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THE ENUMERATION OF MAPS ON THE TORUS AND THE PROJECTIVE PLANE

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ABSTRACT. The enumeration of rooted maps (embedded graphs), by number of edges, on the torus and projective plane, is studied. Explicit expressions for the generating functions are obtained. From these are derived asymptotic expressions and recurrence relations. Numerical tables for the numbers with up to 20 edges are presented.

1. **Introduction.** A *map* is a connected graph which has been embedded in a surface. A map is *rooted* if an edge is distinguished together with a vertex on the edge and a side of the edge. Techniques originated by Tutte [7, 8, 9] for enumerating various classes of rooted maps on the sphere are here applied to the class of all rooted maps on the torus and projective plane. Generating functions, asymptotic expressions, and recurrence relations are derived for the numbers p_n and t_n of rooted maps with n edges on the projective plane and the torus, respectively.

Brown [6] had previously applied Tutte's techniques to obtain formulas for the generating functions by number of edges of rooted non-separable maps and of rooted non-singular maps on the projective plane. He also sketched a method for enumerating such maps on the torus, but did not carry it out because "the proliferation of cases to be considered renders these problems very tedious, if not difficult."

Walsh and Lehman [10, 11, 12] have studied maps on orientable surfaces of arbitrary genus, using an approach different from that of Tutte and Brown. They obtained explicit summations for "tree-rooted" maps, and an algorithm for calculating the number of rooted maps with a specified number of edges. We have used the latter to check the values of t_n for $n \leqq 6$. We have also extended their algorithm to the (non-orientable) projective plane, thereby giving independent confirmation of the values of p_n for $n \leqq 6$.

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Our most striking results are the compact explicit formulas for the generating functions P_1 and T_1 given in (4.21) and (4.22). The paper is organized as follows: Section 2 sets forth definitions; Section 3 defines the relevant generating functions, and derives determining equations for them; Section 4 solves these equations; Section 5 discusses asymptotics and a conjecture; Section 6 describes the computations performed and presents tables of the numerical results.

2. **Definitions.** We shall use the term *surface* to mean a connected, compact 2-manifold without boundary. By a *map* M on a surface S we mean an incidence preserving embedding ϕ of a graph G such that each connected component of $S - \phi(G)$ is simply connected. A component F of $S - \phi(G)$ is a face of M. Each edge of M has two *sides* in an obvious sense. A map is *rooted* by selecting an edge, a direction on the edge and a side of the edge. The face containing the selected side of the root edge is the *root face*. An edge belonging to only one face is called *double*. All others belong to exactly two faces and are called *single*. The *degree* of a face is the number of single edges on its boundary plus twice the number of double edges. It is simply the number of sides of a polygon corresponding to the face in a natural way.

Two maps on a surface may have the same underlying graph. In fact a graph can often be embedded in several non-homeomorphic surfaces. Two maps (S, G, ϕ) (S', G', ϕ') are equivalent if there is a homeomorphism $h: S \to S'$ and a graph isomorphism $g: G \to G'$ such that $h\phi = \phi' g$.

3. **Generating functions, and their equations.** Let s_{ij}, p_{ij}, and t_{ij} be the number of rooted maps on the sphere, the projective plane and the torus, respectively, having i edges and root face degree j. Let $S(x, y)$, $P(x, y)$, and $T(x, y)$ be the corresponding two-variable, ordinary generating functions; for example,

$$T(x, y) = \sum t_{ij} x^i y^j.$$

The arguments (x, y) are usually omitted when S, P, or T appears in an equation. In analyzing T it will be necessary to consider also the generating function

$$D(x, y, z) = \sum d_{ijk} x^i y^j z^k,$$

in which d_{ijk} counts rooted maps on the sphere having i edges, root face degree j and some other distinguished face having degree k. The other distinguished face is also called the second face.

The generating functions S_1, P_1, T_1 and D_1 are the result of setting y equal to 1 in S, P, T and D. Thus P_1 and T_1 are the ordinary generating functions by number of edges for rooted maps on the projective plane and torus – our ultimate goal. We begin by analyzing two different contributions to S, P, and T: terms $x^i y^j$ from maps whose root edge is single, and terms from maps whose root edge is double. This will lead to equations (3.9)-(3.11).

