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Generalized Stack Permutations

M. D. ATKINSON

School of Mathematical and Computational Sciences
North Haugh, St Andrews, Fife KY16 9SS, Scotland
(e-mail: mda@dcs.st-and.ac.uk)

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Stacks which allow elements to be pushed into any of the top r positions and popped from any of the top s positions are studied. An asymptotic formula for the number u_n of permutations of length n sortable by such a stack is found in the cases $r = 1$ or $s = 1$. This formula is found from the generating function of u_n. The sortable permutations are characterized if $r = 1$ or $s = 1$ or $r = s = 2$ by a forbidden subsequence condition.

1. Introduction

Let $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$ be a permutation of $1, 2 \ldots, n$ appearing as the input stream to a stack. If, through an appropriate series of push and pop operations, the stack can discharge the input elements in the order $1, 2 \ldots, n$ then σ is said to be a *stack-sortable* permutation. Stack-sortable permutations were first investigated by Knuth in [4, Section 2.2.1], and it was proved that there are $\binom{2n}{n}/(n+1)$ (the nth Catalan number) stack-sortable permutations of length n. It was also proved that σ is stack-sortable if and only if there are no indices $i < j < k$ with $\sigma_k < \sigma_i < \sigma_j$. The latter fact is nowadays described in the terminology of 'permutation avoidance'.

Two numerical sequences $\pi = [\pi_1, \pi_2, \ldots]$ and $\rho = [\rho_1, \rho_2, \ldots]$ of the same length are said to be *order-isomorphic* if, for all i, j, $\pi_i < \pi_j$ if and only if $\rho_i < \rho_j$. If π and σ are permutations, then π is said to be *involved* in σ if π is order-isomorphic to a subsequence ρ of σ. If π is not involved in σ, then we say that σ *avoids* π. In these terms, a permutation is stack-sortable if and only if it avoids the permutation $[2, 3, 1]$. Many other results about permutation avoidance have been obtained recently (see [6], [10], [8], [3], [2]).

In this paper we use avoidance arguments to generalize Knuth's original results to stacks where the push and pop operations are not confined to a single 'top' position.

Definition. An (r, s)-stack is a container for a sequence admitting an extended push operation and an extended pop operation. The push operation can insert a new element anywhere among the first r places of the current sequence. The pop operation can remove any of the first s elements of the sequence.

Lemma 1.1. *There is a one-to-one correspondence between (r, s)-stack-sortable permutations and (s, r)-stack-sortable permutations.*

Proof. We shall show that, if the permutation γ can be sorted by an (r, s)-stack, then $\rho\gamma^{-1}\rho$ can be sorted by an (s, r)-stack, where ρ is the reversal permutation $[n, n-1, \ldots, 1]$.

Let $A = a_1, \ldots, a_{2n}$ be a sequence of (r, s)-stack operations that sorts γ. Then A will also transform the identity permutation into γ^{-1}. Consider the sequence $A' = a'_{2n}, \ldots, a'_1$ of operations of an (s, r)-stack where a'_i is defined as follows.

(1) If a_i inserts an input symbol into the kth location of the (r, s)-stack then a'_i produces an output symbol by removing the kth element of the (s, r)-stack.
(2) If a_i produces an output symbol by removing the kth element of the (r, s)-stack then a'_i inserts an input symbol into the kth location of the (s, r)-stack.

It is helpful to think of a movie in which A transforms the identity permutation into γ^{-1} step by step. If this movie were to be run in reverse one would see A' transform $\rho\gamma^{-1}$ (the reverse of γ^{-1}) into ρ (the reverse of the identity). In other words, A' sorts $\rho\gamma^{-1}\rho$. □

Of course, a $(1, 1)$-stack is just an ordinary stack, the top element being the first element of the sequence. Modern computers often have a system stack that permits direct access to a small number of elements near the top of the stack. We shall consider the case $s = 1$, use an avoidance criterion to characterize the $(r, 1)$-stack-sortable permutations, and give some enumeration results. These results give an indication of the extra power possessed by stacks with r or s greater than 1. Finally we make a few remarks about the case $r = s = 2$ and indicate that it appears to be significantly harder.

