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Distance Preserving Ramsey Graphs

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We prove the following metric Ramsey theorem. For any connected graph G endowed with a linear order on its vertex set, there exists a graph R such that in every colouring of the t-sets of vertices of R it is possible to find a copy G^* of G inside R satisfying:

- $\mathrm{dist}_{G^*}(x, y) = \mathrm{dist}_R(x, y)$ for every $x, y \in V(G^*)$;
- the colour of each t-set in G^* depends only on the graph-distance metric induced in G by the ordered t-set.

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1. Introduction

In [2], [4] and [22, 23] the following extension of the Ramsey theorem was proved.

Theorem 1.1. *For any graph G there exists a graph R with the property that in any 2-colouring of the edges of R there exists an induced copy $\mathsf{G} \subset R$ which is monochromatic.*[1]

In other words, Theorem 1.1 states that the class of all graphs and induced embeddings has the *edge-Ramsey property*. This theorem, proved in 1973, together with some generalizations and other related results that soon followed, gave rise to the study of the restricted/induced/sparse family of Ramsey theorems (for a survey on these topics see [5, 15]).

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[1] For a graph G, we will use G (typeset in a sans serif font) to denote an isomorphic copy of G.

Remark 1.2. For simplicity we state Ramsey theorems only for 2-colourings, when in fact it is straightforward to extend them to an arbitrary number of colours by applying the 2-colour version inductively.

Theorem 1.1 was generalized by Deuber [3] and Nešetřil and the second author [12] in the sense that the same statement remains true if the colouring of edges (K_2) is replaced by the colouring of cliques (K_k) or induced independent sets ($\overline{K}_k$). Moreover, Theorem 1.1 fails to be true if one colours copies of an arbitrary non-homogeneous graph F. More formally, for any graph $F \neq K_k, \overline{K}_k$ there exists G such that, for every graph H, there is a 2-colouring of the set of all induced copies of F in H such that no induced copy G in H is monochromatic (that is, there must be induced copies of F in G of both colours).

With terminology used in [10] this can be rephrased as follows.

Proposition 1.3. *The class of graphs and induced embeddings has the F-*Ramsey property *if and only if F is a complete graph or an independent set.*

Let us show by means of a simple example that for non-homogeneous unordered graphs F, the class of (unordered) graphs and induced embeddings does not have the F-Ramsey property. Consider the graph $F = P_2$, the path with two edges. Let $G = C_4$ be the cycle of length four and let R be an arbitrary graph. We will now introduce a 2-colouring of the (unordered) induced copies of P_2 in R. First, label the vertices of $V(R)$ with integers $1, 2, \ldots, |V(R)|$. For a path ijk of length two in R, colour ijk red if the middle vertex j is the smallest of the three ($j < i$ and $j < k$); otherwise, colour it blue. Under this colouring, any induced copy of $G = C_4$ in R must contain P_2 of both colours. Indeed, among the four vertices of the C_4, the smallest vertex is the middle vertex of a P_2 coloured red and the largest vertex is the middle vertex of a P_2 coloured blue.

However, it was shown by Nešetřil and Rödl [12] that if one considers graphs with linearly ordered vertex sets and induced monotone embeddings then the theorem becomes true for all graphs $(F, <)$. This is stated in Theorem 1.6 below.

Remark 1.4 (Ordered graphs). Since our result deals with an extension of Theorem 1.6, in this paper we typically assume (as in [1] and [13]) that each graph has a linear order on its vertex set. The example we described above (colouring P_2) shows that this assumption is crucial. All maps between ordered vertex sets are considered to be *monotone*, that is, $\phi(u) < \phi(v)$ whenever $u < v$. In particular, all isomorphisms between ordered graphs are unique.

Definition 1.5 (Subgraphs). We say that the graph G is an *induced subgraph* of the graph H (we write $G \subset H$) if $V(G) \subset V(H)$, $E(G) = \{e \in E(H) \ : \ e \subset V(G)\}$ and the order $<_G$ in $V(G)$ respects the order $<_H$ in $V(H)$, that is, for every $u, v \in V(G)$ we have $u <_G v$ if and only if $u <_H v$.

To avoid cumbersome notation, we will omit the linear orders $<_H$, $<_G$ and denote by $\binom{H}{G}_{\text{ind}}$ the set of all induced subgraphs of H which are (monotone) isomorphic to G.

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With this definition we may now state the Ramsey theorem for graphs with monotone induced embeddings.

Theorem 1.6 ([1, 13]). *For any ordered graphs F and G there exists an ordered graph R such that, for any partition*

$$\binom{R}{F}_{\text{ind}} = \mathcal{A}_1 \cup \mathcal{A}_2,$$

there exists some $\mathsf{G} \in \binom{R}{G}_{\text{ind}}$ *such that* $\binom{\mathsf{G}}{F}_{\text{ind}} \subset \mathcal{A}_i$ *for some* $i \in \{1,2\}$.

In other words, Theorem 1.6 states that the class of ordered graphs and induced monotone embeddings has the $(F,<)$-Ramsey property for any ordered graph $(F,<)$.

Remark 1.7. If a class $\mathcal{K}$ endowed with a set of embeddings has the K-Ramsey property for all $K \in \mathcal{K}$ it is called a *Ramsey class* (see, for instance, [6]). Theorem 1.6 shows that the class of ordered graphs with induced monotone embeddings is a Ramsey class. See [5], [6], [7], [11], [17] and [20] for other examples of Ramsey classes, such as

- finite partially ordered sets (with a fixed linear extension),
- finite vector spaces (over a fixed field F),
- finite labelled partitions,
- finite linearly ordered metric spaces.

Another way to refine Theorem 1.1 is to consider distance preserving embeddings rather than induced ones. (Distance preserving embeddings have been considered in other contexts, for instance, in [8, 24].) For ordered graphs R and G, let $\mathsf{G} \in \binom{R}{G}_{\text{ind}}$ be fixed. If, for all $x, y \in V(\mathsf{G}) \subset V(R)$,

$$\text{dist}_{\mathsf{G}}(x,y) = \text{dist}_R(x,y), \tag{1.1}$$

then G is called a *metric copy* of G in R and the (unique) monotone isomorphism $\phi\colon V(G) \to V(\mathsf{G}) \subset V(R)$ is called a *distance preserving embedding* of G into R. Denote by $\binom{R}{G}_{\text{metric}}$ the set of all metric copies of G in R. Notice that $\binom{R}{G}_{\text{metric}} \subset \binom{R}{G}_{\text{ind}}$.

The following theorem is a consequence of our main result, Theorem 1.11.

Theorem 1.8. *For any ordered connected graphs F and H there exists an ordered graph R such that, for any partition*

$$\binom{R}{F}_{\text{metric}} = \mathcal{A}_1 \cup \mathcal{A}_2,$$

there exists some $\mathsf{H} \in \binom{R}{H}_{\text{metric}}$ *such that* $\binom{\mathsf{H}}{F}_{\text{metric}} \subset \mathcal{A}_i$ *for some* $i \in \{1,2\}$.

In effect, Theorem 1.8 shows that the class of ordered connected graphs with metric embeddings is also a Ramsey class. Our proof of Theorem 1.8 will use a slightly more general setting.

A *discrete metric* ρ on the set $[t] = \{1,2,\ldots,t\}$ is a symmetric function $\rho : [t]^2 \to \mathbb{N} \cup \{\infty\}$ satisfying: $\rho(i,j) = 0$ if and only if $i = j$, and the triangle inequality,

$$\rho(i,j) + \rho(j,k) \geqslant \rho(i,k).$$

In this paper, the metrics considered correspond to the distance given by shortest paths in a graph. For instance, the metric of a clique would satisfy $\rho(i,j) = 1$ for all $i \neq j$ and the metric of an empty graph would satisfy $\rho(i,j) = \infty$ for all $i \neq j$.

Definition 1.9 (Metric induced by a set; (ρ, G)-tuples). Let G be an ordered graph and let $S = \{v_1,\ldots,v_t\} \subset V(G)$, $v_1 < v_2 < \cdots < v_t$, be an arbitrary set. The *metric* ρ *induced by* S *in* G is given by $\rho(i,j) = \mathrm{dist}_G(v_i,v_j)$.

Let ρ be a fixed metric. A set S which induces the metric ρ in G is called a (ρ,G)-*tuple*. The set of all (ρ,G)-tuples of G is denoted $\binom{G}{\rho}$.

We prove a slightly stronger statement from which Theorem 1.8 is derived as a corollary.

Lemma 1.10. *Let $t \in \mathbb{N}$, ρ be a metric on $[t]$ and H be an ordered connected graph. Then there exists an ordered graph R such that, for every 2-colouring of $\binom{R}{\rho}$, there exists $\mathsf{H} \in \binom{R}{H}_{\mathrm{metric}}$ such that $\binom{\mathsf{H}}{\rho}$ is monochromatic.*

We now derive Theorem 1.8 from Lemma 1.10 as follows. Let F and H be given ordered graphs. Take $t = |V(F)|$, and without loss of generality assume that $V(F) = [t]$ (with the usual order $<$). Let ρ be the metric corresponding to dist_F, namely, $\rho(i,j) = \mathrm{dist}_F(i,j)$.

We first obtain an ordered graph R from Lemma 1.10 applied to H and ρ. We claim that the graph R has the Ramsey property of Theorem 1.8.

Notice that $\binom{R}{F}_{\mathrm{metric}} \cong \binom{R}{\rho}$ since the vertex set of a metric copy of F is necessarily a (ρ,R)-tuple. Consequently, we can view any colouring χ of $\binom{R}{F}_{\mathrm{metric}}$ as a colouring of $\binom{R}{\rho}$. By the hypothesis on R, there exists a graph $\mathsf{H} \in \binom{R}{H}_{\mathrm{metric}}$ such that every (ρ,H)-tuple has the same colour c under χ. For every $\mathsf{F} \in \binom{\mathsf{H}}{F}_{\mathrm{metric}}$ the set $V(\mathsf{F})$ is a (ρ,H)-tuple, and therefore $\chi(\mathsf{F}) = c$. It follows that $\binom{\mathsf{H}}{F}_{\mathrm{metric}}$ is monochromatic.

In Section 4 we prove Lemma 1.10 and use it to establish our main result, Theorem 1.11.

Theorem 1.11. *Let $t \in \mathbb{N}$ and H be a connected ordered graph.*

There exists an ordered graph R with the following property. For every 2-colouring of $\binom{V(R)}{t}$ there exists $\mathsf{H} \in \binom{R}{H}_{\mathrm{metric}}$ such that $\binom{\mathsf{H}}{\rho}$ is monochromatic for every metric ρ on $[t]$.

After fixing connected graphs H and F, note that Theorem 1.8 asserts that colouring all metric copies of F in R yields a monochromatic $\binom{H}{F}_{\mathrm{metric}}$. On the other hand, Theorem 1.11 applies to all subgraphs of H on t vertices (even those which are not connected). It guarantees that there exists a copy of H in which the colour of a t element subgraph depends only on its metric within H.

Note that Theorem 1.11 extends Theorem 1.8.

Remark 1.12. The particular case $t = 2$ of Theorem 1.11 implies that for any connected graph H it is possible to find some graph R such that every colouring of the pairs in $\binom{V(R)}{2}$ yields a metric copy $\mathsf{H} \in \binom{R}{H}_{\text{metric}}$ in which the colour of $\{x, y\} \in \binom{V(\mathsf{H})}{2}$ is a function of $\text{dist}_{\mathsf{H}}(x, y)$. (In particular, the edges of H are monochromatic.) This special case $t = 2$ was stated in the survey [15].

Remark 1.13. Notice that for $t = 2$ the linear order on the vertices is irrelevant. In Section 4.2 we show a version of Lemma 1.10 that can be applied to unordered graphs (provided that the metric is 'homogeneous').

Definition 1.14 (ρ_ℓ-metric sets and (ρ_ℓ, G)-tuples). Let $\ell, t \in \mathbb{N}$ be fixed and let ρ be a metric on $[t]$. Let $H = (H, <)$ be a graph and let $S = \{v_1, v_2, \dots, v_t\} \subset V(H)$ be a set with $v_1 < v_2 < \cdots < v_t$. We say that S is *ρ_ℓ-metric with respect to H* if, for all $1 \leqslant i < j \leqslant t$,

- $\text{dist}_H(v_i, v_j) = \rho(i, j)$ whenever $\rho(i, j) \leqslant \ell$,
- $\text{dist}_H(v_i, v_j) \geqslant \ell$ whenever $\rho(i, j) > \ell$.

A set S as above is called a *(ρ_ℓ, H)-tuple*. We denote by $\binom{H}{\rho_\ell}$ the family of all (ρ_ℓ, H)-tuples of H.

A graph G naturally induces a metric $\rho(G)$ over its vertices by defining the distance between pairs of vertices as the length of a shortest path connecting them (when the pair is not connected, their distance is ∞).

Definition 1.15 (ℓ-metric (sub)graph). For a graphs $G \subset R$, the graph G is said to be *ℓ-metric in R* if $V(G)$ is $\rho(G)_\ell$-metric with respect to R. A connected graph G is *metric in R* if it is ℓ-metric in R for all ℓ, namely, $\text{dist}_G(x, y) = \text{dist}_R(x, y)$ for every $x, y \in V(G)$.

Notice that G is ℓ-metric in R if no pair of vertices in G admits a shortcut path in R of length smaller than ℓ. For instance, G is 2-metric in R if and only if it is an induced subgraph of R.

Recalling that all vertex sets are linearly ordered, for $A, B \subset V(G)$ we will write $A \prec B$ if $\max(A) < \min(B)$.

Definition 1.16 (q-partite graphs). For $q \geqslant 2$, the graph G together with the linear order $<$ on $V(G)$ and a partition $V(G) = V_1^q(G) \cup \cdots \cup V_q^q(G)$ is called *q-partite* if

- every edge $e \in G$ is *crossing*, that is, $|e \cap V_i^q(G)| \leqslant 1$ for all $i = 1, \dots, q$,
- the partition satisfies $V_1^q(G) \prec V_2^q \prec \cdots \prec V_q^q(G)$.

Definition 1.17 (Partite embedding/isomorphism). If G and H are ordered q-partite graphs, a *partite embedding* is an injective monotone map $\phi \colon V(G) \to V(H)$ which is edge preserving ($\phi(e) \in E(H)$ for all $e \in E(G)$) and satisfies $\phi(V_j^q(G)) \subset V_j^q(H)$ for all $j = 1, \dots, q$. If, in addition, ϕ is an isomorphism then we call it a *partite isomorphism.*

Definition 1.18 (Notation). We will use the following notation.

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- For a (hyper)graph G we abuse the notation and write $e \in G$ to denote $e \in E(G)$.
- For a (hyper)graph G and a one-to-one map $\phi : V(G) \to X$, set

$$\phi(G) = \big(\phi(V(G)), \{\phi(e) \,:\, e \in G\}\big).$$

- For q-partite graphs G and H we denote by $\binom{H}{G}_{\mathrm{Part}(q)}$ the set of all subgraphs $\phi(G)$ of H, where $\phi : V(G) \to V(H)$ is a partite embedding.
- If G is an isomorphic copy of G with (unique) monotone isomorphism $\sigma : V(G) \to V(\mathsf{G})$ and $\mathcal{I}$ is a hypergraph with $V(\mathcal{I}) \subset V(G)$, then we denote by $\mathcal{I}_{\mathsf{G}}$ the hypergraph $\sigma(\mathcal{I})$.

Lemma 1.19 below is a technical result which will be used in the proof of our main result, Theorem 1.11.

