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Abstract

We show that the Craighero–Gattazzo surface, the minimal resolution of an explicit complex quintic surface with four elliptic singularities, is simply connected. This was conjectured by Dolgachev and Werner, who proved that its fundamental group has a trivial profinite completion. The Craighero–Gattazzo surface is the only explicit example of a smooth simply connected complex surface of geometric genus zero with ample canonical class. We hope that our method will find other applications: to prove a topological fact about a complex surface we use an algebraic reduction mod p technique and deformation theory.

1. Introduction

Simply connected minimal complex surfaces of general type of geometric genus zero, i.e. without global holomorphic 2-forms, occupy a special place in the geography of surfaces; see the excellent survey [BCP11]. These surfaces are homeomorphic (but not diffeomorphic) to del Pezzo surfaces, i.e. blowups of $\mathbb{P}^2$ in $9 - K^2$ points where $1 \leqslant K^2 \leqslant 8$. Describing their Gieseker moduli space of canonically polarized surfaces, or even finding explicit examples, is difficult. The first example was found by Barlow [Bar85]. Her surface has $K^2 = 1$ and contains four (-2)-curves. Contracting them gives a canonically polarized surface with four A_1 singularities. One can show by deformation theory that the local Gieseker moduli space of the Barlow surface is smooth and eight-dimensional, and there exist nearby surfaces which are smooth (see [LC97, Theorem 7] and [Lee02]).

More examples, including examples for every $1 \leqslant K^2 \leqslant 4$, were found using $\mathbb{Q}$-Gorenstein deformation theory, starting with the pioneering work of Lee and Park [LP07]; see also [PPS09a, PPS09b, SU16]. From the moduli space perspective, the Gieseker moduli space of canonically polarized surfaces with ADE singularities is compactified by the Kollár–Shepherd-Barron–Alexeev (KSBA) moduli space of canonically polarized surfaces with semi log canonical singularities [KS88]. We call the complement of the Gieseker space the *KSBA boundary*. Lee, Park, and others explicitly constructed special points on the KSBA boundary, and proved (using deformation theory) that the local KSBA moduli space is smooth at these points, and that one can find nearby surfaces which are smooth. To compute the fundamental group of the smoothing, one has to look into what happens when the singularity is replaced with the Milnor fiber. In the presence of special curves on the singular surface, one can use Van Kampen's theorem to compute the fundamental group of the smoothing; see the proof of Theorem 6.2.

Another remarkable surface was found by Craighero and Gattazzo [CG94]. Their surface S is the minimal resolution of singularities of an explicit quintic surface (2.1) with four elliptic

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singularities. This surface has $K_S^2 = 1$. It was proved by Dolgachev and Werner [DW99] that S is canonically polarized and that its algebraic fundamental group (i.e. the profinite completion of the fundamental group) is trivial. In addition, it was proved by Catanese and Pignatelli [CP00, Theorem 0.31] that the local moduli space of S is smooth of dimension eight. It was originally claimed in [DW99] that S is simply connected, but a serious flaw was discovered in the proof; see [DW99, Erratum].

The goal of this paper is to prove that S is simply connected using an algebraic reduction mod p technique and deformation theory. We would like to use the Lee–Park argument involving the Milnor fiber of a $\mathbb{Q}$-Gorenstein deformation and Van Kampen's theorem. In order to do that, we need a $\mathbb{Q}$-Gorenstein family of complex surfaces $\mathcal{S} \to U$ over a smooth irreducible complex curve U, such that one of the fibers is the Craighero–Gattazzo surface S and another fiber is a simply connected surface with a cyclic quotient singularity and containing a special curve configuration needed to prove simply connectedness. However, it is not clear how to explicitly construct a family containing the Craighero–Gattazzo as a fiber because no explicit model of the moduli space is known.

Our trick is to work out an integral model of the Craighero–Gattazzo surface over a ring of algebraic integers. One obvious model is given by the quintic equation. In an REU (research experience for undergraduates) directed by the first two authors, Charles Boyd discovered that this arithmetic threefold has a non-reduced fiber in characteristic seven, and its local equation has a very special form. Over the complex disc, analogous families of quintic surfaces were studied by the first author in [Ran14], where it was proved that the KSBA replacement acquires a $\frac{1}{4}(1,1)$ singularity in the special fiber. In fact, it is proved in [Ran14] that numerical quintic surfaces with a $\frac{1}{4}(1,1)$ singularity form a divisor in the KSBA moduli space (and this divisor is explicitly described). The upshot is that, to some degree, it can be hoped that this singularity appears in one-parameter families of surfaces, including families over a ring of algebraic integers. We show that the KSBA limit of S over the 7-adic disc is a surface S_0 with a $\frac{1}{4}(1,1)$ singularity. We use the word 'KSBA limit' somewhat loosely here because existence of the mixed characteristic KSBA moduli space (or even canonical KSBA integral models) is still only conjectural.

The minimal resolution of S_0 turns out to be a very special and beautiful Dolgachev surface, i.e. an elliptic fibration over $\mathbb{P}^1$ with two multiple fibers, one of multiplicity two and one of multiplicity three. We call it the Boyd surface. By pure luck, it carries a special curve, which, if it were a complex surface, would have allowed us to conclude that the Craighero–Gattazzo surface S is simply connected. Of course our degeneration is over the 7-adic unit disc, so we can not use Van Kampen's theorem directly. Our main idea is to use deformation theory to conclude that S admits an analogous (but no longer explicit) degeneration over the complex unit disc to a complex surface D_0 with a $\frac{1}{4}(1,1)$ singularity such that its minimal resolution is a complex Dolgachev surface analogous to the Boyd surface.

As an application of our construction, we show in Theorem 7.2 that there exist simply connected Dolgachev surfaces (with multiple fibers of multiplicity $2,3$) which carry algebraic genus two Lefschetz fibrations, specifically genus two fibrations without multiple components in fibers and such that the only singularities of fibers are nodes. Dolgachev and Werner showed existence of a genus two fibration on the Craighero–Gattazzo surface [DW99, Proposition 3.2]. If this fibration had only nodal singular fibers, then by combining our theorem that the Craighero–Gattazzo surface is simply connected, we would have the existence of a simply connected numerical Godeaux surface with a genus two Lefschetz fibration. By [Fre82], these surfaces are homeomorphic to $\mathbb{P}^2$ blown-up in nine or eight points, respectively. In the symplectic category, Lefschetz fibrations on knot surgered elliptic surfaces in the homotopy class of $\mathbb{P}^2$ blown-up at

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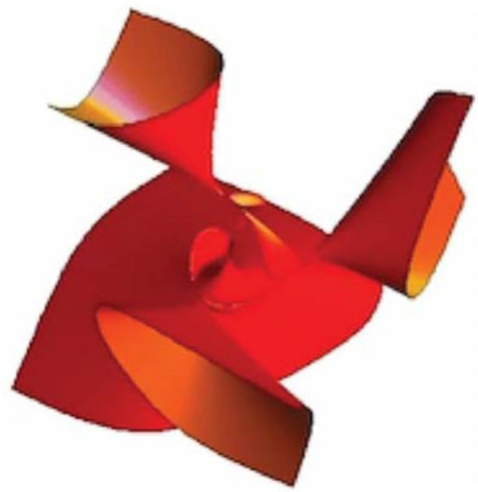

Figure 1. The Craighero–Gattazzo quintic.

nine points were constructed in [FS04] and in the homotopy classes of $\mathbb{P}^2$ blown-up at eight or seven points in [BK15].

2. Stable limit of the Craighero–Gattazzo surface in characteristic seven

Let $X \subset \mathbb{P}^3_{\mathbb{C}}$ be the quintic surface (see Figure 1)

$$\begin{aligned} &a^2(x^2y^3+x^3t^2+y^2z^3+z^2t^3)+m^2(x^3z^2+x^2z^3+y^3t^2+y^2t^3)\\ &\quad +2am(xyz^3+xy^3t+x^3zt+yzt^3)+14m(x^3yz+y^3zt+xz^3t+xyt^3)\\ &\quad +7b(x^2y^2z+y^2z^2t+x^2yt^2+xz^2t^2)+14a(xy^3z+x^3yt+yz^3t+xzt^3)\\ &\quad +c(x^2yz^2+x^2z^2t+xy^2t^2+y^2zt^2)+7e(xy^2z^2+x^2y^2t+x^2zt^2+yz^2t^2)\\ &\quad +f(x^2yzt+xy^2zt+xyz^2t+xyzt^2)+49(x^3y^2+y^3z^2+z^3t^2+x^2t^3)=0. \end{aligned} \tag{2.1}$$

The coefficients are (from [CP00, p. 25], multiplied by 49)

$$a=7r^2,\quad b=-2r^2+13r+18,\quad c=73r^2+75r+92,$$
$$e=-r^2+24r+9,\quad f=181r^2+241r+163,\quad m=3r^2+5r+1,$$

where r is a complex root of the equation

$$r^3+r^2-1=0. \tag{2.2}$$

The surface is invariant under the μ_4 action which cyclically permutes the variables as follows: $x\to y\to z\to t\to x$. It is singular at the points

$$P_1=[1:0:0:0],\quad P_2=[0:1:0:0],\quad P_3=[0:0:1:0],\quad P_4=[0:0:0:1].$$

Its minimal resolution is the *Craighero–Gattazzo surface* S. Exceptional divisors over $P_1,\dots,P_4$ are elliptic curves $\mathcal{E}_1,\dots,\mathcal{E}_4$ such that $\mathcal{E}_i^2=-1$ for each i. These singularities are sometimes called *singularities of type* $\tilde{E}_8$.

Equation (2.1) gives an integral model of X over $\operatorname{Spec}\mathbb{Z}[r]$. Since 3 is a simple root of (2.2) in $\mathbb{Z}/(7)$, by Hensel's lemma we have a section $\operatorname{Spec}\mathbb{Z}_7\to\operatorname{Spec}\mathbb{Z}_7[r]$, where $\mathbb{Z}_7$ is the ring of 7-adic integers. Pulling back the integral model with respect to the base change $\operatorname{Spec}\mathbb{Z}_7 \to \operatorname{Spec}\mathbb{Z}_7[r] \to \operatorname{Spec}\mathbb{Z}[r]$ gives the family $\mathcal{X}$ over $\operatorname{Spec}\mathbb{Z}_7$. The corresponding root of (2.2) modulo 7^3 is equal to 143 and after some manipulations the equation of $\mathcal{X}$ to the order of 7^3 takes the form

$$f_1 f_2^2 + 7 f_2 f_3 + 7^2 f_5 + (\text{higher order terms}), \tag{2.3}$$

where $f_1, f_2, f_3, f_5 \in \mathbb{Z}/(7)[x,y,z,t]$ are the following forms (the subscript indicates the degree):

$$\begin{aligned} f_1 &= x+y+z+t,\\ f_2 &= xz+yt,\\ f_3 &= 2(x^2y+y^2z+z^2t+xt^2)+x^2z+xz^2+y^2t+yt^2\\ &\quad -3(xy^2+yz^2+x^2t+zt^2+xyz+xyt+xzt+yzt), \end{aligned}$$

and

$$\begin{aligned} f_5 = {} & x^3y^2+x^3z^2+y^3z^2+x^2z^3+y^3t^2+z^3t^2+x^2t^3+y^2t^3\\ &+x^3yz+y^3zt+xz^3t+xyt^3-xy^2z^2-x^2y^2t-x^2zt^2-yz^2t^2\\ &-x^2yzt-xy^2zt-xyz^2t-xyzt^2- -3x^2y^3-3y^2z^3-3x^3t^2-3z^2t^3\\ &-2x^2y^2z-2x^2yz^2-2x^2z^2t-2y^2z^2t-2x^2yt^2-2xy^2t^2-2y^2zt^2-2xz^2t^2\\ &-3xy^3z-3x^3yt-3yz^3t-3xzt^3. \end{aligned}$$

This expansion shows that the special fiber of $\mathcal{X}$ is the union of the plane $L = (f_1 = 0)$ and the quadric surface $Q = (f_2 = 0)$ with multiplicity two. In particular, it is not reduced.

Let k be an algebraically closed field of characteristic 7 and let $\mathcal{R}$ be its ring of Witt vectors. We denote the pull-back of $\mathcal{X}$ to $\operatorname{Spec}\mathcal{R}$ (with respect to the canonical inclusion $\mathbb{Z}_7 \hookrightarrow \mathcal{R}$) by the same letter $\mathcal{X}$. We also pullback L and Q to k.

We would like to compute the stable limit of the generic fiber of $\mathcal{X}$. Over the complex disc, stable $\mathbb{Q}$-Gorenstein limits of families of the form (2.3) were computed by the first author [Ran14], and semi-stable Gorenstein limits of sufficiently general families by Ashikaga and Konno [AK91]. In our case the disc is 7-adic but the computation is the same. We now describe what the stable limit is, postponing the proof to Lemma 2.4.

Let $\Delta = L \cap Q \subset Q \simeq \mathbb{P}^1_k \times \mathbb{P}^1_k$. It is a curve in the linear system $|\mathcal{O}(1,1)|$. The curve

$$Q \cap (f_3^2 - 4 f_1 f_5 = 0) \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$$

is the union of two curves in the linear system $|\mathcal{O}(3,3)|$:

$$B_1 = Q \cap (xy^2 + 3x^2z - 3y^2z + 3xz^2 - 3xt^2 + zt^2 = 0) \tag{2.4}$$

and

$$B_2 = Q \cap (yz^2 + 3y^2t - 3z^2t + 3yt^2 - 3yx^2 + tx^2 = 0). \tag{2.5}$$

Figure 2 shows how these curves intersect, where $A_1, \ldots, A_4$ are rulings of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ and $\{Q_1, Q_2\} = \Delta \cap B_1 \cap B_2$.