When the root edge is a single edge, it may be merged with the two faces incident with it, the result being one face of a new map on the same surface but having one less edge. Assuming that the root face of the original map has at least two edge incidences, the new map may be rooted by letting the root edge be the first edge before the old root edge in the bounding sequence of the old root face; the new root edge inherits a direction and a preferred side in a natural manner from the old root edge. If the root face of the original map has one edge only, then the edge immediately after this edge in the bounding sequence of the adjacent face, directed toward the loop's vertex, may be selected as the new root. It is conceivable that both the root face and its neighboring face have only one edge in their bounding sequence: in this case one has the *link map* on the sphere; the result of removing the root edge is a *vertex map* on the sphere, which is considered rooted by convention. See Figure 1.

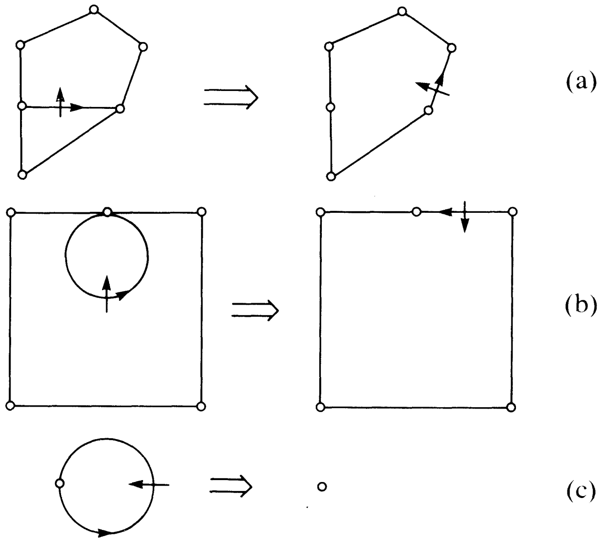

FIGURE 1. Removing a single root edge. (a) The usual case; (b) a root face having one edge only; (c) the link and vertex maps.

This operation is reversible: given a rooted map whose root face has j edge incidences, there are $(j + 1)$ ways to attach a new root edge with direction and preferred side specified; these various choices produce a new root face with $1, 2, \ldots,$ or $(j + 1)$ edge incidences. See Figure 2.

It follows that the contribution to $S(x, y)$ from terms of this type is

$$x \cdot \sum s_{ij} x^i (y + y^2 + \ldots + y^{j+1}), \tag{3.1}$$

that the contribution to $P(x, y)$ from such terms is

$$x \cdot \sum p_{ij} x^i (y + y^2 + \ldots + y^{j+1}), \tag{3.2}$$

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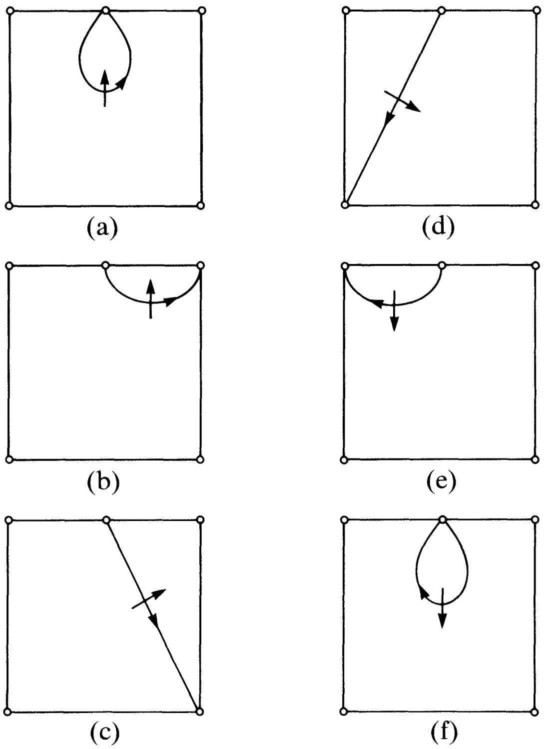

FIGURE 2. The ways to reverse the operation shown in Figure 1 (b). Option (a) recovers the original map.

and that the contribution to $T(x, y)$ is

$$x \cdot \sum t_{ij} x^i (y + y^2 + \dots + y^{j+1}). \tag{3.3}$$

Now we consider those terms in which the root edge is a double edge. Since it is a double edge there is a simple closed loop which begins on the root edge, stays always in the root face, returning to the initial point of the root edge on the opposite side from which it departed. Remove the root edge and cut the surface along this loop producing one or two surfaces with boundary. By attaching a disk to each boundary, we produce one or two surfaces with rooted maps.