2. Avoidance

Theorem 2.1. *A permutation is $(r, 1)$-stack-sortable if and only if it avoids all $r!$ permutations of the form $[a_1, a_2, \ldots, a_r, r+2, 1]$.*

Proof. For convenience we refer, temporarily, to the permutations defined in the statement of the theorem as *impeding* permutations. Let $\alpha = [a_1, a_2, \ldots, a_r, r+2, 1]$ be any impeding permutation. If it can be sorted then all its elements must be pushed onto the stack before any are popped. The result of pushing $a_1, a_2, \ldots, a_r$ results in a stack content that can be any reordering of these elements. Let $x = a_i$ be the bottom element. When the element $r+2$ is pushed onto the stack, x remains the bottom element. At this point the final element 1 can be pushed and then popped, but it is then impossible to pop the element x at its appropriate point because $r+2$ lies above x in the stack. Thus α is not $(r, 1)$-stack-sortable, and therefore no permutation involving α can be $(r, 1)$-stack-sortable either.

To prove the converse – that a permutation α which is not $(r, 1)$-stack-sortable must involve one of the impeding permutations – we consider how α can fail to be $(r, 1)$-stack-sortable. Obviously, if α can be sorted at all, the stack elements must remain sorted

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decreasingly from top to bottom. Thus, when a push operation is carried out there will be at most one stack position into which the new element can be inserted. It is evident that, if α can be sorted, it can be sorted by a sequence of pushes and pops in which, if $1, 2, \ldots, i-1$ have been output already and i is at the top of the stack, then i should be popped before any further pushes. Such a sequence of pushes and pops is called *canonical*.

If the canonical sequence of pushes and pops is incapable of sorting α it must first fail on a push operation. Specifically, the next element z to output is not in the stack and the element y being pushed (which precedes z in the input stream) must be greater than the top r elements $x_1, x_2, \ldots, x_r$ of the stack. Therefore the input stream must originally have had a subsequence Xyz, where X is some arrangement of $x_1, x_2, \ldots, x_r$, and this is order-isomorphic to an impeding permutation. □

Remark.

(1) By using a similar argument, or the correspondence of Lemma 1.1, one can show that a permutation is $(1, s)$-stack-sortable if and only if it avoids all permutations of length $s+2$ of the form $[2, a_1, \ldots, a_s, 1]$.

(2) The class of $(r, 1)$-stack-sortable permutations is an example of a class of permutations $\mathscr{X}$ with a *closure* property: $x \in \mathscr{X}$ and y involved in x implies $y \in \mathscr{X}$. Every such class is characterized by a set of avoided permutations (the minimal permutations not belonging to $\mathscr{X}$). In general this set may not be finite: as shown in [5], deque-sortable permutations cannot be characterized by any finite avoided set.

3. Enumeration

The main aim of this section is to give enumeration results for the number u_n of $(r, 1)$-stack-sortable permutations. Of course, u_n depends on r but, since r will be fixed throughout the section, we suppress a notational reference to it.

Lemma 3.1. *u_n is the number of permutations of length n that avoid all $r!$ permutations of the form $[r+2, a_1, \ldots, a_r, r+1]$.*

Proof. It is easy to see that if α avoids β then α^{-1} avoids β^{-1}. Consequently, a permutation avoids all the permutations in the set $R = \{[a_1, a_2, \ldots, a_r, r+2, 1]\}$ if and only if its inverse avoids every permutation in the set $R^{-1} = \{[r+2, a_1, \ldots, a_r, r+1]\}$. The lemma now follows from Theorem 2.1. □

We define a permutation to be *r-maximal* if it avoids all $r!$ permutations of length $r+1$ of the form $[a_1, \ldots, a_r, r+1]$. We also define *tail*(σ) for any permutation σ to be the longest r-maximal suffix of σ. From the definition of r-maximal, Lemma 3.2 follows.