Lemma 2.1. *Let*

$$\pi : Z \to \mathbb{P}^1_k \times \mathbb{P}^1_k$$

be the double cover branched along $B_1 \cup B_2$. The surface Z has four simple elliptic singularities of type $\tilde{E}_8$ over $P_1, \ldots, P_4$, and two A_1 singularities over Q_1 and Q_2. It is smooth elsewhere.

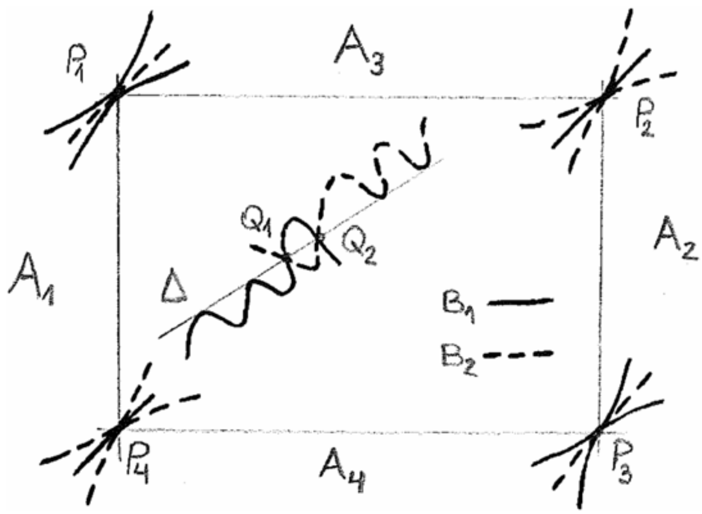

FIGURE 2. Data in $Q \simeq \mathbb{P}^1_k \times \mathbb{P}^1_k$.

Proof. Direct calculation. □

We denote the ramification curves in Z by B_1 and B_2, and we denote the singular points of Z by the same letters as their images in $\mathbb{P}^1 \times \mathbb{P}^1$. Finally, $\pi^{-1}(\Delta)$ is the union of two smooth rational curves: Δ_1 and Δ_2.

Unless it causes confusion, we adopt the following convention throughout this paper: we use the same letter to denote an irreducible curve and its proper transform after some birational transformation.

DEFINITION 2.2. We call the minimal resolution Y of Z *the Boyd surface.*

The Boyd surface contains elliptic curves $E_1, \ldots, E_4$ of self-intersection -1 (preimages of elliptic singularities of Z), (-2)-curves N_1 and N_2 (preimages of A_1 singularities of Z), and (-4)-curves Δ_1 and Δ_2.

DEFINITION 2.3. Let S_0 be the surface obtained by contracting the (-4)-curve Δ_1.

LEMMA 2.4 (Cf. [Ran14]). *There exists a flat family $S \to \operatorname{Spec} \mathcal{R}$ with special fiber S_0 and generic fiber the Craighero–Gattazzo surface S (after pull-back to $\mathbb{C}$). Near the singular point of the special fiber, the family is formally isomorphic to*

$$(xy = z^2 + 7) \subset \tfrac{1}{2}(1,1,1)_{\mathcal{R}} := \operatorname{Spec} \mathcal{R}[x,y,z]^{\mu_2},$$

where μ_2 acts by $x \mapsto -x$, $y \mapsto -y$, $z \mapsto -z$.

Proof. We first produce the stable limit of the Craighero–Gattazzo quintic X in characteristic seven. Let $\mathcal{X}^0$ be the generic fiber of $\mathcal{X}$ given by (2.3). Consider the family $\hat{\mathcal{X}} \to \operatorname{Spec} \mathcal{R}$ given by equations

$$(f_1 w^2 + f_3 w + f_5 + \text{h.o.t.} = 0,\ f_2 = 7w) \subset \mathbb{P}^4_{[x:y:z:t:w]}(1,1,1,1,2)_{\mathcal{R}}$$

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obtained by substituting f_2 for $7w$ in the first three terms of (2.3) and dividing by 343. Here, and throughout, 'h.o.t.' refers to higher-order terms with respect to the 7-adic valuation. The generic fiber of $\hat{\mathcal{X}}$ is clearly isomorphic to $\mathcal{X}^0$.

The special fiber $\hat{\mathcal{X}}_0$ is given by

$$(f_1w^2 + f_3w + f_5 = 0,\ f_2 = 0) \subset \mathbb{P}^4_{[x:y:z:t:w]}(1,1,1,1,2)_k.$$

We claim that it is isomorphic to the surface Z' obtained by blowing down four elliptic (-1)-curves on S_0 to $\tilde{E}_8$-singularities.

The point $(0:0:0:0:1)$ is an isolated singularity with equation, in a local chart,

$$(f_1 + f_3 + f_5 = 0,\ f_2 = 0) \subset \tfrac{1}{2}(1,1,1,1).$$

The singularity is formally isomorphic to

$$(xy = z^2) \subset \tfrac{1}{2}(1,1,1)_k,$$

which has a (-4)-curve as the resolution graph. Moreover, the equation of the whole family $\hat{\mathcal{X}}$ near this point is formally isomorphic to

$$(xy = z^2 + 7) \subset \tfrac{1}{2}(1,1,1)_{\mathcal{R}}.$$

Next we analyze $\hat{\mathcal{X}}_0$ away from $t_0 = (0:0:0:0:1)$. We use the generically two-to-one map $\pi : S_0\backslash\{t_0\} \to Q$ given by $[x:y:z:t:w] \to [x:y:z:t]$. Away from $\Delta = L \cap Q$, π is a double cover branched along $(f_3^2 - 4f_1f_5 = 0) = B_1 \cup B_2$. Thus, it can be identified with $Z'\backslash(\Delta_2 \cup N_1 \cup N_2)$. Over Δ, but away from t_0 (which includes Q_1 and Q_2) the map π is one-to-one. The preimages of Q_1 and Q_2 are lines (with coordinate w). The preimages of the other four points where $f_3 = 0$ are empty; in Figure 2 these are the points where B_1 and B_2 are tangent to Δ. It follows that $\hat{\mathcal{X}}_0$ and Z' are normal surfaces isomorphic in codimension one, and therefore isomorphic.

It remains to note that the family $\hat{\mathcal{X}}$ has $\tilde{E}_8$ singularities along the sections $(1:0:0:0:0)$, $(0:1:0:0:0)$, $(0:0:1:0:0)$, and $(0:0:0:1:0)$. Resolving them gives a family $\mathcal{S} \to \operatorname{Spec}\mathcal{R}$ with special fiber S_0 and generic fiber (after pulling back to $\operatorname{Spec}\mathbb{C}$) the Craighero–Gattazzo surface S. □

3. Study of the Boyd surface: vanishing of obstructions

We have a commutative diagram,

$$\begin{tikzcd} W \arrow[r, "\tau"] \arrow[d, "\pi'"'] & Y \arrow[r] & Z \arrow[d, "\pi"] \\ \mathbb{P} \arrow[rr, "\sigma"] & & Q = \mathbb{P}^1_k \times \mathbb{P}^1_k \end{tikzcd}$$

where the vertical maps are double covers and the horizontal maps are birational. Here $\mathbb{P}$ is obtained by blowing up Q_1 and Q_2 (let $\bar{N}_1$ and $\bar{N}_2$ be the exceptional divisors), blowing up $P_1, \ldots, P_4$ (let $\bar{G}_1, \ldots, \bar{G}_4$ be the exceptional divisors), and then blowing up these four points again in the direction of the tangent cone to $B_1 \cup B_2$ (let $\bar{E}_1, \ldots, \bar{E}_4$ be the exceptional divisors).

Since

$$B_1 + B_2 + 2\bar{N}_1 + 2\bar{N}_2 \sim 6\sigma^*(\mathcal{O}_Q(1,1)) - 3\sum_{i=1}^{4}\bar{G}_i - 6\sum_{i=1}^{4}\bar{E}_i,$$

we have

$$B_1 + B_2 + 2\bar{N}_1 + 2\bar{N}_2 \sim 3\biggl(\sum_{i=1}^{4} A_i + \sum_{i=1}^{4} \bar{G}_i\biggr) \tag{3.1}$$

as well as

$$B_1 + B_2 + \sum_{i=1}^{4} \bar{G}_i \sim 2\biggl(3\sigma^*(\mathcal{O}_Q(1,1)) - \bar{N}_1 - \bar{N}_2 - 3\sum_{i=1}^{4} \bar{E}_i - \sum_{i=1}^{4} \bar{G}_i\biggr). \tag{3.2}$$

We define W to be the double cover of $\mathbb{P}$ branched along the smooth curve

$$B = B_1 + B_2 + \bar{G}_1 + \cdots + \bar{G}_4.$$

Let $N_i, E_i, G_i \subset W$ be the preimages of $\bar{N}_i, \bar{E}_i, \bar{G}_i$, respectively. The curves $G_1, \ldots, G_4$ are (-1)-curves, and contracting them gives the Boyd surface Y. The curves N_1 and N_2 are (-2)-curves on Y, while $E_1, \ldots, E_4$ are elliptic (-1)-curves (i.e. elliptic curves with self-intersection -1).

Theorem 3.1. *We have* $H^2(Y, T_Y(-\log(\Delta_1 + N_1))) = 0$.

Proof. We follow [Ran14, 4.8, 4.10] closely. It suffices to show that

$$H^2(W, T_W(-\log(\Delta_1 + N_1))) = 0. \tag{3.3}$$

Indeed, if this is the case, then Serre duality implies

$$\begin{aligned} 0 &= H^0(W, \Omega^1_W(\log(\Delta_1 + N_1))(K_W)) \\ &= H^0(Y, \tau_*[\Omega^1_W(\log(\Delta_1 + N_1))(G_1 + \cdots + G_4)](K_Y)) \end{aligned}$$

(by Lemma 3.3)

$$= H^0(Y, \Omega^1_Y(\log(\Delta_1 + N_1))(K_Y)) = H^2(Y, T_Y(-\log(\Delta_1 + N_1)))^\vee.$$

Arguing as in [Ran14, 4.8], (3.3) will follow if we can show that

$$H^2(W, T_W(-\log(\Delta_1 + \Delta_2 + N_1)))_- = 0, \tag{3.4}$$

and

$$H^2(W, T_W(-\log(N_1)))_+ = 0, \tag{3.5}$$

where $+/-$ denotes the symmetric/skew-symmetric part with respect to the μ_2-action on the double cover. Explicitly, and using Serre duality multiple times, if $\alpha \in H^0(W, \Omega^1_W(\log(\Delta_1 + N_1))(K))$, then since

$$\Omega^1_W(\log(\Delta_1 + N_1))(K) \subset \Omega^1_W(\log(\Delta_1 + \Delta_2 + N_1))(K)$$

the one-form α must be invariant. But μ_2 interchanges Δ_1 and Δ_2, so that α does not have a pole along Δ_1. Thus, $\alpha \in \Omega^1_W(\log N_1)(K)$ is an invariant one-form. Equation (3.5) implies that $\alpha = 0$.

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Proof of (3.4). At each of the points $Q_3, \ldots, Q_6$ (the remaining points of $B_i \cap \Delta$) we blowup twice to obtain a surface $\mathbb{P}_1$ where Δ and B_i have normal crossings. Let $\bar{C}_i$, $\bar{F}_i$, $i = 3, \ldots, 6$ be the exceptional divisors of these blowups, so that on $\mathbb{P}_1$ we have $\bar{C}_i^2 = -2$ and $\bar{F}_i^2 = -1$. Let $\sigma' : \mathbb{P}_1 \to Q$ be the composition of these blowups, and let $f : W_1 \to \mathbb{P}_1$ be the double cover branched over $B_1 + B_2 + \sum \bar{G}_i + \sum \bar{C}_i$.

The surface W_1 contains (-1)-curves C_i and (-2)-curves F_i which contract to give the surface W. By the (-1)- and (-2)-curve principles [PSU13, Proposition 4.3, Theorem 4.4] (here we only need the (-1)-curve principle), we have

$$H^2(W_1, T_{W_1}(-\log(\Delta_1 + \Delta_2 + N_1))) \simeq H^2(W, T_W(-\log(\Delta_1 + \Delta_2 + N_1))).$$

Note that the double cover f is defined by (see (3.2))

$$B_1 + B_2 + \sum \bar{G}_i + \sum \bar{C}_i \sim 2L,$$

where

$$L \sim 3\sigma'^*(\mathcal{O}_Q(1,1)) - \sum \bar{G}_i - 3\sum \bar{E}_i - \sum \bar{N}_i - \sum \bar{F}_i.$$

Also we have

$$K_{\mathbb{P}_1} = -2\sigma'^*(\mathcal{O}_Q(1,1)) + \sum \bar{N}_i + \sum \bar{G}_i + 2\sum \bar{E}_i + \sum \bar{C}_i + 2\sum \bar{F}_i,$$

and so

$$K_{\mathbb{P}_1} + L \sim \sigma'^*(\mathcal{O}_Q(1,1)) - \sum \bar{E}_i + \sum \bar{C}_i + \sum \bar{F}_i.$$

By Lemma 3.2, we have

$$f_*(T_{W_1}(-\log(\Delta_1 + \Delta_2 + N_1)))_- = T_{\mathbb{P}_1}(-\log(\Delta + \bar{N}_1))(-L).$$

By Serre duality, it suffices to prove vanishing of

$$H^0(\mathbb{P}_1, \Omega^1_{\mathbb{P}_1}(\log(\Delta + \bar{N}_1))(K_{\mathbb{P}_1} + L)),$$

or

$$H^0\Big(\mathbb{P}_1, \Omega^1_{\mathbb{P}_1}(\log(\Delta + \bar{N}_1))\Big(\sigma'^*(\mathcal{O}_Q(1,1)) - \sum \bar{E}_i + \sum \bar{C}_i + \sum \bar{F}_i\Big)\Big).$$

By Lemma 3.3, we have

$$\begin{aligned}
&\sigma'_*\Big(\Omega^1_{\mathbb{P}_1}(\log(\Delta + \bar{N}_1))\Big(\sigma'^*(\mathcal{O}_Q(1,1)) - \sum \bar{E}_i + \sum \bar{C}_i + \sum \bar{F}_i\Big)\Big) \\
&\quad \subset \sigma'_*\Big(\Omega^1_{\mathbb{P}_1}\Big(\Delta + \bar{N}_1 + \sigma'^*(\mathcal{O}_Q(1,1)) - \sum \bar{E}_i + \sum \bar{C}_i + \sum \bar{F}_i\Big)\Big) \\
&\quad = \sigma'_*\Big(\Omega^1_{\mathbb{P}_1}\Big(\sigma'^*(\Delta) + \sigma'^*(\mathcal{O}_Q(1,1)) - \bar{N}_2 - \sum \bar{E}_i - \sum \bar{F}_i\Big)\Big) \\
&\quad \subset \Omega^1_Q \otimes \mathcal{O}_Q(2,2) \otimes \mathcal{I}_{Q_2} \bigotimes_{i=1}^{4} \mathcal{I}_{P_i}.
\end{aligned}$$

Since $\Omega^1_Q = \mathcal{O}_Q(-2,0) \oplus \mathcal{O}_Q(0,-2)$, we have

$$\Omega^1_Q \otimes \mathcal{O}_Q(2,2) = \mathcal{O}_Q(0,2) \oplus \mathcal{O}_Q(2,0).$$

Thus, any global section of $\Omega^1_Q \otimes \mathcal{O}_Q(2,2) \otimes \mathcal{I}_{Q_2} \bigotimes_{i=1}^4 \mathcal{I}_{P_i}$ is a global section of $\mathcal{O}_Q(0,2) \oplus \mathcal{O}_Q(2,0)$ vanishing at the points $Q_2, P_1, \ldots, P_4$. Since these points are in three distinct horizontal and vertical fibers of Q, any such global section must be zero. This completes the proof of (3.4).