On the sphere such a cut always produces two separate components, both of which are spheres (once their one dimensional circular boundaries are capped with disks). Thus the original map decomposes into an ordered pair of spherical maps, each of which can be rooted in a natural manner relative to the original rooting. The contribution to S from such terms is

$$xy^2S^2, \tag{3.4}$$

since the edge incidences of the two components combine with two occurrences of the new root to create the new bounding sequence. See Figure 3.

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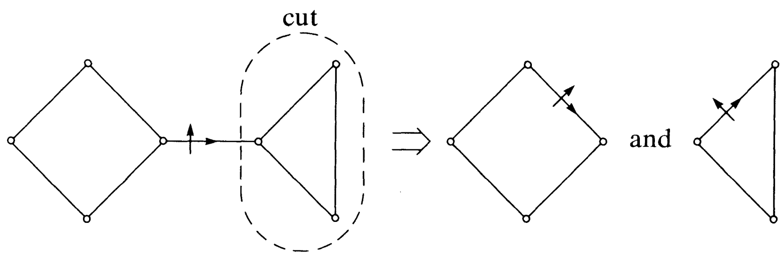

FIGURE 3. Removing a double root edge from the sphere.

The resulting pair of maps from this cut is independent of the simple closed loop chosen, because if λ_1 and λ_2 are two such loops, $\lambda_1\lambda_2^{-1}$ is homotopic to a loop lying within the simply-connected root face and so λ_1 and λ_2 are homotopic. This same remark also applies to the following analysis of cuts on the projective plane and torus.

Removing the root edge and cutting the projective plane along the closed loop may result in either two pieces or one. In the former case one obtains, after capping the cuts, an ordered pair of maps, one of which is on the sphere and the other on the projective plane. See Figure 4. The order is determined by the direction of the original root edge, and so we find a contribution to P of

(3.5) $$xy^2PS + xy^2SP.$$

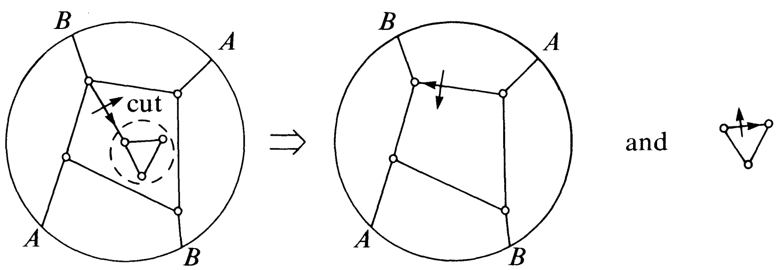

FIGURE 4. Removing a double root edge from the projective plane, when the cut produces two surfaces.

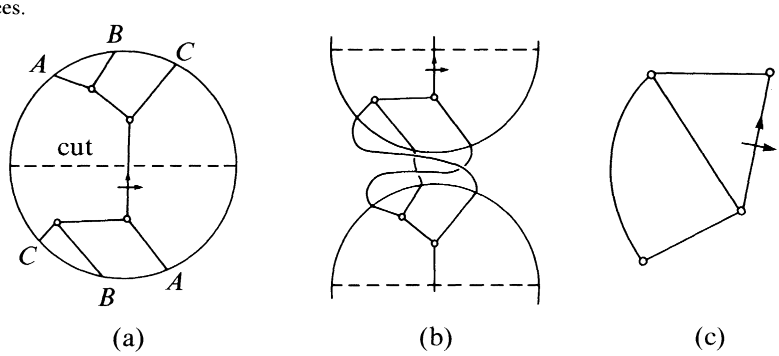

FIGURE 5. Removing a double root edge from the projective plane, when the cut leaves one surface only. (a) The original map and cut; (b) making identifications; (c) the resulting spherical map, after rooting and untwisting.

In the latter case, Figure 5, one obtains a spherical rooted map, and the reverse of the construction may be described as follows. Let M be a spherical map with i edges and j root face edge incidences. Push M into one hemisphere so that the root face is the one containing an entire hemisphere; when this hemisphere is removed and proper identifications made on the equator, the projective plane results. The tail of a new root edge should be attached to the head of M's root edge and a preferred side selected consistently. After "crossing the cross-cap," there are $(j + 1)$ places where the head of the new root edge can be attached. Regardless of where the edge is attached, the new root face has $(j + 2)$ edges. See Figure 6. The contribution to P is thus:

