J. Aust. Math. Soc. **110** (2021), 260–265
doi:10.1017/S1446788719000533

DERIVATION RELATION FOR FINITE MULTIPLE ZETA VALUES IN $\widehat{\mathcal{A}}$

HIDEKI MURAHARA and TOMOKAZU ONOZUKA

(Received 8 September 2018; accepted 24 September 2019; first published online 8 January 2020)

Communicated by M. Coons

Abstract

Ihara *et al.* proved the derivation relation for multiple zeta values. The first-named author obtained its counterpart for finite multiple zeta values in $\mathcal{A}$. In this paper, we present its generalization in $\widehat{\mathcal{A}}$.

2010 *Mathematics subject classification*: primary 11M32.

Keywords and phrases: multiple zeta values, finite multiple zeta values, derivation relation.

1. Introduction

For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$ with $k_r \geq 2$, the multiple zeta values (MZVs) are defined by

$$\zeta(k_1, \dots, k_r) = \sum_{1 \leq n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

For $n \in \mathbb{Z}_{\geq 1}$, we define the $\mathbb{Q}$-algebra $\mathcal{A}_n$ by

$$\mathcal{A}_n := \Big(\prod_p \mathbb{Z}/p^n\mathbb{Z}\Big)\Big/\Big(\bigoplus_p \mathbb{Z}/p^n\mathbb{Z}\Big) = \{(a_p)_p \mid a_p \in \mathbb{Z}/p^n\mathbb{Z}\}/\sim,$$

where $(a_p)_p \sim (b_p)_p$ are identified if and only if $a_p = b_p$ for all but finitely many primes p. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 1}$, the finite multiple zeta values (FMZVs) in $\mathcal{A}_n$ are defined by

$$\zeta_{\mathcal{A}_n}(k_1, \dots, k_r) := \Big(\sum_{1 \leq n_1 < \cdots < n_r \leq p-1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \bmod p^n\Big)_p \in \mathcal{A}_n.$$

Recently, Rosen [8] introduced the $\mathbb{Q}$-algebra $\widehat{\mathcal{A}}$. By natural projections $\mathcal{A}_n \to \mathcal{A}_{n-1}$, we define $\widehat{\mathcal{A}} := \lim_{\leftarrow n} \mathcal{A}_n$, where we put the discrete topology on each $\mathcal{A}_n$.

We also define the natural projections $\pi\colon \prod_p \mathbb{Z}_p \to \widehat{\mathcal{A}}$ and $\pi_n\colon \widehat{\mathcal{A}} \to \mathcal{A}_n$ for each n, where $\mathbb{Z}_p$ is the ring of p-adic integers. Then FMZVs in $\widehat{\mathcal{A}}$ are given by

$$\zeta_{\widehat{\mathcal{A}}}(k_1,\ldots,k_r) := \pi\Bigl(\Bigl(\sum_{1\le n_1<\cdots<n_r\le p-1} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}}\Bigr)_p\Bigr) \in \widehat{\mathcal{A}}.$$

We can easily check that $\pi_n(\zeta_{\widehat{\mathcal{A}}}(\mathbf{k})) = \zeta_{\mathcal{A}_n}(\mathbf{k})$ for $\mathbf{k} \in \mathbb{Z}_{\ge1}^r$. Furthermore, we define $\mathbf{p} := \pi((p)_p) \in \widehat{\mathcal{A}}$ and we also use the notation $\mathbf{p} := \pi_n \circ \pi((p)_p) \in \mathcal{A}_n$ (for details, see Rosen [8] and Seki [10]).

We recall Hoffman's algebraic setup with a slightly different convention (see Hoffman [2]). Let $\mathfrak{H} := \mathbb{Q}\langle x, y\rangle$ be the noncommutative polynomial ring in two indeterminates x, y and $\mathfrak{H}^1$ (respectively $\mathfrak{H}^0$) its subring $\mathbb{Q} + y\mathfrak{H}$ (respectively $\mathbb{Q} + y\mathfrak{H}x$). Set $z_k := yx^{k-1}$ $(k \in \mathbb{Z}_{\ge1})$. We define the $\mathbb{Q}$-linear map $Z\colon \mathfrak{H}^0 \to \mathbb{R}$ by $Z(1) := 1$, $Z(z_{k_1}\cdots z_{k_r}) := \zeta(k_1,\ldots,k_r)$.