Lemma 1.19 (Partite Lemma). *Let $\ell, t, q \in \mathbb{N}$, $t \leqslant q$. Suppose that*

- *ρ is a fixed metric on $[t]$,*
- *G is a q-partite (ordered) graph with partition $V(G) = V_1^q(G) \cup \cdots \cup V_q^q(G)$,*
- *for some $1 \leqslant j_1 < j_2 < \cdots < j_t \leqslant q$, $\mathcal{I} \subset \binom{G}{\rho_\ell}$ is a t-partite t-uniform hypergraph with classes $\{V_{j_i}^q(G)\}_{i=1}^t$ consisting of selected (ρ_ℓ, G)-tuples.*

Then there exists a q-partite ordered graph R and $\mathcal{G} \subset \binom{R}{G}_{\mathrm{Part}(q)}$ satisfying the following properties.

(L1) *For any 2-colouring of the (ρ_ℓ, R)-tuples in $\bigcup_{\mathsf{G} \in \mathcal{G}} \mathcal{I}_{\mathsf{G}}$, there exists $\mathsf{G} \in \mathcal{G}$ such that $\mathcal{I}_{\mathsf{G}} \subset \binom{\mathsf{G}}{\rho_\ell} \subset \binom{R}{\rho_\ell}$ is monochromatic.*
(L2) *Every $\mathsf{G} \in \mathcal{G}$ is ℓ-metric in R.*

Remark 1.20. Note that $\bigcup_{\mathsf{G} \in \mathcal{G}} \mathcal{I}_{\mathsf{G}}$ is a t-partite t-uniform hypergraph with classes $\{V_{j_i}^q(R)\}_{i=1}^t$. This is because by the definition of $\binom{R}{G}_{\mathrm{Part}(q)}$ every $\mathsf{G} \in \mathcal{G} \subset \binom{R}{G}_{\mathrm{Part}(q)}$ is the image of G under a partite embedding into R (and thus $V_j^q(\mathsf{G}) \subset V_j^q(R)$ for all $j = 1, \ldots, q$).

Moreover, it will follow from our proof that for any pair of distinct $\mathsf{G}, \mathsf{G}' \in \mathcal{G}$ we have $V(\mathsf{G}) \cap V(\mathsf{G}') \subset \bigcup_{i=1}^t V_{j_i}^q(R)$.

The proof of Lemma 1.19 uses the *partite construction* method, which was introduced by Nešetřil and Rödl [16], and has been a successful tool for proving the existence of several Ramsey structures such as metric spaces [11], systems of sets [19], Steiner systems [18], *etc*. Perhaps a novelty here is that the Partite Lemma, which was usually proved using the Hales–Jewett theorem directly, is proved here by induction using the partite construction as well.

2. Proof of Lemma 1.19

We will prove a slightly stronger statement by double induction. The main induction is over ℓ. The base case ($\ell = 2$) is presented in Section 3 (Lemma 3.1). In this section we will prove the induction step from ℓ to $\ell + 1$. For easy reference, we describe in detail the induction hypotheses below (these are repeated in Appendix B for the reader who wishes

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to keep on hand the extensive list of hypotheses satisfied by the graphs and families of graphs constructed here).

Remark 2.1. The somewhat complicated intersection conditions (A) and (B) below, serve the purpose of imposing useful constraints on how the copies in the family may intersect while at the same time being weak enough to be carried by the induction. The condition (B) is later used to guarantee that when two vertices are shared by two copies then the distances with respect to each copy are 'compatible'. More precisely, if we wish to obtain a family of ℓ-metric subgraphs then it is obvious that any pair of vertices at distance $\ell' < \ell$ in some copy should not have distance $\neq \ell'$ in another copy.

Induction over ℓ: Hypothesis for R_ℓ and $\mathcal{G}_\ell$. For a q-partite graph G, a metric ρ on $[t]$ and a t-partite t-uniform hypergraph $\mathcal{I} \subset \binom{G}{\rho_\ell}$, there is a graph $R_\ell = R_\ell(q, G, \rho, \mathcal{I})$ and $\mathcal{G}_\ell = \mathcal{G}_\ell(q, G, \rho, \mathcal{I}) \subset \binom{R_\ell}{G}_{\mathrm{Part}(q)}$ satisfying conditions (L1) and (L2) of Lemma 1.19 and

(L3) $E(R_\ell) = \bigcup_{\mathsf{G} \in \mathcal{G}_\ell} E(\mathsf{G})$.

Moreover, $\mathcal{G}_\ell$ satisfies the conditions (A) and (B) below.

Intersection conditions for a family $\mathcal{G}$ of copies of G.

(A) If $\mathsf{G}_1, \mathsf{G}_2 \in \mathcal{G}$ and $u \in V(\mathsf{G}_1) \cap V(\mathsf{G}_2)$, then there are $(\rho_\ell, \mathsf{G}_j)$-tuples $I^j \in \mathcal{I}_{\mathsf{G}_j}$, $j = 1, 2$, such that $u \in I^1 \cap I^2$.

(B) If $\mathsf{G}_1, \mathsf{G}_2 \in \mathcal{G}$ and $u, v \in V(\mathsf{G}_1) \cap V(\mathsf{G}_2)$, then either

- (B1) there exist $(\rho_\ell, \mathsf{G}_j)$-tuples $I^j \in \mathcal{I}_{\mathsf{G}_j}$, $j = 1, 2$, such that $\{u, v\} \subset I^1 \cap I^2$ or
- (B2) the (unique) isomorphisms $\sigma_j : V(\mathsf{G}_j) \to V(G)$, $j = 1, 2$, satisfy $\sigma_1(u) = \sigma_2(u)$ and $\sigma_1(v) = \sigma_2(v)$.

Remark 2.2. The condition (B2) above is an artifact of the induction base, which is then propagated by induction. It should be possible to prove a stronger base assumption which would entirely eliminate the need for (B2). This would simplify the proof of the induction step at the cost of making the base case more involved.

Given q, G, ρ and $\mathcal{I} \subset \binom{G}{\rho_{\ell+1}} \subset \binom{G}{\rho_\ell}$ as in the statement of the lemma, we obtain $R_\ell = R_\ell(q, G, \rho, \mathcal{I})$ and $\mathcal{G}_\ell = \mathcal{G}_\ell(q, G, \rho, \mathcal{I})$ from the induction hypothesis over ℓ. Our goal is to construct $R_{\ell+1}$ and $\mathcal{G}_{\ell+1}$ satisfying the hypothesis for $\ell + 1$.

Consider the family

$$\bigcup_{\mathsf{G} \in \mathcal{G}_\ell} \mathcal{I}_{\mathsf{G}} = \{I_1, I_2, \ldots, I_m\} \subset \binom{R_\ell}{\rho_\ell}. \tag{2.1}$$

This family is a t-partite t-uniform hypergraph with partition $\{V^q_{j_i}(R_\ell)\}_{i=1}^t$ (see Figure 1).

We will construct a sequence of $|V(R_\ell)|$-partite graphs $P_0, P_1, \ldots, P_m$, which we will call *pictures*,[2] and families $\mathcal{G}(P_k) \subset \binom{P_k}{G}_{\mathrm{Part}(q)}$, $k = 0, 1, \ldots, m$. We will then show that $R_{\ell+1} = P_m$

[2] The name 'pictures' has been used before, *e.g.*, in [21].

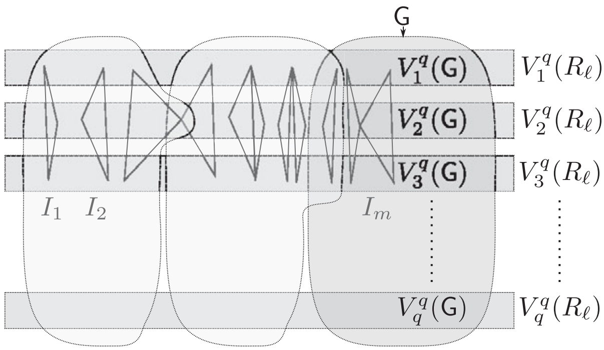

Figure 1. An illustration of R_ℓ and $\mathsf{G} \in \mathcal{G}_\ell$. Here we assume $t = 3$, $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$. The triples of (2.1) are represented by the crossing triangles.

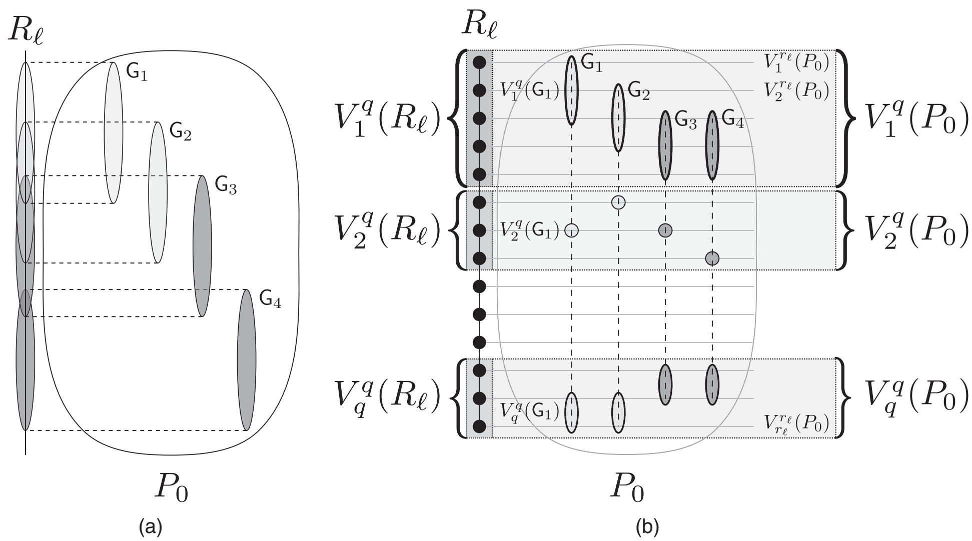

Figure 2. (a) P_0 is a disjoint union of copies of G where each copy is projected by π_0 into a copy of G in $\mathcal{G}_\ell$. (b) P_0 with its coarse q-partition and the refined r_ℓ-partition (see (2.2) and (2.3)). Notice that the copies of G are partite embedded in the q-partite graph P_0 (see Definition 1.17).

and $\mathcal{G}_{\ell+1} = \mathcal{G}(P_m)$ satisfy conditions (L1), (L2), (L3), (A), and (B). This will establish the induction step and conclude the proof of Lemma 1.19.

Let us start by constructing P_0 (see Figure 2). For convenience, let $r_\ell = |V(R_\ell)|$. For each $u \in V(R_\ell)$, let

$$V_u^{r_\ell}(P_0) = \{(u, \mathsf{G}) \ : \ \mathsf{G} \in \mathcal{G}_\ell, V(\mathsf{G}) \ni u\}. \tag{2.2}$$

Recalling the total order on $V(R_\ell)$ we may assume in fact that $V(R_\ell) = \{1, 2, \ldots, r_\ell\}$. We then impose a total order in $V(P_0)$ that satisfies $V_j^{r_\ell}(P_0) \prec V_{j+1}^{r_\ell}(P_0)$ for all $j = 1, \ldots, r_\ell - 1$.

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The edges of P_0 are of the form $\{(u, \mathsf{G}), (w, \mathsf{G})\}$, where $uw \in E(\mathsf{G})$, $\mathsf{G} \in \mathcal{G}_\ell$. Notice that the r_ℓ-partition of P_0 given by (2.2) is indeed such that every edge of P_0 is crossing. We set $\mathcal{G}(P_0)$ to be the set of copies of G in correspondence with $\mathcal{G}_\ell$. In particular, $|\mathcal{G}(P_0)| = |\mathcal{G}_\ell|$. Moreover, the projection $\pi_0(u, G) = u$ defines a monotone homomorphism from P_0 to R_ℓ.

Assuming that the hypothesis holds for some $\ell \geqslant 2$, we will now describe the induction over k.

Induction over k: Hypothesis for P_k and $\mathcal{G}(P_k)$.

(K1) The picture P_k is r_ℓ-partite with classes $V_j^{r_\ell}(P_k)$, $j = 1, \ldots, r_\ell$. The projection map $\pi_k : V(P_k) \to V(R_\ell) = [r_\ell]$ given by $\pi_k(x) = j$ if and only if $x \in V_j^{r_\ell}(P_k)$ is a homomorphism of P_k into R_ℓ. Moreover, $\pi_k(\mathsf{G}) \in \mathcal{G}_\ell$ for every $\mathsf{G} \in \mathcal{G}(P_k)$.
(K2) The family $\mathcal{G}(P_k)$ is contained in $\binom{P_k}{G}_{\mathrm{Part}(q)}$.
(K3) The family $\mathcal{G}(P_k)$ satisfies conditions (A) and (B).
(K4) Every $\mathsf{G} \in \mathcal{G}(P_k)$ is $(\ell + 1)$-metric in P_k.

Claim 2.3. *The graph P_0 satisfies the induction hypothesis for $k = 0$.*

Since the copies of G in P_0 are vertex-disjoint (and thus metric) and are projected by π_0 into copies of G in R_ℓ it is clear that (K1), (K3) and (K4) hold for P_0 and $\mathcal{G}(P_0)$. It remains to check (K2), namely, that $\mathcal{G}(P_0)$ is contained in $\binom{P_0}{G}_{\mathrm{Part}(q)}$.

We now observe that the q-partition of $V(P_0)$ may be expressed in terms of π_0 as

$$V_j^q(P_0) = \pi_0^{-1}(V_j^q(R_\ell)) = \bigcup_{u \in V_j^q(R_\ell)} V_u^{r_\ell}(P_0) \tag{2.3}$$

for $j = 1, \ldots, q$ (see Figure 2). For every $\mathsf{G} \in \mathcal{G}(P_0)$, we have $\mathsf{G}' = \pi_0(\mathsf{G}) \in \mathcal{G}_\ell$. From the induction hypothesis over ℓ we have $\mathsf{G} \in \mathcal{G}(P_0) \subset \binom{R_\ell}{G}_{\mathrm{Part}(q)}$, and hence the isomorphism $\sigma : V(G) \to V(\mathsf{G}')$ must be a partite isomorphism. Then $\pi_0^{-1} \circ \sigma : V(G) \to V(\mathsf{G})$ is a partite isomorphism of G into G by our choice of $V_j^q(P_0)$, $j = 1, \ldots, q$.

Hence P_0 satisfies the induction hypothesis for $k = 0$ and Claim 2.3 is proved.

Suppose that P_k, $\mathcal{G}(P_k)$, and π_k, $k \geqslant 0$, are constructed and satisfy the induction hypothesis. Since every $\mathsf{G} \in \mathcal{G}(P_k)$ is $(\ell + 1)$-metric in P_k, it follows that $\mathcal{I}_{\mathsf{G}} \subset \binom{\mathsf{G}}{\rho_{\ell+1}} \subset \binom{P_k}{\rho_{\ell+1}}$ for every $\mathsf{G} \in \mathcal{G}(P_k)$. Define

$$\mathcal{I}^{(k)} = \left\{ I \in \bigcup_{\mathsf{G} \in \mathcal{G}(P_k)} \mathcal{I}_{\mathsf{G}} \; : \; \pi_k(I) = I_{k+1} \right\} \subset \binom{P_k}{\rho_{\ell+1}}, \tag{2.4}$$

where the (ρ_ℓ, R_ℓ)-tuple $I_{k+1} = \{w_1, w_2, \ldots, w_t\}$ is the $(k + 1)$th tuple from (2.1).

Observe that by construction, $\mathcal{I}^{(k)}$ is a t-partite t-uniform hypergraph. Indeed, every tuple in $\mathcal{I}^{(k)}$ is crossing with respect to the sets $\{\pi_k^{-1}(u) = V_u^{r_\ell}(P_k)\}_{u \in I_{k+1}}$. To construct P_{k+1} we invoke our induction assumption over ℓ with

- r_ℓ in place of q,
- P_k in place of G,
- $\mathcal{I}^{(k)} \subset \binom{P_k}{\rho_{\ell+1}} \subset \binom{P_k}{\rho_\ell}$ in place of $\mathcal{I}$.

