A hierarchy of nonlocal models for the radiative transfer equation

J.-L. FEUGEAS

Centre Lasers Intenses et Applications, UMR Commissariat à l'Energie Atomique–Centre National de la Recherche
Scientifique–Université de Bordeaux 1, Talence Cedex, France

Abstract

For the classic diffusion description of radiative transfer, the specific intensity can be represented by a small angular deviation of the local Planckian equilibrium. In a transparent media, the angular anisotropy becomes strong and one has to solve the general transfer equation. We propose a hierarchy of models that can describe the regime that lies between those two limits. Every member of this family is hyperbolic, flux-limited, and possesses a locally dissipated entropy. This hierarchy also formally recovers the diffusion limit. This study demonstrates that the two-polynomial model is already capable of capturing strong anisotropies.

Keywords: Angular anisotropy; Bose-Einstein theory; Photon emission; Planckian equilibrium; Radiative transfer equation

1. INTRODUCTION

A numerical approximation of the radiative transfer equation usually involves macroscopic methods in regions where the medium is opaque, coupled with methods for solving the photon transport kinetic equation in a transparent medium. In an opaque medium, the radiative intensity is close to the radiative equilibrium. The diffusion approximation or the P_1 method, based on the hypothesis of a directional quasi-isotropy of photon emission distribution provides a rather accurate approximation (Mihalas & Mihalas, 1984; Pomraning, 1992). For the outside diffusion regime, in the case of transparent media, the macroscopic methods are no longer valid. The microscopic description of the transfer equation is then preferred even if it is complicated and time consuming. Also, coupling between diffusion and kinetic solutions in intermediate regions is very complicated.

The author purposed in previous works a new macroscopic hierarchy of models that are capable of restituting high radiative nonequilibria (Dubroca & Feugeas, 1999). The method is based on the entropy minimization principle applied to the radiative transfer equation. This idea comes from the approximation of transitional flows at high altitude introduced by Levermore (1996) for the Maxwell–Boltzmann

theory. The first step of this idea, introduced by Grad (1949), lies on the building of new, continuous equations taking account additional unknowns. These additional equations are obtained by considering an adequate closure. However this approach does not assure the hyperbolic properties of the system.

Levermore (1996) purposed another closure that ensures the hyperbolicity of the moment equations systems and dissipate the entropy locally. This approach appears as a general method to close an infinite hierarchy of macroscopic equations constructed by taking the moments of a microscopic equation (Feugeas, 1997; Charrier *et al.*, 1998).

In previous work (Dubroca & Feugeas, 1999), we applied the idea of the entropy minimization principle to the Bose–Einstein theory to build a hierarchy of models for the radiative transfer, the M-hierarchy. Each member of this hierarchy is a hyperbolic system that has a flux limited by the speed of light and which dissipates entropy locally. The limiting property ensures that any signal is propagated at a velocity below the speed of light, contrary to diffusion models that allowed arbitrary and non physical high velocities. The first member of this hierarchy, the M_1 model, is very attractive because of its analytic formulation. A first version of this model has been analyzed by Fort (1997) in the simple case where the radiative flux is always parallel to a given axis. It has been verified that the M_1 model describes a rather wide class of situations where anisotropy can be characterized by only one direction. This model has been used in several

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physics domains (Audit *et al.*, 2002). In order to enlarge the capabilities of the M_1 model to describe more complicated anisotropies, several modifications are purposed (Dubroca & Klar, 2002; Klar & Dubroca, 2002; Klar *et al.*, 2003). However those are not satisfactory because of losing interesting properties of the hierarchy. At first, one of the very attractive characteristics of the M_1 model is the analytic formulation. Besides, every member of the M-hierarchy is built in the respect of the Galilean invariance of the underlying distribution. The other members of this hierarchy have no known simple explicit formulation. However, they conserve other properties of the M-models: Their flux is limited, they are hyperbolic, and they locally dissipate the entropy. In the present article we explore the anisotropy potentialities of the M_2 model.

Having recalled the main properties of radiative equilibrium and entropy in Section 2, we obtain a hierarchy of moment systems as presented in Section 3. In Section 4, we analyze in detail the first member of the M-hierarchy, the M_1 model. Section 5 addresses the properties of the M_2 model. Section 6 is devoted to this anisotropy analysis of the two first members of the hierarchy. Section 7 concludes the article and proposes some perspectives.