A derivation ∂ on $\mathfrak{H}$ is a $\mathbb{Q}$-linear map $\partial\colon \mathfrak{H} \to \mathfrak{H}$ satisfying Leibniz's rule $\partial(ww') = \partial(w)w' + w\partial(w')$. Such a derivation is uniquely determined by its images of generators x and y. Set $z := x + y$. For each $l \in \mathbb{Z}_{\ge1}$, the derivation ∂_l on $\mathfrak{H}$ is defined by $\partial_l(x) := yz^{l-1}x$ and $\partial_l(y) := -yz^{l-1}x$. We note that $\partial_l(1) = 0$ and $\partial_l(z) = 0$. In addition, R_x is the $\mathbb{Q}$-linear map given by $R_x(w) := wx$ for any $w \in \mathfrak{H}$.

Theorem 1.1 (Derivation relation for MZVs; Ihara *et al.* [5, Theorem 3]). *For $l \in \mathbb{Z}_{\ge1}$ and $w \in \mathfrak{H}^0$,*

$$Z(\partial_l(w)) = 0.$$

Similar to the definition of Z, we define two $\mathbb{Q}$-linear maps $Z_{\mathcal{A}_n}\colon \mathfrak{H}^1 \to \mathcal{A}_n$ and $Z_{\widehat{\mathcal{A}}}\colon \mathfrak{H}^1 \to \widehat{\mathcal{A}}$ by $Z_{\mathcal{A}_n}(1) = Z_{\widehat{\mathcal{A}}}(1) := 1$, $Z_{\mathcal{A}_n}(z_{k_1}\cdots z_{k_r}) := \zeta_{\mathcal{A}_n}(k_1,\ldots,k_r)$ and $Z_{\widehat{\mathcal{A}}}(z_{k_1}\cdots z_{k_r}) := \zeta_{\widehat{\mathcal{A}}}(k_1,\ldots,k_r)$. We write $\mathcal{A} := \mathcal{A}_1$. The derivation relation for FMZVs in $\mathcal{A}$ was conjectured by Oyama and proved by the first-named author.

Theorem 1.2 (Derivation relation for FMZVs in $\mathcal{A}$; Murahara [7, Theorem 2.1]). *For $l \in \mathbb{Z}_{\ge1}$ and $w \in y\mathfrak{H}x$,*

$$Z_{\mathcal{A}}(R_x^{-1}\partial_l(w)) = 0$$

in the ring $\mathcal{A}$.

In this paper, we prove a generalization of the above theorem in the ring $\widehat{\mathcal{A}}$. For nonnegative integers m and n, we define $\beta_{m,n}\colon \mathbb{Q}\langle\langle x, y\rangle\rangle[[u, v]] \to \mathfrak{H}$ by setting $\beta_{m,n}(w)$ to be the coefficients of $u^m v^n$ in w.

Theorem 1.3 (Main theorem). *For $m \in \mathbb{Z}_{\ge0}$ and $w \in y\mathfrak{H}$,*

$$\sum_{n=0}^{\infty} Z_{\widehat{\mathcal{A}}}\Bigl(\beta_{m,n}R_x^{-1}\Delta_u R_x\Bigl(w - wyu\frac{1}{1+xu}\cdot\frac{xv}{1-xv}\Bigr)\Bigr)\mathbf{p}^n = Z_{\widehat{\mathcal{A}}}(w)Z_{\widehat{\mathcal{A}}}\Bigl(\beta_{m,0}\Bigl(\frac{1}{1-yu}\Bigr)\Bigr)$$

in the ring $\widehat{\mathcal{A}}$, where Δ_u is an automorphism on $\mathbb{Q}\langle\langle x, y\rangle\rangle[[u, v]]$ given by

$$\Delta_u := \exp\Bigl(\sum_{l=1}^{\infty} \frac{\partial_l}{l}(-u)^l\Bigr).$$

https://doi.org/10.1017/S1446788719000533 Published online by Cambridge University Press

REMARK 1.4. Since $Z_{\mathcal{A}}(1,\dots,1)=0$ (see, for example, Hoffman [3, Equation (15)]), Theorem 1.3 is a generalization of the equality

$$Z_{\mathcal{A}}(\beta_{m,0}R_x^{-1}\Delta_u R_x(w))=0$$

for $m\in\mathbb{Z}_{\geq 0}$, which was obtained by Ihara (see Horikawa *et al.* [4, Section 5.3]). We note that this is equivalent to Theorem 1.2.

As a corollary of our main theorem, we have Hoffman's relation (see Hoffman [1, Theorem 5.1] for the original formula) for FMZVs in $\widehat{\mathcal{A}}$.