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We then obtain the graph $P_{k+1} = R_\ell(r_\ell, P_k, \rho, \mathcal{I}^{(k)})$ and a family $\mathcal{P}_{k+1} = \mathcal{G}_\ell(r_\ell, P_k, \rho, \mathcal{I}^{(k)}) \subset \binom{P_{k+1}}{P_k}_{\mathrm{Part}(r_\ell)}$ satisfying conditions (L1), (L2), (L3), (A) and (B). More specifically, the following holds.

$(1)_{k+1}$ For every 2-colouring of the (ρ_ℓ, P_{k+1})-tuples in $\bigcup_{\mathsf{P}\in\mathcal{P}_{k+1}} \mathcal{I}^{(k)}_{\mathsf{P}}$, there exists $\mathsf{P} \in \mathcal{P}_{k+1}$ such that $\mathcal{I}^{(k)}_{\mathsf{P}} \subset \binom{\mathsf{P}}{\rho_{\ell+1}} \subset \binom{P_{k+1}}{\rho_\ell}$ is monochromatic (recall that the hypergraph $\mathcal{I}^{(k)}_{\mathsf{P}}$ is an isomorphic copy of $\mathcal{I}^{(k)}$ in P).

$(2)_{k+1}$ Every $\mathsf{P} \in \mathcal{P}_{k+1}$ is ℓ-metric in P_{k+1}.

$(3)_{k+1}$ $E(P_{k+1}) = \bigcup_{\mathsf{P}\in\mathcal{P}_{k+1}} E(\mathsf{P})$.

$(\mathrm{A})_{k+1}$ If $\mathsf{P}^1, \mathsf{P}^2 \in \mathcal{P}_{k+1}$ are distinct and $u \in V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$, then there are $(\rho_{\ell+1}, \mathsf{P}^j)$-tuples $I^j_* \in \mathcal{I}^{(k)}_{\mathsf{P}^j}$, $j = 1, 2$, such that $u \in I^1_* \cap I^2_*$.

$(\mathrm{B})_{k+1}$ If $\mathsf{P}^1, \mathsf{P}^2 \in \mathcal{P}_{k+1}$ are distinct and $u, v \in V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$, then either

- $(\mathrm{B1})_{k+1}$ there exist $(\rho_{\ell+1}, \mathsf{P}^j)$-tuples $I^j_* \in \mathcal{I}^{(k)}_{\mathsf{P}^j}$, $j = 1, 2$, such that $\{u, v\} \subset I^1_* \cap I^2_*$ or
- $(\mathrm{B2})_{k+1}$ the isomorphisms $\phi_j \colon V(\mathsf{P}^j) \to V(P_k)$, $j = 1, 2$, satisfy $\phi_1(u) = \phi_2(u)$ and $\phi_1(v) = \phi_2(v)$.

Remark 2.4. The graph P_{k+1} is obtained by *amalgamating* copies of P_k in a particular way determined by the induction over ℓ. For instance, due to $(\mathrm{A})_{k+1}$, only vertices in $V^{r_\ell}_j(P_{k+1})$, with $j \in I_{k+1}$, may be shared by distinct copies of P_k in P_{k+1}.

See Figure 3 for an illustration of the amalgamation.

The projection $\pi_{k+1} \colon V(P_{k+1}) \to V(R_\ell)$ is defined in terms of the partition $\{V^{r_\ell}_j(P_{k+1})\}^{r_\ell}_{j=1}$ given by the induction hypothesis over ℓ. More concretely, $\pi_{k+1}(u) = j$ if and only if $u \in V^{r_\ell}_j(P_{k+1})$. For any $\mathsf{P} \in \mathcal{P}_{k+1}$, with isomorphism $\phi \colon V(P_k) \to V(\mathsf{P})$, we claim that the following diagram commutes:

$$\begin{array}{ccccc} P_k & \xleftrightarrow{\ \phi\ } & \mathsf{P} & \xrightarrow{\ \iota\ } & P_{k+1} \\ {\scriptstyle \pi_k}\big\downarrow & & & \swarrow {\scriptstyle \pi_{k+1}} & \\ R_\ell & & & & \end{array} \tag{2.5}$$

Indeed, because ϕ is a partite embedding, we have $\phi\big(V^{r_\ell}_j(P_k)\big) \subset V^{r_\ell}_j(P_{k+1})$ for all $j = 1, \dots, r_\ell$. Hence, for $u \in V(P_k)$, $\pi_k(u) = j$ if and only if $u \in V^{r_\ell}_j(P_k)$ if and only if $\phi(u) \in V^{r_\ell}_j(P_{k+1})$ if and only if $\pi_{k+1} \circ \phi(u) = j$. This shows that $\pi_k = \pi_{k+1} \circ \phi$ and thus the diagram (2.5) commutes.

Constructing the q-partition of P_{k+1}. The graph P_{k+1} is q-partite, with partition given by the classes

$$V^q_j(P_{k+1}) = \pi^{-1}_{k+1}\big(V^q_j(R_\ell)\big) = \bigcup_{u \in V^q_j(R_\ell)} V^{r_\ell}_u(P_{k+1}), \quad j = 1, \dots, q. \tag{2.6}$$

Notice that because $V^q_1(R_\ell) \prec V^q_2(R_\ell) \prec \cdots \prec V^q_q(R_\ell)$ and $V^{r_\ell}_1(P_{k+1}) \prec \cdots \prec V^{r_\ell}_{r_\ell}(P_{k+1})$ we also have $V^q_1(P_{k+1}) \prec \cdots \prec V^q_q(P_{k+1})$: see Figure 4.

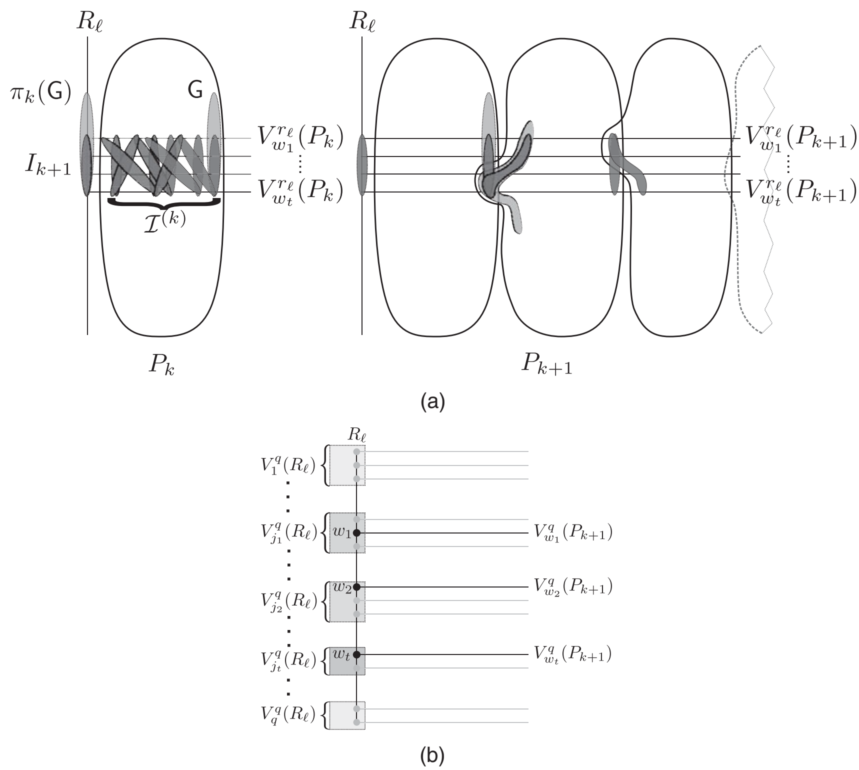

Figure 3. (a) The picture P_{k+1} is obtained from picture P_k by applying the induction hypothesis over ℓ. To simplify the figure, the vertical order of the vertices in the illustration does not coincide with the order of $V(R_\ell) = \{1, \dots, r_\ell\}$. (b) The tuple $I_{k+1} = \{w_1, \dots, w_t\}$ and the corresponding classes $V_{w_i}^q(P_{k+1})$ are drawn according to the order of $V(R_\ell)$. NB It is rather cumbersome to draw the elements of (ρ_ℓ, R_ℓ)-tuples in their correct order. For this reason we will refrain from having $V(R_\ell)$ vertically ordered in the next figures.

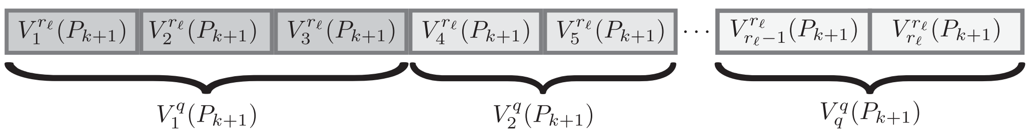

Figure 4. The linearly ordered vertices of P_{k+1} (from left to right) and both q- and r_ℓ-partitions. Note that the r_ℓ-partition of P_{k+1} is a refinement of its q-partition.

Constructing the family $\mathcal{G}(P_{k+1}) \subset \binom{P_{k+1}}{G}_{\mathrm{Part}(q)}$. For any $\mathsf{P} \in \mathcal{P}_{k+1} \subset \binom{P_{k+1}}{P_k}_{\mathrm{Part}(r_\ell)}$, given the (unique monotone) isomorphism $\phi : V(P_k) \to V(\mathsf{P})$, set

$$\mathcal{G}(\mathsf{P}) = \{\phi(\mathsf{G}) \ : \ \mathsf{G} \in \mathcal{G}(P_k)\}.$$

Define

$$\mathcal{G}(P_{k+1}) = \bigcup_{\mathsf{P} \in \mathcal{P}_{k+1}} \mathcal{G}(\mathsf{P}). \tag{2.7}$$

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Observe that there is a rich structure of copies of G in P_{k+1} which is inherited by the many overlapping copies of P_k in P_{k+1}.

We will now start the proof of the induction step over k. The proof is divided into several claims, one for each of the conditions (K1)–(K4) of the induction over k (see above).

Claim 2.5. *Condition* (K1) *holds for* P_{k+1}*, namely, the projection map* π_{k+1} *is a homomorphism of* P_{k+1} *into* R_ℓ *satisfying* $\pi_{k+1}(\mathsf{G}) \in \mathcal{G}_\ell$ *for every* $\mathsf{G} \in \mathcal{G}(P_{k+1})$.

We will start by showing that the projection map π_{k+1} is a homomorphism of P_{k+1} into R_ℓ. To this end we must prove that $\pi_{k+1}(E(P_{k+1})) \subset E(R_\ell)$.

By the induction hypothesis over P_k, the projection $\pi_k : V(P_k) \to V(R_\ell)$ is a homomorphism. Consequently, the diagram (2.5) shows that for every $\mathsf{P} \in \mathcal{P}_{k+1}$, the map $\pi_{k+1}|_{V(\mathsf{P})}$ is a homomorphism of P into R_ℓ and thus $\pi_{k+1}(E(\mathsf{P})) \subset E(R_\ell)$. Since by $(3)_{k+1}$ we have $E(P_{k+1}) = \bigcup_{\mathsf{P}\in\mathcal{P}_{k+1}} E(\mathsf{P})$, it follows that $\pi_{k+1}(E(P_{k+1})) \subset E(R_\ell)$.

It remains to show that $\pi_{k+1}(\mathsf{G}) \in \mathcal{G}_\ell$ for every $\mathsf{G} \in \mathcal{G}(P_{k+1}) = \bigcup_{\mathsf{P}\in\mathcal{P}_{k+1}} \mathcal{G}(\mathsf{P})$. For any $\mathsf{G} \in \mathcal{G}(\mathsf{P})$, $\mathsf{P} \in \mathcal{P}_{k+1}$, we have $\phi^{-1}(\mathsf{G}) \in \mathcal{G}(P_k)$, where $\phi\colon V(P_k) \to V(\mathsf{P})$ is the unique isomorphism. By the induction hypothesis (K1) over P_k it follows that $\pi_k\big(\phi^{-1}(\mathsf{G})\big) \in \mathcal{G}_\ell$. Since the diagram (2.5) commutes,

$$\pi_k\big(\phi^{-1}(\mathsf{G})\big) = \pi_{k+1} \circ \phi\big(\phi^{-1}(\mathsf{G})\big) = \pi_{k+1}(\mathsf{G})$$

and thus $\pi_{k+1}(\mathsf{G}) \in \mathcal{G}_\ell$. This concludes the proof that (K1) holds for P_{k+1}.

Claim 2.6. *Condition* (K2) *holds for* P_{k+1}*, namely,* $\mathcal{G}(P_{k+1}) \subset \binom{P_{k+1}}{G}_{\mathrm{Part}(q)}$.

First observe that for every $\mathsf{P} \in \mathcal{P}_{k+1} \subset \binom{P_{k+1}}{P_k}_{\mathrm{Part}(r_\ell)}$ we have $V_j^{r_\ell}(\mathsf{P}) \subset V_j^{r_\ell}(P_{k+1})$ for all $j = 1, \ldots, r_\ell$. Consequently,

$$V_j^q(\mathsf{P}) \overset{(2.3),(2.6)}{=} \bigcup_{u\in V_j^q(R_\ell)} V_u^{r_\ell}(\mathsf{P}) \subset \bigcup_{u\in V_j^q(R_\ell)} V_u^{r_\ell}(P_{k+1}) \overset{(2.6)}{=} V_j^q(P_{k+1}) \tag{2.8}$$

for all $j = 1, \ldots, q$.

For all $\mathsf{G} \in \mathcal{G}(\mathsf{P}) \subset \binom{\mathsf{P}}{G}_{\mathrm{Part}(q)}$, we have $V_j^q(\mathsf{G}) \subset V_j^q(\mathsf{P}) \subset V_j^q(R_\ell)$. It follows that

$$\mathcal{G}(P_{k+1}) = \bigcup_{\mathsf{P}\in\mathcal{G}(\mathcal{P}_{k+1})} \mathcal{G}(\mathsf{P}) \subset \binom{P_{k+1}}{G}_{\mathrm{Part}(q)}.$$

Therefore the claim is proved.

Claim 2.7 (Auxiliary). *If* $\mathsf{P}^1, \mathsf{P}^2 \in \mathcal{P}_{k+1}$ *are distinct and* $u \in V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$*, then* $\pi_{k+1}(u) \in I_{k+1}$*. Consequently, for each* $\mathsf{G} \in \mathcal{G}(P_{k+1})$ *there is a unique* $\mathsf{P} \in \mathcal{P}_{k+1}$ *such that* $\mathsf{G} \subset \mathsf{P}$.

From condition $(\mathrm{A})_{k+1}$ there exist $I_*^j \in \mathcal{I}_{\mathsf{P}^j}^{(k)}$, $j = 1, 2$, such that $u \in I_*^1 \cap I_*^2$. From diagram (2.5) we conclude that the isomorphism $\phi_1 : V(P_k) \to V(\mathsf{P}^1)$ satisfies $\pi_k =$

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$\pi_{k+1} \circ \phi_1$. Because $I^1 = \phi_1^{-1}(I_*^1) \in \mathcal{I}^{(k)}$, we have

$$\pi_{k+1}(I_*^1) = \pi_{k+1} \circ \phi_1(I^1) = \pi_k(I^1) \overset{(2.4)}{=} I_{k+1}.$$

Consequently, $\pi_{k+1}(u) \in I_{k+1}$.

Since each $\mathsf{G} \in \mathcal{G}(P_{k+1})$ is mapped by π_{k+1} onto a member of $\mathcal{G}_\ell$, the projection must be one-to-one over $V(\mathsf{G})$. Therefore $|\pi_{k+1}(V(\mathsf{G}))| = |V(G)| > t$ and thus $\pi_{k+1}(V(\mathsf{G})) \not\subset I_{k+1}$. It follows that $V(\mathsf{G}) \not\subset V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$.