2. RADIATIVE TRANSFER: EQUILIBRIUM AND ENTROPY

2.1. Radiative transfer equations

The radiative transfer equation governs the radiative field and its interaction with matter. It describes the evolution of the *radiative intensity I* coupled to the matter energy equation. For purposes of simplification, we shall take the hypothesis of the local thermodynamic equilibrium (LTE) and assume that the medium is at rest. In order to simplify the description, we shall subsequently assume that the absorption coefficient is independent of the frequency. This hypothesis is not essential, but it lightens the reasoning pursued in this article. For the same reason, the scattering effects will also be neglected. The radiative transfer equations are then written as follows:

$$\frac{1}{c}\partial_t I + \mathbf{\Omega}\cdot\mathbf{\nabla}I = \sigma(B(T_m) - I), \tag{1}$$

$$\partial_t(\rho C_v T_m) = \int_0^\infty \int_{S_2} \sigma(I - B(T_m))\, d\nu\, d\mathbf{\Omega}. \tag{2}$$

Here, $I = I(\mathbf{r}, \nu, \mathbf{\Omega}, t)$ is a function of the *position* $\mathbf{r}$, the *frequency* ν, the *unitary propagation direction* $\mathbf{\Omega}$, and of the time t, c is the speed of light in a vacuum; T_m is the matter temperature, ρ is the matter specific density, σ is the absorption coefficient, and C_v is the specific heat at a constant volume. The radiative collision operator $\sigma(B(T_m) - I)$ establishes a balance between the emission and absorption of the photons and couples the two equations; S_2 is the unit

sphere. The function $B = B(\nu, T_m)$ is the Planck distribution defined by

$$B(\nu, T_m) = \frac{2h\nu^3}{c^2}\left(\exp\left(\frac{h\nu}{kT_m}\right) - 1\right)^{-1}, \tag{3}$$

where k is the Boltzmann constant and h is the Planck constant. $j = \sigma B(T_m)$ represents the matter emission isotropic source at the temperature of T_m. The moment of a scalar or vector-valued function $g = g(\nu, \mathbf{\Omega})$ in the space of the frequencies and directions is written

$$\langle g \rangle = \frac{1}{c}\int_0^\infty \int_{S_2} g(\nu, \mathbf{\Omega})\, d\nu\, d\mathbf{\Omega}. \tag{4}$$

The *radiative energy* is defined by $E_R = \langle I \rangle$, the *radiative flux vector* by $\mathbf{F}_R = \langle c\mathbf{\Omega}I \rangle$, and the *radiative pressure tensor* by $\overleftrightarrow{P}_R = \langle \mathbf{\Omega} \otimes \mathbf{\Omega}I \rangle$. The normalized flux is written as $\mathbf{f} = \mathbf{F}_R/cE_R$. In the particular case of a Planck radiative intensity, we have $E_R(T_m) = aT_m^4$, $\mathbf{F}_R(T_m) = 0$, and the pressure tensor reduces to the scalar $P_R(T_m) = aT_m^4/3$ with $a = 8\pi^5 k^4/15h^3c^3$. In the case of a given radiative intensity, the radiative temperature T_R is defined as from the radiative energy: $E_R = aT_R^4$. In the case of a Planck radiative intensity, naturally we have $T_R = T_m$.

2.2. Radiative entropy and equilibrium

If the matter is supposed to behave as a perfect gas, the total entropy of the system (1) and (2) is the sum of the matter entropy and of the purely radiative entropy, $H_R^*(I) = \langle h_R^*(I) \rangle$, where $h_R^*(I)$ is a strictly convex function (Fort, 1997):

$$h_R^*(I) = [n\log n - (n + 1)\log(n + 1)]\frac{2k\nu^2}{c^3}, \tag{5}$$

where n is the occupation number density defined as $I = (2h\nu^3/c^2)n$. The total entropy of the system is then $(H_R^*(I) + S)/c$ where $S = -\rho\, C_v \log(T_m)$ is the matter entropy. We have the following properties on the entropy of the system. The total entropy of model (1) and (2) is locally dissipated (decreases in time):

$$\frac{1}{c}\partial_t(\langle h_R^* \rangle + S) + \mathbf{\nabla}\cdot\langle \mathbf{\Omega}h_R^* \rangle \le 0.$$

Moreover, at the thermodynamic equilibrium, $T = T_R = T_m$, the Planck distribution (3) or the so-called equilibrium distribution, produces the minimum radiative entropy under the constraint of a good reconstruction of the energy:

$$H_R^*(\mathcal{B}_0) = \min_I \{H_R^*(I) : \langle I \rangle = E_R\}. \tag{6}$$

This problem of minimization is equivalent to solve the Lagrangian equation:

$$\mathcal{L}(I, \alpha) = H_R^*(I) - \alpha(E_R - \langle I \rangle),$$

where α is the Lagrange multiplier vector.

The constraint $\langle I \rangle = E_R$ being achieved by definition, the solution $\mathcal{B}_m$ to this minimization problem must satisfy, for all I, the following condition:

$$\partial_I \mathcal{L}(\mathcal{B}_m, \alpha)(I) = \left\langle \left(-\frac{k}{hc\nu} \ln\left(1 + \frac{1}{n}\right) + \alpha \right) I \right\rangle = 0.$$

The unique solution to this equation is solution of

$$\alpha = \frac{k}{hc\nu} \ln\left(1 + \frac{1}{n}\right). \tag{7}$$

in other words

$$\mathcal{B}_0 = \frac{2h\nu^3}{c^2} \left(\exp\left(\frac{hc\alpha\nu}{k}\right) - 1 \right)^{-1}. \tag{8}$$

The value of α is determined by the constraint $\langle I \rangle = aT_m^4$. Thus, $\alpha = 1/T_m c$ and $T_m = T_R$ at the Planckian equilibrium.

