THE JOURNAL OF SYMBOLIC LOGIC
Volume 85, Number 4, December 2020

THE FUNDAMENTAL THEOREM OF CENTRAL ELEMENT THEORY

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Abstract. We give a short proof of the fundamental theorem of central element theory (see: Sanchez Terraf and Vaggione, *Varieties with definable factor congruences*, T.A.M.S. 361). The original proof is constructive and very involved and relies strongly on the fact that the class be a variety. Here we give a more direct nonconstructive proof which applies for the more general case of a first-order class which is both closed under the formation of direct products and direct factors.

§1. Introduction. In this note we give a short proof of the fundamental theorem of central element theory [7, Theorem 1]. The proof given in [7] is constructive and very involved and relies strongly on the fact that the class be a variety (i.e. equationally definable class of algebras). Here we shall give a more direct nonconstructive proof which applies for the more general case of a first-order class which is both closed under the formation of direct products and direct factors.

The theorem links several natural concepts of very different natures which we shall describe in the following lines. Let $\mathcal{L}$ be a first-order language and let $\mathbf{A}$ be an $\mathcal{L}$-structure. Let $\Delta^{\mathbf{A}} = \{(a,a) : a \in A\}$ and $\nabla^{\mathbf{A}} = A \times A$. By a *pair of complementary factor relations of* $\mathbf{A}$ we understand a pair (θ_1,θ_2) of equivalence relations on A satisfying:

- $\theta_1 \cap \theta_2 = \Delta^{\mathbf{A}}$ and $\theta_1 \circ \theta_2 = \nabla^{\mathbf{A}}$.
- For every n-ary function symbol $f \in \mathcal{L}$ and $i = 1,2$ we have that
$$(a_1,b_1),\dots,(a_n,b_n) \in \theta_i \text{ implies } (f^{\mathbf{A}}(a_1,\dots,a_n), f^{\mathbf{A}}(b_1,\dots,b_n)) \in \theta_i.$$
- For every n-ary relation symbol $R \in \mathcal{L}$, if $(a_1,\dots,a_n),(b_1,\dots,b_n) \in R^{\mathbf{A}}$ and $(z_1,\dots,z_n)$ is such that
$$(z_1,a_1),\dots,(z_n,a_n) \in \theta_1 \qquad (z_1,b_1),\dots,(z_n,b_n) \in \theta_2$$
then $(z_1,\dots,z_n) \in R^{\mathbf{A}}$.

There is a well known correspondence between pairs of complementary factor relations and direct product representations. More concretely, if $\sigma : \mathbf{A} \to \mathbf{A}_1 \times \mathbf{A}_2$ is an isomorphism, then $(\ker(\pi_1 \circ \sigma), \ker(\pi_2 \circ \sigma))$ is a pair of complementary factor relations of $\mathbf{A}$, where $\pi_i : A_1 \times A_2 \to A_i$ is the canonical projection, and, reciprocally, if (θ_1,θ_2) is a pair of complementary factor relations of $\mathbf{A}$, we can define $\mathbf{A}/\theta_i$ to be the structure with universe A/θ_i and the interpretations given by:

Received January 3, 2020.
2020 *Mathematics Subject Classification*. 03C05, 03C40, 08B25.
Key words and phrases. Boolean factor relations, definable factor relations, central element, direct product, factor congruence.

0022-4812/20/8504-0010
DOI:10.1017/jsl.2020.41

- $f^{\mathbf{A}/\theta_i}(a_1/\theta_i, \ldots, a_n/\theta_i) = f^{\mathbf{A}}(a_1, \ldots, a_n)/\theta_i$, for every n-ary function symbol f of $\mathcal{L}$
- $R^{\mathbf{A}/\theta_i} = \{(a_1/\theta_i, \ldots, a_n/\theta_i) : (a_1, \ldots, a_n) \in R^{\mathbf{A}}\}$, for every n-ary relation symbol R of $\mathcal{L}$,

obtaining that the canonical map $\mathbf{A} \to \mathbf{A}/\theta_1 \times \mathbf{A}/\theta_2$ is an isomorphism. These constructions are mutually inverse.

For an $\mathcal{L}$-structure $\mathbf{A}$ define

$$FR_p(\mathbf{A}) = \{(\theta,\delta) : (\theta,\delta) \text{ is a pair of complementary factor relations of } \mathbf{A}\}.$$

In virtue of the importance of the direct product construction, the above described correspondence between direct product decompositions of $\mathbf{A}$ and elements of $FR_p(\mathbf{A})$ indicates about the importance of studying the set $FR_p(\mathbf{A})$.

