{\xi}{\sqrt{N}} + \frac{2}{N}\right)\left(\frac{\partial \Pi}{\partial \xi}\frac{2}{\sqrt{N}} + \frac{\partial^2 \Pi}{\partial \xi^2}\frac{2}{N}\right) + O(N^{-1})$$

$$= -[(1-\phi)aG - \phi b]\frac{\partial \Pi}{\partial \xi}\sqrt{N} + \Big\{[aG(c) + b + \phi b' - (1-\phi)aG'c']\Pi$$

$$+ [aG + b + \phi b' - (1-\phi)aG'c']\xi\frac{\partial \Pi}{\partial \xi} + \left(\frac{1-\phi}{2}aG + \phi b\right)\frac{\partial^2 \Pi}{\partial \xi^2}\Big\}N^0 + O(N^{-1/2}).$$

The comparison of their term with order $\sqrt{N}$ gives us the aggregate law of motion for ϕ:

$$\frac{d\phi}{dt} = (1-\phi)aG[c(\phi)] - \phi b(\phi).$$

The Fokker–Planck equation is the coefficient of the term with order N^0 and is given by

$$A(\phi)\Pi + A(\phi)\frac{\partial \Pi}{\partial \xi} + C(\phi)\frac{\partial^2 \Pi}{\partial \xi^2} = 0$$

at the stationary state, where $A(\phi) = aG(c) + b + \phi b' - (1-\phi)aG'c'$ and $C(\phi) = (1-\phi)aG/2 + \phi b$.

DERIVATION OF EXPECTED VALUE FUNCTIONS AROUND ϕ

Let us move on to show the system of equations (5), (6), and (7) in terms of average ϕ of n/N. With new notations for value functions introduced in Section 5, equation (5) can be expanded as follows:

$$r \cdot \left[V_e(\phi) + V_e'(\phi)\frac{\xi}{\sqrt{N}}\right]$$

$$= \left(b + b'\frac{\xi}{\sqrt{N}}\right)\left[b + V_u + V_u' \cdot \left(\frac{\xi}{\sqrt{N}} - \frac{2}{N}\right) - V_e - V_e'\frac{\xi}{\sqrt{N}}\right]$$

$$+ aN\left(1 - \phi - \frac{\xi}{\sqrt{N}}\right)\left[G(c^*) + G'(c^*)c^{*\prime}\frac{\xi}{\sqrt{N}}\right]$$

$$\times \left[V_e + V_e' \cdot \left(\frac{\xi}{\sqrt{N}} + \frac{1}{N}\right) - V_e - V_e'\frac{\xi}{\sqrt{N}}\right]$$

$$+ \frac{N}{2}\left(\phi + \frac{\xi}{\sqrt{N}} - \frac{2}{N}\right)\left(b + b'\frac{\xi}{\sqrt{N}}\right)$$

$$\times \left[V_e + V_e' \cdot \left(\frac{\xi}{\sqrt{N}} - \frac{2}{N}\right) - V_e - V_e'\frac{\xi}{\sqrt{N}}\right],$$

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where $c^* = c^*(\phi)$ and $c^{*\prime} = c^{*\prime}(\phi)$. Taking the above expectations over ξ, we get the following equation (note that $E\xi = 0$ and $E\xi^2 = \sigma^2$):

$$rV_e = b(v + V_u - V_e) + [(1-\phi)aG - \phi b]V_e' + \left[b'V_u' - (aG'c^{*\prime} + 2b')V_e'\right]\frac{\sigma^2}{N} + \frac{2b\cdot(V_e' - V_u')}{N}.$$

Similarly, for equation (6), we get

$$r\cdot\left[V_u(\phi) + V_u'(\phi)\frac{\xi}{\sqrt{N}}\right] = a\left[V_e + V_e'\left(\frac{\xi}{\sqrt{N}} + \frac{1}{N}\right) - V_u - V_u'\frac{\xi}{\sqrt{N}}\right]\left(G + G'c^{*\prime}\frac{\xi}{\sqrt{N}}\right) - a\int_{\underline{c}}^{c^* + c^{*\prime}\xi/\sqrt{N}} z\,dG(z) + \left(1 - \phi - \frac{\xi}{\sqrt{N}} - \frac{1}{N}\right)a\left(G + G'c^{*\prime}\frac{\xi}{\sqrt{N}}\right)V_u' - \left(\phi + \frac{\xi}{\sqrt{N}}\right)\left(b + b'\frac{\xi}{\sqrt{N}}\right)V_u'.$$

Hence,

$$rV_u = aG(V_e - V_u) - a\hat{c} + [(1-\phi)aG - \phi b]V_u' + \left[aG'c^{*\prime}V_e' - (2aG'c^{*\prime} + b')V_u'\right]\frac{\sigma^2}{N} + \frac{aG\cdot(V_e' - V_u')}{N},$$

where $\hat{c} = \int_{\underline{c}}^{c^*(\phi)} z\,dG(z)$.

For equation (10), we have

$$c^* + c^{*\prime}\frac{\xi}{\sqrt{N}} = V_e - V_u + V_e'\cdot\left(\frac{\xi}{\sqrt{N}} + \frac{1}{N}\right) - V_u'\frac{\xi}{\sqrt{N}}.$$

Again, by taking the expectation about ξ, we get

$$c^*(\phi) = V_e(\phi) - V_u(\phi) + O(N^{-1}).$$

Now, taking the difference between rV_e and rV_u, we get

$$r(V_e - V_u) = b[v - (V_e - V_u)] - aG(V_e - V_u) + a\hat{c} + [(1-\phi)aG - \phi b](V_e' - V_u') - 2(aG'c^{*\prime} + b')(V_e' - V_u')\frac{\sigma^2}{N} + \frac{2b - aG}{N}(V_e' - V_u').$$

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Substituting the relationships $c^* = V_e - V_u + O(N^{-1})$ and $c^{*\prime} = V_e' - V_u' + O(N^{-1})$ into the above, we get

$$rc^* = b(v - c^*) - a(Gc^* - \hat{c}) + [(1 - \phi)aG(c^*) - \phi b(\phi)]c^{*\prime},$$

where the terms of order less than N^{-1} are omitted. The pair $[c^*(\phi), \phi]$ at the critical point is determined by the equations

$$rc^* = b(\phi)(v - c^*) - a[G(c^*)c^* - \hat{c}] + [(1 - \phi)aG(c^*) - \phi b(\phi)]c^{*\prime}$$

and

$$0 = (1 - \phi)aG[c^*(\phi)] - \phi b(\phi).$$

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