$$x \cdot \sum (j + 1) s_{ij} x^i y^{j+2}. \tag{3.6}$$

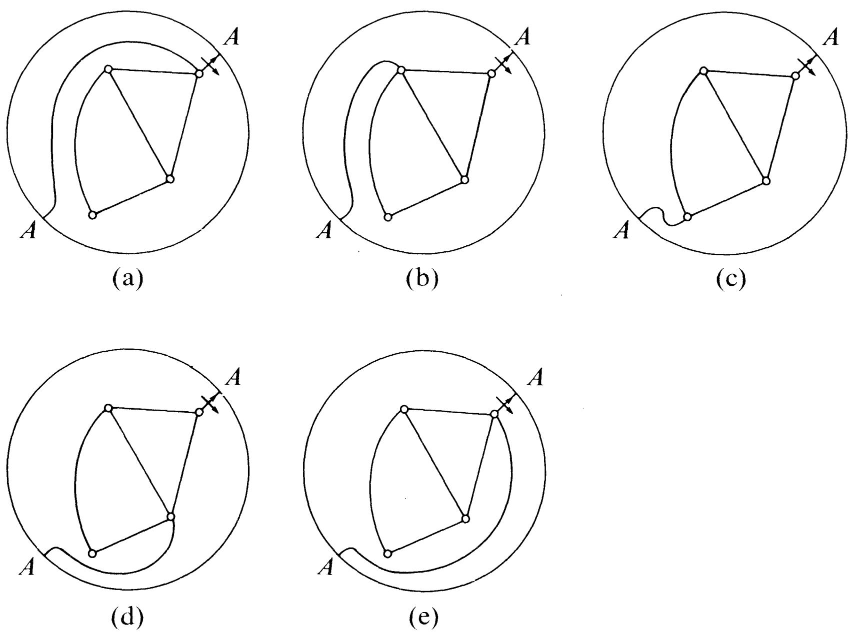

FIGURE 6. The ways to reverse the operation shown in Figure 5. Option (c) recovers the original map.

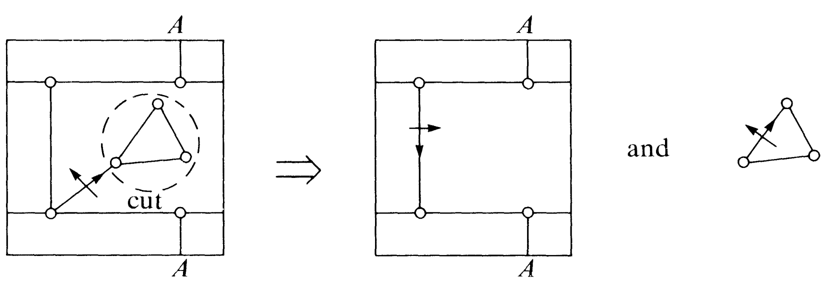

FIGURE 7. Removing a double root edge from the torus, when the cut leaves one toroidal and one spherical map.

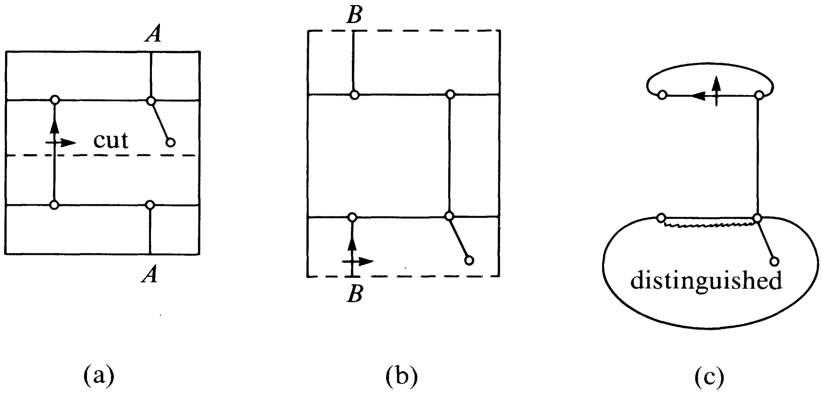

FIGURE 8. Removing a double root edge from the torus, when the cut leaves a spherical map with a second distinguished face. (a) The original map and cut; (b) some rearrangement; (c) the resulting spherical map, with distinguished face and edge incidence.