It is important to remark that relation (7) allows us to show, before any analytic computation, that $\alpha > 0$ (because n is always positive). This result is very important because it insures that B is strictly positive. We will see further that this property is true for every member of the M-hierarchy.

3. HIERARCHY OF MOMENT MODELS

Moment models were introduced by Grad (1949) and developed by Levermore (1996; see also Feugeas, 1997; Charrier *et al.*, 1998). Their construction starts by the choice of a vector-valued subspace, $\mathbb{M}$, of a finite dimension m of functions of $\mathbf{\Omega}$ with real values. $\mathbb{M}$ is usually chosen as being a space of Legendre polynomials or of the spheric functions. For a given m, the vector $\boldsymbol{m}$ is composed of the basic functions of $\mathbb{M}$. For example, for $m = 1$, the vector $\boldsymbol{m} = (1, \mathbf{\Omega})$. This is a combination of a scalar and a vector. Similarly, for $m = 2$, $\boldsymbol{m} = [\mathbf{\Omega}, \mathbf{\Omega} \otimes \mathbf{\Omega}]^T$.

3.1. Closure of moment equations

Considering the moments of the radiative transfer equation multiplied by the vector $\boldsymbol{m}$ leads to the construction of radiative moment equations:

$$\partial_t \langle \boldsymbol{m}I \rangle + c\nabla \cdot \langle \mathbf{\Omega}\boldsymbol{m}I \rangle = c\langle \boldsymbol{m}\sigma(B(T_m) - I) \rangle. \tag{9}$$

The moment vector is written as $\mathbf{E_R} = \langle \boldsymbol{m}I \rangle$. An additional relation is now needed to close the chosen moment model. The choice of closure relation is not unique, and is subject to only one constraint: The radiative intensity of the model

should be nonnegative. As was shown by Levermore (1996) for the Maxwell–Boltzmann theory, the closure of the radiative moment model (9) can be obtained by finding the distribution that minimizes the entropy under the constraint of a good construction of the moments generated in the vectorial space $\mathbb{M}$ (Muller & Ruggeri, 1993; Fort, 1997; Dubroca & Feugeas, 1999):

$$H_R^*(\mathcal{B}_m) = \min_I \{ H_R^*(I) : \langle \boldsymbol{m}I \rangle = \mathbf{E_R} \}. \tag{10}$$

This minimization problem is equivalent to solve the following Lagrangian equation:

$$\mathcal{L}(I, \boldsymbol{\alpha}) = H_R^*(I) - \boldsymbol{\alpha} \cdot (\mathbf{E_R} - \langle \boldsymbol{m}I \rangle),$$

where $\boldsymbol{\alpha}$ is the vector of the Lagrange multipliers. This vector is defined such that $\boldsymbol{\alpha} \cdot \boldsymbol{m} > 0$, which insures the positiveness of the solution ($\mathcal{B}_m > 0$). The solution $\mathcal{B}_m$ to this minimization problem must satisfy, for all I, the following condition:

$$\partial_{\mathcal{B}_m} \mathcal{L}(\mathcal{B}_m, \boldsymbol{\alpha})(I) = \left\langle \left(-\frac{k}{hc\nu} \ln\left(1 + \frac{1}{n}\right) + \boldsymbol{\alpha} \cdot \boldsymbol{m} \right) I \right\rangle,$$

$$= 0.$$

The solution to this equation is

$$\boldsymbol{\alpha} \cdot \boldsymbol{m} = \frac{k}{hc\nu} \ln\left(1 + \frac{1}{n}\right), \tag{11}$$

which leads to

$$\mathcal{B}_m(\boldsymbol{\alpha}) \equiv \frac{2h\nu^3}{c^2} \left[\exp\left(\frac{h\nu}{kT_m} \boldsymbol{\alpha} \cdot \boldsymbol{m}(\mathbf{\Omega})\right) - 1 \right]^{-1}. \tag{12}$$

Here, $\boldsymbol{\alpha}$ is determined by the constraint $\langle \boldsymbol{m}I \rangle = \mathbf{E_R}$.

Besides, the previous relation (11) insures that $\boldsymbol{\alpha} \cdot \boldsymbol{m} > 0$, because $n > 0$. This result is very important because it guarantees that, for every member of the hierarchy, $B > 0$.

The moment models are therefore written in the following form, if $\mathcal{B}_0 = \mathcal{B}_0(T_m)$ is given by (8):

$$\frac{1}{c}\partial_t \langle \boldsymbol{m}\mathcal{B}_m \rangle + \nabla \cdot \langle \mathbf{\Omega}\boldsymbol{m}\mathcal{B}_m \rangle = \langle \boldsymbol{m}\sigma(\mathcal{B}_0 - \mathcal{B}_m) \rangle, \tag{13}$$

where $\mathcal{B}_m$ is given by (12).