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Lemma 3.2. *Let Y be a smooth projective surface defined over an algebraically closed field of characteristic $\neq 2$. Let $f : X \to Y$ be a double cover with a smooth branch divisor $B \subset Y$. Let $C = f^{-1}(D)$ be the preimage of a smooth curve D on Y, and suppose that D intersects B transversally. Then*

$$f_*(\Omega^1_X(\log C)) = \Omega^1_Y(\log(D)) \oplus \Omega^1_Y(\log(D+B))(-L)$$

and

$$f_*(T_X(-\log C)) = T_Y(-\log(D+B)) \oplus T_Y(-\log(D))(-L),$$

where $B \sim 2L$. Moreover, these decompositions break the sheaves into their invariant and anti-invariant subspaces under the action of μ_2 by deck transformations.

Proof. The surface X is defined in the total space of the line bundle L by the equation $z^2 = x$ where x is a global section of $\mathcal{O}_Y(2L)$. This allows us to work étale-locally, using the argument of [Ran14, 4.6]. $\square$

Lemma 3.3. *Let Y be a smooth projective surface defined over an algebraically closed field. Let $\sigma : X \to Y$ be the blowup of $p \in Y$ with exceptional divisor E. Then for every integer $m \geqslant 0$, we have $\sigma_*(\Omega^1_X(mE)) = \Omega^1_Y$. Moreover, $\sigma_*(\Omega^1_X(-E)) = \Omega^1_Y \otimes \mathcal{I}_p$, where $\mathcal{I}_p$ is the ideal sheaf of the point p.*

Proof. Let η be the generic point of Y. The sheaves $\sigma_*(\Omega^1_X(mE))$ and Ω^1_Y are subsheaves of the constant sheaf with stalk $\Omega^1_{Y,\eta}$ (the sheaf of rational differentials). A local section of $\sigma_*(\Omega^1_X(mE))$ is regular outside of p and therefore regular at p since Ω^1_Y is locally free. Thus, we have an injective map $i : \sigma_*(\Omega^1_X(mE)) \to \Omega^1_Y$. It is surjective because given a local 1-form $\alpha \in \Omega^1_Y(U)$, the 1-form $\sigma^*(\alpha) \in \Omega^1_X(\sigma^{-1}(U)) \subset \Omega^1_X(mE)(\sigma^{-1}(U))$ maps to α.

For the second part, we have an injective map $i : \sigma_*(\Omega^1_X(-E)) \to \Omega^1_Y$, as above. Moreover, any one-form $i(\alpha)$ in the image of i vanishes at p, since α vanishes along E. Thus, the image of i is the sheaf $\Omega^1_Y \otimes \mathcal{I}_p$. $\square$

Proof of (3.5). Note that we have the short exact sequence

$$0 \to T_W(-\log(\Delta_1 + \Delta_2 + N_1)) \to T_W(-\log, N_1) \to \mathcal{N}_{\Delta_1/W} \oplus \mathcal{N}_{\Delta_2/W} \to 0.$$

Since $H^2(W, \mathcal{N}_{\Delta_1/W} \oplus \mathcal{N}_{\Delta_2/W}) = 0$, it suffices to prove that

$$H^2(W, T_W(-\log(\Delta_1 + \Delta_2 + N_1)))_+ = 0.$$

This part is more delicate and the proof occupies the rest of the section.

By Lemma 3.2, we have

$$f_*(T_{W_1}(-\log(\Delta_1 + \Delta_2 + N_1)))_+ = T_{\mathbb{P}_1}\left(-\log\left(\Delta + \bar{N}_1 + B_1 + B_2 + \sum \bar{G}_i + \sum \bar{C}_i\right)\right).$$

Again applying the (-1) and (-2)-curve principles, it suffices to show that

$$H^2(\mathbb{P}_1, T_{\mathbb{P}_1}(-\log(\Delta + B_1 + B_2))) = 0.$$

To begin with, we claim that

$$H^2(\mathbb{P}_1, T_{\mathbb{P}_1}(-\log(B_1 + B_2))) = 0. \tag{3.6}$$

Because B_1 and B_2 have simple normal crossings after contracting the curves $\bar{C}_i$ and $\bar{F}_i$, it suffices to show that

$$H^2(\mathbb{P}, T_{\mathbb{P}}(-\log(B_1+B_2))) = 0$$

or equivalently (by Serre duality)

$$H^0\left(\mathbb{P}, \Omega^1_{\mathbb{P}}(\log(B_1+B_2))\left(-2D + \sum \bar{N}_i + \sum \bar{G}_i + 2\sum \bar{E}_i\right)\right) = 0.$$

Letting $\mathcal{F} = \mathcal{O}_{\mathbb{P}}(-2D + \sum \bar{N}_i + \sum \bar{G}_i + 2\sum \bar{E}_i)$, we have the short exact sequence

$$0 \to \Omega^1_{\mathbb{P}_1} \otimes \mathcal{F} \to \Omega^1_{\mathbb{P}}(\log(B_1+B_2)) \otimes \mathcal{F} \to (\mathcal{O}_{B_1} \oplus \mathcal{O}_{B_2}) \otimes \mathcal{F} \to 0.$$

The products $B_j \cdot \mathcal{F} = -4 < 0$ for $j = 1, 2$ and thus

$$H^0((\mathcal{O}_{B_1} \oplus \mathcal{O}_{B_2}) \otimes \mathcal{F}) = 0.$$

The projection formula and Lemma 3.3 give

$$H^0(\mathbb{P}_1, \Omega^1_{\mathbb{P}_1} \otimes \mathcal{F}) \simeq H^0(Q, \Omega^1_Q(-2D)).$$

The sheaf $\Omega^1_Q(-2D) = \mathcal{O}_Q(-4,-2) \oplus \mathcal{O}_Q(-2,-4)$ has no global sections, completing the proof of claim (3.6).

Now consider the short exact sequence

$$0 \to T_{\mathbb{P}_1}(-\log(\Delta + B_1 + B_2)) \to T_{\mathbb{P}_1}(-\log(B_1+B_2)) \to \mathcal{N}_{\Delta/\mathbb{P}_1} \to 0.$$

By claim (3.6), vanishing of $H^2(\mathbb{P}_1, T_{\mathbb{P}_1}(-\log(\Delta + B_1 + B_2)))$ will be complete once we show that the map

$$H^1(\mathbb{P}_1, T_{\mathbb{P}_1}(-\log(B_1+B_2))) \to H^1(\mathbb{P}_1, \mathcal{N}_{\Delta/\mathbb{P}_1}) \tag{3.7}$$

is surjective. We identify $H^1(\mathbb{P}_1, T_{\mathbb{P}_1}(-\log(B_1+B_2)))$ with the space of first-order infinitesimal deformations of $\mathbb{P}_1$ which contain an embedded first-order deformation of $B_1 \cup B_2$. We identify $H^1(\mathbb{P}_1, \mathcal{N}_{\Delta/\mathbb{P}_1})$ with the space of obstructions to deforming Δ in $\mathbb{P}_1$. Thus, the map (3.7) factors through the natural map

$$H^1(\mathbb{P}_1, T_{\mathbb{P}_1}) \to H^1(\mathbb{P}_1, \mathcal{N}_{\Delta/\mathbb{P}_1}) \tag{3.8}$$

which sends an infinitesimal first-order deformation of $\mathbb{P}_1$ to the obstruction to deforming Δ in this first-order deformation of $\mathbb{P}_1$. We have to show that given any such obstruction, there is a deformation of the pair $(\mathbb{P}_1, B_1 + B_2)$ that maps to the given obstruction.

Recall that $\mathbb{P}_1$ is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up once at each of $Q_1, \ldots, Q_6; P_1, \ldots, P_4$, and again at each of $Q_3, \ldots, Q_6$ in the direction of the proper transform of Δ and at each of $P_1, \ldots, P_4$ in the direction of tangent cone of $B_1 \cup B_2$. We denote by $\sigma_2 : \mathbb{P}_1 \to \tilde{Q}$ the 'intermediate' blowup, i.e. the map which contracts the last eight (-1)-curves on $\mathbb{P}_1$.

We have the following exact sequence of sheaves on $\tilde{Q}$

$$0 \to (\sigma_2)_* T_{\mathbb{P}_1} \to T_{\tilde{Q}} \to \bigoplus_{i=3}^{6} k^2_{Q_i} \oplus \bigoplus_{i=1}^{4} k^2_{P_i} \to 0.$$

Looking at the corresponding exact sequence in cohomology, we see that every infinitesimal first-order deformation of $\mathbb{P}_1$ arises from either an infinitesimal first-order deformation of $\tilde{Q}$ (corresponding to an element of $H^1(\tilde{Q}, T_{\tilde{Q}})$) or from an infinitesimal first-order deformation of

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the points $Q_3, \dots, Q_6, P_1, \dots, P_4$ on $\tilde{Q}$, or both. This latter space is isomorphic to a vector space $V = (k^2)^8$. We note that V has a linear subspace $V_1 \simeq k^8$ corresponding to infinitesimal first-order deformations of the points $Q_3, \dots, Q_6$, $P_1, \dots, P_4$ to points along the exceptional divisors of σ_2, i.e. changing the tangent direction of the infinitely-near blowup.

Similarly, because $\mathbb{P}^1 \times \mathbb{P}^1$ is rigid, every first-order infinitesimal deformation of $\tilde{Q}$ arises from a first-order infinitesimal deformation of the points $Q_1, \dots, Q_6; P_1, \dots, P_4$ in $\mathbb{P}^1 \times \mathbb{P}^1$. This latter deformation space is isomorphic to the vector space $W = (k^2)^{10}$. Thus, we have short exact sequences

$$0 \to V \to H^1(\mathbb{P}_1, T_{\mathbb{P}_1}) \to H^1(\tilde{Q}, T_{\tilde{Q}}) \to 0$$
$$0 \to H^0(\mathbb{P}^1 \times \mathbb{P}^1, T_{\mathbb{P}^1 \times \mathbb{P}^1}) \to W \to H^1(\tilde{Q}, T_{\tilde{Q}}) \to 0$$

signifying that every first-order infinitesimal deformation of $\mathbb{P}_1$, and therefore of $(\mathbb{P}_1, B_1 \cup B_2)$, arises from a first-order infinitesimal deformation of the points $Q_1, \dots, Q_6$; $P_1, \dots, P_4$ in $\mathbb{P}^1 \times \mathbb{P}^1$ (i.e. an element of W) or a first-order deformation of $Q_3, \dots, Q_6, P_1, \dots, P_4$ in $\tilde{Q}$, or both.

We note that (3.8), and even $V_1 \to H^1(\mathbb{P}_1, \mathcal{N}_{\Delta/\mathbb{P}_1})$, is surjective, i.e. each obstruction in $H^1(\mathbb{P}_1, \mathcal{N}_{\Delta/\mathbb{P}_1})$ arises from a first-order infinitesimal deformation of $Q_1, \dots, Q_6$ and the tangent directions of $Q_3, \dots, Q_6$ in $\mathbb{P}^1 \times \mathbb{P}^1$ that fails to induce a first-order embedded deformation of Δ.

LEMMA 3.4. *The space $H^1(\mathbb{P}_1, \mathcal{N}_{\Delta/\mathbb{P}_1})$ has dimension seven and has the following distinguished basis. Each basis element comes from a first-order deformation of $\mathbb{P}_1$ which fixes Q_1, Q_2, Q_3 (this takes care of infinitesimal automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$) and either:*

- *I_k for $k = 1, 2, 3$ leaves the tangent direction at Q_3 fixed, i.e. parallel to Δ, and moves $Q_{k+3} \in \{Q_4, Q_5, Q_6\}$ off Δ while keeping the remaining points and their tangent directions fixed, i.e. parallel to Δ; or*
- *I_k for $k = 4, 5, 6, 7$ fixes $Q_{k-1} \in \{Q_3, Q_4, Q_5, Q_6\}$ and changes the tangent direction at Q_{k-1}, moving the remaining points of Q_4, Q_5, Q_6 along Δ and keeping the tangent directions at these remaining points fixed, i.e. parallel to Δ.*

Proof. Simple calculation. □

To show that the map (3.7) is surjective, it suffices to show that for each deformation type listed, there exists an equisingular deformation of $B_1 \cup B_2$ in $\mathbb{P}^1 \times \mathbb{P}^1$ which passes through the points to which $Q_1, \dots, Q_6$ deform and which has the desired tangent direction at each point.