When the torus is cut we obtain either two maps, one spherical and one toroidal, leading to a contribution of

$$xy^2TS + xy^2ST; \tag{3.7}$$

or one piece with two circles to be capped. See Figures 7 and 8. The latter case may be viewed as a rooted spherical map, with a second face and incident edge distinguished. The tail of the root edge induces the rooting of the spherical map,

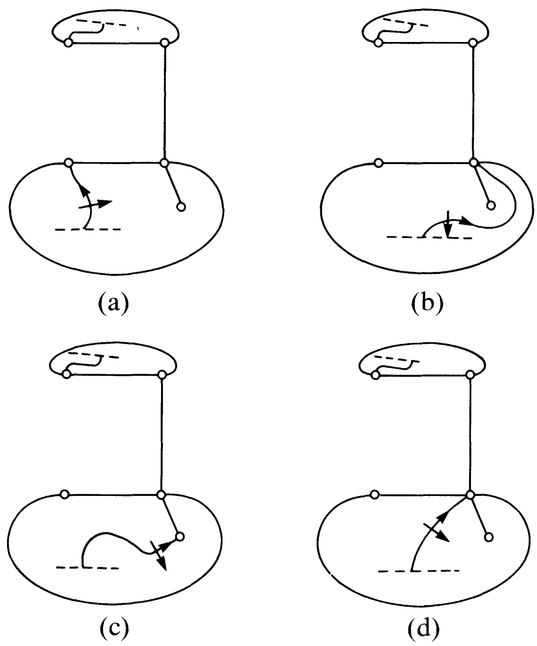

FIGURE 9. The ways to reverse the operation shown in Figure 8. Option (a) recovers the original map.

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and the head selects the second face and incident edge. Given a spherical map counted by d_{ijk} there are k ways to produce a map counted by $t_{i+1,j+k+2}$; thus we obtain a contribution to T of (see Figure 9)

$$x \cdot \sum k d_{ijk} x^i y^{j+k+2}. \tag{3.8}$$

We may now combine the preceding to produce equations for S, P, and T. After some simple algebra and including the edgeless vertex map for S from (3.1) and (3.4),

$$S = 1 + \frac{xy}{1-y}(S_1 - yS) + xy^2S^2; \tag{3.9}$$

from (3.2), (3.5) and (3.6),

$$P = \frac{xy}{1-y}(P_1 - yP) + 2xy^2SP + (xy^2S + xy^3S_y); \tag{3.10}$$

and from (3.3), (3.7) and (3.8),

$$T = \frac{xy}{1-y}(T_1 - yT) + 2xy^2ST + xy^3D_z(x, y, y). \tag{3.11}$$

The generating function $D(x, y, z)$ can be analyzed in the same manner as S with two new features: we must keep up with which component contains the second face and, when the root edge is single, we must distinguish between when the adjacent face is the second face, and when it is not.

We find the following contributions to D:

(i) when the root edge is single and the second face is adjacent to the root,

$$x \sum s_{ij} x^i (yz^{j+1} + y^2z^j + \ldots + y^{j+1}z); \tag{3.12}$$

(ii) when the root edge is single and the second face is not adjacent to the root, as in (3.1),

$$x \sum d_{ijk} x^i (y + y^2 + \ldots + y^{j+1}) z^k; \tag{3.13}$$

(iii) when the root edge is double, as in (3.4),

$$xy^2SD + xy^2DS. \tag{3.14}$$

Combining these last three, we obtain our final equation,

$$D = \frac{xy}{1-y/z}(zS(x, z) - yS) + \frac{xy}{1-y}(D_1 - yD) + 2xy^2SD. \tag{3.15}$$

4. **Solving the equations.** Our method of solving the previous equations, via an auxiliary power series f introduced below, follows Tutte and Brown. We first rewrite (3.9)-(3.11) and (3.15) as:

(4.1) $$S\left(1 + \frac{xy^2}{1-y} - xy^2S\right) = 1 + \frac{xy}{1-y}S_1;$$

(4.2) $$P\left(1 + \frac{xy^2}{1-y} - 2xy^2S\right) = \frac{xy}{1-y}P_1 + xy^2(S + yS_y);$$

(4.3) $$T\left(1 + \frac{xy^2}{1-y} - 2xy^2S\right) = \frac{xy}{1-y}T_1 + xy^3D_z(x, y, y);$$

(4.4) $$D\left(1 + \frac{xy^2}{1-y} - 2xy^2S\right) = \frac{xy}{1-y}D_1 + \frac{xyz}{z-y}(zS(x, z) - yS).$$

Let us introduce A ($=A(x, y, S)$) to be $2xy^2(y-1)S + xy^2 - y + 1$, and B ($=B(x, y)$) to be $4x^2y^3(y-1)S_1 - 4xy^2(y-1)^2 + (xy^2 - y + 1)^2$. When the last four equations are multiplied by $(1 - y)$, and completion of the square is used on (4.1), they may be written as