There is a wide extent of classes $\mathcal{K}$ in which the structure of $FR_p(\mathbf{A})$, for $\mathbf{A} \in \mathcal{K}$, can be studied via *central elements*. A *class with* $\vec{0}$ *and* $\vec{1}$ is a first-order class $\mathcal{K}$ of $\mathcal{L}$-structures for which there are 0-ary terms $0_1, \ldots, 0_N$, $1_1, \ldots, 1_N$ such that

$$\mathcal{K} \models \vec{0} = \vec{1} \to x = y$$

where $\vec{0} = (0_1, \ldots, 0_N)$ and $\vec{1} = (1_1, \ldots, 1_N)$. Classical examples of this type of classes are:

- The class of bounded lattices ($\mathcal{L} = \{\vee, \wedge, 0, 1\}$).
- The class of rings with identity ($\mathcal{L} = \{+, ., 0, 1\}$).
- The class of bounded posets ($\mathcal{L} = \{\leq, 0, 1\}$).

($N = 1$, in the three cases.) We note that the existence of the terms $0_1, \ldots, 0_N$, $1_1, \ldots, 1_N$ is a natural condition, since a simple compactness argument shows that a first-order class $\mathcal{K}$ is a class with $\vec{0}$ and $\vec{1}$ iff the language has at least a constant symbol and the members of $\mathcal{K}$ with at least two elements have no one-element substructure.

Let $\mathcal{K}$ be a class with $\vec{0}$ and $\vec{1}$. If $\vec{a} \in A^N$ and $\vec{b} \in B^N$, then we use $[\vec{a}, \vec{b}]$ to denote the N-tuple $((a_1, b_1), \ldots, (a_N, b_N)) \in (A \times B)^N$. Let $\mathbf{A} \in \mathcal{K}$. We say that $(\vec{e}, \vec{f})$ is a *pair of complementary central elements of* $\mathbf{A}$ if there exists an isomorphism $\mathbf{A} \to \mathbf{A}_1 \times \mathbf{A}_2$, such that $\vec{e} \to [\vec{0}, \vec{1}]$ and $\vec{f} \to [\vec{1}, \vec{0}]$. Let

$$CE_p(\mathbf{A}) = \{(\vec{e}, \vec{f}) : (\vec{e}, \vec{f}) \text{ is a pair of complementary central elements of } \mathbf{A}\}.$$

Of course, if for $(\theta,\delta) \in FR_p(\mathbf{A})$ we define[1]

$$\vec{e}_{(\theta,\delta)} = \text{unique } \vec{u} \in A^N \text{such that } \vec{u} \equiv \vec{0}(\theta) \text{and } \vec{u} \equiv \vec{1}(\delta),$$
$$\vec{f}_{(\theta,\delta)} = \text{unique } \vec{u} \in A^N \text{such that } \vec{u} \equiv \vec{1}(\theta) \text{and } \vec{u} \equiv \vec{0}(\delta)$$

then $(\vec{e}_{(\theta,\delta)}, \vec{f}_{(\theta,\delta)}) \in CE_p(\mathbf{A})$ and also we note that for every $(\vec{e}, \vec{f}) \in CE_p(\mathbf{A})$, there is $(\theta,\delta) \in FR_p(\mathbf{A})$ such that $(\vec{e}, \vec{f}) = (\vec{e}_{(\theta,\delta)}, \vec{f}_{(\theta,\delta)})$. This relation between $CE_p(\mathbf{A})$ and $FR_p(\mathbf{A})$ is very well behaved in the classical examples of classes with $\vec{0}$ and $\vec{1}$. For example if $\mathcal{K}$ is the class of bounded distributive lattices and $\mathbf{L} \in \mathcal{K}$, then

$$CE_p(\mathbf{L}) = \{(e, f) : e \vee f = 1 \text{ and } e \wedge f = 0\}$$

[1]We write $\vec{a} \equiv \vec{b}(\theta)$ to express that $(a_i, b_i) \in \theta$, $i = 1, \ldots, N$.

https://doi.org/10.1017/jsl.2020.41 Published online by Cambridge University Press

and the maps

$$(e,f) \to (\theta^{\mathbf{L}}(0,e),\theta^{\mathbf{L}}(0,f)),$$
$$(\theta,\delta) \to (e_{(\theta,\delta)},f_{(\theta,\delta)})$$

are a pair of mutually inverse bijections between $CE_p(\mathbf{L})$ and $FR_p(\mathbf{L})$ (here $\theta^{\mathbf{L}}(a,b)$ denotes the least congruence collapsing a and b). The same occurs with the other classical examples, that is, always we obtain a natural bijection between $CE_p(\mathbf{A})$ and $FR_p(\mathbf{A})$ and hence we can replace factor relations by central elements. Of course this is an advantage since factor relations are second-order objects and central elements are first-order ones.