Claim 2.8. *Condition* (K3) *holds for* P_{k+1}, *namely,* $\mathcal{G}(P_{k+1})$ *satisfies the intersection conditions* (A) *and* (B).

Let $\mathsf{G}_1, \mathsf{G}_2 \in \mathcal{G}(P_{k+1})$ be distinct and arbitrary. By Claim 2.7 there are unique $\mathsf{P}^1, \mathsf{P}^2 \in \mathcal{P}_{k+1}$ such that $\mathsf{G}_j \subset \mathsf{P}^j$, $j = 1, 2$. If $\mathsf{P}^1 = \mathsf{P}^2$ then the induction hypothesis over $\mathsf{P}^1 = \mathsf{P}^2 \cong P_k$ implies that both conditions (A) and (B) hold for G_1 and G_2. Hence let us suppose that $\mathsf{P}^1 \neq \mathsf{P}^2$.

Proof of (A). Since $\mathcal{P}_{k+1}$ satisfies $(\mathrm{A})_{k+1}$, it follows that for any $u \in V(\mathsf{G}_1) \cap V(\mathsf{G}_2) \subset V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$ there exist $(\rho_{\ell+1}, \mathsf{P}^j)$-tuples $I_*^j \in \mathcal{I}_{\mathsf{P}^j}^{(k)}$, $j = 1, 2$, such that $u \in I_*^1 \cap I_*^2$. Let $\mathsf{G}_j^* \in \mathcal{G}(\mathsf{P}^j)$ be such that $I_*^j \in \mathcal{I}_{\mathsf{G}_j{}^*}$. For each $j = 1, 2$ we are going to obtain $I^j \in \mathcal{I}_{\mathsf{G}_j}$ with $u \in I^1 \cap I^2$.

First we show that there exists $I^1 \in \mathcal{I}_{\mathsf{G}_1}$ such that $u \in I^1$. If $\mathsf{G}_1 = \mathsf{G}_1'$, we are done by taking $I^1 = I_*^1$ so let us assume that $\mathsf{G}_1 \neq \mathsf{G}_1'$. The induction hypothesis (K3) applied to $\mathsf{P}^1 \cong P_k$ implies that $\mathcal{G}(\mathsf{P}^1)$ satisfies condition (A): since $u \in V(\mathsf{G}_1) \cap V(\mathsf{G}_1')$ there exists $I^1 \in \mathcal{I}_{\mathsf{G}_1}$ such that $u \in I^1 \cap I_*^1$. Similarly we find $I^2 \in \mathcal{I}_{\mathsf{G}_2}$ such that $u \in I^2$ and hence $u \in I^1 \cap I^2$, thus proving that condition (A) holds for $\mathcal{G}(P_{k+1})$. □

Proof of (B). Suppose that there are two distinct $u, v \in V(\mathsf{G}_1) \cap V(\mathsf{G}_2) \subset V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$. Condition $(\mathrm{B})_{k+1}$ applies to $\mathcal{P}_{k+1}$, implying that either $(\mathrm{B1})_{k+1}$ or $(\mathrm{B2})_{k+1}$ holds for $u, v, \mathsf{P}^1, \mathsf{P}^2$.

If $(\mathrm{B1})_{k+1}$ holds for $u, v, \mathsf{P}^1, \mathsf{P}^2$ we will show that (B1) holds for $u, v, \mathsf{G}^1, \mathsf{G}^2$. Consider the $(\rho_{\ell+1}, \mathsf{P}^j)$-tuples $I_*^j \in \mathcal{I}_{\mathsf{P}^j}^{(k)}$, $j = 1, 2$, such that $u, v \in I_*^1 \cap I_*^2$. Let $\mathsf{G}_j^* \in \mathcal{G}(\mathsf{P}^j)$ be such that $I_*^j \in \mathcal{I}_{\mathsf{G}_j^*}$, $j = 1, 2$.

First we will show that there exists $I^1 \in \mathcal{I}_{\mathsf{G}_1}$ such that $u, v \in I^1$. If $\mathsf{G}_1^* = \mathsf{G}_1$, set $I^1 = I_*^1$. Otherwise, observe that $u, v \in V(\mathsf{G}_1) \cap V(\mathsf{G}_1^*)$ and $\mathsf{G}_1, \mathsf{G}_1^* \in \mathcal{G}(\mathsf{P}^1)$. We may now use the induction hypothesis (K3) on $\mathsf{P}^1 \cong P_k$, which states that condition (B) holds for $\mathcal{G}(\mathsf{P}^1)$. In particular, either (B1) applies and we immediately obtain $I^1 \in \mathcal{I}_{\mathsf{G}_1}$ satisfying $u, v \in I^1 \cap I_*^1$, or (B2) applies and the isomorphisms σ_1, σ_1^* from $\mathsf{G}_1, \mathsf{G}_1^*$ to G are such that $\sigma_1(u) = \sigma_1^*(u)$ and $\sigma_1(v) = \sigma_1^*(v)$. However, in the latter case, set $I^1 = \sigma_1^{-1} \circ \sigma_1^*(I_*^1) \in \mathcal{I}_{\mathsf{G}_1}$ and observe that $u, v \in I^1$.

In the same way we obtain $I^2 \in \mathcal{I}_{\mathsf{G}_2}$ such that $u, v \in I^2$, and thus establish that (B1) holds for $u, v, \mathsf{G}^1, \mathsf{G}^2$.

Consider now the case that $(\mathrm{B2})_{k+1}$ holds for $u, v, \mathsf{P}^1, \mathsf{P}^2$. In other words, for the (unique) isomorphisms $\phi_j : V(\mathsf{P}^j) \to V(P_k)$, $j = 1, 2$, we have $\phi_1(u) = \phi_2(u)$ and $\phi_1(v) = \phi_2(v)$. Let

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$\mathsf{G}_j^* = \phi_j(\mathsf{G}_j) \in \mathcal{G}(P_k)$, $j = 1, 2$ and set $x = \phi_1(u)$, $y = \phi_1(v)$. Since $x, y \in V(\mathsf{G}_1^*) \cap V(\mathsf{G}_2^*)$ and $\mathcal{G}(P_k)$ satisfies condition (B), one of the following must hold.

- *There exist* $I_*^j \in \mathcal{I}_{\mathsf{G}_j^*}$, $j = 1, 2$, *such that* $x, y \in I_*^1 \cap I_*^2$. Letting $I^j = \phi_j^{-1}(I_*^j) \in \mathcal{I}_{\mathsf{G}_j}$ for $j = 1, 2$, we have $u, v \in I^1 \cap I_2$. Hence condition (B1) holds for $u, v, \mathsf{G}^1, \mathsf{G}^2$.
- *The isomorphisms* $\sigma_j^* : V(\mathsf{G}_j^*) \to V(G)$ *satisfy* $\sigma_1^*(x) = \sigma_2^*(x)$, $\sigma_1^*(y) = \sigma_2^*(y)$. Since the (unique) isomorphisms $\sigma_j : V(\mathsf{G}_j) \to V(G)$ satisfy

$$\sigma_j = \sigma_j^* \circ \phi_j,$$

  we have

$$\sigma_1(u) = \sigma_1^*(\phi_1(u)) = \sigma_1^*(x) = \sigma_2^*(x) = \sigma_2^*(\phi_2(u)) = \sigma_2(u)$$

  and, similarly, $\sigma_1(v) = \sigma_2(v)$. Consequently, condition (B2) holds for $u, v, \mathsf{G}^1, \mathsf{G}^2$.

This concludes the proof that $\mathcal{G}(P_{k+1})$ satisfies condition (B). □

Before showing that condition (K4) holds we will prove two auxiliary claims.

Claim 2.9 (Auxiliary). *Suppose that* $\mathsf{P}^1, \mathsf{P}^2 \in \mathcal{P}_{k+1}$, $u, v \in V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$, $d_1 = \mathrm{dist}_{\mathsf{P}^1}(u, v)$ *and* $d_2 = \mathrm{dist}_{\mathsf{P}^2}(u, v)$. *Then either* $\min\{d_1, d_2\} \geqslant \ell + 1$ *or* $d_1 = d_2$.

Without loss of generality assume that $\mathsf{P}^1 \neq \mathsf{P}^2$, $d_1 = \min\{d_1, d_2\} \leqslant \ell$, and $u \neq v$. Since $\mathcal{P}_{k+1}$ satisfies condition $(\mathrm{B})_{k+1}$, either condition $(\mathrm{B1})_{k+1}$ or condition $(\mathrm{B2})_{k+1}$ applies to $u, v \in V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$.

Suppose first that $(\mathrm{B2})_{k+1}$ holds for $u, v, \mathsf{P}^1, \mathsf{P}^2$, namely, the isomorphisms $\phi_j : V(\mathsf{P}^j) \to V(P_k)$ are such that $\phi_1(u) = \phi_2(u)$ and $\phi_1(v) = \phi_2(v)$. In this case, $\phi = \phi_2^{-1} \circ \phi_1 : V(\mathsf{P}^1) \to V(\mathsf{P}^2)$ is the isomorphism from P^1 to P^2. Moreover, ϕ satisfies $\phi(u) = u$ and $\phi(v) = v$. It follows that

$$\mathrm{dist}_{\mathsf{P}^1}(u, v) = \mathrm{dist}_{\mathsf{P}^2}(\phi(u), \phi(v)) = \mathrm{dist}_{\mathsf{P}^2}(u, v).$$

The equality in this case holds even for arbitrary distances d_1, d_2.

Suppose now that condition $(\mathrm{B1})_{k+1}$ holds for $u, v, \mathsf{P}^1, \mathsf{P}^2$, namely, there exist $(\rho_{\ell+1}, \mathsf{P}^j)$-tuples $I^j \in \mathcal{I}_{\mathsf{P}^j}^{(k)} \subset \binom{\mathsf{P}^j}{\rho_{\ell+1}}$, $j = 1, 2$, such that $u, v \in I^1 \cap I^2$.

Let $\mathsf{G}_j \in \mathcal{G}(\mathsf{P}^j)$ be such that $I^j \in \mathcal{I}_{\mathsf{G}_j}$ for $j = 1, 2$. By the induction hypothesis over $\mathsf{P}^j \cong P_k$, the graph G_j is $(\ell + 1)$-metric in P^j. In particular, $\mathrm{dist}_{\mathsf{P}^1}(u, v) = d_1 \leqslant \ell$ implies that $\mathrm{dist}_{\mathsf{G}_1}(u, v) = d_1$.

Recall that

$$\pi_{k+1}(I^1) = \pi_{k+1}(I^2) = I_{k+1} = \{w_1 < w_2 < \cdots < w_t\} \subset V(R_\ell).$$

In particular, $\pi_{k+1}(u) = w_a$ and $\pi_{k+1}(v) = w_b$, for some $1 \leqslant a, b \leqslant t$. Consequently, u is the ath element of I^j $(j = 1, 2)$ and v is the bth element of I^j $(j = 1, 2)$. Because $\mathrm{dist}_{\mathsf{G}_1}(u, v) =$

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$d_1 \leqslant \ell$,

$$d_1 = \underbrace{\mathrm{dist}_{\mathsf{G}_1}(u,v) = \overbrace{\rho(a,b)}^{I^2 \text{ is a } (\rho_{\ell+1},\mathsf{G}_2)\text{-tuple}}}_{I^1 \text{ is a } (\rho_{\ell+1},\mathsf{G}_1)\text{-tuple}} \overbrace{= \mathrm{dist}_{\mathsf{G}_2}(u,v)} \geqslant \mathrm{dist}_{\mathsf{P}^2}(u,v) = d_2 = \max\{d_1,d_2\}$$

and thus $d_1 = d_2$. Hence, Claim 2.9 follows.

Claim 2.10 (Auxiliary). *Suppose that $\mathsf{G}_1, \mathsf{G}_2 \in \mathcal{G}_\ell$ and there are distinct $u,v \in V(\mathsf{G}_1) \cap V(\mathsf{G}_2)$. Moreover, assume that there exists $I^1 \in \mathcal{I}_{\mathsf{G}_1}$ such that $u,v \in I^1$. Then there exists $I^2 \in \mathcal{I}_{\mathsf{G}_2}$ such that $u,v \in I^2$.*

If $\mathsf{G}_1 = \mathsf{G}_2$ then the claim is trivial, so let us assume the graphs are distinct. By assumption, $\mathcal{G}_\ell$ satisfies condition (B). If (B1) holds then the existence of I^2 is immediate.

If, on the other hand, (B2) holds, then the isomorphisms $\sigma_j : V(\mathsf{G}_j) \to V(G)$ satisfy $\sigma_1(u) = \sigma_2(u)$ and $\sigma_1(v) = \sigma_2(v)$. The map $\sigma = \sigma_2^{-1} \circ \sigma_1 : V(\mathsf{G}_1) \to V(\mathsf{G}_2)$ is clearly the isomorphism from G_1 to G_2. Since $\sigma(u) = u$ and $\sigma(v) = v$, it follows that $I^2 = \sigma(I^1) \in \mathcal{I}_{\mathsf{G}_2}$ satisfies the conditions of the claim.

Claim 2.11. *Condition* (K4) *holds for P_{k+1}, namely, every $\mathsf{G} \in \mathcal{G}(P_{k+1})$ is $(\ell+1)$-metric.*

For an arbitrary $\mathsf{G} \in \mathcal{G}(P_{k+1})$ and $u,v \in V(\mathsf{G})$ we will show the following.

(i) If $\mathrm{dist}_{\mathsf{G}}(u,v) \leqslant \ell$ then $\mathrm{dist}_{P_{k+1}}(u,v) = \mathrm{dist}_{\mathsf{G}}(u,v)$.

(ii) If $\mathrm{dist}_{\mathsf{G}}(u,v) \geqslant \ell+1$ then $\mathrm{dist}_{P_{k+1}}(u,v) \geqslant \ell+1$.

The two conditions above imply that G is $(\ell+1)$-metric in P_{k+1}. Indeed, when $\mathrm{dist}_{\mathsf{G}}(u,v) = \ell+1$ we have

$$\ell + 1 \overset{\text{(ii)}}{\leqslant} \mathrm{dist}_{P_{k+1}}(u,v) \leqslant \mathrm{dist}_{\mathsf{G}}(u,v) = \ell+1,$$

and equality holds. Consequently, for all $u,v \in V(\mathsf{G})$ we have $\mathrm{dist}_{P_{k+1}}(u,v) = \mathrm{dist}_{\mathsf{G}}(u,v)$ whenever $\mathrm{dist}_{\mathsf{G}}(u,v) \leqslant \ell+1$ and $\mathrm{dist}_{P_{k+1}}(u,v) \geqslant \ell+1$ whenever $\mathrm{dist}_{\mathsf{G}}(u,v) > \ell+1$.

We start by proving (i). Assume that $\mathrm{dist}_{\mathsf{G}}(u,v) \leqslant \ell$. If $\mathrm{dist}_{P_{k+1}}(u,v) < \mathrm{dist}_{\mathsf{G}}(u,v)$, consider a shortest path $\mathrm{P}(u,v)$ in P_{k+1}. The projection of this path, $\pi_{k+1}(\mathrm{P}(u,v))$, is a trail in R_ℓ starting at $x = \pi_{k+1}(u)$ and ending at $y = \pi_{k+1}(v)$. Since $\mathsf{G}' = \pi_{k+1}(\mathsf{G}) \in \mathcal{G}_\ell$ and π_{k+1} is an isomorphism between G and G', it follows that $\mathrm{dist}_{\mathsf{G}'}(x,y) = \mathrm{dist}_{\mathsf{G}}(u,v) \leqslant \ell$. On the other hand, the trail $\pi_{k+1}(\mathrm{P}(u,v))$ shows that

$$\begin{aligned} \mathrm{dist}_{R_\ell}(x,y) &\leqslant |\pi_{k+1}(\mathrm{P}(u,v))| \leqslant |\mathrm{P}(u,v)| \\ &= \mathrm{dist}_{P_{k+1}}(u,v) < \mathrm{dist}_{\mathsf{G}}(u,v) = \mathrm{dist}_{\mathsf{G}'}(x,y). \end{aligned} \tag{2.9}$$

However, this contradicts the fact that G' is ℓ-metric in R_ℓ.