(4.5) $$A^2 = B;$$

(4.6) $$AP = xyP_1 - (y-1)xy^2(S + yS_y);$$

(4.7) $$AT = xyT_1 - (y-1)xy^3D_z(x, y, y);$$

(4.8) $$AD = xyD_1 - \frac{(y-1)xyz}{(z-y)}(zS(x, z) - yS).$$

Let f ($=f(x)$) be the power series in x which gives the identity:

$$A(x, f, S(x, f)) \equiv 0.$$

We then have:

(4.9) $$S_1 = \frac{4xf^2(f-1)^2 - (xf^2 - f + 1)^2}{4x^2f^3(f-1)};$$

(4.10) $$P_1 = \frac{(f-1)xf^2(S(x, f) + fS_y(x, f))}{xf};$$

(4.11) $$T_1 = \frac{(f-1)xf^3D_z(x, f, f)}{xf};$$

(4.12) $$D_1 = \frac{(f-1)xfz(zS(x, z) - fS(x, f))}{(x-f)xf}.$$

Thus, we must find $f(x)$, $S(x, f(x))$, $S_y(x, f(x))$, and $D_z(x, f(x), f(x))$. From $A(x, f, S(x, f)) = 0$ we obtain

(4.13) $$S(x, f) = \frac{f - 1 - xf^2}{2xf^2(f-1)}.$$

There are two ways to obtain $f(x)$. One is from the previously known parameterization of S_1. Another is Brown's method [5]: from $A^2 = B$, we have

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$B = B_y = 0$ at $y = f$. This leads to a quadratic equation in x, the proper solution of which may be found by examining x near zero. Thus,

$$x = (f - 1)(3 - 2f)/f^2, \tag{4.14}$$

which gives, letting $R = \sqrt{1 - 12x}$,

$$f = \frac{6}{5 + R}. \tag{4.15}$$

From (4.9) and (4.14) we see

$$S_1 = \frac{(4 - 3f)f}{(3 - 2f)^2},$$

a previously known result, [7, p. 254].

Let $A^{(k)}$ denote the kth partial with respect to y of $A(x, y, S(x, y))$ at f and likewise for other functions. Since $A(x, f, S(x, f)) = 0$, by (4.8) and (4.12)

$$D(x, f, z) = \lim_{y \to f} \frac{xyz(C(x, y, z) - C(x, f, z))}{A(x, y, S(x, y)) - A(x, f, S(x, f))} = \frac{xfzC^{(1)}}{A^{(1)}},$$

where

$$C = \frac{1 - y}{z - y}(zS(x, z) - yS(x, y)).$$

By differentiating,

$$\frac{\partial C}{\partial y} = \left(\frac{-1}{z - y} + \frac{1 - y}{(z - y)^2}\right)(zS(x, z) - yS(x, y)) - \frac{1 - y}{z - y}\frac{\partial}{\partial y}(yS(x, y)).$$

To get $D_z(x, f, f)$, we can set $z = f + \epsilon$, and extract the coefficient of ϵ^1; thus, using $[\epsilon^m]$ to denote "extract the coefficient of ϵ^m,"

$$\begin{aligned} \frac{A^{(1)}}{xf}D_z(x, f, f) &= [\epsilon^1](f + \epsilon)\left\{\left(\frac{-1}{\epsilon} + \frac{1 - f}{\epsilon^2}\right)\right. \\ &\quad \left. \times ((f + \epsilon)S(x, f + \epsilon) - fS(x, f)) - \frac{1 - f}{\epsilon}(yS)^{(1)}\right\} \\ &= [\epsilon^1](f + \epsilon)\left(\frac{-1}{\epsilon} + \frac{1 - f}{\epsilon^2}\right)(f + \epsilon)S(x, f + \epsilon) \\ &= [\epsilon^3](f + \epsilon)^2(1 - f - \epsilon)S(x, f + \epsilon) \\ &= \frac{1}{6}(y^2(1 - y)S)^{(3)}. \end{aligned} \tag{4.16}$$