3.2. Radiative moment models

The hierarchy of the moment models (13) thus generated, offers interesting structural properties: The normalized flux $\mathbf{f}$ is limited ($\|\mathbf{f}\| \leq 1$); the system (13) is hyperbolic; and the system (13) locally dissipates the entropy.

The first property is extremely important as it guarantees that the propagation of a perturbation in the radiative medium

is limited by the speed of light. This property is not satisfied in the classical diffusion, or Eddington, description (P_1 model; Castor, 1981; Pomraning, 1992). It lies on the basic assumption that the angular dependence of the specific intensity is represented by the first two terms in the spherical harmonic expansion ($I = I_0 + \mathbf{\Omega} \cdot \mathbf{I}$). This specific intensity I is then almost isotropic as the angular term is supposed to be a small perturbation $\|\mathbf{I}\| \ll I_0$. If this condition is not respected, it leads to nonphysical negative emission in the flux direction. More generally, the P_n models fail to respect the property of limitation of the normalized flux, and require the introduction of a procedure for the limitation of the radiative flow in the nonopaque zones.

In turn for the M-models, the second and third properties guarantee that the system (13) has been properly set mathematically. Limitation properties on the radiative pressure $\overleftrightarrow{P}_R$ tensor can therefore be established. If the normalized flux f and the Eddington tensor $\overleftrightarrow{D}_R = \overleftrightarrow{P}_R / E_R$ are moments of the radiative intensity $\mathcal{B}_m$ defined by (12), they satisfy the following constraints:

$$tr(\overleftrightarrow{D}_R) = 1, \qquad \overleftrightarrow{D}_R - \mathbf{f} \otimes \mathbf{f} \geq 0.$$

This results in the flux limitation. For an extreme non-equilibrium, one finds $\|\mathbf{f}\| = 1$ and $\overleftrightarrow{D}_R = \mathbf{f} \otimes \mathbf{f}$.

4. THE FIRST MODEL OF THE HIERARCHY

The M_1 model is the first of the hierarchy, and it corresponds to the vector-valued space, $\mathbb{M}$, spanned by the vectors 1 and $\mathbf{\Omega}$. This is the smallest vector-valued subspace to restitute an angular anisotropy. The distribution function has the following form:

$$\mathcal{B}_1 = \frac{2h\nu^3}{c^2} \left[\exp\left(\frac{h\nu}{kT_m} B(1 - \mathbf{A} \cdot \mathbf{\Omega}) \right) - 1 \right]^{-1}. \quad (14)$$

The vector $\mathbf{A}$ and the constant B are defined by the conditions $E_R = \langle \mathcal{B}_1(A, B) \rangle$ and $\mathbf{F}_R = \langle c\mathbf{\Omega}\mathcal{B}_1(A, B) \rangle$:

$$\mathbf{A} = \frac{2 - \sqrt{4 - 3\|\mathbf{f}\|^2}}{\|\mathbf{f}\|^2} \mathbf{f},$$

$$B = \frac{T_m}{T_R} \left(\frac{3 + \|\mathbf{A}\|^2}{3(1 - \|\mathbf{A}\|^2)^3} \right)^{1/4}.$$

The details of this calculation are presented in Dubroca and Feugeas (1999). The radiative M_1 model can now be closed, and the tensor $\overleftrightarrow{P}_R = \overleftrightarrow{D}_R E_R$ can be expressed analytically as a function of the moments E_R and $\mathbf{F}_R$ of the model:

$$\partial_t E_R + \mathbf{\nabla} \cdot \mathbf{F}_R = c\sigma[aT_m^4 - E_R], \quad (15)$$

$$\frac{1}{c} \partial_t \mathbf{F}_R + c\mathbf{\nabla}(\overleftrightarrow{D}_R E_R) = -\sigma \mathbf{F}_R. \quad (16)$$

where $10\overleftrightarrow{D}_R$ is the Eddington tensor, calculated as a function of the Eddington factor χ and $\mathbf{n} = \mathbf{f}/\|\mathbf{f}\|$:

$$\overleftrightarrow{D}_R = \frac{\overleftrightarrow{P}_R}{E_R} = \frac{1 - \chi}{2} \overleftrightarrow{Id} + \frac{3\chi - 1}{2} \mathbf{n} \otimes \mathbf{n}. \quad (17)$$

The Eddington factor is a function of $\|\mathbf{f}\|$:

$$\chi = \frac{3 + 4\|\mathbf{f}\|^2}{5 + 2\sqrt{4 - 3\|\mathbf{f}\|^2}}. \quad (18)$$

It is an increasing function of $\|\mathbf{f}\|$ for which $\chi \leq 1$ and $\chi(0) = \frac{1}{3}$. The intrinsic values of the hyperbolic M_1 model at equilibrium (when $\mathbf{f} = 0$) are equal in absolute value to $c/\sqrt{3}$, and have the opposite signs. This situation clearly illustrates the emission isotropy of the photons that prevails at radiative equilibrium. On the other hand, in the case of extreme nonequilibrium (when $\|\mathbf{f}\|$ tends toward 1), the eigenvalues of the Jacobian tend towards the speed of light in absolute value and are of the same sign. In the case of extreme nonequilibrium (the angular distribution of the radiative intensity moves towards a δ-function), it is clearly found that the photons all move at the same speed and in the same direction.