Now let us prove (ii). Suppose for the sake of contradiction that there exists a path $\mathrm{P}(u,v)$ in P_{k+1} with

$$|\mathrm{P}(u,v)| \leqslant \ell \quad \text{and} \quad \mathrm{dist}_{\mathsf{G}}(u,v) \geqslant \ell+1. \tag{2.10}$$

By Claim 2.7, there exists a unique $\mathsf{P}^1 \in \mathcal{P}_{k+1} \subset \binom{P_{k+1}}{P_k}_{\mathrm{Part}(r_\ell)}$ such that $\mathsf{G} \subset \mathsf{P}^1$.

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Fact 2.12. *The path* $\mathrm{P}(u,v)$ *satisfies the following:*

(a) $\mathrm{P}(u,v) \not\subset \mathsf{P}^1$,

(b) *there is no internal vertex of* $\mathrm{P}(u,v)$ *in* $V(\mathsf{P}^1)$, *hence* $E\big(\mathrm{P}(u,v)\big) \cap E(\mathsf{P}^1) = \emptyset$,

(c) $\pi_{k+1}(u), \pi_{k+1}(v) \in I_{k+1}$,

(d) $\mathrm{P}(u,v) \not\subset \mathsf{P}^2$ *for every* $\mathsf{P}^2 \in \mathcal{P}_{k+1}$.

By the induction hypothesis over the picture $\mathsf{P}^1 \cong P_k$, the graph G must be $(\ell+1)$-metric in P^1, and thus

$$\mathrm{dist}_{\mathsf{P}^1}(u,v) \geqslant \ell + 1. \tag{2.11}$$

In particular, (a) holds, that is, the path $\mathrm{P}(u,v)$ cannot be entirely contained in P^1.

Suppose that the path $\mathrm{P}(u,v)$ contains an internal vertex $w \in V(\mathsf{P}^1)$. Then the (non-trivial) induced sub-paths $\mathrm{P}(u,w)$ and $\mathrm{P}(w,v)$ have length strictly shorter than ℓ. Our assumption that P^1 is ℓ-metric in P_{k+1} implies that $|\mathrm{P}(u,w)| \geqslant \mathrm{dist}_{\mathsf{P}^1}(u,w)$ and $|\mathrm{P}(w,v)| \geqslant \mathrm{dist}_{\mathsf{P}^1}(w,v)$. Therefore

$$\begin{aligned} |\mathrm{P}(u,v)| &= |\mathrm{P}(u,w)| + |\mathrm{P}(w,v)| \geqslant \mathrm{dist}_{\mathsf{P}^1}(u,w) + \mathrm{dist}_{\mathsf{P}^1}(w,v) \\ &\geqslant \mathrm{dist}_{\mathsf{P}^1}(u,v) \overset{(2.11)}{\geqslant} \ell + 1, \end{aligned} \tag{2.12}$$

which contradicts the fact that $|\mathrm{P}(u,v)| \leqslant \ell$. Therefore (b) holds.

Because of (b), the edge of the path incident to u, say $e = \{u,w\}$, must be contained in some $\mathsf{P}^2 \in \mathcal{P}_{k+1}$, $\mathsf{P}^2 \neq \mathsf{P}^1$, otherwise w would be an internal vertex of $\mathrm{P}(u,v)$. In particular, $u \in V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$. From Claim 2.7 we conclude that $\pi_{k+1}(u) \in I_{k+1}$. For the same reason we conclude that $\pi_{k+1}(v) \in I_{k+1}$ and therefore (c) holds.

To show that (d) is satisfied, suppose that $\mathrm{P}(u,v) \subset \mathsf{P}^2$ for some $\mathsf{P}^2 \in \mathcal{P}_{k+1}$, $\mathsf{P}^2 \neq \mathsf{P}^1$. Then $d_2 = \mathrm{dist}_{\mathsf{P}^2}(u,v) \leqslant \ell$. From Claim 2.9 we conclude that

$$\mathrm{dist}_{\mathsf{P}^1}(u,v) = d_1 = d_2 = \ell,$$

which contradicts (2.11). Therefore (d) holds.

We now return to the proof of Claim 2.11(ii). From (a)–(d) we conclude that the path $\mathrm{P}(u,v)$ can be decomposed into sub-paths contained in at least two distinct copies of P_k in $\mathcal{P}_{k+1}$. Therefore we may find vertices $u = x_1, x_2, \ldots, x_r = v$, $r \geqslant 3$, belonging to $\mathrm{P}(u,v)$ such that each (non-trivial) sub-path $\mathrm{P}(x_j, x_{j+1})$, $j = 1, \ldots, r-1$, is entirely contained in some $\mathsf{P}^{j+1} \in \mathcal{P}_{k+1}$, and $\mathsf{P}^{j+1} \neq \mathsf{P}^{j+2}$ for $j = 1, \ldots, r-2$ (see the illustration in Figure 5).

Note that each $\mathrm{P}(x_j, x_{j+1})$ has length at most $\ell - 1$ since the sum of the lengths of each sub-path equals $|\mathrm{P}(u,v)| \leqslant \ell$. From Claim 2.7 and (c) we infer that $\pi_{k+1}(x_j) \in I_{k+1} = \{w_1 < w_2 < \cdots < w_t\}$ for $j = 1, \ldots, r$. For each $j = 1, \ldots, r$, let $a_j \in [t]$ be such that $\pi_{k+1}(x_j) = w_{a_j}$.

For every $j = 1, \ldots, r-1$, the projection $\pi_{k+1}\big(\mathrm{P}(x_j, x_{j+1})\big)$ is a trail connecting w_{a_j} and $w_{a_{j+1}}$ of length $|\mathrm{P}(x_j, x_{j+1})| \leqslant \ell - 1$. Consequently, $\mathrm{dist}_{R_\ell}(w_{a_j}, w_{a_{j+1}}) \leqslant \ell - 1$. Let $\mathsf{G}_{I_{k+1}} \in \mathcal{G}_\ell \subset \binom{R_\ell}{G}_{\mathrm{Part}(q)}$ be such that $I_{k+1} \in \mathcal{I}_{\mathsf{G}_{I_{k+1}}} \subset \binom{\mathsf{G}_{I_{k+1}}}{\rho_{\ell+1}}$. Since $\mathsf{G}_{I_{k+1}}$ is ℓ-metric in R_ℓ, it follows that

$$\mathrm{dist}_{\mathsf{G}_{I_{k+1}}}(w_{a_j}, w_{a_{j+1}}) = \mathrm{dist}_{R_\ell}(w_{a_j}, w_{a_{j+1}}) \leqslant |\mathrm{P}(x_j, x_{j+1})| \leqslant \ell - 1.$$

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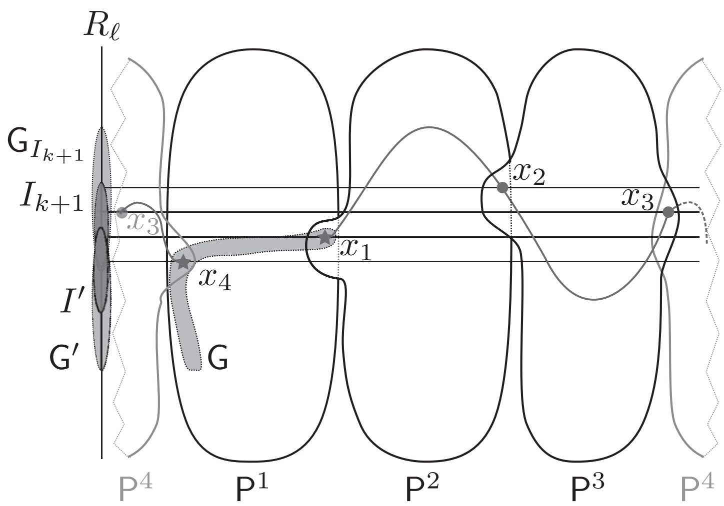

Figure 5. An illustration of a path $\mathrm{P}(u,v)$ and its sub-paths from case (ii) of Claim 2.11 with $u = x_1$ and $v = x_4$. We also have $t = 4$, $a_1 = 3$, $a_2 = 1$, $a_3 = 2$ and $a_4 = 4$. The vertex x_3 is repeated because P^4 is wrapped around and effectively intersects both P^3 and P^1. Note that $\mathsf{G}' = \pi_{k+1}(\mathsf{G})$ and that $\mathsf{G}_{I_{k+1}}$ contains I_{k+1}.

Because $I_{k+1} \in \binom{\mathsf{G}_{I_{k+1}}}{\rho_{\ell+1}}$, we must have $\mathrm{dist}_{\mathsf{G}_{I_{k+1}}}(w_{a_j}, w_{a_{j+1}}) = \rho(a_j, a_{j+1})$ and thus

$$|\mathrm{P}(u,v)| = \sum_{j=1}^{r-1} |\mathrm{P}(x_j, x_{j+1})| \geqslant \sum_{j=1}^{r-1} \mathrm{dist}_{\mathsf{G}_{I_{k+1}}}(w_{a_j}, w_{a_{j+1}})$$
$$= \sum_{j=1}^{r-1} \rho(a_j, a_{j+1}) \geqslant \rho(a_1, a_r), \tag{2.13}$$

where in the last part we used the triangle inequality.

Let $\mathsf{G}' = \pi_{k+1}(\mathsf{G}) \in \mathcal{G}_\ell$. Notice that $w_{a_1} = \pi_{k+1}(u), w_{a_r} = \pi_{k+1}(v) \in V(\mathsf{G}') \cap V(\mathsf{G}_{I_{k+1}})$. From Claim 2.10 applied to G' and $\mathsf{G}_{I_{k+1}}$ we conclude that there exists $I' \in \mathcal{I}_{\mathsf{G}'}$ such that $w_{a_1}, w_{a_r} \in I' \cap I_{k+1}$. Moreover, by the induction hypothesis (over ℓ) every graph in $\mathcal{G}_\ell$ is partite embedded into R_ℓ, that is, $\mathcal{G}_\ell \subset \binom{R_\ell}{G}_{\mathrm{Part}(q)}$. In particular, $V_j^q(\mathsf{G}'), V_j^q(\mathsf{G}_{I_{k+1}}) \subset V_j^q(R_\ell)$ for all $j = 1, \ldots, q$. Because $\mathcal{I} \subset \binom{G}{\rho_{\ell+1}}$ is a t-partite hypergraph with classes $\{V_{j_i}^q(G)\}_{i=1}^t$, it follows that $\mathcal{I}_{\mathsf{G}'}$ is t-partite with classes $\{V_{j_i}^q(\mathsf{G}') \subset V_{j_i}^q(R_\ell)\}_{i=1}^t$ and $\mathcal{I}_{\mathsf{G}_{I_{k+1}}}$ is t-partite with classes $\{V_{j_i}^q(\mathsf{G}_{I_{k+1}}) \subset V_{j_i}^q(R_\ell)\}_{i=1}^t$. This ensures that both $I' \in \mathcal{I}_{\mathsf{G}'}$ and $I_{k+1} \in \mathcal{I}_{\mathsf{G}_{I_{k+1}}}$ are crossing with respect to $\{V_{j_i}^q(R_\ell)\}_{i=1}^t$. Therefore, the a_1th element in I' is w_{a_1} and the a_rth element in I' is w_{a_r}. Because $I' \in \binom{\mathsf{G}'}{\rho_{\ell+1}}$ and $\rho(a_1, a_r) \leqslant \ell$, we have $\mathrm{dist}_{\mathsf{G}'}(w_{a_1}, w_{a_r}) = \rho(a_1, a_r) \leqslant \ell$

Since π_{k+1} is the isomorphism of G into G' we have

$$\mathrm{dist}_{\mathsf{G}}(u,v) = \mathrm{dist}_{\mathsf{G}'}(w_{a_1}, w_{a_r}) = \rho(a_1, a_r) \leqslant \ell,$$

which is a contradiction to the original assumption (2.10) that $\mathrm{dist}_{\mathsf{G}}(u,v) \geqslant \ell + 1$. This finishes the proof of Claim 2.11.

Remark 2.13. A subtle point in the proof Claim 2.11(ii) is that while the copies of G in $\mathcal{G}_\ell$ are only guaranteed to be ℓ-metric in R_ℓ, for $\mathsf{G}^1, \mathsf{G}^2 \in \mathcal{G}_\ell$ and $u, v \in V(\mathsf{G}^1) \cap V(\mathsf{G}^2)$ – as in Claim 2.9 – we have either $\mathrm{dist}_{\mathsf{G}^1}(u,v) = \mathrm{dist}_{\mathsf{G}^2}(u,v)$ or $\min\{\mathrm{dist}_{\mathsf{G}^1}(u,v), \mathrm{dist}_{\mathsf{G}^2}(u,v)\} \geqslant \ell + 1$. In other words, if $\mathrm{dist}_{\mathsf{G}^1}(u,v) = \ell + 1$ there may exist a path $\mathrm{P}(u,v)$ in R_ℓ of length ℓ but this path cannot be entirely contained in any $\mathsf{G}^2 \in \mathcal{G}_\ell$.

We have proved the induction step over k by establishing Claims 2.5, 2.6, 2.8 and 2.11. In order to prove that

$$R_{\ell+1} = P_m \quad \text{and} \quad \mathcal{G}_{\ell+1} = \mathcal{G}(P_m) \tag{2.14}$$

satisfy the induction hypothesis for $\ell + 1$, it remains to show that (L1) and (L3) hold.

The property (L3) follows from $(3)_m$, $(3)_{m-1}$, ..., $(3)_1$ since every edge $e \in E(P_m)$ must belong to some copy P^0 of P_0, and thus $e \in E(\mathsf{G})$ for some $\mathsf{G} \in \mathcal{G}(\mathsf{P}^0) \subset \mathcal{G}(P_m) = \mathcal{G}_{\ell+1}$. More formally,

$$\begin{aligned} E(R_{\ell+1}) = E(P_m) &= \bigcup_{\mathsf{P}^{m-1} \in \mathcal{P}_m} E(\mathsf{P}^{m-1}) \\ &= \bigcup_{\mathsf{P}^{m-1} \in \mathcal{P}_m} \bigcup_{\mathsf{P}^{m-2} \in \mathcal{P}_{m-1}(\mathsf{P}^{m-1})} \cdots \bigcup_{\mathsf{P}^0 \in \mathcal{P}_1(\mathsf{P}^1)} E(\mathsf{P}^0) \\ &= \bigcup_{\mathsf{P}^{m-1},\dots,\mathsf{P}^0} \bigcup_{\mathsf{G} \in \mathcal{G}(\mathsf{P}^0)} E(\mathsf{G}) \\ &\overset{(2.7)}{=} \bigcup_{\mathsf{G} \in \mathcal{G}(P_m)} E(\mathsf{G}). \end{aligned} \tag{2.15}$$

To prove[3] that the condition (L1) is satisfied by $R_{\ell+1}$ and $\mathcal{G}_{\ell+1}$, we first show that under certain assumptions on a colouring of P_0 one can obtain $\mathsf{G} \in \mathcal{G}(P_0)$ with $\mathcal{I}_\mathsf{G}$ monochromatic. Our goal is then reduced to finding some $\mathsf{P}^0 \subset R_{\ell+1}$, $\mathsf{P}^0 \cong P_0$, which is coloured in such a way.