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Let $H(x, y) = (1 - y)y^2S(x, y)$, so that, by definition of A,

$$A(x, y, S(x, y)) = xy^2 - y + 1 - 2xH(x, y).$$

By repeated differentiation with respect to y of the equation

$$A(x, y, S(x, y))^2 = B(x, y),$$

$$\text{(4.17)} \qquad (A^{(1)})^2 = B^{(2)}/2;$$

$$A^{(2)} = B^{(3)}/6A^{(1)};$$

$$A^{(3)} = \left(B^{(4)} - (B^{(3)})^2/3B^{(2)}\right)/8A^{(1)}.$$

On the other hand,

$$A^{(3)} = -2x\frac{\partial^3}{\partial y^3}H|_{y=f}.$$

By (4.11) and (4.16),

$$\text{(4.18)} \qquad T_1 = f^2(f - 1)D_z(x, f, f)$$

$$= \frac{f^2(f-1)xf}{A^1} \cdot \frac{1}{6} \cdot \frac{A^{(3)}}{-2x}$$

$$= \frac{f^3(1-f)(B^{(4)} - (B^{(3)})^2/3B^{(2)})}{48B^{(2)}}.$$

By (4.10),

$$P_1 = (f - 1) \cdot f \cdot (yS)^{(1)}$$

$$= \frac{A^{(1)}}{2x} - f + \frac{1}{2x} + (1 - 2f)fS(x, f),$$

where we have used $A^{(1)}$ to eliminate $S^{(1)}$.

Then by (4.13), (4.14) and (4.15)

$$\text{(4.19)} \qquad P_1 = \frac{\dfrac{1 + 2R}{3} + A^{(1)}}{2x}.$$

The formulas for T_1 and P_1 will be put entirely in terms of R. By differentiating

$$B = 4x^2y^3(y - 1)S_1 - 4xy^2(y - 1)^2 + (xy^2 - y + 1)^2$$

with respect to y, evaluating at $y = f$, and using (4.14) and (4.15), we find

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$$B^{(2)} = \frac{2R(R+2)}{3},$$

$$B^{(3)} = \frac{5(R-1)(R+2)(R+5)}{9},$$

$$B^{(4)} = \frac{(R-1)(R+5)^2(3R+5)}{18}.$$

From (4.9),

$$S_1 = \frac{4(2R+1)}{3(R+1)^2}; \tag{4.20}$$

from (4.17) and (4.19), with consideration to square root sign,

$$P_1 = \frac{\frac{2R+1}{3} - \sqrt{R(R+2)/3}}{2x}; \tag{4.21}$$

from (4.18),

$$T_1 = \frac{(R-1)^2}{12R^2(R+2)}. \tag{4.22}$$

5. **Asymptotics, and a conjecture.** Let now s_n, p_n, t_n be the coefficients of S_1, P_1, and T_1. For s_n, one may obtain asymptotics from Tutte's [9; Eq. (5.1)] explicit

$$s_n = \frac{2(2n)!3^n}{n!(n+2)!}, \tag{5.1}$$

or from the generating function:

$$S_1 = \frac{4}{3}(1 + 2R)(1 - 2R + 3R^2 - 4R^3 + \ldots)$$

$$= \frac{4}{3}(1 - R^2 + 2R^3 - \ldots),$$

in which the dominant part is $(8/3)R^3 = (8/3)(1 - 12x)^{3/2}$. Thus we obtain the known formula

$$s_n \sim \frac{8 \cdot 12^n}{3 \cdot n^{5/2}\Gamma(-3/2)} \tag{5.2}$$

$$= \frac{2 \cdot 12^n}{\sqrt{\pi} n^{5/2}}.$$

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For P_1 the asymptotics is determined by the singularity $R^{1/2}$ at $x = 1/12$; thus, [1, Theorem 4],

$$(5.3) \qquad p_n \sim \frac{1}{2\left(\frac{1}{12}\right)}\left(-\sqrt{\frac{2}{3}}\right)\frac{12^n}{n^{5/4}\Gamma\left(-\frac{1}{4}\right)} = \frac{\sqrt{\frac{3}{2}}\,12^n}{n^{5/4}\Gamma(3/4)}.$$

For T_1 the asymptotics is determined by the singularity R^{-2} at $x = 1/12$; thus, [1, Theorem 4],

$$t_n \sim \frac{12^n}{24}.$$

It has been conjectured [2, sec. 7] that asymptotic enumeration of a class of maps on a surface leads to a formula of the form

$$C \cdot n^{\alpha} r^{-n},$$

in which α depends only on the Euler characteristic of the surface, r depends only on the class of maps, and C depends on both. We are only now beginning to obtain non-planar results related to this conjecture.