The M_1 model, like every member of the M-hierarchy, has many desirable properties. It has been used in several domains (Audit *et al.*, 2002). However, this first model of the M-hierarchy is not able to describe more than one beam anisotropy. To capture more anisotropic effects, various modifications of the M_1 model are purposed (Dubroca & Klar, 2002; Klar & Dubroca, 2002; Klar *et al.*, 2003). However they lose interesting properties of the hierarchy. For example, the M-hierarchy models respect the Galilean invariance property of the underlying distribution. Besides, one of the very attractive characteristic of the first member of the hierarchy is the explicit analytic formulation. The other members of this M-hierarchy have no known simple explicit formulation. However, they conserve all the interesting properties of the M-models (hyperbolicity, flux is limited, entropic, Galilean invariance). For this reason, I propose here to explore and describe the potentialities of the second model of the M-hierarchy.

5. THE M_2 MODEL

The M_2 model corresponds to the vector-valued space, $\mathbb{M}s$, spanned by the vector $\mathbf{\Omega}$ and the tensor $\mathbf{\Omega} \otimes \mathbf{\Omega}$. The distribution function is then expressed as follows:

$$\mathcal{B}_2 \equiv \frac{2h\nu^3}{c^2} \left[\exp\left(\frac{h\nu}{kT_m} (\overleftrightarrow{C} : (\mathbf{\Omega} - \mathbf{A}) \otimes (\mathbf{\Omega} - \mathbf{A})) \right) - 1 \right]^{-1},$$

where the vector $\mathbf{A}$ and the tensor $\overleftrightarrow{C}$ are defined by the conditions : $\mathbf{F}_R = \langle c\mathbf{\Omega}\mathcal{B}_2 \rangle$ and $\overleftrightarrow{P}_R = \langle \mathbf{\Omega} \otimes \mathbf{\Omega}\mathcal{B}_2 \rangle$. The distribution $\mathcal{B}_2$ achieves the minimum radiative entropy under the

constraint of a good reconstruction of the moments $\mathbf{E_R} = [\mathbf{F}_R, \overset{\leftrightarrow}{P}_R]^T$:

$$H_R^*(\mathcal{B}_2) = \min_I \{H_R^*(I) : \langle mI \rangle = \mathbf{E_R}\}.$$

where $\boldsymbol{\alpha} = [\mathbf{A}, \overset{\leftrightarrow}{C}C]^T$ is the vector of the Lagrange multipliers. $\boldsymbol{m} = [\boldsymbol{\Omega}, \boldsymbol{\Omega} \otimes \boldsymbol{\Omega}]^T$. As we have shown, $\boldsymbol{\alpha}$ is defined such that, for any $\boldsymbol{\Omega}$:

$$\boldsymbol{\alpha} \cdot \boldsymbol{m}(\boldsymbol{\Omega}) > 0,$$

which insure that $\mathcal{B}_2 > 0$. For any $\boldsymbol{\Omega}$,

$$\overset{\leftrightarrow}{C} : (\boldsymbol{\Omega} - A) \otimes (\boldsymbol{\Omega} - A) > 0.$$

That means that C is a symmetric positively defined matrix for any $\boldsymbol{\Omega}$.

One can readily see that the pressure relation recovers also the energy relation because $Tr(\boldsymbol{\Omega} \otimes \boldsymbol{\Omega}) = 1$. That is the reason why the M_2 model only uses those two relations:

$$\frac{1}{c}\partial_t \mathbf{F}_R + c\boldsymbol{\nabla}\overset{\leftrightarrow}{P}_R = -\sigma\mathbf{F}_R,$$

$$\partial_t \overset{\leftrightarrow}{P}_R + \boldsymbol{\nabla}\overset{\leftrightarrow}{Q}_R = c\sigma[aT_m^4 Id - \overset{\leftrightarrow}{P}_R],$$

where the tensor $\overset{\leftrightarrow}{Q}_R$ is defined as $\langle \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} \otimes \boldsymbol{\Omega}\mathcal{B}_2 \rangle$.

We can note that M_2 required the resolution of nine equations in the three-dimensional case and five equations in the two-dimensional case. Besides, the M_2 model formally recovers the M_1 model. However, one cannot obtain an analytic formulation of this closure. This is not a big constraint to solving such models. Kinetic schemes offer very elegant and effective solutions (Dubroca & Klar, 2002).