The above situation naturally leads us to the following philosophical question:

- For which classes $\mathcal{K}$ with $\vec{0}$ and $\vec{1}$ we have a satisfactory central element theory in the sense that for every $\mathbf{A} \in \mathcal{K}$, we can codify the elements of $FR_p(\mathbf{A})$ with elements of $CE_p(\mathbf{A})$?

Suppose that for some $\mathbf{A} \in \mathcal{K}$ there exist distinct $(\theta_1,\delta_1),(\theta_2,\delta_2) \in FR_p(\mathbf{A})$ such that $(\vec{e}_{(\theta_1,\delta_1)},\vec{f}_{(\theta_1,\delta_1)}) = (\vec{e}_{(\theta_2,\delta_2)},\vec{f}_{(\theta_2,\delta_2)})$. In other words, there are two essentially distinct representations $\sigma : \mathbf{A} \to \mathbf{A}_1 \times \mathbf{A}_2$ and $\eta : \mathbf{A} \to \mathbf{B}_1 \times \mathbf{B}_2$, which have associated the same pair $(\vec{e},\vec{f})$ of complementary central elements. Of course in this case it seems somewhat unprovable that we can replace pairs of $FR_p(\mathbf{A})$ by central elements since, in some sense, we have less pairs of complementary central elements than pairs of complementary factor relations.

The prohibition of the above situation is called the *determining property*. That is to say:

DP For every $\mathbf{A} \in \mathcal{K}$, if $(\vec{e},\vec{f}) \in CE_p(\mathbf{A})$, then there exists only one $(\theta,\delta) \in FR_p(\mathbf{A})$ such that $(\vec{e},\vec{f}) = (\vec{e}_{(\theta,\delta)},\vec{f}_{(\theta,\delta)})$.

Note that DP is equivalent to the fact that for every $\mathbf{A} \in \mathcal{K}$, the map $(\theta,\delta) \to (\vec{e}_{(\theta,\delta)},\vec{f}_{(\theta,\delta)})$ is a bijection between $FR_p(\mathbf{A})$ and $CE_p(\mathbf{A})$. So, a reasonable answer to the above question is that the class $\mathcal{K}$ satisfies DP. This property guarantees that, just as it is done in the classical cases, we can replace factor relations by central elements and the gain is obvious.

In the theorem which is the object of this paper we shall prove that if $\mathcal{K}$ is a class with $\vec{0}$ and $\vec{1}$ satisfying

(1) If $\mathbf{A},\mathbf{B} \in \mathcal{K}$, then $\mathbf{A} \times \mathbf{B} \in \mathcal{K}$;
(2) If $\mathbf{A} \times \mathbf{B} \in \mathcal{K}$, then $\mathbf{A},\mathbf{B} \in \mathcal{K}$

then DP is equivalent to several natural conditions which we shall describe next. Our first one deserves the name of *definable factor relations* (*DFR*, for short).

DFR There is a first-order formula $\omega(\vec{z},\vec{w},x,y)$ such that for every $\mathbf{A} \in \mathcal{K}$ and $(\theta,\delta) \in FR_p(\mathbf{A})$ we have that

$$\theta = \{(a,b) : \mathbf{A} \vDash \omega(\vec{e}_{(\theta,\delta)},\vec{f}_{(\theta,\delta)},a,b)\}.$$

Since $\vec{e}_{(\delta,\theta)} = \vec{f}_{(\theta,\delta)}$ and $\vec{f}_{(\delta,\theta)} = \vec{e}_{(\theta,\delta)}$, when DFR holds we have that $\delta = \{(a,b) : \mathbf{A} \vDash \omega(\vec{f}_{(\theta,\delta)},\vec{e}_{(\theta,\delta)},a,b)\}$. Hence DFR implies DP, since (θ,δ) is determined by $(\vec{e}_{(\theta,\delta)},\vec{f}_{(\theta,\delta)})$ via ω. An extrinsic version of DFR can be written as follows.

https://doi.org/10.1017/jsl.2020.41 Published online by Cambridge University Press

(W) There is a first-order formula $\omega(\vec{z},\vec{w},x,y)$ such that for every $\mathbf{A},\mathbf{B}\in\mathcal{K}$,

$$\mathbf{A}\times\mathbf{B}\vDash\omega([\vec{0},\vec{1}],[\vec{1},\vec{0}],(a,b),(a',b'))\text{ iff }a=a'.$$

Note that, since $\mathcal{K}$ satisfies (1) and (2) above, DFR and (W) are trivially equivalent. However, the classical examples satisfy the following strengthening of (W):