To begin, let us choose bi-homogeneous coordinates $((\alpha : \alpha'), (\beta : \beta'))$ on $Q = \mathbb{P}^1 \times \mathbb{P}^1$ so that $\alpha = x/y = -t/z$ and $\beta = x/t = -y/z$. Let g_1 and g_2 be the equations (bihomogeneous of degree $(3, 3)$) of B_1 and B_2, respectively. Referring to (2.4) and (2.5), we have

$$g_1 = -\alpha\alpha'^2\beta^3 + 3\alpha^2\alpha'\beta^2\beta' - 3\beta^2\beta'\alpha'^3 - 3\alpha\alpha'^2\beta\beta'^2 + 3\alpha^3\beta\beta'^2 + \alpha^2\alpha'\beta'^3$$
$$g_2 = -\beta\beta'^2\alpha'^3 - 3\alpha\alpha'^2\beta^2\beta' + 3\alpha\alpha'^2\beta'^3 - 3\beta\beta'^2\alpha^2\alpha' + 3\beta^3\alpha^2\alpha' - \beta^2\beta'\alpha^3.$$

Global first-order deformations $\tilde{B}_1$ and $\tilde{B}_2$ of B_1 and B_2 are given by equations

$$g_1 + \varepsilon\bar{g}_1 = g_1 + \varepsilon \sum_{0 \leqslant i,j \leqslant 3} a_{ij}\alpha^i\alpha'^{3-i}\beta^j\beta'^{3-j}$$

and

$$g_2 + \varepsilon\bar{g}_2 = g_2 + \varepsilon \sum_{0 \leqslant i,j \leqslant 3} b_{ij}\alpha^i\alpha'^{3-i}\beta^j\beta'^{3-j},$$

respectively. In order to describe equisingular first-order deformations of $B_1 \cup B_2$, we move the singularities of B_1 and B_2 at $P_1, \dots, P_4$ to the points $(\varepsilon c_1, \varepsilon d_1), \dots, (\varepsilon c_4, \varepsilon d_4)$, given in local

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coordinates on $U_1, \dots, U_4 \subset Q$, respectively, where

$$U_1 = \{\alpha = \beta = 1\}, \quad U_2 = \{\alpha' = \beta = 1\},$$
$$U_3 = \{\alpha = \beta' = 1\}, \quad U_4 = \{\alpha' = \beta' = 1\}.$$

To simplify calculations, we change coordinates on $U_1, \dots, U_4$, so that the points $(\varepsilon c_1, \varepsilon d_1), \dots, (\varepsilon c_4, \varepsilon d_4)$ are at the origin.

Letting $g_{ij} + \varepsilon \bar{g}_{ij}$ be the degree j part of the equation $g_i + \varepsilon \bar{g}_i$ with respect to the new coordinates, we have the following conditions. These ensure that $B_1 \cup B_2$ maintains the singularities, with possibly different tangent cones, at the points to which $P_1, \dots, P_4$ deform. For simplicity we use the same notation for $P_1, \dots, P_4$ and the points to which they deform:

(1) $g_{10} + \varepsilon \bar{g}_{10} = 0$ on each U_i; this forces $\tilde{B}_1$ to pass through $P_1, \dots, P_4$;
(2) $g_{11} + \varepsilon \bar{g}_{11} = 0$ on U_1, U_4; this forces $\tilde{B}_1$ to be singular at P_1 and P_4;
(3) $g_{12} + \varepsilon \bar{g}_{12} = (m + m_1 \varepsilon)(g_{21} + \varepsilon \bar{g}_{21})^2$, for some constants m, m_1, on U_1, U_4 (where m, m_1 may differ on U_1, U_4); this forces the tangent cones of $\tilde{B}_1$ at P_1 and P_4 to be the same as those of $\tilde{B}_2$ at P_1 and P_4;
(4) $g_{13} + \varepsilon \bar{g}_{13} = (g_{21} + \varepsilon \bar{g}_{21})(h + \varepsilon h_1)$, where h and h_1 are quadratic forms; by Lemma 3.6, this forces $\tilde{B}_1$ to have tacnodes at the points P_1 and P_4;
(5) $g_{20} + \varepsilon \bar{g}_{20} = 0$ on each U_i; this forces and $\tilde{B}_2$ to pass through $P_1, \dots, P_4$;
(6) $g_{21} + \varepsilon \bar{g}_{21} = 0$ on U_2, U_3; this forces $\tilde{B}_2$ to be singular at P_2 and P_3;
(7) $g_{22} + \varepsilon \bar{g}_{22} = (n + n_1 \varepsilon)(g_{11} + \varepsilon \bar{g}_{11})^2$, for some constants n, n_1, on U_2, U_3 (where n, n_1 may differ on U_2, U_3); this forces the tangent cones of $\tilde{B}_2$ at P_2 and P_3 to be the same as those of $\tilde{B}_1$ at P_2 and P_3;
(8) $g_{23} + \varepsilon \bar{g}_{23} = (g_{11} + \varepsilon \bar{g}_{11})(h + \varepsilon h_1)$, where h and h_1 are quadratic forms; by Lemma 3.6, this forces $\tilde{B}_2$ to have tacnodes at the points P_2 and P_3.

Returning to original coordinates, and after simple algebraic manipulations, this gives the following system of 28 linear equations in c_i, d_i, a_{ij}, b_{ij} (four blocks for four charts).

Equations 3.5.

$$\begin{gathered}
a_{33} = 0 \\
b_{33} = d_1 - 3c_1 \\
a_{32} = -3c_1 - 6d_1 \\
a_{23} = 2c_1 - 3d_1 \\
a_{22} = a_{31} + b_{23} + 3b_{32} \\
a_{13} = 2a_{31} - 2b_{32} + 4b_{23} \\
2c_1 - d_1 + 3a_{12} + a_{03} + 2a_{21} - a_{30} = 0
\end{gathered}$$

$$\begin{gathered}
a_{30} = -c_2 - 3d_2 \\
b_{30} = 0 \\
b_{31} = 3c_2 + 2d_2 \\
b_{20} = 3d_2 + c_2 \\
b_{32} = 2b_{10} + 4a_{31} + 2a_{20} \\
b_{21} = 6b_{10} + 6a_{31} + 3a_{20} \\
5c_2 + 3d_2 + 5b_{00} + 3b_{11} + 6b_{22} + 5b_{33} = 0
\end{gathered}$$

$$\begin{gathered}
a_{03} = c_3 + 3d_3 \\
b_{03} = 0 \\
b_{02} = 2d_3 + 3c_3 \\
b_{13} = 3d_3 + c_3 \\
b_{01} = 5a_{13} + 2b_{23} + 3a_{02} \\
b_{12} = 4a_{13} + 6b_{23} + a_{02} \\
c_3 + 2d_3 - 3b_{11} + b_{33} + 2b_{22} + b_{00} = 0
\end{gathered}$$

$$\begin{gathered}
a_{00} = 0 \\
b_{00} = d_4 - 3c_4 \\
a_{01} = 3c_4 + 6d_4 \\
a_{10} = 3d_4 - 2c_4 \\
a_{20} = 2a_{02} + 2b_{01} + 3b_{10} \\
a_{11} = a_{02} + 4b_{01} + 6b_{10} \\
4c_4 + 5d_4 + 2a_{03} + 3a_{12} + a_{21} + 5a_{30} = 0.
\end{gathered}$$

Next, we determine all additional conditions on a_{ij}, b_{ij}, c_i, d_i which ensure that $\tilde{B}_1$ and $\tilde{B}_2$ pass through the points to which $Q_1, \ldots, Q_6$ deform, with the desired multiplicities at each point. To do so, we look in the chart U_4. Here, the equation of Δ is

$$\alpha(1+\beta) + \beta - 1 = 0.$$

Solving for α gives

$$\alpha = \frac{1-\beta}{1+\beta}.$$

Thus, the points at which Δ intersects B_1 and B_2 are the roots of the following polynomials:

$$(\beta^2+1)(\beta^2+4\beta+6)^2$$

and

$$(\beta^2+1)(\beta^2+6\beta+6)^2.$$

This gives the six points at which B_1 and B_2 intersect Δ:

$$\begin{aligned}
Q_1 &= (-i, i), & Q_2 &= (i, -i), \\
Q_3 &= (3-5i, -2+4i), & Q_4 &= (3+5i, -2-4i), \\
Q_5 &= (-5+4i, -3+5i), & Q_6 &= (-5-4i, -3-5i),
\end{aligned}$$

where $i^2 + 1 = 0 \bmod 7$.

The intersections of $\bar{g}_1 = 0$ and $\bar{g}_2 = 0$ with Δ are given by the zeros of the following polynomials:

$$\begin{aligned}
\hat{g}_1 = {} & (1+\beta)^3(a_{00} + a_{01}\beta + a_{02}\beta^2 + a_{03}\beta^3) \\
& + (1+\beta)^2(1-\beta)(a_{10} + a_{11}\beta + a_{12}\beta^2 + a_{13}\beta^3) \\
& + (1+\beta)(1-\beta)^2(a_{20} + a_{21}\beta + a_{22}\beta^2 + a_{23}\beta^3) \\
& + (1-\beta)^3(a_{30} + a_{31}\beta + a_{32}\beta^2 + a_{33}\beta^3)
\end{aligned}$$

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$$\begin{aligned}\hat{g}_2 = &(1+\beta)^3(b_{00}+b_{01}\beta+b_{02}\beta^2+b_{03}\beta^3)\\ &+(1+\beta)^2(1-\beta)(b_{10}+b_{11}\beta+b_{12}\beta^2+b_{13}\beta^3)\\ &+(1+b)(1-\beta)^2(b_{20}+b_{21}\beta+b_{22}\beta^2+b_{23}\beta^3)\\ &+(1-\beta)^3(b_{30}+b_{31}\beta+b_{32}\beta^2+b_{33}\beta^3).\end{aligned}$$

Using these equations, we obtain eight additional linear equations in a_{ij}, b_{ij}, c_i, d_i. These ensure that $\tilde{B}_1$ and $\tilde{B}_2$ pass through Q_1, Q_2, that $\tilde{B}_1$ passes through Q_3, Q_4, and that $\tilde{B}_2$ passes through Q_5, Q_6. Note that each restriction arises from setting β equal to i, $-i$, $-2+4i$, $-2-4i$, $-3+5i$, or $-3-5i$ in the appropriate equation.

(B1Q1)

$$\begin{aligned}&(3c_1-3c_4+3d_1+d_4-2a_{20}-2a_{21}-a_{31}+3a_{02}-a_{12}+3a_{03}-2b_{10}+3b_{32}+2b_{23})i\\ &\quad+3c_1-3c_4+3d_1+d_4+2a_{20}-2a_{21}+a_{31}-3a_{02}-a_{12}+3a_{03}+2b_{10}-3b_{32}-2b_{23}=0\end{aligned}$$

(B1Q2)

$$\begin{aligned}&(-3c_1+3c_4-3d_1-d_4+2a_{20}+2a_{21}+a_{31}-3a_{02}+a_{12}-3a_{03}+2b_{10}-3b_{32}-2b_{23})i\\ &\quad+3c_1-3c_4+3d_1+d_4+2a_{20}-2a_{21}+a_{31}-3a_{02}-a_{12}+3a_{03}+2b_{10}-3b_{32}-2b_{23}=0\end{aligned}$$

(B2Q1)

$$\begin{aligned}&(-3c_2-c_3-3d_3-a_{20}-3a_{31}-a_{02}-2b_{11}+b_{22}+3b_{32}+b_{23})i\\ &\quad+3c_2+c_3+3d_3-a_{20}-3a_{31}-a_{02}+2b_{11}-b_{22}+3b_{32}+b_{23}=0\end{aligned}$$

(B2Q2)

$$\begin{aligned}&(3c_2+c_3+3d_3+a_{20}+3a_{31}+a_{02}+2b_{11}-b_{22}-3b_{32}-b_{23})i\\ &\quad+3c_2+c_3+3d_3-a_{20}-3a_{31}-a_{02}+2b_{11}-b_{22}+3b_{32}+b_{23}=0\end{aligned}$$

(B1Q3)

$$\begin{aligned}&(-c_4+2d_1-2d_4+3a_{20}+3a_{21}-a_{31}+3a_{12}+2b_{10}-2b_{32}-3b_{23})i\\ &\quad+3c_1-c_4-3d_1-2d_4-2a_{20}+3a_{21}-3a_{31}+a_{02}+a_{12}-a_{03}-3b_{10}-3b_{32}+b_{23}=0\end{aligned}$$

(B1Q4)

$$\begin{aligned}&(c_4-2d_1+2d_4-3a_{20}-3a_{21}+a_{31}-3a_{12}-2b_{10}+2b_{32}+3b_{23})i\\ &\quad+3c_1-c_4-3d_1-2d_4-2a_{20}+3a_{21}-3a_{31}+a_{02}+a_{12}-a_{03}-3b_{10}-3b_{32}+b_{23}=0\end{aligned}$$

(B2Q5)

$$\begin{aligned}&(c_1+3c_2+2c_3+2d_1-d_3+a_{20}-3a_{31}+2a_{02}-b_{10}+3b_{11}-b_{22}+b_{23})i\\ &\quad+2c_1-2c_2+2c_3-3d_1+3d_2-d_3-2a_{20}-a_{31}+3a_{02}+b_{10}+3b_{11}-3b_{22}+b_{32}-3b_{23}=0\end{aligned}$$

(B2Q6)

$$\begin{aligned}&(-c_1-3c_2-2c_3-2d_1+d_3-a_{20}+3a_{31}-2a_{02}+b_{10}-3b_{11}+b_{22}-b_{23})i\\ &\quad+2c_1-2c_2+2c_3-3d_1+3d_2-d_3-2a_{20}-a_{31}+3a_{02}+b_{10}+3b_{11}-3b_{22}+b_{32}-3b_{23}=0.\end{aligned}$$

Taking the derivatives of $\hat{g}_1$ and $\hat{g}_2$ with respect to β and setting β equal to $-2+4i$, $-2-4i$, $-3+5i$, or $-3-5i$ as appropriate gives the final four linear equations in a_{ij}, b_{ij}, c_i, d_i. These ensure that $\tilde{B}_1$ and $\tilde{B}_2$ are tangent to Δ at Q_3, Q_4 and Q_5, Q_6, respectively.