6. ANALYSIS OF THE NONEQUILIBRIUM SOLUTIONS

The specific intensity I is a function of the position $\mathbf{r}$, of the frequency ν, and of the unitary propagation direction $\boldsymbol{\Omega}$. For a given coordinate system, this vector can be written as a function of the angle θ: $\boldsymbol{\Omega} = {}^T[\cos(\theta), \sin(\theta)]$. Then, for a given temperature, for a given point of the space, the normalized radiative intensity can be represented in cylindrical coordinates (I, θ, ν). Figure 1b,c presents two views of this function. For a given θ, the distribution $I(\nu)$ is always a Planck distribution (Fig. 1b) with a temperature depending on θ. Because the functional form of $I(\nu)$ is given, it is sufficient to plot an angular dependence for an arbitrary given ν. Therefore, the polar representation provides a useful tool to analyze the anisotropy of emission (Fig. 1a).
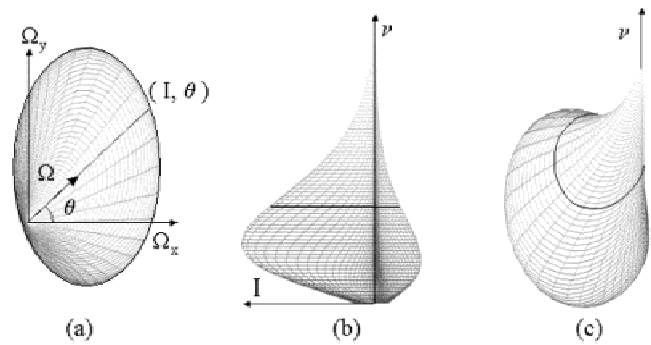

Fig. 1. Specific intensity $I(\theta, \nu)$ for a given temperature: (a) two-dimensional polar coordinates $I(\theta)$ for a given frequency; (b) $I(\theta, \nu)$; (c) cylindrical coordinates (I, θ, ν).

6.1. Analysis of the M_1 model

In an isotropic configuration, the underlying intensities, for the diffusion approximation P_1, and for M_1 are identical (Fig. 2a). This rapidly ceases to be the case with a normalized flux of the order of 0.3, and the difference increases for $\|\mathbf{f}\| = 0.5$ (Fig. 2b). When the normalized flux is of the order of 0.9 (Fig. 2b), that is, in the case of high anisotropies, a non-physical behavior of the P_1 model is evidenced. It would appear that in this case the radiative intensity is positive in the direction of the flux, that is, in the direction of highest energy. But the specific intensity is negative in the opposite direction of the flux. That indicates that over a certain limit the underlying diffusion intensity becomes
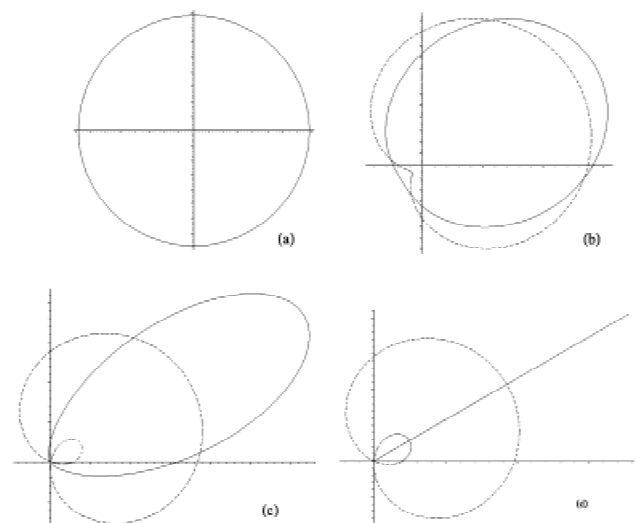

Fig. 2. The M_1 model: the underlying distributions of the M_1 model compared with the P_1 one: The dashed line correspond to the diffusion approximation (P_1) and the full one corresponds to the first member of the M-hierarchy (M_1). a: Planckian equilibrium. b: Medium anisotropy. c: Strong anisotropy; the diffusion approximation distribution can be negative with nonphysical sense. d: Extreme equilibrium: In the case of the light ray, the diffusion is no longer available.

negative. On the contrary, the underlying M_1 intensity remains physically admissible for any flux.

When the norm of the normalized flux tends toward 1 (Fig. 2d), the maximum physically acceptable anisotropy, the difference between the two models increases. The M_1 model tends toward a distribution carried solely by the radiative flux (a light ray), whereas the model P_1 fails to adopt any special direction. The P_1 model, contrary to the M_1 model, does not have any intrinsic flux limitation. For example, for $\|\mathbf{f}\| \geq 1$, the P_1 model is still defined and starts to present a strong region of negative radiative intensity. This analysis confirms the capacity of the underlying radiative intensity of the M_1 model to capture strong one-ray radiative nonequilibrium. For a maximum physically acceptable anisotropy, the M_1 model tends toward a distribution carried solely by the radiative flux (a light ray), whereas the P_1 model does not opt for any direction in particular and predicts a negative radiative intensity in the direction of the flux. This model is not able to describe more than a one-ray regime. The second model of the M-hierarchy offers more anisotropic potentialities.