Claim 2.14 (Auxiliary). *Suppose that the tuples in $\bigcup_{\mathsf{G} \in \mathcal{G}(P_0)} \mathcal{I}_\mathsf{G}$ are coloured in such a way that the colour of any $I \in \bigcup_{\mathsf{G} \in \mathcal{G}(P_0)}$ depends only on the projection $\pi_0(I) \in \bigcup_{\mathsf{G} \in \mathcal{G}_\ell} \mathcal{I}_\mathsf{G}$.*

Then there exists $\mathsf{G} \in \mathcal{G}(P_0)$ with $\mathcal{I}_\mathsf{G}$ monochromatic.

Under the assumptions of the claim there is an induced colouring of the tuples in $\bigcup_{\mathsf{G} \in \mathcal{G}_\ell} \mathcal{I}_G$ given by assigning to each $\mathcal{I}' \in \bigcup_{\mathsf{G} \in \mathcal{G}_\ell} \mathcal{I}_G$ the same colour of the tuples $I \in \bigcup_{\mathsf{G} \in \mathcal{G}(P_0)}$ satisfying $\pi_0(I) = I'$.

By the induction hypothesis (L1) over R_ℓ and $\mathcal{G}_\ell$, there must be some $\mathsf{G}^* \in \mathcal{G}_\ell$ such that $\mathcal{I}_{\mathsf{G}^*}$ is monochromatic under this induced colouring. By construction, $\mathsf{G} = \pi_0^{-1}(\mathsf{G}^*)$ is contained in $\mathcal{G}(P_0)$ (see Figure 2). Since the colour of any tuple $I \in \mathcal{I}_\mathsf{G}$ is given by the colour of $\pi_0(I) \in \mathcal{I}_{\mathsf{G}^*}$, it is clear that $\mathcal{I}_\mathsf{G}$ is monochromatic.

Claim 2.15 below establishes (L1).

[3] This proof closely follows [18].

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Claim 2.15. *For every 2-colouring of* $\bigcup_{\mathsf{G}\in\mathcal{G}_{\ell+1}}\mathcal{I}_{\mathsf{G}} \subset \binom{R_{\ell+1}}{\rho_{\ell+1}}$ *there exists some* $\mathsf{G}\in\mathcal{G}_{\ell+1}$ *such that* $\mathcal{I}_{\mathsf{G}}$ *is monochromatic.*

Let a 2-colouring of $\bigcup_{\mathsf{G}\in\mathcal{G}_{\ell+1}}\mathcal{I}_{\mathsf{G}}$ be given. In view of Claim 2.14 we now look for a copy $\mathsf{P}^0 \subset R_{\ell+1}$ such that the colouring of $\bigcup_{\mathsf{G}\in\mathcal{G}(\mathsf{P}^0)}\mathcal{I}_{\mathsf{G}}$ satisfies the conditions of the claim.

Notice that because of (2.4) and (2.7), we have

$$\bigcup_{\mathsf{P}\in\mathcal{P}_m}\mathcal{I}_{\mathsf{P}}^{(m-1)} \subset \bigcup_{\mathsf{P}\in\mathcal{P}_m}\bigcup_{\mathsf{G}\in\mathcal{G}(\mathsf{P})}\mathcal{I}_{\mathsf{G}} = \bigcup_{\mathsf{G}\in\mathcal{G}(P_m)}\mathcal{I}_{\mathsf{G}}.$$

Hence there is an induced 2-colouring of $\bigcup_{\mathsf{P}\in\mathcal{P}_m}\mathcal{I}_{\mathsf{P}}^{(m-1)}$. By property $(1)_m$, there exist some $\mathsf{P}^{m-1}\in\mathcal{P}_m$ such that $\mathcal{I}_{\mathsf{P}^{m-1}}^{(m-1)}$ is monochromatic. Let $\pi^{m-1}: V(\mathsf{P}^{m-1})\to V(R_\ell)$ be the natural projection/homomorphism of P^{m-1} onto R_ℓ. Notice that because $\mathsf{P}^{m-1}\cong P_{m-1}$, $\mathcal{I}_{\mathsf{P}^{m-1}}^{(m-1)}\cong\mathcal{I}^{(m-1)}$ and π^{m-1} is the map induced by π_{m-1}, the definition in (2.4) translates to

$$\mathcal{I}_{\mathsf{P}^{m-1}}^{(m-1)} = \left\{ I\in\bigcup_{\mathsf{G}\in\mathcal{G}(\mathsf{P}^{m-1})}\mathcal{I}_{\mathsf{G}} \;:\; \pi^{m-1}(I) = I_m\right\}. \tag{2.16}$$

Hence, the colour of all the tuples in $\bigcup_{\mathsf{G}\in\mathcal{G}(\mathsf{P}^{m-1})}\mathcal{I}_G$ projecting onto I_m is the same.

Applying property $(1)_{m-1}$ to $\mathsf{P}^{m-1}\cong P_{m-1}$, we obtain some graph $\mathsf{P}^{m-2}\in\mathcal{P}_{m-1}(\mathsf{P}^{m-1})\subset\binom{\mathsf{P}^{m-1}}{P_{m-2}}_{\mathrm{Part}(r_\ell)}$ such that $\mathcal{I}_{\mathsf{P}^{m-2}}^{(m-2)}$ is monochromatic. As before, the projection π^{m-2} of P^{m-2} onto R_ℓ is such that

$$\mathcal{I}_{\mathsf{P}^{m-2}}^{(m-2)} = \left\{ I\in\bigcup_{\mathsf{G}\in\mathcal{G}(\mathsf{P}^{m-2})}\mathcal{I}_G \;:\; \pi^{m-2}(I) = I_{m-1}\right\}.$$

Moreover, because $\mathsf{P}^{m-2}\in\binom{\mathsf{P}^{m-1}}{P_{m-2}}_{\mathrm{Part}(r_\ell)}$, we have $\pi^{m-2}=\pi^{m-1}|_{V(\mathsf{P}^{m-2})}$. Since $\mathcal{G}(\mathsf{P}^{m-2})\subset\mathcal{G}(\mathsf{P}^{m-1})$, from (2.16) we have

$$\left\{ I\in\bigcup_{\mathsf{G}\in\mathcal{G}(\mathsf{P}^{m-2})}\mathcal{I}_G \;:\; \pi^{m-2}(I) = I_m\right\}\subset\mathcal{I}_{\mathsf{P}^{m-1}}^{(m-1)}.$$

By repeating this argument sequentially (invoking $(1)_{m-2},\dots,(1)_1$) we obtain $\mathsf{P}^{m-1}\supset\mathsf{P}^{m-2}\supset\cdots\supset\mathsf{P}^0$, satisfying the following. For all $k=0,\dots,m-1$, the family $\mathcal{I}_{\mathsf{P}^k}^{(k)}$ is monochromatic and

$$\left\{ I\in\bigcup_{\mathsf{G}\in\mathcal{G}(\mathsf{P}^{0})}\mathcal{I}_G \;:\; \pi^{0}(I) = I_{k+1}\right\}\subset\mathcal{I}_{\mathsf{P}^{k}}^{(k)},$$

where $\pi^0=\pi^1|_{V(\mathsf{P}^0)}=\cdots=\pi^{m-1}|_{V(\mathsf{P}^0)}$ is the projection/homomorphism of P^0 onto R_ℓ.

Consequently, the colour of a tuple $I\in\bigcup_{\mathsf{G}\in\mathcal{G}(\mathsf{P}^0)}\mathcal{I}_G$ depends only on its projection $\pi^0(I)$. This means that the assumptions of Claim 2.14 are satisfied by P^0. The claim then yields $\mathsf{G}\in\mathcal{G}(\mathsf{P}^0)\subset\mathcal{G}_{\ell+1}$ such that $\mathcal{I}_{\mathsf{G}}$ is monochromatic, thus proving that (L1) holds for $R_{\ell+1}$ and $\mathcal{G}_{\ell+1}$.

The conditions (K1)–(K4), which hold for $R_{\ell+1}=P_m$ and $\mathcal{G}_{\ell+1}=\mathcal{G}(P_m)$, together with (2.15) and Claim 2.15, establish that the induction hypothesis holds for $\ell+1$. Lemma 1.19 then follows by induction.

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3. The base of the induction

Here we state Lemma 3.1, the induction base of the proof of Lemma 1.19. The proof of this lemma is based on an application of the Hales–Jewett theorem.

Lemma 3.1. *Let $t, q \in \mathbb{N}$, $t \leqslant q$. Suppose that*

- *ρ is a fixed metric on $[t]$,*
- *G is a q-partite (ordered) graph with partition $V(G) = V_1^q(G) \cup \cdots \cup V_q^q(G)$,*
- *for some $1 \leqslant j_1 < j_2 < \cdots < j_t \leqslant q$, $\mathcal{I} \subset \binom{G}{\rho_2}$ is a t-partite t-uniform hypergraph with classes $\{V_{j_i}^q(G)\}_{i=1}^t$ consisting of selected (ρ_2, G)-tuples.*

Then there exists a q-partite graph R and $\mathcal{G} \subset \binom{R}{G}_{\mathrm{Part}(q)}$ satisfying the following properties.

(L1) *For any 2-colouring of the (ρ_2, R)-tuples in $\bigcup_{\mathsf{G}\in\mathcal{G}} \mathcal{I}_{\mathsf{G}}$, there exists $\mathsf{G} \in \mathcal{G}$ such that every $\mathcal{I}_{\mathsf{G}} \subset \binom{\mathsf{G}}{\rho_2} \subset \binom{R}{\rho_2}$ is monochromatic.*
(L2) *Every $\mathsf{G} \in \mathcal{G}$ is 2-metric in R.*
(L3) *$E(R) = \bigcup_{\mathsf{G}\in\mathcal{G}} E(\mathsf{G})$.*
(L4) *The family $\mathcal{G}$ satisfies conditions* (A) *and* (B).

Remark 3.2. For the fixed (discrete[4]) metric ρ on $[t]$, consider a graph F_ρ with vertex set $[t]$ such that $ij \in F_\rho$ if and only if $\rho(x, y) = 1$. With this definition we have $\binom{G}{\rho_2} \cong \binom{G}{F_\rho}$, *i.e.*, $\binom{G}{\rho_2}$ coincides with the set of all induced copies of F_ρ in G.

Notice also that the fact that every $\mathsf{G} \in \mathcal{G}$ is 2-metric in R implies that G is an induced subgraph of R. Indeed, by the definition, for all $x, y \in V(\mathsf{G})$, when $\mathrm{dist}_R(x, y) \leqslant 2$ we must have $\mathrm{dist}_{\mathsf{G}}(x, y) = \mathrm{dist}_R(x, y)$ and when $\mathrm{dist}_R(x, y) > 2$ we must have $\mathrm{dist}_{\mathsf{G}}(x, y) \geqslant 2$. In particular, $xy \in R$ if and only if $xy \in \mathsf{G}$.

Lemma 3.1 appears in [18] without explicitly stating condition (L4), which is needed here for technical reasons to carry on the induction. For completeness we include in the Appendix the proof of [18] modified to explicitly establish (L4).

4. Proof of Theorem 1.11

In this section we give a sketch of the proof of Lemma 1.10 and later use it to prove Theorem 1.11 in Section 4.1. Since this proof is very similar to the proof of the induction step in Lemma 1.19 (albeit simpler), we avoid repeating some details and instead refer the reader to parts of the proof of Lemma 1.19 that present similar arguments. The main difference between this proof and that of Lemma 1.19 is that here the 'metric' part of the result follows rather trivially from our use of the Partite Lemma 1.19. On the other hand, we are now able to partition (colour) all of $\binom{R}{\rho}$ and not just a t-partite system.

Let H be a given connected graph on n vertices and let ρ be a metric on t elements. Set $N = \mathrm{R}_t(n)$, where $\mathrm{R}_t(n)$ is the smallest number such that for every 2-colouring of the complete t-uniform hypergraph $\binom{[N]}{t}$ there exists a monochromatic $\binom{S}{t}$ with $|S| = n$.

[4] Recall that all metrics in this paper are discrete.

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As in the proof of Lemma 1.19 we construct an N-partite graph P_0 consisting of disjoint copies of H (see Figure 2). Set $V(P_0) = [N] \times \binom{[N]}{n}$. For a set $S \in \binom{[N]}{n}$, let $\phi_S : V(H) \to S$ be the unique monotone map and set H_S to be a graph with vertex set $S \times \{S\}$ and edges given by

$$\{\{(\phi_S(x), S), (\phi_S(y), S)\} \ : \ xy \in H\}.$$

Let

$$E(P_0) = \bigcup_{S \in \binom{[N]}{n}} E(H_S).$$

Notice that P_0 is indeed the disjoint union of the copies of H in the family $\mathcal{H}(P_0) = \{H_S \ : \ S \in \binom{[N]}{n}\}$. Let $\pi_0 : V(P_0) \to [N]$ be the projection onto the first coordinate.

Define

$$\mathcal{H}_0 = \left\{\pi_0(H_S) \ : \ S \in \binom{[N]}{n}\right\}.$$

Consider the hypergraph

$$\bigcup_{\mathsf{H} \in \mathcal{H}_0} \binom{\mathsf{H}}{\rho} = \{I_1, \ldots, I_m\} \subset \binom{[N]}{t},$$

and set

$$\mathcal{I}^{(0)} = \left\{I \in \bigcup_{\mathsf{H} \in \mathcal{H}(P_0)} \binom{\mathsf{H}}{\rho} \ : \ \pi_0(I) = I_1\right\} \subset \binom{P_0}{\rho}.$$

(Note that $\mathcal{I}^{(0)}$ is defined in a similar way as the hypergraph in (2.4).) Observe that the t-uniform hypergraph $\mathcal{I}^{(0)}$ is t-partite with respect to $\{V_j^N(P_0) = \pi_0^{-1}(j)\}_{j \in I_1}$.

Set $\ell = \max\{\mathrm{dist}_H(x, y) \ : \ x, y \in V(H)\} < \infty$ and apply Lemma 1.19 to the N-partite graph P_0 (instead of a q-partite G) and the family $\mathcal{I}^{(0)} \subset \binom{P_0}{\rho_\ell}$. We then obtain the Ramsey N-partite graph P_1 and $\mathcal{P}_1 \subset \binom{P_1}{P_0}_{\mathrm{Part}(N)}$ for which (L1) and (L2) hold. In particular, (L2) ensures that every $\mathsf{P} \in \mathcal{P}_1$ is ℓ-metric in P_1. By our choice of ℓ, this implies that every $\mathsf{H} \in \mathcal{H}(\mathsf{P})$ is metric in P_1.

In general, we obtain P_{k+1} from P_k, $k = 0, \ldots, m-1$, by applying Lemma 1.19 to the N-partite graph P_k and the t-partite t-uniform hypergraph

$$\mathcal{I}^{(k)} = \left\{I \in \bigcup_{\mathsf{H} \in \mathcal{H}(P_k)} \binom{\mathsf{H}}{\rho} \ : \ \pi_k(I) = I_{k+1}\right\} \subset \binom{P_k}{\rho_\ell}.$$

The graph P_{k+1} and the family $\mathcal{P}_{k+1} \subset \binom{P_{k+1}}{P_k}_{\mathrm{Part}(N)}$ we obtain are such that every $\mathsf{H} \in \mathcal{H}(P_{k+1}) = \bigcup_{\mathsf{P} \in \mathcal{P}_{k+1}} \mathcal{H}(\mathsf{P})$ is metric in P_{k+1} and $\pi_{k+1}(\mathsf{H}) \in \mathcal{H}_0$ (where $\pi_{k+1} : V(P_{k+1}) \to [N]$ is defined as the projection that maps every $v \in V_j^N(P_{k+1})$ to j for all $j = 1, \ldots, N$).