(dB1Q3)

$$\begin{aligned}&(3c_1-2c_4-3d_1+3d_4+a_{21}+2a_{31}+2a_{02}+a_{03}-b_{10}-2b_{32})i\\&\quad-c_1+c_4-2d_1+2d_4+2a_{20}+a_{21}+3a_{02}+a_{03}-3b_{10}-2b_{32}-b_{23}=0\end{aligned}$$

(dB1Q4)

$$\begin{aligned}&(-3c_1+2c_4+3d_1-3d_4-a_{21}-2a_{31}-2a_{02}-a_{03}+b_{10}+2b_{32})i\\&\quad-c_1+c_4-2d_1+2d_4+2a_{20}+a_{21}+3a_{02}+a_{03}-3b_{10}-2b_{32}-b_{23}=0\end{aligned}$$

(dB2Q5)

$$\begin{aligned}&(c_1-3c_2-c_3+2d_1+d_2-3d_3+3a_{20}+2a_{31}+3a_{02}+2b_{10}-2b_{11}-3b_{22}-2b_{23})i\\&\quad-c_2-3c_3-3d_2-2d_3-3a_{31}-3a_{02}-b_{10}-b_{22}-3b_{32}=0\end{aligned}$$

(dB2Q6)

$$\begin{aligned}&(-c_1+3c_2+c_3-2d_1-d_2+3d_3-3a_{20}-2a_{31}-3a_{02}-2b_{10}+2b_{11}+3b_{22}+2b_{23})i\\&\quad-c_2-3c_3-3d_2-2d_3-3a_{31}-3a_{02}-b_{10}-b_{22}-3b_{32}=0.\end{aligned}$$

Consider a basis element in $H^1(\mathbb{P}_1,\mathcal{N}_{\Delta/\mathbb{P}_1})$ corresponding via Lemma 3.4 to some deformation of the points $P_1,\dots,P_4,Q_1,\dots,Q_6$ together with the tangent directions of $P_1,\dots,P_4,Q_4,\dots,Q_6$ in $\mathbb{P}^1\times\mathbb{P}^1$. There are two cases, as in Lemma 3.4.

Consider for example the basis element I_1. The existence of an equisingular deformation of $(\mathbb{P}_1,B_1\cup B_2)$ mapping to I_1 is equivalent to the existence of a_{ij},b_{ij},c_i,d_i which satisfy (3.5), as well as B1Q1, B1Q2, B2Q1, B2Q2, B1Q3, dB1Q3, B1Q4-1, B2Q5, B2Q6. Here, we use Lemma 3.7.

Next we consider the basis element I_4. The existence of an equisingular deformation of $(\mathbb{P}_1,B_1\cup B_2)$ mapping to I_4 is equivalent to the existence of a_{ij},b_{ij},c_i,d_i which satisfy (3.5), as well as B1Q1, B1Q2, B2Q1, B2Q2, B1Q3, dB1Q3, dB1Q3-1, B2Q4, B2Q5, B2Q6. Here, we use Lemma 3.7.

Thus, each of the seven basis elements corresponds to finding a non-trivial solution of a large system of linear equations. As working with such large matrices is unwieldy, we use Macaulay2 to check this (see the code included in the Appendix). In each case, we find that solutions indeed form either a three- or four-dimensional vector space, depending on the basis element, completing the proof. □

Lemma 3.6. *The singularity $(h_1^2+h_1h_2+\text{h.o.t.}=0)\subset\mathbb{A}^2$, where h_i is a form of degree i, is a tacnode (or a degeneration of a tacnode).*

Proof. Completing the square, the equation becomes $((h_1+\frac{1}{2}h_2)^2+\text{h.o.t.}=0)\subset\mathbb{A}^2$. Letting $h=h_1+\frac{1}{2}h_2$, the singularity becomes $(h^2+\text{h.o.t.}=0)\subset\mathbb{A}^2$. As there are no terms of degree 3, this is a tacnode. □

Lemma 3.7. *Let $B=(g=0)$ be the germ of a smooth curve in $\mathbb{A}^2$ which is simply tangent to the x-axis at the origin, and let $\tilde{B}=(g+\varepsilon\bar{g}=0)$ be its first-order infinitesimal embedded deformation. Then $\tilde{B}$ is tangent to the x-axis if and only if $\bar{g}(0,0)=0$.*

Proof. Suppose $\bar{g}(0,0)=0$. We have to show that there exists x_0 with

$$g(\varepsilon x_0,0)+\varepsilon\bar{g}(\varepsilon x_0,0)=0$$

and that

$$\frac{d}{dx}(g(\varepsilon x_0, 0) + \varepsilon \bar{g}(\varepsilon x_0, 0)) = 0.$$

Taking the Taylor expansion of these with respect to ε, the first of these obviously holds. The second holds for

$$x_0 = \frac{-\bar{g}_1'(0,0)}{g_1''(0,0)}. \qquad \square$$

4. The Boyd surface is a Dolgachev surface

Lemma 4.1. *Blowing-down $\bar{N}_1$ and $\bar{N}_2$ on $\mathbb{P}$ gives a Halphen surface of index three [CD89, ch. V, § 6] with a multiple fiber $A_1 + \cdots + A_4 + \bar{G}_1 + \cdots \bar{G}_4$ of type I_8.*

Proof. By 3.1, $\mathbb{P}$ has a fibration $\mathbb{P} \to \mathbb{P}^1_k$ with connected fibers such that the general fiber is smooth of genus one; see [Băd01, § 7]. Moreover, the I_8 fiber $\sum_{i=1}^4 A_i + \sum_{i=1}^4 \bar{G}_i$ has multiplicity three. Thus, this elliptic fibration is a Halphen surface of index three (after one blows down $\bar{N}_1$ and $\bar{N}_2$); see [CD89, ch. V, Theorem 5.6.1]. $\square$

Lemma 4.2. *The Boyd surface Y is a Dolgachev surface in characteristic seven. The elliptic fibration $Y \to \mathbb{P}^1_k$ has four singular fibers: one I_4 with multiplicity three, one I_4 with multiplicity two, and two reduced I_2.*

Proof. We denote by α the composition $W \to \mathbb{P} \to \mathbb{P}^1_k$. Since this is a projective morphism, we have a Stein factorization for α, i.e. maps $\beta : W \to C$ with connected fibers and $\gamma : C \to \mathbb{P}^1_k$ a finite morphism such that $\alpha = \gamma \circ \beta$. Note that the multiplicity of the fiber $B_1 + B_2 + \bar{N}_1 + \bar{N}_2$ of $\mathbb{P} \to \mathbb{P}^1_k$ is 1, and so $\gamma : C \to \mathbb{P}^1_k$ is a finite separable morphism. Note also that the fibers $B_1 + B_2 + \bar{N}_1 + \bar{N}_2$ and I_8 in $\mathbb{P} \to \mathbb{P}^1_k$ pull back to connected fibers of α with multiplicities two and three, respectively. Since these multiplicities are coprime, we must have that the degree of γ is one, and so γ is an isomorphism. In this way α has connected fibers. In addition, since it has two multiple fibers, the Kodaira dimension of Y is non-negative [CD89].

The double cover $W \to \mathbb{P}$ induces a connected étale cover between the non-multiple fibers of α. Note that $\mathbb{P} \to \mathbb{P}^1_k$ can only have irreducible singular fibers apart from $B_1 + B_2 + \bar{N}_1 + \bar{N}_2$ and I_8, because the Picard number of $\mathbb{P}$ is 12. Therefore, we can have either two I_1 or one II as extra singular fibers. But a fiber of type II is étale simply connected, and so it does not have a connected étale cover of degree two. Thus, $\mathbb{P} \to \mathbb{P}^1_k$ has precisely two extra I_1 singular fibers, and their pre-images under $W \to \mathbb{P}$ give two I_2 reduced fibers for α. This elliptic fibration induces a relatively minimal elliptic fibration $Y \to \mathbb{P}^1_k$, after we blow-down the curves $G_1, \ldots, G_4$.

Using well-known facts on double covers, one can easily verify that $K_Y^2 = 0$, $\chi(\mathcal{O}_Y) = 1$, and

$$p_g(Y) = h^2(-L) = h^0(K_{\mathbb{P}} + L) = 0, \tag{4.1}$$

where

$$L = 3\sigma^*(\Delta) - \sum_{i=1}^4 \bar{E}_i - 2\sum_{i=1}^4 \bar{G}_i - \bar{N}_1 - \bar{N}_2$$

is the line bundle defining the double cover π'. Thus, $q(Y) = 0$. $\square$

The previous lemma shows the canonical class of Y has the form

$$K_Y \sim -F + \Gamma_2 + 2\Gamma_3 \equiv 1/6F, \tag{4.2}$$

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where F is a general fiber, Γ_2 is the I_4 with multiplicity two, and Γ_3 is the I_4 with multiplicity three.

Lemma 4.3. *We have that K_{S_0} is nef.*

Proof. The Boyd surface Y is the minimal resolution of the surface S_0, which has log terminal singularities. Therefore, it suffices to show that K_Y is nef, which follows from (4.2). $\square$

5. Some mixed characteristic deformation theory

In this section we show that the Craighero–Gattazzo surface can be degenerated to a special complex surface with a $\frac{1}{4}(1,1)$ singularity. Our argument is based on the following simple fact.

Lemma 5.1. *Let $\mathcal{R}$ be a DVR with residue field k and quotient field K. Let $\bar{K}$ be the algebraic closure of K. Let T be a smooth $\mathcal{R}$-scheme. Let $o \in T$ be a k-point. Let $\sigma_1, \sigma_2 : \operatorname{Spec} \mathcal{R} \to T$ be two sections passing through o. Then there exists an irreducible smooth $\bar{K}$-curve C and a morphism $C \to T_{\bar{K}}$ such that its image contains $\sigma_1(\eta)$ and $\sigma_2(\eta)$, where $\eta \in \operatorname{Spec} \mathcal{R}$ is the generic point.*

Remark 5.2. For the proof we only need σ_1 to be a section; σ_2 can be a section $\operatorname{Spec} \mathcal{R}' \to T_{\mathcal{R}'}$ after a finite surjective base change $\operatorname{Spec} \mathcal{R}' \to \operatorname{Spec} \mathcal{R}$.

Proof. We can substitute T with an affine connected component $\operatorname{Spec} A$ of o. By [Mum70, p. 56], it suffices to prove that T_K is geometrically connected. Since it is smooth over $\operatorname{Spec} K$ and has a K-point $\sigma_1(\eta)$, it suffices to prove that it is connected. Arguing by contradiction, suppose it is disconnected. Then $H^0(T_K, \mathcal{O}_{T_K})$ contains a non-trivial idempotent e. Let $\pi \in \mathcal{R}$ be a uniformizer. Since T is flat over $\operatorname{Spec} \mathcal{R}$, π is not a zero-divisor in A, and so $e \in A[1/\pi]$. Let n be the minimal non-negative integer such that e can be written as a/π^n with $a \in A$. Then $a^2 = \pi^n a$. Since T is smooth over $\operatorname{Spec} \mathcal{R}$, its special fiber is reduced. It follows that $n = 0$ because otherwise $a^2 = 0 \mod (\pi)$ and therefore $a = 0 \mod (\pi)$, which implies that n is not minimal. So $e \in A$, which contradicts the connectedness of T. $\square$

Lemma 5.3. *Let $\mathcal{R}$ be a complete DVR with residue field k and quotient field K. Let $\bar{K}$ be the algebraic closure of K. Let F be a limit-preserving contravariant functor from the category of $\mathcal{R}$-schemes to the category of sets.*

Fix $\zeta_0 \in F(\operatorname{Spec} k)$. Let F_{ζ_0} be its 'deformation functor', i.e. a functor from the category of pointed $\mathcal{R}$-schemes (X, x_0), where x_0 is a closed point with residue field k, to sets. Specifically, $F_{\zeta_0}(X, x_0) = \{\xi \in F(X) \mid F(i)\xi = \zeta_0\}$, where $i : \operatorname{Spec} k = \operatorname{Spec} k(x_0) \hookrightarrow X$ is the inclusion.

Suppose the restriction of F_{ζ_0} to the category of spectra of local artinian $\mathcal{R}$-algebras with residue field k is smooth and satisfies Schlessinger's conditions [Sch68]. Suppose also that the natural map

$$F_{\zeta_0}(\operatorname{Spec} A) \to \varprojlim F_{\zeta_0}(\operatorname{Spec} A/\mathfrak{m}^n) \tag{5.1}$$

is bijective for every complete local Noetherian $\mathcal{R}$-algebra $(A, \mathfrak{m})$ with residue field k.

Let $\Sigma_1, \Sigma_2 \in F_{\zeta_0}(\operatorname{Spec} \mathcal{R})$ and let $\bar{\Sigma}_1, \bar{\Sigma}_2 \in F(\operatorname{Spec} \bar{K})$ be their pull-backs to $\operatorname{Spec} \bar{K}$. Then there exists an irreducible smooth $\bar{K}$-curve C, $\bar{K}$-points $y_1, y_2 \in C$, and an element $\Sigma \in F(C)$ which restricts to $\bar{\Sigma}_1$ and $\bar{\Sigma}_2$ at y_1 and y_2, respectively.