(L) There is a first-order formula $\lambda(\vec{z},x,y)$ such that for every $\mathbf{A},\mathbf{B}\in\mathcal{K}$,

$$\mathbf{A}\times\mathbf{B}\vDash\lambda([\vec{0},\vec{1}],(a,b),(a',b'))\text{ iff }a=a'.$$

For example, if $\mathcal{K}$ is the class of bounded lattices, we can take $\lambda(z_1,x,y):=x\vee z_1=y\vee z_1$ and if $\mathcal{K}$ is the class of bounded posets (here $\mathcal{L}=\{\leq,0,1\}$), we can take $\lambda(z_1,x,y)$ to be the conjunction of the two following formulas

$$\forall u\ ((z_1\leq u\ \&\ x\leq u)\to y\leq u)\qquad\forall u\ ((z_1\leq u\ \&\ y\leq u)\to x\leq u).$$

Since $\mathbf{A}\times\mathbf{B}$ is isomorphic to $\mathbf{B}\times\mathbf{A}$ via the map $(a,b)\to(b,a)$, it is trivial that a formula λ satisfying (L) also satisfies

$$\mathbf{A}\times\mathbf{B}\vDash\lambda([\vec{1},\vec{0}],(a,b),(a',b'))\text{ iff }b=b',$$

for any $\mathbf{A},\mathbf{B}\in\mathcal{K}$. Observe that this condition not only states the equality of the second coordinate but also $\vec{0}$ and $\vec{1}$ have been interchanged in the formula λ. Since in general $\vec{0}$ and $\vec{1}$ are not interchangeable, it is not obvious that (L) be equivalent to the following condition.

(R) There is a first-order formula $\rho(\vec{z},x,y)$ such that for every $\mathbf{A},\mathbf{B}\in\mathcal{K}$,

$$\mathbf{A}\times\mathbf{B}\vDash\rho([\vec{0},\vec{1}],(a,b),(a',b'))\text{ iff }b=b'.$$

Condition (R) is present in a lot of cases, since when there is a binary term $\times$ such that $\mathcal{K}\models x\times 0_1=0_1\wedge x\times 1_1=x$, the formula $\rho:=x\times z_1=y\times z_1$ trivially satisfies (R).

Next we state the last two properties involved in the theorem. Define

$$FR(\mathbf{A})=\{\theta:\exists\delta\ (\theta,\delta)\in FR_p(\mathbf{A})\}.$$

The elements of $FR(\mathbf{A})$ are called *factor relations*. In general $FR(\mathbf{A})$ is not closed under intersection, nor relational product. Moreover, even in case that $FR(\mathbf{A})$ is closed under $\circ$ and $\cap$, the resulting bounded lattice $(FR(\mathbf{A}),\circ,\cap,\Delta^{\mathbf{A}},\nabla^{\mathbf{A}})$ can be not uniquely complemented. We say that the class $\mathcal{K}$ has *Boolean factor relations* (*BFR*, for short) if the following property holds.

BFR If $\mathbf{A}\in\mathcal{K}$, then $(FR(\mathbf{A}),\circ,\cap,\Delta^{\mathbf{A}},\nabla^{\mathbf{A}})$ is a bounded distributive lattice (and hence a Boolean lattice).

The property BFR grew out of the work of A. Tarski and others on the existence of unique direct product decompositions of groupoids with identity by directly indecomposable groupoids [4, 5, 8] . They proved that BFR is equivalent to the strict refinement property (a strengthening of the property stating that every two direct product representations have a common refinement), which implies the existence of at most one direct product decomposition with directly indecomposable factors. Several years later C. C. Chang, B. Jónsson, and A. Tarski [2] generalized the definitions of strict refinement property and BFR to the setting of arbitrary first-order structures and prove that they are equivalent.

It is also noteworthy that in several important works on sheaf representations BFR has played a key role. For example, S. Comer published a seminal article [3] showing that the Pierce sheaf construction for rings with identity [6] extends to any algebra having BFR, and in Bigelow and Burris [1] it is shown that in a variety with BFR the weak Boolean product representations with directly indecomposable factors are unique and coincide with the Pierce sheaf.

To state the last property involved in the theorem we need the following notation. If r_1 is a binary relation on A_1 and r_2 is a binary relation on A_2, then we use $r_1 \times r_2$ to denote the binary relation on $A_1 \times A_2$ given by

$$((a_1,a_2),(b_1,b_2)) \in r_1 \times r_2 \text{ iff } (a_i,b_i) \in r_i, \text{ for } i = 1,2.$$

We say that the class $\mathcal{K}$ has *factorable factor relations* (*FFR*, for short) if the following property holds.