COROLLARY 1.5. *For $w\in y\mathfrak{H}$,*

$$Z_{\widehat{\mathcal{A}}}(R_x^{-1}\partial_1(wx))=-\sum_{n=1}^{\infty}Z_{\widehat{\mathcal{A}}}(wyx^n)p^n-Z_{\widehat{\mathcal{A}}}(w)Z_{\widehat{\mathcal{A}}}(y)$$

in the ring $\widehat{\mathcal{A}}$.

2. Proof of the main theorem

2.1. Notation. The harmonic product $*$ and the shuffle product ш on $\mathfrak{H}^1$ are defined by

$$\begin{aligned}1*w &= w*1 := w,\\ z_kw_1*z_lw_2 &:= z_k(w_1*z_lw_2)+z_l(z_kw_1*w_2)+z_{k+l}(w_1*w_2),\\ 1\,\text{ш}\,w &= w\,\text{ш}\,1 := w,\\ uw_1\,\text{ш}\,vw_2 &:= u(w_1\,\text{ш}\,vw_2)+v(uw_1\,\text{ш}\,w_2)\end{aligned}$$

($k,l\in\mathbb{Z}_{\geq 1}$, $u,v\in\{x,y\}$ and w, w_1, w_2 are words in $\mathfrak{H}^1$), together with $\mathbb{Q}$-bilinearity. The harmonic product $*$ and the shuffle product ш are commutative and associative; therefore, $\mathfrak{H}^1$ is a $\mathbb{Q}$-commutative algebra with respect to $*$ and ш, respectively (see Hoffman [2]).

2.2. Propositions and lemmas. In this subsection, we prepare some propositions which will be used later.

PROPOSITION 2.1. *For w_1, $w_2\in\mathfrak{H}^1$,*

$$Z_{\widehat{\mathcal{A}}}(w_1*w_2)=Z_{\widehat{\mathcal{A}}}(w_1)Z_{\widehat{\mathcal{A}}}(w_2)$$

in the ring $\widehat{\mathcal{A}}$.

PROOF. This is obtained by the definition of the harmonic product. □

PROPOSITION 2.2 (Jarossay [6], Seki [9, Theorem 6.4]). *For w_1, $w_2\in\mathfrak{H}^1$ with $w_2=z_{k_1}\cdots z_{k_r}$,*

$$Z_{\widehat{\mathcal{A}}}(w_1\,\text{ш}\,w_2)=(-1)^{k_1+\cdots+k_r}\sum_{l_1,\dots,l_r\in\mathbb{Z}_{\geq 0}}\Bigg[\prod_{i=1}^{r}\binom{k_i+l_i-1}{l_i}\Bigg]Z_{\widehat{\mathcal{A}}}(w_1z_{k_r+l_r}\cdots z_{k_1+l_1})\boldsymbol{p}^{l_1+\cdots+l_r}$$

in the ring $\widehat{\mathcal{A}}$.

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PROPOSITION 2.3 (Ihara *et al.* [5, Corollary 3]). *For* $w \in \mathfrak{H}^1$,

$$\frac{1}{1-yu} * w = \frac{1}{1-yu} \sqcup\!\sqcup \Delta_u(w).$$

2.3. Proof of Theorem 1.3. By Proposition 2.1,

$$Z_{\widehat{\mathcal{A}}}\Big(\beta_{m,0}\Big(\frac{1}{1-yu} * w\Big)\Big) = Z_{\widehat{\mathcal{A}}}\Big(\beta_{m,0}\Big(\frac{1}{1-yu}\Big)\Big) Z_{\widehat{\mathcal{A}}}(w). \tag{2-1}$$

On the other hand,

$$Z_{\widehat{\mathcal{A}}}(w \sqcup\!\sqcup y^r) = \sum_{n=0}^{\infty} Z_{\widehat{\mathcal{A}}}\Big(\beta_{0,n}\Big(w\Big(-y\frac{1}{1-xv}\Big)^r\Big)\Big)\boldsymbol{p}^n$$

holds by Proposition 2.2. Then

$$\begin{aligned}
Z_{\widehat{\mathcal{A}}}\Big(\beta_{m,0}\Big(\frac{1}{1-yu} \sqcup\!\sqcup \Delta_u(w)\Big)\Big) &= Z_{\widehat{\mathcal{A}}}\Big(\beta_{m,0}\Big(\Delta_u(w) \sqcup\!\sqcup \frac{1}{1-yu}\Big)\Big) \\
&= Z_{\widehat{\mathcal{A}}}\Big(\beta_{m,0}\Big(\Delta_u(w) \sqcup\!\sqcup \sum_{i=0}^{\infty} y^i u^i\Big)\Big) \\
&= \sum_{n=0}^{\infty} Z_{\widehat{\mathcal{A}}}\Big(\beta_{m,n}\Big(\sum_{i=0}^{\infty} \Delta_u(w)\Big(-yu\frac{1}{1-xv}\Big)^i\Big)\Big)\boldsymbol{p}^n \\
&= \sum_{n=0}^{\infty} Z_{\widehat{\mathcal{A}}}\Big(\beta_{m,n}\Big(\Delta_u(w)\frac{1}{1+yu(1-xv)^{-1}}\Big)\Big)\boldsymbol{p}^n.
\end{aligned}$$