Lemma 3.2. *A permutation σ of length m is r-maximal if and only if the following conditions hold.*

(1) *The largest element m occurs among positions $1, 2, \ldots, r$ of σ.*

(2) *If m is deleted from σ the resulting sequence is an r-maximal permutation of length $m-1$.* □

Let U_n be the set of permutations of length n that avoid all the permutations of the form $[r+2, a_1, \ldots, a_r, r+1]$ (*i.e.*, those given in Lemma 3.1). Let U_{ni} be the set of permutations σ in U_n with $|\mathrm{tail}(\sigma)| = i$, and let $u_{ni} = |U_{ni}|$.

Lemma 3.3. *The numbers u_{ni} satisfy the following conditions.*

D1. *If $n \leqslant r$, then $u_{ni} = 0$ if $n \neq i$ and $u_n = u_{nn} = n!$*
D2. *If $n \geqslant r$, then $u_{ni} = 0$ if $i < r$.*
D3. *If $n \geqslant r$, then $u_{ni} = ru_{n-1,i-1} + \sum_{j \geqslant i} u_{n-1,j}$ if $i \geqslant r$.*

These conditions determine the numbers u_{ni} uniquely.

Proof. An $(r,1)$-stack is obviously capable of sorting every permutation of length r or less. Moreover, for such permutations their longest r-maximal suffix is themselves. This proves D1. For permutations of length r or more the longest r-maximal suffix is of length at least r and so D2 holds.

For D3 we begin by noting that every permutation of U_{ni} arises from inserting n into a permutation of $U_{n-1,j}$ for some j. Consider any $\sigma \in U_{n-1,j}$ and write it as $\sigma = \alpha x \beta$ where $\beta = \mathrm{tail}(\sigma)$. If n were to be inserted in σ before the element x the result could not be in U_n, since $x\beta$ has a subsequence order-isomorphic to a permutation of the form $[a_1, \ldots, a_r, r+1]$ and $nx\beta$ would have a subsequence order-isomorphic to $[r+2, a_1, \ldots, a_r, r+1]$. On the other hand, if n is inserted after element x the resulting permutation is certainly in U_n.

If n were to be inserted in one of the r places after x and before the rth element of β we would obtain a permutation of $U_{n,j+1}$ by Lemma 3.2. On the other hand, if n were to be inserted immediately after the $(r+p)$th element of β, for any $p \geqslant 0$, we would obtain a permutation of $U_{n,j-p}$ (since, by Lemma 3.2, the longest r-maximal suffix would begin $r-1$ places before n).

Thus an element of U_{ni} can arise in r different ways from inserting n into a permutation of $U_{n-1,i-1}$ and, for each $j \geqslant i$, can arise in one way only from inserting n into a permutation of $U_{n-1,j}$. This proves D3. □

Lemma 3.4. *The conditions of Lemma 3.3 are equivalent to these conditions.*

E1. *If $n \leqslant r$, then $u_{ni} = 0$ if $n \neq i$ and $u_n = u_{nn} = n!$*
E2. *If $n \geqslant r$, then $u_{ni} = 0$ if $i < r$.*
E3. *$u_{ni} = 0$ for all $n < i$.*
E4. *If $n \geqslant r$, then $u_{n,i} = u_{n,i-1} + (r-1)u_{n-1,i-1} - ru_{n-1,i-2}$ if $i > r$.*

Proof. E3 follows easily from D1, D2 and D3 by induction. E4 follows by differencing the two equations (from D3)

$$u_{ni} = ru_{n-1,i-1} + \sum_{j \geqslant i} u_{n-1,j} \quad \text{and} \quad u_{n,i-1} = ru_{n-1,i-2} + \sum_{j \geqslant i-1} u_{n-1,j}.$$