FFR If $\mathbf{A},\mathbf{B} \in \mathcal{K}$, then for every $\theta \in FR(\mathbf{A} \times \mathbf{B})$ there exist $\theta_1 \in FR(\mathbf{A})$ and $\theta_2 \in FR(\mathbf{B})$ such that $\theta = \theta_1 \times \theta_2$.

Now we can state the theorem.

Theorem 1. *Let $\mathcal{K}$ be a class with $\vec{0}$ and $\vec{1}$ and suppose that*

(1) *If $\mathbf{A},\mathbf{B} \in \mathcal{K}$, then $\mathbf{A} \times \mathbf{B} \in \mathcal{K}$.*
(2) *If $\mathbf{A} \times \mathbf{B} \in \mathcal{K}$, then $\mathbf{A},\mathbf{B} \in \mathcal{K}$.*

Then DP, DFR, (L), (R), (W), BFR, and FFR are equivalent.

Of course any *variety* (i.e. equationally definable class of algebras) satisfies (1) and (2) and hence the above theorem applies to every variety with $\vec{0}$ and $\vec{1}$. In this category are the classes of bounded semilattices, bounded lattices, rings with identity, etc. Furthermore, if $\mathcal{K}$ is a class with $\vec{0}$ and $\vec{1}$, which can be axiomatized by sentences of the form $\forall \vec{x} (\bigwedge_{i=1}^{n} \alpha_i \to \alpha)$, where $\alpha_1, \dots, \alpha_n, \alpha$ are atomic formulas such that $\mathcal{K} \models \exists \vec{x} \bigwedge_{i=1}^{n} \alpha_i$, then $\mathcal{K}$ satisfies (1) and (2) and hence the above theorem applies. This is because the sentences of the form $\exists \vec{x} \bigwedge_{i=1}^{n} \alpha_i \wedge \forall \vec{x} (\bigwedge_{i=1}^{n} \alpha_i \to \alpha)$, where $\alpha_1, \dots, \alpha_n, \alpha$ are atomic formulas, are preserved by direct products and direct factors. The class of bounded posets falls in this category. The equivalence of BFR and FFR is due to Chang, J ónsson and Tarski [2].

§2. Proof of Theorem 1.

In order to prove the main theorem first we state the following basic results.

Lemma 2. *Suppose θ is a binary relation on $A \times B$. The following are equivalent*

(i) *$\theta = \theta_1 \times \theta_2$, for some binary relations θ_1 and θ_2*
(ii) *$((a,b),(a',b')), ((c,d),(c',d')) \in \theta$ implies $((a,d),(a',d')) \in \theta$.*

Lemma 3. *Let θ_i,δ_i be binary relations on A_i, for $i = 1,2$. Then*

- $(\theta_1 \times \theta_2) \cap (\delta_1 \times \delta_2) = (\theta_1 \cap \delta_1) \times (\theta_2 \cap \delta_2)$,
- $(\theta_1 \times \theta_2) \circ (\delta_1 \times \delta_2) = (\theta_1 \circ \delta_1) \times (\theta_2 \circ \delta_2)$.

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LEMMA 4. *Let* $\mathbf{A}_1,\mathbf{A}_2$ *be* $\mathcal{L}$*-structures. Let* θ_i,δ_i *be binary relations on* A_i *, for* $i=1,2$. *Then* $((\theta_1\times\theta_2),(\delta_1\times\delta_2))\in FR_p(\mathbf{A}_1\times\mathbf{A}_2)$ *iff* $(\theta_i,\delta_i)\in FR_p(\mathbf{A}_i)$, *for* $i=1,2$.

Let $\mathcal{K}$ be a class with $\vec{0}$ and $\vec{1}$ and suppose that $\mathcal{K}$ satisfies DP. If $(\vec{e},\vec{f})\in CE_p(\mathbf{A})$, we denote by $(\theta^{\mathbf{A}}_{(\vec{e},\vec{f})},\delta^{\mathbf{A}}_{(\vec{e},\vec{f})})$ the only pair $(\theta,\delta)\in FR_p(\mathbf{A})$ satisfying $(\vec{e},\vec{f})=(\vec{e}_{(\theta,\delta)},\vec{f}_{(\theta,\delta)})$.