Proof. By [Sch68], F_{ζ_0} admits a hull, and by (5.1) we can assume that the hull is induced by an element $\bar{\zeta} \in F_{\zeta_0}(\operatorname{Spec} \mathcal{H})$, where $(\mathcal{H}, \mathfrak{m})$ is a complete local Noetherian $\mathcal{R}$-algebra with residue field k. By Artin's algebraization theorem [Art69, Theorem 1.6], there exists an $\mathcal{R}$-scheme of finite type T, a closed k-point $o \in T$, an element $\zeta \in F_{\zeta_0}(T, o)$, and an isomorphism $\sigma : \hat{\mathcal{O}}_{T,o} \to \mathcal{H}$ such that $F(\sigma)\zeta$ and $\bar{\zeta}$ agree on $\mathcal{H}/\mathfrak{m}^n$ for all $n \geqslant 1$. By (5.1), in fact $F(\sigma)\zeta = \bar{\zeta}$.

Since F_{ζ_0} is smooth, $T \to \operatorname{Spec} \mathcal{R}$ is formally smooth at o, and therefore we can assume that T is a smooth $\mathcal{R}$-scheme after shrinking it if necessary.

Since $\mathcal{R}$ is complete, we can find sections $\sigma_1, \sigma_2 : \operatorname{Spec} \mathcal{R} \to T$ such that $F(\sigma_i)(\zeta)$ and Σ_i agree on $\mathcal{R}/\mathfrak{n}^n$ for any $n \geqslant 1$, where $\mathfrak{n} \subset \mathcal{R}$ is the maximal ideal. By (5.1), $F(\sigma_i)(\zeta) = \Sigma_i$. It remains to apply Lemma 5.1. □

In our application F will be a functor of $\mathbb{Q}$-Gorenstein deformations, as worked out in [Hac04] in characteristic zero and [AH11] in general. For simplicity, we allow only Cohen–Macaulay surfaces. Following [AH11], let $\mathcal{K}^\omega$ be the category of Kollár families fibered in groupoids over the category of schemes. An object of $\mathcal{K}^\omega$ over a scheme B is a triple $(f : X \to B, F, \phi)$, where f is a proper flat family of connected reduced Cohen–Macaulay surfaces, F is a coherent sheaf, and $\phi : F \to \omega_{X/B}$ is an isomorphism. Moreover, we assume that the formation of every reflexive power $F^{[n]}$ commutes with arbitrary base change (we call this the Kollár condition) and that for every geometric point s of B there exists a positive integer N_s such that $F^{[N_s]}|_{X_s}$ is invertible and ample. See [AH11] for the description of morphisms in $\mathcal{K}^\omega$ and for the proof that it is an algebraic stack. The functor $\operatorname{Def}^{\mathbb{Q}G}$ of $\mathbb{Q}$-Gorenstein deformations is the associated set-valued functor of isomorphism classes of Kollár families.

Theorem 5.4. *Let $\mathcal{R}$ be a complete DVR with algebraically closed residue field k and quotient field K. Let $\bar{K}$ be the algebraic closure of K. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two $\mathbb{Q}$-Gorenstein families over* $\operatorname{Spec} \mathcal{R}$. *Suppose their special fibers are both isomorphic to a k-surface X. Let $\mathcal{K}^\omega_{\mathcal{R}}$ be the restriction of $\mathcal{K}^\omega$ to the category of $\mathcal{R}$-schemes. Suppose it is $\mathcal{R}$-smooth at $X \to \operatorname{Spec} k$. Then there exists an irreducible smooth $\bar{K}$-curve C, $\bar{K}$-points $y_1, y_2 \in C$, and a $\mathbb{Q}$-Gorenstein family over C with fibers at y_1 and y_2 isomorphic to $(\mathcal{X}_1)_{\bar{K}}$ and $(\mathcal{X}_2)_{\bar{K}}$, respectively.*

Proof. Since $\mathcal{K}^\omega_{\mathcal{R}}$ is an algebraic $\mathcal{R}$-stack, its associated set-valued functor $\operatorname{Def}^{\mathbb{Q}G}_{\mathcal{R}}$ satisfies the conditions of Lemma 5.3 by Artin's criterion [Art74]. □

In our situation, $\mathcal{X}_1$ will be a degeneration of the Craighero–Gattazzo surface to the contraction S_0 of the Boyd surface Y. To construct the second family, we will need the following well-known fact.

Lemma 5.5. *Let k be an algebraically closed field, let $\mathcal{R}$ be a complete DVR with residue field k, let Y be a smooth projective surface over k and let $C_1, \ldots, C_r \subset Y$ be smooth curves intersecting transversally. Suppose*

$$H^2(Y, T_Y(-\log(C_1 + \cdots + C_r))) = H^2(Y, \mathcal{O}_Y) = 0.$$

Then there exists a smooth projective family of surfaces $\mathcal{Y} \to \operatorname{Spec} \mathcal{R}$ with closed subschemes $\mathcal{C}_1, \ldots, \mathcal{C}_r \subset \mathcal{Y}$ smooth and proper over $\operatorname{Spec} \mathcal{R}$ such that the special fiber is $(Y; C_1, \ldots, C_r)$.

Proof. This is well-known but we sketch a proof for completeness. Let $\mathfrak{m} \subset \mathcal{R}$ be the maximal ideal and let $\mathcal{R}_n = \mathcal{R}/\mathfrak{m}^{n+1}$ for each $n = 0, 1, \ldots$. We first lift $(Y; C_1, \ldots, C_r)$ to a scheme and a

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collection of subschemes flat over $\operatorname{Spec}\mathcal{R}_n$ for each n by induction on n. So assume we already have a lift $(Y^n; C_1^n, \dots, C_r^n)$ to $\operatorname{Spec}\mathcal{R}_n$. We have an exact sequence

$$0 \to T_Y(-\log(C_1 + \dots + C_r)) \to T_Y \to i_{1*}N_{C_1/Y} \oplus \dots \oplus i_{r*}N_{C_r/Y} \to 0 \tag{5.2}$$

of sheaves on Y, where $i_j : C_j \to Y$ denotes the embedding for each j. Since $H^2(Y, T_Y(-\log(C_1 + \dots + C_r))) = 0$, we have $H^2(Y, T_Y) = 0$ as well. Therefore, we can lift Y^n to a scheme Y^{n+1} flat (and then automatically smooth and proper) over $\operatorname{Spec}\mathcal{R}_{n+1}$. Moreover, all possible lifts form an affine space with underlying vector space $H^1(Y, T_Y)$. Since

$$H^1(Y, T_Y) \to H^1(C_1, N_{C_1/Y}) \oplus \dots \oplus H^1(C_r, N_{C_r/Y})$$

is surjective by $H^2(Y, T_Y(-\log(C_1 + \dots + C_r))) = 0$, we can choose a lift such that the corresponding class in $H^1(C_i, N_{C_i/Y})$ vanishes for each i. This class can be interpreted as an obstruction to lifting $C_i^n \subset Y^n$ to a subscheme $C_i^{n+1} \subset Y^{n+1}$ flat over $\operatorname{Spec}\mathcal{R}_{n+1}$. So we can lift all C_i to subschemes $C_i^{n+1} \subset Y^{n+1}$ flat (and automatically smooth and proper) over $\operatorname{Spec}\mathcal{R}_{n+1}$. The projective limit $\hat{\mathcal{Y}} = \varprojlim Y^n$ is a formal scheme smooth and proper over $\operatorname{Spf}\mathcal{R}$. The projective limits $\hat{\mathcal{C}}_i = \varprojlim C_i^n$ for $i = 1, \dots, n$ are closed formal subschemes smooth and proper over $\operatorname{Spf}\mathcal{R}$.

Since $H^2(Y, \mathcal{O}_Y) = 0$, we can lift any ample invertible sheaf on Y to an (automatically ample) invertible sheaf on $\hat{\mathcal{Y}}$. By Grothendieck's existence theorem [EGAIII, 5.4.5], there exists a scheme $\mathcal{Y}$ projective and flat (and then automatically smooth) over $\operatorname{Spec}\mathcal{R}$ such that $\hat{\mathcal{Y}}$ is a completion of its special fiber. By [EGAIII, 5.1.8], there exist closed subschemes $\mathcal{C}_1, \dots, \mathcal{C}_r \subset \mathcal{Y}$ such that $\hat{\mathcal{C}}_1, \dots, \hat{\mathcal{C}}_r$ are completions of their special fibers. They are flat (and automatically smooth and proper) over $\operatorname{Spec}\mathcal{R}$. □

Notation 5.6. We revert to the notation of the previous sections; in particular, $\mathcal{R}$ will denote the ring of Witt vectors of an algebraically closed field k of characteristic seven. We denote by Y the Boyd surface over k. The (-4)-curve Δ_1 and the (-2)-curve N_1 of Y intersect transversally and in one point.

Lemma 5.7. *There exists a smooth projective family of surfaces $\mathcal{Y} \to \operatorname{Spec}\mathcal{R}$ with closed subschemes $\mathcal{C}, \mathcal{N} \subset \mathcal{Y}$ smooth and proper over $\operatorname{Spec}\mathcal{R}$ such that their geometric fibers are transversal rational curves of self-intersection -4 and -2, respectively. The special fiber is the Boyd surface (Y, Δ_1, N_1).*

Proof. This follows from Theorem 3.1, (4.1), preservation of intersection numbers, and Lemma 5.5. □

We need a few facts about the $\frac{1}{4}(1,1)$ singularity. Let μ_4 be the $\mathbb{Z}$-group scheme $\operatorname{Spec}\mathbb{Z}[\iota]/(\iota^4 - 1)$ with comultiplication $\iota \to \iota \otimes \iota$. Let

$$\mathbb{X} = \operatorname{Spec}\mathbb{Z}[u,v]^{\mu_4} = \operatorname{Spec}\mathbb{Z}[u^4, u^3v, u^2v^2, uv^3, v^4],$$

where μ_4 acts on $\mathbb{A}^2$ with weights (ι, ι). For any scheme S, we say that $\mathbb{X}_S \to S$ is the standard family of surfaces with $\frac{1}{4}(1,1)$ singularity. If k is a field, then $\mathbb{X}_k$ is isomorphic to the cone over the rational normal curve in $\mathbb{P}^4_k$.

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Definition 5.8. Let S be a locally Noetherian scheme and let $\mathcal{X} \to S$ be a flat family of geometrically connected reduced surfaces smooth outside of a section $\Sigma : S \to \mathcal{X}$. We say that $\mathcal{X} \to S$ has a $\frac{1}{4}(1,1)$ singularity along Σ if there exists a (not necessarily cartesian) commutative diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g"] \arrow[d] & \mathcal{X} \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

of morphisms with commuting sections Σ and $\Sigma' : S' \to \mathcal{X}'$ such that f is surjective étale, g is étale, and $\mathcal{X}'$ is isomorphic to an étale neighborhood of the section in the standard family $\mathbb{X}_{S'}$.

Lemma 5.9. *Let $\mathcal{X} \to S$ be a flat family of geometrically connected reduced surfaces with a section $\Sigma : S \to \mathcal{X}$ over a locally Noetherian base scheme S and smooth outside of Σ. Then $\mathcal{X}$ has $\frac{1}{4}(1,1)$ singularity along Σ if and only if there exists a morphism $\pi : \mathcal{Y} \to \mathcal{X}$ over S such that $\mathcal{Y} \to S$ is smooth, π is an isomorphism outside of Σ, and $\mathbb{P} = \pi^{-1}(\Sigma)$ is a $\mathbb{P}^1$-bundle over S such that all geometric fibers have self-intersection -4. In this case $\mathcal{X} \to S$ satisfies the Kollár condition.*

Proof. In one direction, we obtain $\mathcal{Y}$ by blowing up Σ. In the opposite direction, since the question is étale-local on S and $\mathcal{X}$, we can assume that $\mathcal{X}$ and S are spectra of Henselian local rings. By [LN13, Theorem 2.13], it suffices to find relative Cartier divisors D_1 and D_2 of $\mathcal{X} \to S$ such that their scheme-theoretic intersections with $\mathbb{P}$ are disjoint sections of the $\mathbb{P}^1$-bundle. As in the proof of [LN13, Theorem 2.11], their existence follows from surjectivity of $\operatorname{Pic} \mathcal{X} \to \operatorname{Pic} \mathbb{P}^1_s$ (see [EGAIV, Corollary 21.9.12]), where $s \in S$ is the closed point. Finally, $\mathbb{X}_{S'}$ (being toric) and hence $\mathcal{X}$ satisfy the Kollár condition. □

Recall that we have a contraction $Y \xrightarrow{\alpha} S_0$ of Δ_1 to a $\frac{1}{4}(1,1)$ singularity.

Lemma 5.10. *We can 'blow down' the deformation $\mathcal{Y} \to \operatorname{Spec} \mathcal{R}$ of Y to the deformation $\bar{\mathcal{Y}} \to \operatorname{Spec} \mathcal{R}$ of S_0, i.e. there exists a morphism $\mathcal{Y} \to \bar{\mathcal{Y}}$ of deformations over $\operatorname{Spec} \mathcal{R}$ which on the special fiber gives α.*

This morphism contracts $\mathcal{C}$ to a section Σ of $\bar{\mathcal{Y}} \to \operatorname{Spec} \mathcal{R}$ and it is an isomorphism outside Σ. The family $\bar{\mathcal{Y}} \to \operatorname{Spec} \mathcal{R}$ has a $\frac{1}{4}(1,1)$ singularity along Σ and is smooth elsewhere. It is $\mathbb{Q}$-Gorenstein.