TABLE 1

Numbers of rooted maps on the projective plane

p_n	n
1	1
10	2
98	3
982	4
10062	5
105024	6
1112757	7
11934910	8
129307100	9
1412855500	10
15548498902	11
172168201088	12
1916619748084	13
21436209373224	14
240741065193282	15
2713584138389838	16
30687358107371442	17
348061628432108352	18
3958320704050442886	19
45125127833890889940	20

TABLE 2

Numbers of rooted maps on the torus

t_n	n
1	2
20	3
307	4
4280	5
56914	6
736568	7
9370183	8
117822512	9
1469283166	10
18210135416	11
224636864830	12
2760899996816	13
33833099832484	14
413610917006000	15
5046403030066927	16
61468359153954656	17
747672504476150374	18
9083423595292949240	19
110239596847544663002	20

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TABLE 3

Polynomials $K_{1/2,b}(y)$ for enumerating maps on the projective plane by vertices and edges

b	y^0	y^1	y^2	y^3	y^4	y^5
1	1					
2	5	9				
3	22	112	118			
4	93	899	2346	1773		
5	386	5940	27446	48426	28650	
6	1586	35138	247752	745180	995290	484578

Above we see that for all rooted (connected) maps one has $r = 1/12$ and $\alpha = (5/2)(g - 1)$ for the sphere, the projective plane, and the torus, where g denotes "type," defined by $g = 1 - (1/2)\chi$, with χ the Euler characteristic. (For orientable surfaces this agrees with the usual genus.) In a later paper [3] we shall extend this to arbitrary surfaces, and also show [4] that for the class of tree-rooted maps on arbitrary surfaces one has $\alpha = 3(g - 1)$. We also find the pattern confirmed for a fixed surface (projective plane) and several classes: all rooted (above), 2-connected [13] and non-singular [13].

6. **Computations.** Table 1, showing $p_n = \sum_j p_{nj}$, and Table 2, showing $t_n = \sum_j t_{nj}$, were calculated from equations (4.21) and (4.22) respectively. In fact, using (4.21, 4.22), the following two recurrences were found:

$$\begin{aligned}(6.1)\qquad & (n+1)n(n-1)p_n - 4n(n-1)(8n-13)p_{n-1} \\ & + 144(n-1)(n-2)(2n-5)p_{n-2} \\ & + 216(2n-5)(4n-9)p_{n-3} - 1728(2n-5) \\ & \times (2n-7)(n-3)p_{n-4} = 0,\ n \geqq 4;\end{aligned}$$

$$\begin{aligned}(6.2)\qquad & nt_n - 22(n-1)t_{n-1} + 4(22n-65)t_{n-2} \\ & + 96(5n-4)t_{n-3} - 576(2n-7)t_{n-4} = 0,\ n \geqq 4.\end{aligned}$$

These are not so short as the recursion

$$(n+2)s_n = 6 \cdot (2n-1)s_{n-1},\ n \geqq 1,$$

satisfied by $s_n = \sum_j s_{nj}$, which yields the simple product formula (5.1).

Walsh and Lehman [11, p. 123] gave the formula

$$(6.3)\quad A_{g,b}(x) = \frac{1-\sqrt{1-4x}}{2x}(1-4x)^{-b}K_{g,b}\left(\frac{(1-4x)^{-(1/2)}-1}{2}\right)$$

for the ordinary generating function $\sum_p a_{gbp}x^p$ in which a_{gbp} enumerates rooted maps on an orientable surface of genus g, having $(p+1)$ vertices and $(b+p)$

edges. They showed that $K_{g,b}(y)$ is a polynomial of degree $(b - 1)$, and gave an algorithm for calculating $K_{g,b}(y)$ as well as a table for $g \leqq 3$, $b \leqq 6$. Using this table, the values of t_n for $n \leqq 6$ were recalculated and found to agree with those given in our Table 2. Our calculation of $K_{1,6}(y)$ gave 326,496 as the coefficient of y^5, indicating a digit transposition in [11; Table I; p. 124]. To compare the efficiency of the "K_{gb} method" to recursion (6.2), we mention that the calculation of $K_{1,6}(y)$ involves a summation over 2,310 matrices.

The method of [11] applies, we have found, to non-orientable surfaces also, again leading to an equation of the form (6.3). We may elaborate on this matter in a later paper, but for now we conclude with Table 3 showing $K_{1/2,b}(y)$ for $1 \leqq b \leqq 6$. The determination of $K_{1/2,6}(y)$ involves a summation over 1586 matrices, not all distinct.

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