Take $R = P_m$ and $\mathcal{H} = \mathcal{H}(P_m) \subset \binom{R}{H}$. Just as in Claim 2.15 one may show that in any 2-colouring of $\bigcup_{\mathsf{H} \in \mathcal{H}(P_m)} \binom{\mathsf{H}}{\rho} \subset \binom{R}{\rho}$ there exists a copy of P_0 in R, say $\mathsf{P}^0 \subset R$, such that the colour of a tuple $I \in \binom{\mathsf{H}}{\rho} \subset \binom{R}{\rho}$, $\mathsf{H} \in \mathcal{H}(\mathsf{P}^0)$, depends only on the projection $\pi^0(I) \in \{I_1, \ldots, I_m\}$, where $\pi^0 : V(\mathsf{P}^0) \to [N]$ is the natural projection of P^0 onto $[N]$. In

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particular, there is an induced 2-colouring of the tuples $I_1, I_2, \ldots, I_m \in \binom{[N]}{t}$. Extend this induced 2-colouring to all of $\binom{[N]}{t}$ arbitrarily.

By the definition of N, there must be a monochromatic $\binom{S}{t}$ with $|S| = n$. Let $\mathsf{H} \in \mathcal{H}(\mathsf{P}^0)$ be the (unique) graph such that $\pi^0(V(\mathsf{H})) = S$. Since the colour of every $I \in \binom{\mathsf{H}}{\rho}$ is the same as the colour of $\pi^0(I) \in \binom{S}{t}$, it follows that $\binom{\mathsf{H}}{\rho}$ is monochromatic. Moreover H is metric in $R = P_m$ since it belongs to $\mathcal{H}(P_m)$.

4.1. Proof of Theorem 1.11

By repeated applications of Lemma 1.10, we will obtain Theorem 1.11.

Let $\mathcal{M} = \{\rho^1, \ldots, \rho^m\}$ be the set of all metrics induced by t vertices of H. Apply Lemma 1.10 to $R_0 = H$ and ρ^1 to obtain a graph R_1. After R_i is constructed, $1 \leqslant i \leqslant m-1$, obtain R_{i+1} by applying Lemma 1.10 to R_i and ρ^{i+1}.

We claim that $R = R_m$ satisfies the conditions of Theorem 1.11. Indeed, given any 2-colouring of $\binom{V(R)}{t}$, we can find a metric copy R^{m-1} of R_{m-1} in which every (ρ^m, t)-tuple in $\binom{\mathsf{R}^{m-1}}{\rho^m}$ is coloured by c_m. Iterating this argument yields a sequence $\mathsf{R}^0 \subset \mathsf{R}^1 \subset \cdots \subset \mathsf{R}^{m-1} \subset R$ such that $\mathsf{R}^i \cong R_i$ is metric in R^{i+1} and every (ρ^{i+1}, t)-tuple in $\binom{\mathsf{R}^i}{\rho^{i+1}}$ has the same colour c_{i+1}. The graph $\mathsf{H} = \mathsf{R}^0 \cong H$ is metric in R and is such that $\binom{\mathsf{H}}{\rho^i} \subset \binom{\mathsf{R}^{i-1}}{\rho^i}$ is monochromatic (with colour c_i) for $i = 1, \ldots, m$.

4.2. An unordered version of Lemma 1.10

We now address the question of what could be an 'unordered version' of Lemma 1.10. Let (M, ρ) be a finite unordered metric space with $|M| = t$ and integer distances. For any connected graph H, let $\binom{H}{\rho}$ be the set of all t-sets $T \subset V(H)$ such that the metric spaces (T, dist_H) and (M, ρ) are isometric.

Analogously to Proposition 1.3, one can show the following characterization of the metric spaces (M, ρ) for which the class of unordered graphs with metric embeddings has the (M, ρ)-Ramsey property.

Proposition 4.1. *Let (M, ρ) be a finite metric space with integer distances. The following statements are equivalent.*

(a) *For any unordered connected graph H there exists an unordered graph R such that, for any partition*

$$\binom{R}{\rho} = \mathcal{A}_1 \cup \mathcal{A}_2,$$

there exists $i \in \{1, 2\}$ and $\mathsf{H} \in \binom{R}{H}_{\mathrm{metric}}$ satisfying

$$\binom{\mathsf{H}}{\rho} \subset \mathcal{A}_i.$$

(b) *ρ is homogeneous, that is, there exists a positive integer c such that for any pair of distinct elements $m, m' \in M$ we have $\rho(m, m') = c$.*

The proof of (b) $\Longrightarrow$ (a) is a direct consequence of Theorem 1.11. Indeed, due to the symmetry of homogeneous metrics the ordering is irrelevant.

The proof of (a) $\Longrightarrow$ (b) closely follows the arguments from [12] and [14] and therefore we omit it.

Appendix A: Proof of Lemma 3.1

Before proving the lemma, we recall some definitions relevant to the Hales–Jewett theorem.

Suppose that $\mathcal{I} \subset \binom{G}{\rho_2}$ is a t-partite t-uniform hypergraph with vertex set V and classes $V_1 = V^q_{j_1}(G), \ldots, V_t = V^q_{j_t}(G)$. Let $\mathcal{I}^n$ be the set of n-tuples of elements of $\mathcal{I}$. A *combinatorial line* L in $\mathcal{I}^n$ associated with a partition $[n] = M_L \cup F_L$, $M_L \neq \emptyset$, and an $|F_L|$-tuple $(I^L_k)_{k \in F_L} \in \mathcal{I}^{F_L}$ is given by

$$L = \{(I_1, I_2, \ldots, I_n) \in \mathcal{I}^n \ : \ I_r = I_s \text{ for } r, s \in M_L \text{ and } I_k = I^L_k \text{ for } k \in F_L\}.$$

The set M_L is called the set of *moving* coordinates, while F_L is called the set of *fixed* coordinates. Notice that every combinatorial line has precisely $|\mathcal{I}|$ elements.

The Hales–Jewett theorem is stated as follows. For a proof, see for instance [7].

Theorem A.1 ([9]). *For any integer $r \geqslant 2$ and finite set $\mathcal{I}$ there exists n such that in every r-colouring of $\mathcal{I}^n$ there exists a monochromatic line.*

For our purposes it will be useful to interpret an element $I \in \mathcal{I}$ as a vector with t coordinates, where the jth coordinate is simply the unique vertex in $I \cap V_j$. In this way, an element in $\mathcal{I}^n$ may be viewed as a $t \times n$ matrix. Consequently, a line L of $\mathcal{I}^n$ may be described as a collection of size $|\mathcal{I}|$ consisting of $t \times n$ matrices Q^L_I, $I \in \mathcal{I}$, where the columns of Q^L_I indexed by F_L are fixed and independent of I, while every column indexed by M_L is precisely I. For example, for $n = 4$, $M_L = \{1, 2\}$, $F_L = [4] \setminus M_L = \{3, 4\}$ and $L = \{(I, I, I^L_3, I^L_4) \ : \ I \in \mathcal{I}\}$, the elements of L are the matrices

$$Q^L_I = \begin{bmatrix} | & | & | & | \\ I & I & I^L_3 & I^L_4 \\ | & | & | & | \end{bmatrix} \tag{A.1}$$

for all $I \in \mathcal{I}$.

Proof of Lemma 3.1. Suppose that G and $\mathcal{I}$ are given as in the statement of the lemma. Let $J = \{j_1, \ldots, j_t\}$ be the set of indices with the property of the assumption, namely, $\mathcal{I}$ is a t-partite t-uniform hypergraph with classes $\{V^q_j(G)\}_{j \in J}$. Let n be given by Theorem A.1 (with $r = 2$) applied to $\mathcal{I}$. Let $\{L_1, \ldots, L_N\}$ denote the set of all lines in $\mathcal{I}^n$

Let $W = \bigcup_{I \in \mathcal{I}} I$ and $W_j = V^q_j(G) \cap W$. (Notice that $W_j = \emptyset$ when $j \notin J$.) The vertex set of R is given by

$$V(R) = \big([N] \times (V(G) \setminus W)\big) \cup \bigcup_{j \in J} W^n_j.$$

The edge set of R will be defined later (see (A.3) below).

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In our construction, the family $\mathcal{G}$ will be in direct correspondence with the set of lines in $\mathcal{I}^n$, namely, to each line L_j there will be a corresponding $\mathsf{G}_j \in \mathcal{G}$. In order to guarantee that $\mathcal{G}$ satisfies (A) we will have $V(\mathsf{G}_j) \setminus \bigcup_{I \in \mathcal{I}_{\mathsf{G}_j}} I = \{j\} \times (V(G) \setminus W)$ for $j = 1, \ldots, N$.

For a line L_a determined by the values $\left(I_k^a\right)_{k \in F_a}$ of its fixed coordinates F_a, we represent $I_k^a = \{I_{k,j}^a \in W_j\}_{j \in J}$ as a column-vector $[I_{k,j}^a]_{j \in J}$. Let us define the map $\psi_a : V(G) \to V(R)$ as follows:

$$\psi_a(v) = \begin{cases} (a, v) & \text{for } v \in V(G) \setminus W, \\ (v_1, v_2, \ldots, v_n) & \text{for } v \in W_j, j \in J, \text{ where} \\ \quad v_k = v \text{ for } k \in M_a \text{ and } v_k = I_{k,j}^a \text{ for } k \in F_a. \end{cases} \tag{A.2}$$

Fix some $I = \{u_1 < u_2 < \cdots < u_t\} \in \mathcal{I}$. Because $\mathcal{I}$ is t-partite with classes $\{V_{j_i}^q(G)\}_{i=1}^t$, we have $u_i \in W_{j_i}$ and thus $\psi_a(u_i)$ is an n-tuple for all $i = 1, \ldots, t$. Therefore, in view of (A.1) and (A.2),

$$Q_I^{L_a} = \psi_a(I) = \begin{bmatrix} \psi_a(u_1) \\ \psi_a(u_2) \\ \vdots \\ \psi_a(u_t) \end{bmatrix}.$$

Indeed, the equality above is true because

- for $k \in M_a$ we have $\psi_a(u_i)_k = u_i$ for all i, and hence the kth column of the matrix on the right is simply I,
- for $k \in F_a$, we have $\psi_a(u_i)_k = I_{k,j_i}^a$ for all i, and hence the kth column of the matrix on the right is simply I_k^a.

Observe that the rows of the matrices $Q_I^{L_a}$ correspond to vertices of R.

Claim A.2. *The map $\psi_a : V(G) \to V(R)$ is one-to-one.*

Suppose for the sake of contradiction that two distinct $u, v \in V_j^q(G)$, $1 \leqslant j \leqslant q$, are such that $\psi_a(u) = \psi_a(v)$. We cannot have $\psi_a(u) = (a, u)$ since that would imply $u = v$. Consequently, $u, v \in W_j$ with $j \in J$. Hence both $\psi_a(u)$ and $\psi_a(v)$ must be n-tuples such that $\psi_a(u)_k = u \neq v = \psi_a(v)_k$ for all $k \in M_a$. Therefore u cannot be distinct from v and hence Claim A.2 holds.

Set

$$E(R) = \bigcup_{a=1}^{N} E\left(\psi_a(G)\right) \tag{A.3}$$

and let $\mathcal{G} = \{\mathsf{G}_a = \psi_a(G) \ : \ a = 1, \ldots, N\}$. Observe that by our definition of $\mathcal{G}$, (L3) follows directly from (A.3).

We now must prove that the conclusions of the lemma hold for R and $\mathcal{G}$. This will be accomplished by the following steps.

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Step I. Define a total order on $V(R)$ and a q-partition $V(R) = V_1^q(R) \cup V_2^q(R) \cup \cdots \cup V_q^q(R)$ such that every ψ_a is a monotone map satisfying $\psi_a(V_j^q(G)) \subset V_j^q(R)$ for every j.

Step II. Show that $\mathcal{G}$ satisfies the intersection conditions (A) and (B) and thus prove (L4).

Step III. Use Step II to show that every $\mathsf{G}_a \in \mathcal{G}$ is an induced subgraph of R and thus prove (L2).

Step IV. Show that the family $\mathcal{G}$ is Ramsey in R, namely, prove (L1).

Proof of Step I. For all j, define

$$V_j^q(R) = \big([N] \times (V_j^q(G) \setminus W)\big) \cup W_j^n. \tag{A.4}$$

Observe that $V(R) = V_1^q(R) \cup V_2^q(R) \cup \cdots \cup V_q^q(R)$. Moreover, it is simple to check that $\psi_a(V_j^q(G)) \subset V_j^q(R)$ for all j. Let us now define a total order on $V(R)$ for which every map ψ_a is monotone. It is enough to define the order for each $V_j^q(R)$ since we require $V_1^q(R) \prec V_2^q(R) \prec \cdots \prec V_q^q(R)$.

For $j \notin J$, we have $W_j = \emptyset$ and thus $V_j^q(R) = [N] \times V_j^q(G)$. Order the vertices lexicographically and observe that for every $a \in [N]$, $\psi_a(v) < \psi_a(w)$ if and only if $v < w$.

Since for $j \in J$ the class $V_j^q(R)$ may contain both pairs and n-tuples as elements, our ordering is somewhat more complicated than a simple lexicographical order on tuples.

Let $f : V_j^q(R) \to V_j^q(G)^n \times \{0, 1, \ldots, N\}$ be defined as follows. For a tuple $(v_1, \ldots, v_n) \in W_j^n$, set $f(v_1, \ldots, v_n) = (v_1, \ldots, v_n, 0)$; for $(a, v) \in [N] \times (V_j^q(G) \setminus W)$ set $f(a, v) = (v_1, \ldots, v_n, a)$, where $v_k = v$ for all $k \in M_a$ and $v_k = I_{k,j}^a$ for all $k \in F_a$. The ordering on $V_j^q(R)$ is induced by f and the lexicographic order on the image of f, namely, we set $x < y$ if and only if $f(x) < f(y)$.

Let $v, w \in V_j^q(G)$ be such that $v < w$. By definition, for every $a \in [N]$, $\psi_a(v) < \psi_a(w)$ if and only if $f(\psi_a(v)) < f(\psi_a(w))$. Since $f(\psi_a(v))_k = f(\psi_a(w))_k = I_{k,j}^a$ for every $k \in F_a$, the first coordinate where the elements $f(\psi_a(v))$ and $f(\psi_a(v))$ differ is in M_a. On the other hand, for $k \in M_a$ we have

$$f(\psi_a(v))_k = v < w = f(\psi_a(w))_k.$$

We conclude that $f(\psi_a(v)) < f(\psi_a(w))$ if and only if $v < w$. Hence $\psi_a(v) < \psi_a(w)$ if and only if $v < w$.

Proof of Step II. Suppose that $x \in V(\mathsf{G}_a) \cap V(\mathsf{G}_b)$ with $a \neq b$. We must have $x \in W_j^n$ for some $j \in J$ since otherwise for some $v \in V(G) \setminus W$, we have $x = (a, v) = (b, v)$ which contradicts $a \neq b$. It follows therefore that $\psi_a^{-1}(x), \psi_b^{-1}(x) \in W_j$. Since $W_j \subset W = \bigcup_{I \in \mathcal{I}} I$, there exists $I'_a, I'_b \in \mathcal{I}$ such that $\psi_a^{-1}(x) \in I'_a$ and $\psi_b^{-1}(x) \in I'_b$. Consequently, $x \in I_a = \psi_a(I'_a) \in \mathcal{I}_{\mathsf{G}_a}$ and $x \in I_b = \psi_b(I'_b) \in \mathcal{I}_{\mathsf{G}_b}$. This establishes the intersection condition (A) for members of $\mathcal{G}$.

Now let us prove condition (B). Suppose that there are distinct $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in V(\mathsf{G}_a) \cap V(\mathsf{G}_b)$, $a \neq b$.

We distinguish between two cases:

(i) $M_a \cap M_b \neq \emptyset$,

(ii) $M_a \cap M_b = \emptyset$ (then $M_a \subset F_b$ and $M_b \subset F_a$).