LEMMA 5. *Let* $\mathcal{K}$ *be a class with* $\vec{0}$ *and* $\vec{1}$ *satisfying DP. Let* $\mathbf{A},\mathbf{B}\in\mathcal{K}$ *be such that* $\mathbf{A}\times\mathbf{B}\in\mathcal{K}$. *If* $(\vec{e},\vec{f})\in CE_p(\mathbf{A})$ *and* $(\vec{g},\vec{h})\in CE_p(\mathbf{B})$, *then* $([\vec{e},\vec{g}],[\vec{f},\vec{h}])\in CE_p(\mathbf{A}\times\mathbf{B})$ *and* $\theta^{\mathbf{A}\times\mathbf{B}}_{([\vec{e},\vec{g}],[\vec{f},\vec{h}])}=\theta^{\mathbf{A}}_{(\vec{e},\vec{f})}\times\theta^{\mathbf{B}}_{(\vec{g},\vec{h})}$.

Now we can give the proof.

PROOF OF THEOREM 1. (L) $\Rightarrow$(W). Take $\omega(\vec{z},\vec{w},x,y)=\lambda(\vec{z},x,y)$.

(W) $\Rightarrow$DFR. It is trivial.

DFR $\Rightarrow$DP. Suppose $\mathcal{K}$ satisfies DFR. If $(\vec{e},\vec{f})\in CE_p(\mathbf{A})$ and $(\theta,\delta)\in FR_p(\mathbf{A})$ is such that $(\vec{e},\vec{f})=(\vec{e}_{(\theta,\delta)},\vec{f}_{(\theta,\delta)})$, then DFR says us that

$$\theta=\{(a,b):\mathbf{A}\vDash\omega(\vec{e},\vec{f},a,b)\},$$
$$\delta=\{(a,b):\mathbf{A}\vDash\omega(\vec{f},\vec{e},a,b)\}$$

(note that $\vec{e}_{(\delta,\theta)}=\vec{f}_{(\theta,\delta)}$ and $\vec{f}_{(\delta,\theta)}=\vec{e}_{(\theta,\delta)}$). Thus the pair (θ,δ) is determined by $(\vec{e},\vec{f})$ and hence $\mathcal{K}$ satisfies DP.

DP $\Rightarrow$FFR. Let $\mathbf{A},\mathbf{B}\in\mathcal{K}$ and suppose $\theta\in FR(\mathbf{A}\times\mathbf{B})$. Let δ be such that $(\theta,\delta)\in FR_p(\mathbf{A}\times\mathbf{B})$. Let $\vec{e}=\vec{e}_{(\theta,\delta)}$ and $\vec{f}=\vec{f}_{(\theta,\delta)}$. We observe that $\theta^{\mathbf{A}\times\mathbf{B}}_{(\vec{e},\vec{f})}=\theta$. Suppose $((a,b),(a',b'))\in\theta$ and $((c,d),(c',d'))\in\theta$. In order to apply Lemma 2, we want to prove that $((a,d),(a',d'))\in\theta$. Since $(\vec{e},\vec{f})\in CE_p(\mathbf{A}\times\mathbf{B})$, Lemma 5 says that

$$([\vec{e},\vec{e}],[\vec{f},\vec{f}])\in CE_p((\mathbf{A}\times\mathbf{B})\times(\mathbf{A}\times\mathbf{B}))$$

and

$$\theta^{(\mathbf{A}\times\mathbf{B})\times(\mathbf{A}\times\mathbf{B})}_{([\vec{e},\vec{e}],[\vec{f},\vec{f}])}=\theta^{\mathbf{A}\times\mathbf{B}}_{(\vec{e},\vec{f})}\times\theta^{\mathbf{A}\times\mathbf{B}}_{(\vec{e},\vec{f})}.$$

Hence we obtain that

(a) $\theta^{(\mathbf{A}\times\mathbf{B})\times(\mathbf{A}\times\mathbf{B})}_{([\vec{e},\vec{e}],[\vec{f},\vec{f}])}=\theta\times\theta$.

In particular this says that

(b) $(((a,b),(c,d)),((a',b'),(c',d')))\in\theta^{(\mathbf{A}\times\mathbf{B})\times(\mathbf{A}\times\mathbf{B})}_{([\vec{e},\vec{e}],[\vec{f},\vec{f}])}$.

Let $\alpha\colon (A\times B)\times(A\times B)\to(A\times B)\times(A\times B)$ be the natural automorphism given by $\alpha((a,b),(c,d))=((a,d),(c,b))$. Of course (b) says that

$$(\alpha((a,b),(c,d)),\alpha(((a',b'),(c',d')))\in\theta^{(\mathbf{A}\times\mathbf{B})\times(\mathbf{A}\times\mathbf{B})}_{(\alpha([\vec{e},\vec{e}]),\alpha([\vec{f},\vec{f}]))}.$$