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Conversely, D3 follows from E3 and E4 by summing, from $j = i+1$ to $n+1$, the rewritten equations $u_{nj} - u_{n,j-1} = ru_{n-1,j-1} - ru_{n-1,j-2} - u_{n-1,j-1}$ of E4. □

Lemma 3.5. *$u_{r-1,r}, u_{r,r}, u_{r+1,r}, \ldots$ are the coefficients $v_0, v_1, v_2, \ldots$ in*

$$\frac{(r-1)!}{2}(1+(r-1)x - \sqrt{(r-1)^2x^2 - 2(r+1)x + 1}) = \sum_{n=0}^{\infty} v_n x^n.$$

Proof. We make the substitution $t_{ni} = u_{n+r-1,i+r-1}$ and translate the conditions of Lemma 3.4 to give:

F1. $t_{00} = (r-1)!$ and $t_{n0} = 0$ if $n > 0$.
F2. $t_{0i} = 0$ if $i > 0$.
F3. $t_{ni} = 0$ for all $n < i$.
F4. $t_{ni} = t_{n,i-1} + (r-1)t_{n-1,i-1} - rt_{n-1,i-2}$ for all $i \geqslant 2$ and $n \geqslant 1$.

If we put $v(x) = \sum_{n=0}^{\infty} v_n x^n = \sum_{n=0}^{\infty} t_{n1} x^n$ and $T(x,y) = \sum t_{ni} x^n y^i$, conditions F1, F2, and F4 give rise to an equation satisfied by $v(x)$ and $T(x,y)$ whose solution is

$$\begin{aligned} T(x,y) &= \frac{(r-1)! + yv(x) - y(r-1)! - xy(r-1)!(r-1)}{1-(r-1)xy - y + rxy^2} \\ &= (r-1)! + \frac{y(v(x) - r!xy)}{1-(r-1)xy - y + rxy^2}. \end{aligned}$$

In order to satisfy condition F3 too, we must choose the power series $v(x)$ appropriately. We factor the denominator of $T(x,y)$ as

$$1-(r-1)xy - y + rxy^2 = (1-\rho(x)y)(1-\sigma(x)y),$$

where $\rho(x)\sigma(x) = rx$ and

$$\begin{aligned} \rho(x) &= \frac{1}{2}\left(1+(r-1)x + \sqrt{(r-1)^2x^2 - 2(r+1)x + 1}\right), \\ \sigma(x) &= \frac{1}{2}\left(1+(r-1)x - \sqrt{(r-1)^2x^2 - 2(r+1)x + 1}\right). \end{aligned}$$

Note that $\sigma(x)$ is a power series with $\sigma(0) = 0$. In fact, $(r-1)!\sigma(x)$ is the sought-for power series for $v(x)$ since, when $(r-1)!\sigma(x)$ is substituted for $v(x)$, $T(x,y)$ becomes

$$\begin{aligned} (r-1)! + \frac{y((r-1)!\sigma(x) - r!xy)}{(1-\rho(x)y)(1-\sigma(x)y)} &= (r-1)! + \frac{y((r-1)!\sigma(x) - (r-1)!\rho(x)\sigma(x)y)}{(1-\rho(x)y)(1-\sigma(x)y)} \\ &= (r-1)! + \frac{y(r-1)!\sigma(x)}{1-\sigma(x)y} \\ &= (r-1)! + y(r-1)!\sigma(x)\sum_{m=0}^{\infty}\sigma(x)^m y^m, \end{aligned}$$

which does have a power series expansion satisfying condition F3. Since condition F3 uniquely determines $v(x)$, the proof is complete. □

Theorem 3.6.