6.2. Analysis of the M_2 model

At first, as was expected, the underlying distribution of the M_2 model, confirms its ability to recover formally the M_1 description. That is the case for weakly anisotropic situations (Fig. 3a) and also for extreme one-ray angular distributions (Fig. 3b). However, this model has much greater potentialities. Figure 4 presents the symmetric bipolar situation that the M_2 model is able to represent. The matrix $\overset{\leftrightarrow}{C}$ has a diagonal asymmetry in the first example (Fig. 4a), and a off-diagonal component for the other one (Fig. 4b). Figure 5 introduces nonsymmetric bipolarity characteristics associated with the energy flux. Figure 6 shows various two-ray emission configurations. Those kind of emissions could be longitudinal (Fig. 6a), oblique (Fig. 6b), and nonsymmetric (Fig. 6c).

The analysis of the emission anisotropic potentiality of the underlying radiative intensity of the M_2 model confirms its capacity to capture strong and various anisotropies. Those strong potentialities show that this model could be used in the inertial confinement fusion (ICF) domain.
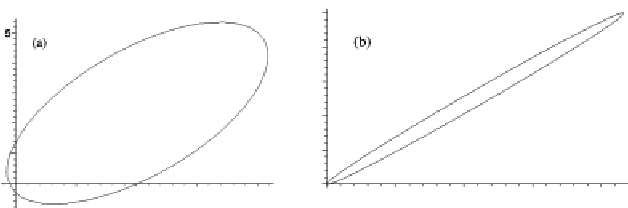

Fig. 4. The M_2 model: the symmetric bipolarity characteristic in the main (a) and secondary (b) directions; this could be the case of the emission seen by a point between two sources, ton their axe.

Let us consider, for example, the case of an emissing sphere (Fig. 7a). Usually, this kind of configuration cannot be represent by classical diffusive models or by the M_1 model. A punctual source, very far from point M, could be represent by this first model. However, when M get closer from the spherical source, the emission becomes bipolar as we can see in the Figure 7 in the Cartesian description (b) or in the polar one (c). This configuration can of course be represented by the second model of the hierarchy (Fig. 7c). This ability to get such bipolar situations can be extended in the multidimensional cases. Besides, if the source loses its symmetry, the second member of the hierarchy is able to take it into account. To illustrate the range of potentiality of this model, we could also choose the case of a punctual intense source in a gray middle, or the case of two far and punctual sources. For those cases, the M_2 model can get good solutions.

More generally, this second member seems to be able to get a solution for the treatment of boundary conditions between gray and transparent middle.

Fig. 5. The M_2 model: symmetric bipolarity characteristic; this could be the case of the emission of two sources seen from a point near (a) or far (b) from their axes or of two far sources (c).

Fig. 3. The M_2 model: The one-ray configuration shows that the second member. If this hierarchy formally recovers the M_1 model for medium dequilibrium (a) or for a stronger one (b).
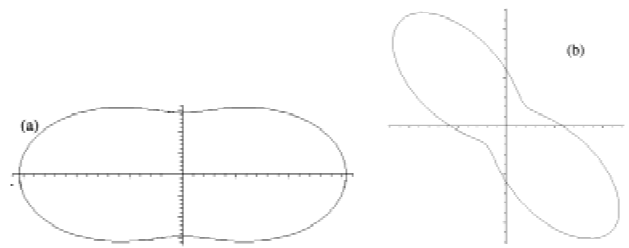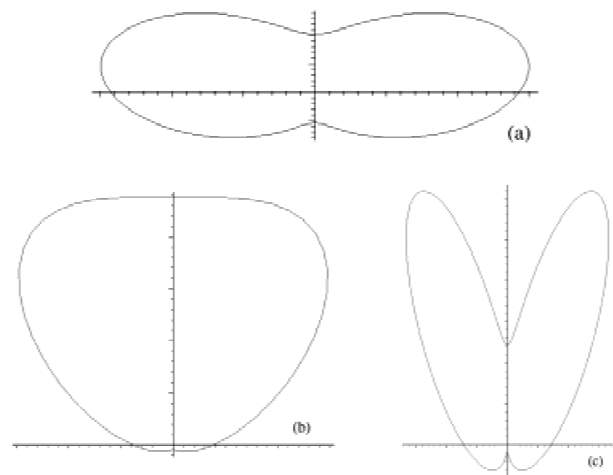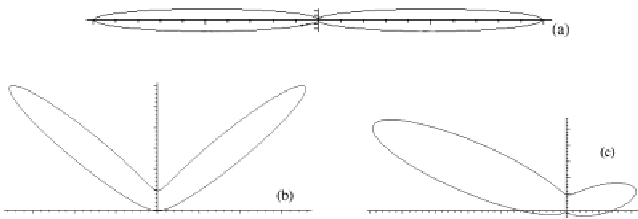
Fig. 6. The M_2 model can represent various extreme two-ray emission configurations; (a) symmetric and longitudinal; (b) symmetric; (c) non-symmetric.