But $\alpha([\vec{e},\vec{e}])=[\vec{e},\vec{e}]$ and $\alpha([\vec{f},\vec{f}])=[\vec{f},\vec{f}]$, which implies that

$$(((a,d),(c,b)),((a',d'),(c',b')))\in\theta^{(\mathbf{A}\times\mathbf{B})\times(\mathbf{A}\times\mathbf{B})}_{([\vec{e},\vec{e}],[\vec{f},\vec{f}])}.$$

https://doi.org/10.1017/jsl.2020.41 Published online by Cambridge University Press

Thus (a) produces $((a,d),(a',d')) \in \theta$. Hence Lemma 2 says that $\theta = \theta_1 \times \theta_2$, for some relations θ_1, θ_2. Similarly we can prove that $\delta = \delta_1 \times \delta_2$. By Lemma 4 we have that $\theta_1 \in FR(\mathbf{A})$ and $\theta_2 \in FR(\mathbf{B})$.

FFR $\Rightarrow$BFR. See [2] and [1].

BFR $\Rightarrow$(L). First we will prove that the following strong form of DP holds.

SDP For every $\mathbf{A} \in \mathcal{K}$, if $(\theta_1,\delta_1),(\theta_2,\delta_2) \in FR_p(\mathbf{A})$ and $\vec{e}_{(\theta_1,\delta_1)} = \vec{e}_{(\theta_2,\delta_2)}$, then $(\theta_1,\delta_1) = (\theta_2,\delta_2)$.

Let $\vec{e} = \vec{e}_{(\theta_1,\delta_1)} = \vec{e}_{(\theta_2,\delta_2)}$. We observe that $\vec{0} \equiv \vec{e}(\theta_1 \cap \theta_2)$ and $\vec{1} \equiv \vec{e}(\delta_1 \cap \delta_2)$. Thus we have that $\vec{0} \equiv \vec{1}((\theta_1 \cap \theta_2) \circ (\delta_1 \cap \delta_2))$. Let $\gamma = (\theta_1 \cap \theta_2) \circ (\delta_1 \cap \delta_2)$. Since $\mathbf{A}$ has BFR we have that $\gamma \in FR(\mathbf{A})$. Hence $\mathbf{A}/\gamma \in \mathcal{K}$, because $\mathcal{K}$ satisfies (2). Since $\vec{0}^{\mathbf{A}/\gamma} \equiv \vec{1}^{\mathbf{A}/\gamma}$ and $\mathcal{K} \models \vec{0} = \vec{1} \rightarrow x = y$, we have that $|A/\gamma| = 1$. Hence $\gamma = \nabla^{\mathbf{A}}$. Thus we have

$$\begin{aligned}\nabla^{\mathbf{A}} &= \gamma \\ &= (\theta_1 \cap \theta_2) \circ (\delta_1 \cap \delta_2) \\ &= (\theta_1 \circ \delta_1) \cap (\theta_1 \circ \delta_2) \cap (\theta_2 \circ \delta_1) \cap (\theta_2 \circ \delta_2).\end{aligned}$$

Hence we obtain that $\theta_1 \circ \delta_2 = \nabla^{\mathbf{A}}$, which produces

$$\begin{aligned}\theta_2 &= \theta_2 \cap \nabla^{\mathbf{A}} \\ &= \theta_2 \cap (\theta_1 \circ \delta_2) \\ &= (\theta_2 \cap \theta_1) \circ (\theta_2 \cap \delta_2) \\ &= (\theta_2 \cap \theta_1) \circ \Delta^{\mathbf{A}} \\ &= \theta_2 \cap \theta_1\end{aligned}$$

that is, $\theta_2 \subseteq \theta_1$. In a similar manner it can be proved the other inclusions completing that $\theta_1 = \theta_2$ and $\delta_1 = \delta_2$. This concludes the proof of SDP.

Let $\mathcal{L}$ be the language of $\mathcal{K}$ and let $\mathcal{L}_e = \mathcal{L} \cup \{c_1, \ldots, c_N\} \cup \{R\}$, where $c_1, \ldots, c_N$ are new distinct constant symbols and R is a new 4-ary relation symbol. If $\mathbf{A}$ is an $\mathcal{L}_e$-structure, we use $\mathbf{A}_{\mathcal{L}}$ to denote the reduct of $\mathbf{A}$ to $\mathcal{L}$. Let $\mathcal{K}_e$ be the class of all $\mathcal{L}_e$-structures $\mathbf{A}$ satisfying:

(I) $\mathbf{A}_{\mathcal{L}} \in \mathcal{K}$.