Proof. This follows from the fact that $R^1\alpha_*(\mathcal{O}_Y) = 0$ as in [Wah76] (where the equi-characteristic local case is worked out). Specifically, let $\hat{\mathcal{Y}}$ be the formal completion of the special fiber in $\mathcal{Y}$. Let $\hat{\bar{\mathcal{Y}}}$ be a formal scheme with underlying topological space S_0 and sheaf of rings $\alpha_*\mathcal{O}_{\hat{\mathcal{Y}}}$. The vanishing of $R^1\alpha_*(\mathcal{O}_Y)$ implies that $\hat{\bar{\mathcal{Y}}}$ is flat over $\operatorname{Spf} \mathcal{R}$ by [Wah76, 0.4.4]. Since $H^2(S_0, \mathcal{O}_{S_0}) = 0$ and S_0 is projective, $\hat{\bar{\mathcal{Y}}}$ carries an ample line bundle, and therefore is a formal fiber of a scheme $\bar{\mathcal{Y}}$ projective and flat over $\operatorname{Spec} \mathcal{R}$, by Grothendieck's existence theorem [EGAIII, 5.4.5]. Since the formal fiber functor is fully faithful [EGAIII, 5.4.1], the morphism $\hat{\mathcal{Y}} \to \hat{\bar{\mathcal{Y}}}$ is induced by the morphism $\alpha : \mathcal{Y} \to \bar{\mathcal{Y}}$. The rest follows from Lemma 5.9. □

Lemma 5.11. *The $\mathcal{R}$-stack $\mathcal{K}^{\omega}_{\mathcal{R}}$ is smooth at $S_0 \to \operatorname{Spec} k$.*

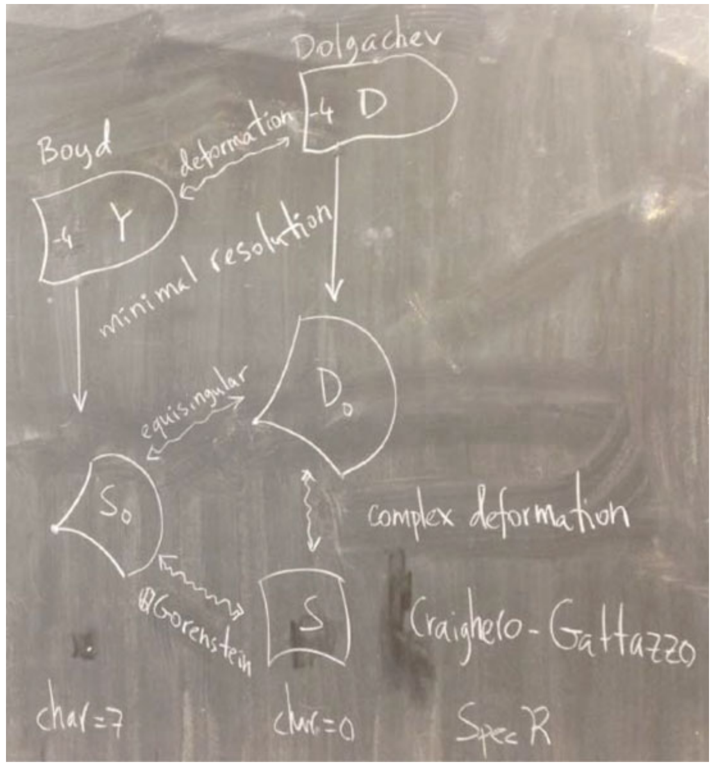

Figure 3. The big picture.

Proof. It suffices to prove that the special fiber of $\mathcal{K}^{\omega}_{\mathcal{R}}$, i.e. the algebraic stack of Kollár families over k, is smooth at $S_0 \to \operatorname{Spec} k$. There are several ways to deduce this from Theorem 3.1. One is to use the theory of index one covers as in [Hac04, § 3] (which assumes characteristic zero but in our case this is not important because the index of the singularity two is not divisible by the characteristic seven). One can also mimic calculations in [Hac04] in the setting of [AH11]. Finally, one can apply [Wah81, Proposition 6.4] (or [LN13, Theorem 4.6]), which shows that the morphism of deformation functors of artinian rings $\operatorname{Def} X \to \operatorname{Def}^{\mathrm{loc}} X$ is smooth and that local $\mathbb{Q}$-Gorenstein deformations of a $\frac{1}{4}(1,1)$-singularity are unobstructed. □

Let $(D; \Gamma, N)$ be the general fiber of the family $(\mathcal{Y}; \mathcal{C}, \mathcal{N}) \to \operatorname{Spec} \mathcal{R}$ after pull-back to $\operatorname{Spec} \mathbb{C}$. Let $D \to D_0$ be the contraction of Γ. Here D_0 is the general fiber of $\bar{\mathcal{Y}} \to \operatorname{Spec} \mathcal{R}$ (after pull-back to $\operatorname{Spec} \mathbb{C}$). Figure 3 shows the big picture.

Theorem 5.12. *There exists a $\mathbb{Q}$-Gorenstein family of complex surfaces $\mathbb{S} \to U$ over a smooth irreducible complex curve such that one of the fibers is D_0 and another fiber is the Craighero–Gattazzo surface S.*

Proof. This follows from Theorem 5.4 and Lemmas 5.10, 2.4, and 5.11. □

The following corollary (of the proof) was first proved in [CP00, Theorem 0.31].

Porism 5.13. *The Craighero–Gattazzo surface is unobstructed and its local moduli space is smooth of dimension eight.*

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Proof. Since the stack of Kollár families $\mathcal{K}^{\omega}_{\mathcal{R}}$ is $\mathcal{R}$-smooth at $S_0 \to \operatorname{Spec} k$, the stack $\mathcal{K}^{\omega}_{\mathbb{C}}$ is $\mathbb{C}$-smooth at $S \to \operatorname{Spec} \mathbb{C}$. But in the neighborhood of a smooth surface such as S, $\mathcal{K}^{\omega}_{\mathbb{C}}$ can be identified with the Deligne–Mumford stack of Gieseker families of canonically polarized surfaces with canonical singularities. □

6. Calculation of the fundamental group

Proposition 6.1. *The surface D is a complex Dolgachev surface with multiple fibers of multiplicity two and three. In particular, $\pi_1(D) = 1$.*

Proof. We first claim that

$$\pi_1^{\mathrm{alg}}(D_0) = 1. \tag{6.1}$$

We are going to use that $\pi_1^{\mathrm{alg}}(S) = 1$ (see [DW99]). Since this is the only fact about S that we need, we can shrink the curve U from Theorem 5.12 and without loss of generality assume that U is a complex disc. Since $\mathcal{S}$ contracts onto D_0, we have $\pi_1(\mathcal{S}) = \pi_1(D_0)$. Now using the same argument as in [Xia91, p. 601], we have an exact sequence

$$\pi_1(S) \to \pi_1(\mathcal{S}) \to \pi_1(U) \to 1,$$

and so $\pi_1(S)$ surjects onto $\pi_1(D_0)$. The right exactness of profinite completions [RZ10, Proposition 3.2.5] implies that $\pi_1^{\mathrm{alg}}(S)$ surjects onto $\pi_1^{\mathrm{alg}}(D_0)$, which implies (6.1). Alternatively, surjectivity of $\pi_1^{\mathrm{alg}}(S) \to \pi_1^{\mathrm{alg}}(D_0)$ follows from the Grothendieck's specialization theorem [SGA1, Corollary 2.3].

We have $K_D^2 = 0$. By Lemma 4.3 and Corollary 5.12, K_{D_0} is nef. Therefore, D is not rational. Indeed, if D is rational, then by Riemann–Roch $h^0(D, -K_D) \geqslant 1$ and so $-K_D \sim E \geqslant 0$. Since $K_D \cdot \Gamma = 2$, we have $\Gamma \subset E$. We know that $f^*(2K_{D_0}) \sim -2E + \Gamma$ where $f : D \to D_0$ is the minimal resolution. But $E \neq \Gamma$, and so $f^*(2K_{D_0})$ cannot be nef. Also, the Kodaira dimension of D cannot be zero because of the Enriques classification and $K_D \cdot \Gamma = 2$, and cannot be two because of Kawamata's argument [Kaw92] (see [Ran14, Lemma 2.4]). Therefore, the Kodaira dimension is one, and so D is an elliptic fibration over $\mathbb{P}^1$ (since $q(D) = 0$).

Say we have r multiple fibers of multiplicities $m_1, \ldots, m_r$. By [Xia91, p. 601],

$$\pi_1(D) \simeq \langle a_1, \ldots, a_r : a_1 \cdots a_r = a_1^{m_1} = \cdots = a_r^{m_r} = 1 \rangle.$$

But this group is residually finite (see [LS77, p. 126] and [LS77, p. 141 last paragraph]). We also have $\pi_1^{\mathrm{alg}}(D) = \pi_1^{\mathrm{alg}}(D_0)$ (see [Kol93]), and so by the above we get $\pi_1(D) = 1$. This implies that there are only two multiple fibers m_1F_1, m_2F_2 with coprime multiplicities m_1, m_2. Let F be a general fiber of $D \to \mathbb{P}^1$, and let $\Gamma \cdot F = d$. Then, since $K_D \sim -F + (m_1 - 1)F_1 + (m_2 - 1)F_2$, we have $\Gamma \cdot K_D = d - d/m_1 - d/m_2 = 2$. In addition, since $\Gamma \cdot F_1 = d/m_1$ and $\Gamma \cdot F_2 = d/m_2$, we have $d = \lambda m_1 m_2$, and so $\lambda(m_1m_2 - m_1 - m_2) = 2$. The only possible solutions, up to permuting one and two, are $\lambda = 2$, $m_1 = 2$, $m_2 = 3$. □

Theorem 6.2. *We have $\pi_1(S) = 1$.*

Proof. Here we use the method of [LP07], which applies Van Kampen's theorem and the Milnor fiber of the $\mathbb{Q}$-Gorenstein smoothing of $\frac{1}{4}(1, 1)$. We only need $\pi_1(D\backslash\Gamma) = 1$. By Van Kampen's theorem, we have $\pi_1(D) \simeq \pi_1(D\backslash\Gamma)/\overline{\langle\alpha\rangle}$ where α is a loop around Γ, and $\overline{\langle\alpha\rangle}$ is the smallest normal subgroup of $\pi_1(D\backslash\Gamma)$ containing $\langle\alpha\rangle$. We can and do consider α as given by a loop

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around N, since N and Γ intersect transversally. As $N \cdot \Gamma = 1$, the set $N' := N \cap (D \backslash \Gamma)$ is simply connected, and so $\alpha \subset N' \subset D \backslash \Gamma$ is homotopically trivial. Therefore, $\overline{\langle \alpha \rangle} = 1$, and so $\pi_1(D \backslash \Gamma) = 1$ since by Proposition 6.1 we have $\pi_1(D) = 1$. After this, one directly applies [LP07, pp. 493 and 499]. □

7. Genus two Lefschetz fibration on a Dolgachev surface

In § 5 we constructed a lifting of the Boyd surface Y (a Dolgachev surface in characteristic seven) to some Dolgachev surface D in characteristic zero. Using results of § 3 we can be much more explicit.

Theorem 7.1. *The Boyd surface Y can be lifted to a complex Dolgachev surface D of type $2, 3$, which possesses an I_4 fiber of multiplicity two, two (-4) curves, and four elliptic (-1) curves $E_1, \dots, E_4$. This surface has a Campedelli-type description as the minimal resolution of singularities of the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ with four elliptic singularities and two A_1 singularities.*

Proof. In § 3, the main point was to prove that

$$H^2(\mathbb{P}_1, T_{\mathbb{P}_1}(-\log(\Delta + B_1 + B_2))) = 0.$$

By applying the (-1) and (-2) principles as before, we have

$$H^2\left(\mathbb{P}_1, T_{\mathbb{P}_1}\left(-\log\left(\Delta + B_1 + B_2 + \sum \bar{N}_i + \sum \bar{G}_i + \sum \bar{E}_i + \sum \bar{C}_i + \sum \bar{F}_i\right)\right)\right) = 0.$$

By Lemma 5.5, preservation of intersection numbers, and $H^2(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}) = 0$, we have that the configuration of curves $\Delta + B_1 + B_2 + \sum \bar{N}_i + \sum \bar{G}_i + \sum \bar{E}_i + \sum \bar{C}_i + \sum \bar{F}_i$ exists in $\mathbb{P}_1$ over $\mathbb{C}$. We will use the same notation as in characteristic seven. Then, by contracting $\sum \bar{N}_i + \sum \bar{G}_i + \sum \bar{E}_i + \sum \bar{C}_i + \sum \bar{F}_i$, we obtain curves $\Delta + B_1 + B_2$ in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ with the corresponding singularities. In this way, we can check that $\Delta \sim (1,1)$ and $B_i \sim (3,3)$ in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$. Note that the two singularities of B_1 and the two singularities of B_2 may not be located at the special position we had in characteristic seven. Let us call these points $P_1, \dots, P_4$ as before.

The linear system $|\mathcal{O}(2,2)|$ contains a member, which we call Γ, that passes through $P_1, \dots, P_4$ with the direction of the tangent cone to $B_1 \cup B_2$. Indeed, Γ exists because $h^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2)) = 9$ and passing through four points with four given directions imposes eight conditions.

Then one easily checks in $\mathbb{P}_1$ that

$$B_1 + B_2 + 2\bar{N}_1 + 2\bar{N}_2 \sim 3\Gamma \tag{7.1}$$

as well as

$$B_1 + B_2 + \sum_{i=1}^{4} \bar{G}_i \sim 2\left(3\sigma^*(\Delta) - \bar{N}_1 - \bar{N}_2 - 3\sum_{i=1}^{4} \bar{E}_i - \sum_{i=1}^{4} \bar{G}_i\right). \tag{7.2}$$

In this way, (7.1) gives an elliptic fibration $\mathbb{P}_1 \to \mathbb{P}^1_{\mathbb{C}}$ with one multiple fiber Γ of multiplicity three, and (7.2) gives a double cover $W \to \mathbb{P}_1$ of $\mathbb{P}_1$ branched along $B_1 + B_2 + \sum_{i=1}^{4} \bar{G}_i$, just as before. Again the pre-images of $\bar{G}_1, \dots, \bar{G}_4$ give (-1)-curves in W, which we contract to obtain a surface D. Using the standard formulas for double covers, as before, we get $K_D^2 = 0$ and $\chi(\mathcal{O}_D) = 1$. Also, we can directly compute $p_g(D) = 0$ using the defining line bundle of the double cover, and so $q(D) = 0$. The pull-back of the elliptic fibration $\mathbb{P}_1 \to \mathbb{P}^1_{\mathbb{C}}$ gives an elliptic fibration $D \to \mathbb{P}^1_{\mathbb{C}}$ with two multiple fibers: the pre-images of Γ and $B_1 + B_2 + \bar{N}_1 + \bar{N}_2$, with multiplicities three and two, respectively. The two (-4) curves are pre-images of Δ. □

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Moreover, we note that the pull-backs of the two rulings of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ give two distinct genus two fibrations $D \to \mathbb{P}^1_{\mathbb{C}}$.