7. CONCLUSIONS AND PERSPECTIVES

The models of this M-hierarchy offer very attractive properties. The hyperbolicity allowed us to use effective numerical tools. The limitation of the flux, and the entropy dissipation insure the physical credibility of the modelization. The construction of those models respects particularly the Galilean invariance of the underlying specific intensity. The first model (M_1) is very effective in the one-ray cases. One very interesting and useful property of this first member is the analytic formulation of its closure. However, it can be used in its multigroup formulation (Turpault, 2002) that is not analytic. Besides, various nonanalytic modifications also have been proposed by different authors (Dubroca & Klar, 2002; Klar & Dubroca, 2002; Klar *et al.*, 2003) since the original one (Dubroca & Feugeas, 1999) to get more anisotropy. The other members of this M-hierarchy have no known simple analytic formulation of their closure. However, they all respect the basic properties of the hierarchy and particularly the Galilean invariance property of the underlying distribution. Besides, the second model provides enough degrees of freedom to reach a large range of Des-equilibrium regime. The resolution of this model requires solving only nine moments equations (3D).

The main difficulty of the moments method is the *realizability* of the distribution function: for any moments set $\mathbf{E}_m$, one has to insure that it is always possible to find a distribution (12). We can show that all models of this hierarchy are realizable.

We can also note here another very useful characteristic of this family. All models have the same hyperbolic structure. This property allows us to use the same numerical tools for each one. Beside, the boundary conditions treatment is
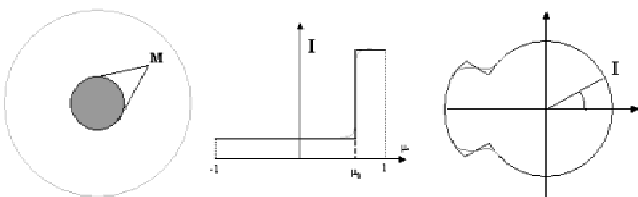

Fig. 7. M_2 model: Application of a spherical emitting source (a); for a point M close from the spherical source, the specific intensity becomes bipolar as we can see in the Cartesian description (b) or in the polar one (c).

one point very important for those kinds of radiative regime. For example, we could have to couple different kind of methods (continuous, kinetic) between gray and transparent middle. If we want to exchange kinetic informations such as the specific intensity, we have to respect the kinetic properties.

REFERENCES

AUDIT, E., CHARRIER, P., CHIÈZE, J.-P. & DUBROCA, B. (2002). A radiation-hydrodynamics scheme valid from the transport to the diffusion limit. *J. Comp. Phys.* **20**, 263.

CASTOR, J.I., LUTZ, J.H. & SEATON, M.J. (1981). Ultraviolet spectra of planetary nebulae. *Royal Astronomical Society, Monthly Notices* **194**, 547–567.

CHARRIER, P., DUBROCA, B., FEUGEAS, J.L. & MIEUSSENS, L. (1998). Modèles à vitesse discrètes pour le calcul d'ćoulements hors équilibre cinétique. *C.R. Acad. Sci., Série 1* **326**, 1347–1352.

CHARRIER, P., DUBROCA, B. & FEUGEAS, J.L. (1998). Levermore's Moment Method of Boltzmann Equation for Non-Equilibrium Kinetic Flows. R. Brun, R. Campargue, R. Gatignol, J.-C. Lengrand (Eds.). *21th International Symposium On Rarefied Gas Dynamic*, **103**, 1110. Marseille: Cépaduès Editions, Toulouse.

DUBROCA, B. & FEUGEAS, J.-L. (1999). Etude théorique et numérique d'une hiérarchie de modèles aux moments pour le transfert radiatif. *C.R. Acad. Sci., Série 1* **329**, 915–920.

DUBROCA, B. & KLAR, A. (2002). Half moment closure for radiative transfer equations. *J. Comp. Phys.* **180**, 1–13.

FEUGEAS, J.-L. (1997). Etude numérique des systèmes aux moments de Levermore pour la modélisation d'écoulements hors equilibre cinétique. Thèse de l'Université de Bordeaux I.

FORT, J. (1997). Information-theoretical approach to radiative transfer. *Physica A* **243**, 275–303.

GRAD, H. (1949). On the kinetic theory of rarefied gases. *Comm. Pure and Appl. Math.* **Ii**, 331–407.

KLAR, A., DUBROCA, B. FRANK, M. & THOEMMES, G. (2003). A half space moment approximation to the radiative heat transfer equations. *Z. Angew. Math. Mech.* **83**, 1–6.

KLAR, A. & DUBROCA, B. (2002). A half moment model to take into account strong kinetic non-equilibrium. *C.R. Acad. Sci., Série 1* **335**, 1–6.

LEVERMORE, D. (1996). Moment closure hierarchies for kinetic theories. *J. Stat. Phys.* **83**, 1021–1065.

MIHALAS, D. & MIHALAS, B.W. (1984). Foundations of radiation hydrodynamics. New York: Oxford University Press.

MULLER, I. & RUGGERI, T. (1993). *SprinTracts on Natural Philosophy*, Vol. 37. New York: Springer-Verlag.

POMRANING, G.C. (1992). *The Equations Of Radiation Thermodynamics*. New York: Pergamon Press.

STRUCHTRUP, H. (1997). Extended moment method in radiative transfer. *Annals of Physics* **257**, 111–135.

TURPAULT, R. (2002). Multigroup half space moment approximations. *C.R. Acad. Sci., Série 1* **334**, 331.