(1) *If $n \leqslant r$, $u_n = n!$.*

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(2) *If $n \geqslant r$, u_n is the coefficient of x^{n-r+2} in*

$$q(x) = -\frac{(r-1)!}{2}\sqrt{(r-1)^2x^2 - 2(r+1)x + 1}.$$

Proof. Part (1) is clear. For part (2), note that $q(x)$ and $v(x)$ differ only in their linear terms and so it is sufficient to prove that $u_{n+1,r} = u_n$ if $n \geqslant r$ (since, by Lemma 3.5, $u_{n+1,r}$ is the coefficient of x^{n-r+2} in $q(x)$). So let $\sigma \in U_{n+1,r}$. Since tail(σ) has length r, the final $r+1$ symbols of σ must be order-isomorphic to a permutation $[a_1, \ldots, a_r, r+1]$. If the last symbol of σ was not $n+1$ itself, then $n+1$ would occur before the last $r+1$ symbols of σ and, with them, produce a subsequence that was order-isomorphic to $[r+2, a_1, \ldots, a_r, r+1]$, a contradiction. Thus σ is the result of appending $n+1$ to a permutation in U_n and so $U_{n+1,r}$ and U_n are in one-to-one correspondence. □

The asymptotic behaviour of u_n can be found from Theorem 3.6 using an observation in [4], p. 534: the coefficient of w^n in $\sqrt{1-w}\sqrt{1-\alpha w}$ (with $0 < \alpha < 1$) is asymptotic to $-\frac{1}{2}\sqrt{(1-\alpha)/\pi}\; n^{-3/2}$. We can write

$$\begin{aligned}\sqrt{(r-1)^2x^2 - 2(r+1)x + 1} &= \sqrt{(1-(\sqrt{r}+1)^2x)(1-(\sqrt{r}-1)^2x)} \\ &= \sqrt{1-(\sqrt{r}+1)^2x}\sqrt{1-\frac{(\sqrt{r}-1)^2}{(\sqrt{r}+1)^2}(\sqrt{r}+1)^2x}.\end{aligned}$$

Putting $w = (\sqrt{r}+1)^2x$ and $\alpha = (\sqrt{r}-1)^2/(\sqrt{r}+1)^2$, we can therefore deduce that the coefficient of x^n in $\sqrt{(r-1)^2x^2 - 2(r+1)x + 1}$ is asymptotic to $-\sqrt{r^{1/2}/(\pi n^3)}(1+\sqrt{r})^{2n-1}$. Theorem 3.6 now gives the following theorem.

Theorem 3.7. *u_n is asymptotic to $\frac{1}{2}(r-1)!\sqrt{r^{1/2}/(\pi n^3)}(1+\sqrt{r})^{2n-2r+3}$.* □

Note that the coefficients of $\sqrt{(r-1)^2x^2 - 2(r+1)x + 1}$ can be calculated rapidly by the following method.

If $p(x) = (r-1)^2x^2 - 2(r+1)x + 1$ and $\sqrt{p(x)} = \sum g_n x^n$, then $p'(x)\sum g_n x^n = 2p(x)\sum g_n n x^{n-1}$.

By equating coefficients of x we find $g_0 = 1, g_1 = -(r+1)$, and

$$ng_n = (r+1)(2n-3)g_{n-1} - (r-1)^2(n-3)g_{n-2} \qquad \text{for all } n \geqslant 2.$$

It is interesting to compare the case $r = 2$ of our results with the analogous results for restricted output deques in Knuth [4] (another structure that permits two possible input operations and one output operation). The numbers of sortable permutations are the same (the Schröder numbers; see West [10]), both sets of permutations are characterized by avoiding a pair of permutations of length 4 ([2, 3, 4, 1] and [3, 2, 4, 1] for the (2, 1)-stack and [2, 4, 3, 1], [4, 2, 3, 1] for the restricted deque; see Pratt [5]), yet there appears to be no elementary connection between these two situations.

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4. (2,2)-stacks

In this section we give an avoidance criterion for $(2, 2)$-stack-sortable permutations. It is somewhat more complicated than those for $r = 1$ or $s = 1$ and this, together with the numerical evidence, indicates that generalizing the results of the previous sections will not be straightforward.