From direct calculation,

$$\begin{aligned}
&\{1+yu(1-xv)^{-1}\}^{-1} \\
&\quad = \{(1-xv+yu)(1-xv)^{-1}\}^{-1} \\
&\quad = (1-xv)(1-xv+yu)^{-1} \\
&\quad = (1-xv)\{(1+yu)(1-(1+yu)^{-1}xv)\}^{-1} \\
&\quad = (1-xv)(1-(1+yu)^{-1}xv)^{-1}(1+yu)^{-1} \\
&\quad = (1-xv)\sum_{i=0}^{\infty}\Big(\frac{1}{1+yu}xv\Big)^i(1+yu)^{-1}.
\end{aligned}$$

Since $\Delta_u(x) = (1+yu)^{-1}x$,

$$\begin{aligned}
&\{1+yu(1-xv)^{-1}\}^{-1} \\
&\quad = (1-xv)\sum_{i=0}^{\infty}(\Delta_u(x)v)^i(1+yu)^{-1} \\
&\quad = (1+yu)^{-1} + \sum_{i=1}^{\infty}(\Delta_u(x)-x)(\Delta_u(x))^{i-1}v^i(1+yu)^{-1}.
\end{aligned}$$

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Since $x = \Delta_u(x + y(1 + xu)^{-1}xu)$,

$$\begin{aligned}&\{1 + yu(1 - xv)^{-1}\}^{-1}\\&\quad = (1 + yu)^{-1} - \sum_{i=1}^{\infty} \Delta_u\Big(y\frac{xu}{1 + xu}\Big)\Delta_u(x^{i-1})v^i(1 + yu)^{-1}\\&\quad = R_x^{-1}\Delta_u R_x\Big(1 - yu\frac{1}{1 + xu} \cdot \frac{xv}{1 - xv}\Big).\end{aligned}$$

Hence,

$$\begin{aligned}&Z_{\widehat{\mathcal{A}}}\Big(\beta_{m,0}\Big(\frac{1}{1 - yu} ш \Delta_u(w)\Big)\Big)\\&\quad = \sum_{n=0}^{\infty} Z_{\widehat{\mathcal{A}}}\Big(\beta_{m,n}R_x^{-1}\Delta_u R_x\Big(w - wyu\frac{1}{1 + xu} \cdot \frac{xv}{1 - xv}\Big)\Big)\boldsymbol{p}^n. \qquad (2\text{-}2)\end{aligned}$$

By (2-1), (2-2) and Proposition 2.3, we finally obtain the theorem.

2.4. Proof of Corollary 1.5. When $m = 1$ in Theorem 1.3,

$$\sum_{n=0}^{\infty} Z_{\widehat{\mathcal{A}}}\Big(\beta_{1,n}R_x^{-1}\Delta_u R_x\Big(w - wyu\frac{1}{1 + xu} \cdot \frac{xv}{1 - xv}\Big)\Big)\boldsymbol{p}^n = Z_{\widehat{\mathcal{A}}}(w)Z_{\widehat{\mathcal{A}}}(y).$$

Since

$$\beta_{1,n}R_x^{-1}\Delta_u R_x\Big(w - wyu\frac{1}{1 + xu} \cdot \frac{xv}{1 - xv}\Big) = \begin{cases} -wyx^n & \text{if } n \geq 1,\\ -R_x^{-1}\partial_1 R_x(w) & \text{if } n = 0,\end{cases}$$

we get the result.

Acknowledgement

The authors would like to express their sincere gratitude to Dr. Minoru Hirose for valuable comments.

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HIDEKI MURAHARA, Nakamura Gakuen University Graduate School,
5-7-1, Befu, Jonan-ku, Fukuoka, 814-0198, Japan
e-mail: hmurahara@nakamura-u.ac.jp

TOMOKAZU ONOZUKA, Multiple Zeta Research Center,
Kyushu University, 744, Motooka, Nishi-ku, Fukuoka,
819-0395, Japan
e-mail: t-onozuka@math.kyushu-u.ac.jp