Suppose first that (i) holds and fix $k \in M_a \cap M_b$. We have $\psi_a^{-1}(x) = x_k = \psi_b^{-1}(x)$, and similarly $\psi_a^{-1}(y) = \psi_b^{-1}(y)$. Consequently, in this case condition (B2) holds as the isomorphisms $\sigma_a = \psi_a^{-1} : V(\mathsf{G}_a) \to V(G)$ and $\sigma_b = \psi_b^{-1} : V(\mathsf{G}_b) \to V(G)$ satisfy $\sigma_a(x) = \sigma_b(x)$ and $\sigma_a(y) = \sigma_b(y)$.

Now suppose that (ii) holds; in particular, we must have $M_a \subset F_b$ and $M_b \subset F_a$. Let $\left(I_k^a = [I_{k,j}^a]_{j\in J}\right)_{k\in F_a}$ and $\left(I_k^b = [I_{k,j}^b]_{j\in J}\right)_{k\in F_b}$ be the tuples of fixed elements that define the lines L_a and L_b respectively. Let $j, j' \in J$ be such that $x \in W_j^n$ and $y \in W_{j'}^n$.

For $k \in M_a \subset F_b$, (A.2) implies that

$$\psi_a^{-1}(x) \overset{k\in M_a}{=} x_k \overset{k\in F_b}{=} I_{k,j}^b$$

and similarly $\psi_a^{-1}(y) = y_k = I_{k,j'}^b$. In particular, $\{\psi_a^{-1}(x), \psi_a^{-1}(y)\} = \{I_{k,j}^b, I_{k,j'}^b\} \subset I_k^b \in \mathcal{I}$. Let $I_a = \psi_a(I_k^b) \in \mathcal{I}_{\mathsf{G}_a}$ and notice that

$$\{x, y\} = \psi_a\left(\{\psi_a^{-1}(x), \psi_a^{-1}(y)\}\right) \subset \psi_a\left(I_k^b\right) = I_a.$$

A symmetric argument yields $I_b \in \mathcal{I}_{\mathsf{G}_b}$ such that $\{x, y\} \in I_b$. Hence, condition (B1) follows.

To summarize, case (i) implies condition (B2) and case (ii) implies condition (B1).

Proof of Step III. Let $\mathsf{G}_a \in \mathcal{G}$ be arbitrary. To prove that G_a is an induced subgraph of R, we must check that for every pair of distinct $x, y \in V(\mathsf{G}_a)$, if $x, y \in V(\mathsf{G}_b)$ for some $b \neq a$, then $\{x, y\} \in \mathsf{G}_a$ if and only if $\{x, y\} \in \mathsf{G}_b$. Since $x, y \in V(\mathsf{G}_a) \cap V(\mathsf{G}_b)$, we may invoke the intersection properties of $\mathcal{G}$ proved in Step II.

If condition (B2) holds, the unique isomorphisms σ_a, σ_b of $\mathsf{G}_a, \mathsf{G}_b$ into G satisfy $\sigma_a(x) = \sigma_b(x)$ and $\sigma_a(y) = \sigma_b(y)$. Since σ_a is an isomorphism, $\{x, y\} \in \mathsf{G}_a$ if and only if $e = \{\sigma_a(x), \sigma_a(y)\} \in G$. Similarly, $\{x, y\} \in \mathsf{G}_b$ if and only if $e' = \{\sigma_b(x), \sigma_b(y)\} \in G$. Because $e = e'$, we infer that $\{x, y\} \in \mathsf{G}_a$ if and only if $\{x, y\} \in \mathsf{G}_b$.

If condition (B1) holds, let $I_a \in \mathcal{I}_{\mathsf{G}_a}$ and $I_b \in \mathcal{I}_{\mathsf{G}_b}$ be such that $x, y \in I_a \cap I_b$. Let $j_r, j_s \in J$ $(1 \leqslant r, s \leqslant t)$ be such that $x \in V_{j_r}^q(R)$ and $y \in V_{j_s}^q(R)$. Because $I_a \in \binom{\mathsf{G}_a}{\rho_2}$, it follows that $\mathrm{dist}_{\mathsf{G}_a}(x, y) = \rho(r, s)$ whenever $\rho(r, s) \leqslant 2$ and $\mathrm{dist}_{\mathsf{G}_a}(x, y) \geqslant 2$ whenever $\rho(r, s) > 2$. In particular, $\{x, y\} \in \mathsf{G}_a$ if and only if $\rho(r, s) = 1$. Similarly, $\{x, y\} \in \mathsf{G}_b$ if and only if $\rho(r, s) = 1$. Therefore $\{x, y\} \in \mathsf{G}_a$ if and only if $\{x, y\} \in \mathsf{G}_b$.

Proof of Step IV. We will now show that for any 2-colouring of the (ρ_2, R)-tuples in $\bigcup_{\mathsf{G}\in\mathcal{G}} \mathcal{I}_{\mathsf{G}}$ there exists $\mathsf{G} \in \mathcal{G}$ such that every t-tuple in $\mathcal{I}_{\mathsf{G}} \subset \binom{\mathsf{G}}{\rho_2}$ is monochromatic. It will be convenient to assume that all t-tuples in $V_{j_1}^q(R) \times \cdots \times V_{j_t}^q(R)$ are coloured.

Consider $Q = (I_1, \ldots, I_n) \in \mathcal{I}^n$ as a $t \times n$ matrix with columns $I_1, \ldots, I_n$. The kth row of the matrix is in $V_{j_k}^q(R)$ (recall that $J = \{j_1, \ldots, j_t\}$). In particular, Q is in correspondence with a t-tuple of $V_{j_1}^q(R) \times \cdots \times V_{j_t}^q(R)$. Define the colour of Q as the colour of the corresponding t-tuple.

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By the Hales–Jewett theorem, there is a monochromatic line L_a, $a \in [N]$, in such a colouring. It follows that $\mathsf{G} = \mathsf{G}_a$ is such that $\mathcal{I}_{\mathsf{G}}$ is monochromatic. Indeed, every t-tuple $\psi_a(I) \in \mathcal{I}_{\mathsf{G}_a}$, $I \in \mathcal{I}$, corresponds to the matrix $Q_I^{L_a}$ contained in the line L_a (see (A.2)). □

Appendix B: Induction hypotheses (reprise)

Induction over ℓ: Hypothesis for R_ℓ and $\mathcal{G}_\ell$. For a q-partite graph G, a metric ρ on $[t]$ and a t-partite t-uniform hypergraph $\mathcal{I} \subset \binom{G}{\rho_\ell}$, there is a graph $R_\ell = R_\ell(q, G, \rho, \mathcal{I})$ and

$\mathcal{G}_\ell = \mathcal{G}_\ell(q, G, \rho, \mathcal{I}) \subset \binom{R_\ell}{G}_{\mathrm{Part}(q)}$ satisfying conditions (L1) and (L2) of Lemma 1.19 and

(L3) $E(R_\ell) = \bigcup_{\mathsf{G} \in \mathcal{G}_\ell} E(\mathsf{G})$.

Moreover, $\mathcal{G}_\ell$ satisfies the conditions (A) and (B) below.

Intersection conditions for a family $\mathcal{G}$ of copies of G.

(A) If $\mathsf{G}_1, \mathsf{G}_2 \in \mathcal{G}$ and $u \in V(\mathsf{G}_1) \cap V(\mathsf{G}_2)$, then there are $(\rho_\ell, \mathsf{G}_j)$-tuples $I^j \in \mathcal{I}_{\mathsf{G}_j}$, $j = 1, 2$, such that $u \in I^1 \cap I^2$.

(B) If $\mathsf{G}_1, \mathsf{G}_2 \in \mathcal{G}$ and $u, v \in V(\mathsf{G}_1) \cap V(\mathsf{G}_2)$, then either

- (B1) there exist $(\rho_\ell, \mathsf{G}_j)$-tuples $I^j \in \mathcal{I}_{\mathsf{G}_j}$, $j = 1, 2$, such that $\{u, v\} \subset I^1 \cap I^2$ or
- (B2) the (unique) isomorphisms $\sigma_j : V(\mathsf{G}_j) \to V(G)$, $j = 1, 2$, satisfy $\sigma_1(u) = \sigma_2(u)$ and $\sigma_1(v) = \sigma_2(v)$.

Induction over k: Hypothesis for P_k and $\mathcal{G}(P_k)$.

(K1) The picture P_k is r_ℓ-partite with classes $V_j^{r_\ell}(P_k)$, $j = 1, \ldots, r_\ell$. The projection map $\pi_k : V(P_k) \to V(R_\ell) = [r_\ell]$ given by $\pi_k(x) = j$ if and only if $x \in V_j^{r_\ell}(P_k)$ is a homomorphism of P_k into R_ℓ. Moreover, $\pi_k(\mathsf{G}) \in \mathcal{G}_\ell$ for every $\mathsf{G} \in \mathcal{G}(P_k)$.

(K2) The family $\mathcal{G}(P_k)$ is contained in $\binom{P_k}{G}_{\mathrm{Part}(q)}$.

(K3) The family $\mathcal{G}(P_k)$ satisfies conditions (A) and (B).

(K4) Every $\mathsf{G} \in \mathcal{G}(P_k)$ is $(\ell + 1)$-metric in P_k.

$(1)_{k+1}$ For every 2-colouring of the (ρ_ℓ, P_{k+1})-tuples in $\bigcup_{\mathsf{P} \in \mathcal{P}_{k+1}} \mathcal{I}_{\mathsf{P}}^{(k)}$, there exists $\mathsf{P} \in \mathcal{P}_{k+1}$ such that $\mathcal{I}_{\mathsf{P}}^{(k)} \subset \binom{\mathsf{P}}{\rho_{\ell+1}} \subset \binom{P_{k+1}}{\rho_\ell}$ is monochromatic (recall that the hypergraph $\mathcal{I}_{\mathsf{P}}^{(k)}$ is an isomorphic copy of $\mathcal{I}^{(k)}$ in P).

$(2)_{k+1}$ Every $\mathsf{P} \in \mathcal{P}_{k+1}$ is ℓ-metric in P_{k+1}.

$(3)_{k+1}$ $E(P_{k+1}) = \bigcup_{\mathsf{P} \in \mathcal{P}_{k+1}} E(\mathsf{P})$.

$(A)_{k+1}$ If $\mathsf{P}^1, \mathsf{P}^2 \in \mathcal{P}_{k+1}$ are distinct and $u \in V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$, then there are $(\rho_{\ell+1}, \mathsf{P}^j)$-tuples $I_*^j \in \mathcal{I}_{\mathsf{P}^j}^{(k)}$, $j = 1, 2$, such that $u \in I_*^1 \cap I_*^2$.

$(B)_{k+1}$ If $\mathsf{P}^1, \mathsf{P}^2 \in \mathcal{P}_{k+1}$ are distinct and $u, v \in V(\mathsf{P}^1) \cap V(\mathsf{P}^2)$, then either

- $(B1)_{k+1}$ there exist $(\rho_{\ell+1}, \mathsf{P}^j)$-tuples $I_*^j \in \mathcal{I}_{\mathsf{P}^j}^{(k)}$, $j = 1, 2$, such that $\{u, v\} \subset I_*^1 \cap I_*^2$ or
- $(B2)_{k+1}$ the isomorphisms $\phi_j : V(\mathsf{P}^j) \to V(P_k)$, $j = 1, 2$, satisfy $\phi_1(u) = \phi_2(u)$ and $\phi_1(v) = \phi_2(v)$.

References

[1] Abramson, F. G. and Harrington, L. A. (1978) Models without indiscernibles. *J. Symbolic Logic* **43** 572–600.

[2] Deuber, W. (1975) Generalizations of Ramsey's theorem. In *Infinite and Finite Sets I: Colloq., Keszthely, 1973, dedicated to P. Erdős on his 60th birthday*, Vol. 10 of *Colloq. Math. Soc. János Bolyai*, North-Holland, pp. 323–332.

[3] Deuber, W. (1975) Partitionstheoreme für Graphen. *Comment. Math. Helv.* **50** 311–320.

[4] Erdős, P., Hajnal, A. and Pósa, L. (1975) Strong embeddings of graphs into colored graphs. In *Infinite and Finite Sets I: Colloq., Keszthely, 1973, dedicated to P. Erdős on his 60th birthday*, Vol. 10 of *Colloq. Math. Soc. János Bolyai*, North-Holland, pp. 585–595.

[5] Graham, R. L., Grötschel, M. and Lovász, L., editors (1995) *Handbook of Combinatorics*, Vols 1, 2, Elsevier Science.

[6] Graham, R. L., Leeb, K. and Rothschild, B. L. (1972) Ramsey's theorem for a class of categories. *Proc. Nat. Acad. Sci. USA* **69** 119–120.

[7] Graham, R. L., Rothschild, B. L. and Spencer, J. H. (1990) *Ramsey Theory*, second edition, Wiley-Interscience Series in Discrete Mathematics and Optimization.

[8] Graham, R. L. and Winkler, P. M. (1984) Isometric embeddings of graphs. *Proc. Nat. Acad. Sci. USA* **81** 7259–7260.

[9] Hales, A. W. and Jewett, R. I. (1963) Regularity and positional games. *Trans. Amer. Math. Soc.* **106** 222–229.

[10] Nešetřil, J. (2005) Ramsey classes and homogeneous structures. *Combin. Probab. Comput.* **14** 171–189.

[11] Nešetřil, J. (2007) Metric spaces are Ramsey. *European J. Combin.* **28** 457–468.

[12] Nešetřil, J. and Rödl, V. (1975) Partitions of subgraphs. In *Recent Advances in Graph Theory: Proc. Second Czechoslovak Sympos., Prague, 1974*, Academia, pp. 413–423.

[13] Nešetřil, J. and Rödl, V. (1977) Partitions of finite relational and set systems. *J. Combin. Theory Ser. A* **22** 289–312.

[14] Nešetřil, J. and Rödl, V. (1978) On a probabilistic graph-theoretical method. *Proc. Amer. Math. Soc.* **72** 417–421.

[15] Nešetřil, J. and Rödl, V. (1979) Partition theory and its application. In *Surveys in Combinatorics: Proc. Seventh British Combinatorial Conf., Cambridge, 1979*, Vol. 38 of *London Math. Soc. Lecture Note Ser.*, Cambridge University Press, pp. 96–156.

[16] Nešetřil, J. and Rödl, V. (1979) A short proof of the existence of highly chromatic hypergraphs without short cycles. *J. Combin. Theory Ser. B* **27** 225–227.

[17] Nešetřil, J. and Rödl, V. (1984) Combinatorial partitions of finite posets and lattices: Ramsey lattices. *Algebra Universalis* **19** 106–119.

[18] Nešetřil, J. and Rödl, V. (1987) Strong Ramsey theorems for Steiner systems. *Trans. Amer. Math. Soc.* **303** 183–192.

[19] Nešetřil, J. and Rödl, V. (1989) The partite construction and Ramsey set systems. *Discrete Math.* **75** 327–334.

[20] Nešetřil, J. and Rödl, V. (1990) Introduction: Ramsey theory old and new. In *Mathematics of Ramsey Theory*, Vol. 5 of *Algorithms Combin.*, Springer, pp. 1–9.

[21] Nešetřil, J. and Rödl, V. (1992) On Ramsey graphs without bipartite subgraphs. *Discrete Math.* **101** 223–229.

[22] Rödl, V. (1973) A generalization of Ramsey theorem and dimension of graphs. Master's thesis, Charles University, Prague.

[23] Rödl, V. (1976) A generalization of the Ramsey theorem. In *Graphs, Hypergraphs and Block Systems: Zielona Góra 1976*, pp. 211–219.

[24] Winkler, P. M. (1984) Isometric embedding in products of complete graphs. *Discrete Appl. Math.* **7** 221–225.