(II) If $\theta = \{(a,b) \in A^2 : (a,b,0_1^{\mathbf{A}},0_1^{\mathbf{A}}) \in R^{\mathbf{A}}\}$ and $\delta = \{(a,b) \in A^2 : (0_1^{\mathbf{A}},0_1^{\mathbf{A}},a,b) \in R^{\mathbf{A}}\}$, then $(\theta,\delta) \in FR_p(\mathbf{A}_{\mathcal{L}})$ and $\vec{e}_{(\theta,\delta)} = (c_1^{\mathbf{A}}, \ldots, c_N^{\mathbf{A}})$.

(III) If $(a,b,c,d) \in R^{\mathbf{A}}$, then either $c = d = 0_1^{\mathbf{A}}$ or $a = b = 0_1^{\mathbf{A}}$.

We observe that $\mathcal{K}_e$ is a first-order class. Let $(\mathbf{A},\vec{e},R_1),(\mathbf{A},\vec{e},R_2) \in \mathcal{K}_e$. We will prove that $R_1 = R_2$. By (I) we have that $\mathbf{A} \in \mathcal{K}$. Note that by (II) and SDP we have that

$$\begin{aligned}\{(a,b) \in A^2 : (a,b,0_1^{\mathbf{A}},0_1^{\mathbf{A}}) \in R_1\} &= \{(a,b) \in A^2 : (a,b,0_1^{\mathbf{A}},0_1^{\mathbf{A}}) \in R_2\}, \\ \{(a,b) \in A^2 : (0_1^{\mathbf{A}},0_1^{\mathbf{A}},a,b) \in R_1\} &= \{(a,b) \in A^2 : (0_1^{\mathbf{A}},0_1^{\mathbf{A}},a,b) \in R_2\}.\end{aligned}$$

So (III) implies that $R_1 = R_2$. Thus the relation R is implicitly definable in $\mathcal{K}_e$ and hence Beth's definability theorem says that there is a first-order $(\mathcal{L} \cup \{c_1, \ldots, c_N\})$-formula $\varphi(x,y,z,w)$ satisfying $\mathcal{K}_e \vDash \varphi(x,y,z,w) \leftrightarrow R(x,y,z,w)$. Let $\tilde{\varphi}(z_1, \ldots, z_N, x, y, z, w)$ be an $\mathcal{L}$-formula such that $\varphi = \tilde{\varphi}(c_1, \ldots, c_N, x, y, z, w)$. Thus we

https://doi.org/10.1017/jsl.2020.41 Published online by Cambridge University Press

have that

$$\mathcal{K}_e \vDash \tilde{\varphi}(c_1,\dots,c_N,x,y,z,w) \leftrightarrow R(x,y,z,w)$$

Let $\lambda(z_1,\dots,z_N,x,y) = \tilde{\varphi}(z_1,\dots,z_N,x,y,0_1,0_1)$. Note that

(*) $\mathcal{K}_e \vDash \lambda(c_1,\dots,c_N,x,y) \leftrightarrow R(x,y,0_1,0_1)$.

We will prove that $\mathcal{K}$ satisfies (L) via λ. Let $\mathbf{A},\mathbf{B} \in \mathcal{K}$. Let $R \subseteq (A \times B)^4$ be the union of the following two sets

$$\{((a_1,b_1),(a_2,b_2),(a_3,b_3),(a_4,b_4)) : a_1 = a_2 \text{ and } (a_3,b_3) = (a_4,b_4) = (0_1^{\mathbf{A}},0_1^{\mathbf{B}})\},$$
$$\{((a_1,b_1),(a_2,b_2),(a_3,b_3),(a_4,b_4)) : b_3 = b_4 \text{ and } (a_1,b_1) = (a_2,b_2) = (0_1^{\mathbf{A}},0_1^{\mathbf{B}})\}.$$

It is easy to check that $(\mathbf{A}\times\mathbf{B},(0_1^{\mathbf{A}},1_1^{\mathbf{B}}),(0_2^{\mathbf{A}},1_2^{\mathbf{B}}),\dots,(0_N^{\mathbf{A}},1_N^{\mathbf{B}}),R) \in \mathcal{K}_e$. So (*) says that

$$\begin{aligned}\mathbf{A}\times\mathbf{B} \vDash \lambda([\vec{0}^{\mathbf{A}},\vec{1}^{\mathbf{B}}],(a_1,b_1),(a_2,b_2)) \quad &\text{iff} \quad ((a_1,b_1),(a_2,b_2),(0_1^{\mathbf{A}},0_1^{\mathbf{B}}),(0_1^{\mathbf{A}},0_1^{\mathbf{B}})) \in R \\ &\text{iff} \quad a_1 = a_2\end{aligned}$$

which proves (L).

DP $\Rightarrow$(R). It is similar to the proof of DP $\Rightarrow$(L).

(R) $\Rightarrow$DP. Trivial.

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