Theorem 4.1. *A permutation is* $(2, 2)$*-stack-sortable if and only if it avoids all of the following 8 permutations:* $[2, 3, 4, 5, 1]$, $[2, 3, 5, 4, 1]$, $[3, 2, 4, 5, 1]$, $[3, 2, 5, 4, 1]$, $[2, 4, 5, 1, 6, 3]$, $[2, 4, 6, 1, 5, 3]$, $[4, 2, 5, 1, 6, 3]$, $[4, 2, 6, 1, 5, 3]$.

Proof. It is easy to check that none of the permutations in the statement of the theorem can be sorted by a $(2, 2)$-stack and so any $(2, 2)$-stack-sortable permutation must avoid them. For the converse we need to extend the idea of a canonical sequence of pushes and pops appearing in Theorem 2.1. In this case a canonical sequence of pushes and pops is one that respects the following principles.

(1) If $1, 2, \ldots, i-1$ have been output already and i is at the top of the stack or immediately below the top then i should be popped before any further pushes.
(2) If an element i has to be pushed into one of the top 2 positions of the stack, then it should be pushed so that the top two elements of the stack are sorted in decreasing order.

We argue that, if a permutation is $(2, 2)$-stack-sortable, then it can be sorted by a canonical sequence of pushes and pops. It is immediately evident that the application of the first principle can never be disadvantageous. To see that the second principle can always be applied without loss, consider a permutation that is sortable by way of a sequence of pushes and pops which, at some point, pushes an element onto the stack so that the top element y and its neighbour x satisfy $x < y$ (in violation of the second principle). After this step there will be further pushes and pops and eventually x will be removed from the stack. However, during this part of the algorithm, y must remain on the stack so no element may be pushed below x; moreover, only elements less than x will be encountered in the input permutation. It follows that we could have achieved the same result by applying the second principle.

(Note: this argument is more subtle than it might at first sight appear. It is not valid for $(2, 3)$-stacks, for example: $[2, 4, 5, 1, 6, 7, 3]$ is sortable by a (2,3)-stack but the sorting method must not begin by pushing the elements 2 and 4 with the top element being 2.)

We can now complete the proof of Theorem 4.1. Let σ be a permutation that cannot be sorted by the canonical sequence of pushes and pops (*i.e.*, any unsortable permutation). Then the canonical sequence must reach a point where it needs to output an element p but, although p is in the stack, there are two elements x, y above p in the stack (and greater than p). When x was placed in the stack there must have been an element q already on top of p (or x could have been placed below p). Moreover, neither p nor q can be ready to be output and so there must be an element j, after x, less than both of them. In other words the permutation σ must contain either a subsequence $pqxj$ or a

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Table 1 Numbers of (2, 2)-stack-sortable permutations.

n	1	2	3	4	5	6	7	8	9	10
y_n	1	2	6	24	116	628	3636	21956	136428	865700

subsequence $qpxj$ order-isomorphic to one of $[2, 3, 4, 1]$ or $[3, 2, 4, 1]$. In the same way σ must contain a subsequence $pq'yj'$ or $q'pyj'$ order-isomorphic to $[2, 3, 4, 1]$ or $[3, 2, 4, 1]$.

Thus, because σ is unsortable, it must contain a subsequence on the symbols p, q, q', x, y, j, j' (not necessarily in this order, nor necessarily distinct) with the properties just described. This already shows that σ must contain a subsequence on at most 7 elements ordered in one of a certain number of ways. In fact, an exhaustive search of all the possibilities shows that all of these subsequences involve at least one of the 8 permutations listed in the lemma, thus completing the proof. □

At this point one might hope that an enumeration result for (2, 2)-stack-sortable permutations might be possible. In fact, all we have succeeded in doing is to compute a table of values of y_n (the number of (2, 2)-stack-sortable permutations of length n) for n up to 10. We have no satisfactory explanation of these numbers but, tantalizingly, the Sloane Superseeker program [7] guesses that the ordinary generating function $y = y(x)$ of the sequence satisfies the equation

$$x = \frac{y(3y-1)}{(y+1)(2y^2+2y-1)}.$$

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