Theorem 7.2. *There exist Dolgachev surfaces (with multiple fibers of multiplicity* 2, 3*) which carry genus two Lefschetz fibrations, specifically genus two fibrations without multiple components in fibers and such that the only singularities of fibers are nodes.*

Proof. In characteristic seven, we have two genus two fibrations on the Boyd surface induced by the two rulings in $\mathbb{P}^1 \times \mathbb{P}^1$. We first want to find out the singular fibers of these fibrations. For that, we need to look at the induced morphisms $B_i \subset \mathbb{P} \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ for each i and for each ruling.

Using (2.4) and (2.5) of B_1 and B_2, respectively, we obtain that, for the ruling $\beta = x/t$, the morphism $B_1 \to \mathbb{P}^1$ has branch points at β satisfying $(\beta^2+1)^2 = 0$, and the morphism $B_2 \to \mathbb{P}^1$ is branched at β satisfying $\beta^4 + 4\beta^2 + 1 = 0$. One verifies that in the first case, the points of ramification are $Q_1 = (-i, i)$ and $Q_2 = (i, -i)$, and B_1 is tangent to the ruling with flex points at Q_1 and Q_2. For the second ruling, the roles of B_1 and B_2 are interchanged in relation to ramification, and B_2 is tangent to the ruling with flex points at Q_1 and Q_2 for B_2.

Using the previous observations on the ramification points of $B_1 \to \mathbb{P}^1$ and $B_2 \to \mathbb{P}^1$, we obtain the following singular fibers for the genus two fibrations $Y \to \mathbb{P}^1$ (we take it from one ruling, the other is analogous).

(1) Two reduced singular fibers consisting of $E_1 \cup A_1 \cup E_4$ and $E_2 \cup A_2 \cup E_3$ where E_i are disjoint elliptic (-1) curves, and A_i are (-2) rational curves, each intersecting two E_j at one nodal point.
(2) Two reduced singular fibers over $\beta = i, -i$ consisting of one nodal rational curve together with N_1, and another rational nodal curve with N_2. Each of the N_i passes through the corresponding node, forming a simple triple point for the fiber.
(3) Four reduced singular curves, each consisting of a nodal curve whose resolution is an elliptic curve.

We claim that there exists a lifting of this Dolgachev surface to characteristic zero as in Theorem 7.1 such that case (2) is eliminated. In other words, we have to construct a lifting of $\mathbb{P}_1$ together with the curves $\Delta + B_1 + B_2 + \sum \bar{N}_i + \sum \bar{G}_i + \sum \bar{E}_i + \sum \bar{C}_i + \sum \bar{F}_i$ such that the flex ramification points for $B_1 \to \mathbb{P}^1$ disappear, becoming simple ramification for a degree three morphism $B_1 \to \mathbb{P}^1_{\mathbb{C}}$. Using the Macaulay2 code in the Appendix, we show the existence of a first other deformation of that type. This together with unobstructed deformations, as in the remark above, gives a lifting to $\operatorname{Spec} \mathcal{R}$ such that, over the generic point, the curve B_1 is not flex with respect to any ruling. In this way, at least for one ruling, the corresponding genus two fibration on the complex 2, 3 Dolgachev surface has only singular fibers which are reduced and with nodes as singularities, i.e. it is a Lefschetz fibration. □

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Appendix

This appendix contains the Macaulay2 source code used to compute the rank of matrices in the proof of Theorem 3.1 and Theorem 7.2.

```
--For simplicity, this includes the extra variable y.
R=ZZ/7[t,c1,c2,c3,c4,d1,d2,d3,d4,a20,a21,a31,a02,a12,a03,b10,b11,b22,b32,b23,x,y];

--Adjoin a square root of -1:
R1=R/(t^2+1)

-- Twenty-one of the restrictions on coefficients arising from forcing desired singularities
at the points to which P1, P2, P3, P4 deform. These allow us to reduce the number of variables
from 40 to 19.

a33=0; b33=d1-3*c1; a32=-3*c1-6*d1; a23=2*c1-3*d1;
a22=2*a31+4*b23-2*b32; a13=2*a31-2*b32+4*b23;
a30=2*c1-d1+3*a12+2*a21+a03; b30=0; b31=3*c2+2*d2; b20=3*d2+c2;
b21=6*b10+6*a31+3*a20; b00=-c2-2*d2-2*b11-4*b22-b33;
b03=0; b02=2*d3+3*c3; b13=3*d3+c3; b01=5*a13+2*b23+3*a02;
b12=4*a13+6*b23+a02; a00=0; a01=3*c4+6*d4; a10=3*d4-2*c4;
a11=a02+4*b01+6*b10;

--The intersection of $\bar{B}_1$ and $\bar{B}_2$ with $\Delta$
(Here, x=$\beta$):

g1bar= (1+x)^3*(a00+a01*x+a02*x^2+a03*x^3)
+(1+x)^2*(1-x)*(a10+a11*x+a12*x^2+a13*x^3)
+(1+x)*(1-x)^2*(a20+a21*x+a22*x^2+a23*x^3)
+(1-x)^3*(a30+a31*x+a32*x^2+a33*x^3);

g2bar= (1+x)^3*(b00+b01*x+b02*x^2+b03*x^3)
+(1+x)^2*(1-x)*(b10+b11*x+b12*x^2+b13*x^3)
+(1+x)*(1-x)^2*(b20+b21*x+b22*x^2+b23*x^3)
+(1-x)^3*(b30+b31*x+b32*x^2+b33*x^3);

--The derivatives of g1bar and g2bar:
dg1bar= diff(x, g1bar);

dg2bar=diff(x, g2bar);

-- B1 and B2 pass through Q1 and Q2:
B1Q1=sub(g1bar, x=>t); B1Q2=sub(g1bar, x=>-t);

B2Q1=sub(g2bar, x=>t); B2Q2=sub(g2bar, x=>-t);

-- B1 passes through Q3 (x=-2+4i), Q4 (x=-2-4i):
B1Q3=sub(g1bar, x=>-2+4*t); B1Q4=sub(g1bar, x=>-2-4*t);

-- B2 passes through Q5, Q6;
B2Q5=sub(g2bar, x=>-3+5*t); B2Q6=sub(g2bar, x=>-3-5*t);

-- B1 is tangent at Q3, Q4:
dB1Q3=sub(dg1bar, x=>-2+4*t); dB1Q4=sub(dg1bar, x=>-2-4*t);

-- B2 tangent at Q5, Q6
dB2Q5=sub(dg2bar, x=>-3+5*t); dB2Q6=sub(dg2bar, x=>-3-5*t);

-- Each of the following ideals gives the kernel of one of the seven the systems of linear
```


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equations. Notice that in each, we include the remaining seven restrictions arising from
forcing desired singularities at the points to which
P1, P2, P3, P4 deform.

--move Q4 off Delta, moving Q5, Q6 along, keeping tangent direction at Q3, Q5, Q6

I1=ideal(a30+c2+3*d2, b32-2*b10-4*a31-2*a20,  a03-c3-3*d3,
c3+2*d3-3*b11+b33+2*b22+b00, b00-d4+3*c4,
a20-2*a02-2*b01-3*b10,4*c4+5*d4+2*a03+3*a12+a21+5*a30,
B1Q1, B1Q2, B2Q1, B2Q2, B1Q3, dB1Q3, B1Q4-1,B2Q5, B2Q6);

--move Q5 off Delta, moving Q4, Q6 along, keeping tangent direction at Q3, Q4, Q6:

I2=ideal(a30+c2+3*d2, b32-2*b10-4*a31-2*a20,  a03-c3-3*d3,
c3+2*d3-3*b11+b33+2*b22+b00, b00-d4+3*c4,
a20-2*a02-2*b01-3*b10,4*c4+5*d4+2*a03+3*a12+a21+5*a30,
B1Q1, B1Q2, B2Q1, B2Q2, B1Q3, dB1Q3, B1Q4, B2Q6,B2Q5-1) ;

--move Q6 off Delta, moving Q4, Q5 along, keeping tangent direction at Q3, Q4, Q5:

I3=ideal(a30+c2+3*d2, b32-2*b10-4*a31-2*a20,  a03-c3-3*d3,
c3+2*d3-3*b11+b33+2*b22+b00, b00-d4+3*c4,
a20-2*a02-2*b01-3*b10,4*c4+5*d4+2*a03+3*a12+a21+5*a30,
B1Q1, B1Q2, B2Q1, B2Q2, B1Q3, dB1Q3, B1Q4,B2Q5, B2Q6-1)


--leave all points on Delta, moving Q4, Q5, Q6 along, change tangent direction at Q3:

I4=ideal(a30+c2+3*d2, b32-2*b10-4*a31-2*a20,  a03-c3-3*d3,
c3+2*d3-3*b11+b33+2*b22+b00, b00-d4+3*c4,
a20-2*a02-2*b01-3*b10,4*c4+5*d4+2*a03+3*a12+a21+5*a30,
B1Q1, B1Q2, B2Q1, B2Q2, B1Q3, B1Q4, dB1Q3-1,B2Q5, B2Q6)

--leave all points on Delta, moving Q5, Q6 along, change tangent direction at Q4, keep tangent
direction at Q3:

I5=ideal(a30+c2+3*d2, b32-2*b10-4*a31-2*a20,  a03-c3-3*d3,
c3+2*d3-3*b11+b33+2*b22+b00, b00-d4+3*c4,
a20-2*a02-2*b01-3*b10,4*c4+5*d4+2*a03+3*a12+a21+5*a30,
B1Q1, B1Q2, B2Q1, B2Q2, B1Q3, B1Q4, dB1Q4-1,B2Q5, B2Q6, dB1Q3)

--leave all points on Delta, moving Q4, Q6 along, change tangent direction at Q5, keep tangent
direction at Q3:

I6=ideal(a30+c2+3*d2, b32-2*b10-4*a31-2*a20,  a03-c3-3*d3,
c3+2*d3-3*b11+b33+2*b22+b00, b00-d4+3*c4,
a20-2*a02-2*b01-3*b10,4*c4+5*d4+2*a03+3*a12+a21+5*a30,
B1Q1, B1Q2, B2Q1, B2Q2, B1Q3, B1Q4, dB2Q5-1,B2Q5, B2Q6, dB1Q3)

--leave all points on Delta, moving Q4, Q5 along, change tangent direction at Q6, keep tangent
direction at Q3:

I7=ideal(a30+c2+3*d2, b32-2*b10-4*a31-2*a20,  a03-c3-3*d3,
c3+2*d3-3*b11+b33+2*b22+b00, b00-d4+3*c4,
a20-2*a02-2*b01-3*b10,4*c4+5*d4+2*a03+3*a12+a21+5*a30,
B1Q1, B1Q2, B2Q1, B2Q2, B1Q3, B1Q4, dB2Q6-1,B2Q5, B2Q6, dB1Q3)

-- Check the dimension of each ideal (note: each has one less dimension that Macaulay2 gives,
```

```
because of the extra variable y)

-- four-dimensional
dim1=dim(I1); dim2=dim(I2); dim3=dim(I3); dim4=dim(I4);

--three-dimensional:
dim5=dim(I5); dim6=dim(I6); dim7=dim(I7);

-- The remaining code is used to prove the existence of the Lefschetz fibration. In particular,
we prove existence of a deformation of B1+B2 so that B1+B2 maintains its singularities
at P1,..., P4 and so that B1 is no longer tangent to the fiber x=i or x=-i at Q1, Q2:

-- $\bar{B}_1$ and $\bar{B}_2$ (alpha=y, beta=x):
g1bar= (a00+a01*x+a02*x^2+a03*x^3)
+y*(a10+a11*x+a12*x^2+a13*x^3)
+y^2*(a20+a21*x+a22*x^2+a23*x^3)
+y^3*(a30+a31*x+a32*x^2+a33*x^3);

g2bar=(b00+b01*x+b02*x^2+b03*x^3)
+y*(b10+b11*x+b12*x^2+b13*x^3)
+y^2*(b20+b21*x+b22*x^2+b23*x^3)
+y^3*(b30+b31*x+b32*x^2+b33*x^3);

-- Writing the local equation of B1 along the fiber at Q1 and Q2:
B1Q1bar=sub(sub(g1bar, x=>t), y=>y-t);
B1Q2bar=sub(sub(g1bar, x=>-t), y=>y+t);

-- These ensure B1 vanishes at Q1 and Q2:

van1=sub(B1Q1bar, y=>0); van2=sub(B1Q2bar, y=>0);

-- These (when nonzero) force B1 to be no longer tangent to the fiber x=i, x=-i at Q1, Q2:

dB1Q1=diff(y, B1Q1bar); dB1Q2=diff(y, B1Q2bar);


Lefschetz=ideal(a30+c2+3*d2, b32-2*b10-4*a31-2*a20, a03-c3-3*d3, c3+2*d3-3*b11+b33+2*b22+b00,
b00-d4+3*c4, a20-2*a02-2*b01-3*b10,4*c4+5*d4+2*a03+3*a12+a21+5*a30, van1, van2, dB1Q1-1, dB1Q2-1)

dimL= dim(Lefschetz); -- 10-dimensional
```

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