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GLOBAL WELL-POSEDNESS FOR 3D NAVIER–STOKES EQUATIONS WITH ILL-PREPARED INITIAL DATA

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Abstract We study the global well-posedness of 3D Navier–Stokes equations for a class of large initial data. This type of data slowly varies in the vertical direction (expressed as a function of εx_3), and it is ill-prepared in the sense that its norm in C^{-1} will blow up at the rate $\varepsilon^{-\alpha}$ for $\frac{1}{2} < \alpha < 1$ as ε tends to zero.

1. Introduction

In this paper, we study Navier–Stokes equations with initial data which is slowly varying in the vertical variable. More precisely, we consider the system

$$(NS) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p \quad \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_{0,\varepsilon}, \end{cases}$$

where $\Omega = \mathbb{R}^3$ and $u_{0,\varepsilon}$ is a divergence-free vector field, whose dependence on the vertical variable x_3 will be chosen to be 'slow', meaning that it depends on εx_3, where ε is a small parameter. More precisely, our initial data is of the form

$$u_{0,\varepsilon} = (\varepsilon^{1-\alpha} v_0^h(x_h, \varepsilon x_3), \varepsilon^{-\alpha} v_0^3(x_h, \varepsilon x_3)),$$

for $\alpha \in [0, 1]$. This initial data may be arbitrarily large, with size $\varepsilon^{-\alpha}$, in the space $B^{-1}_{\infty,\infty}$ which is well-adapted space to measure the amplitude of the initial data for the Navier–Stokes equations (see [10, 12]).

The well-prepared case ($\alpha = 0$) was studied by Chemin and Gallagher [10]. The ill-prepared case ($\alpha = 1$) was studied by Chemin *et al.* [12]. For the later case, the data was supposed to evolve in a special domain $\mathbb{T}^2 \times \mathbb{R}$. Recently, Paicu and Zhang [17] considered the intermediate case ($\alpha = \frac{1}{2}$). The aim of this paper is to generalize the result of [12] to the domain $\mathbb{R}^3$ in the more difficult case of $\frac{1}{2} < \alpha < 1$.

By making a change of the scale, one obtains a Navier–Stokes-type system with anisotropic viscosity $-\Delta_\varepsilon \stackrel{\text{def}}{=} \Delta_h + \varepsilon^2\partial_3^2$ and anisotropic pressure gradient $-(\nabla_h p, \varepsilon^2\partial_3 p)$; see the rescaled system (2.1). In the rescaled system, there is a loss of regularity in the vertical variable in Sobolev estimates if we want to obtain estimates which are independent of ε. To overcome this difficulty, one needs to work with analytical initial data. The most important tool developed by Chemin [6] consists in making analytical-type estimates, and at the same time controlling the size of the analyticity band. This is performed by the control of nonlinear quantities which depend on the solution itself. Even in this situation, it is important to take into account very carefully the special structure of Navier–Stokes equations. In fact, a global in time Cauchy–Kowalewskaya-type theorem was obtained in [12]. Some local in time results for Euler and Prandtl equations with analytic initial data can be found in [18]. We mention also some other works [1, 3, 4, 14, 15] where are obtained estimates in analytic spaces for Euler and Navier–Stokes equations with control on the size of the analyticity band and the analytic norm of the solution.

In [12], the fluid is supposed to evolve in a special domain $\Omega = \mathbb{T}^2 \times \mathbb{R}_v$. This choice of domain is justified by the pressure term. Indeed, the pressure verifies the elliptic equation $\Delta_\varepsilon p = \partial_i\partial_j(u^i u^j)$, and, consequently, $\nabla_h p = (-\Delta_\varepsilon)^{-1}\nabla_h\partial_i\partial_j(u^i u^j)$. Because we have that Δ_ε^{-1} converges to Δ_h^{-1}, it is important to control the low horizontal frequencies, while, in the case of the periodic torus in the horizontal variable, we have only zero horizontal frequency and high horizontal frequency.

In order to extend the result of [12] to the domain $\mathbb{R}^3$, one of the key points is to control the low horizontal frequency very precisely. For this purpose, we choose to work in a class of anisotropic Sobolev spaces in [17]. An important observation is that the operator $\Delta_h^{-1}\nabla_h(a\nabla_h b)$ is a bounded operator in these anisotropic Sobolev spaces. In [17], we proved the global well-posedness of (NS) in the intermediate case of $\alpha = \frac{1}{2}$. In such case, there is only one half derivative loss in the vertical variable. Thus, the energy method can be used to gain the smoothing effect of the half-order derivative from the analyticity. We mention that the Cauchy–Kowalewskaya theorem allows one to control in the analytical framework a loss of one full derivative but only locally in time. Here we are interested in a global in time Cauchy–Kowalewskaya-type result.

In this paper, we will consider the case when $\frac{1}{2} < \alpha < 1$. In this case, there is an order-α derivative loss in the vertical variable. Thus, the energy method in [17] does not work. Roughly speaking, we need to deal with three subtle questions in order to obtain a global Cauchy–Kowalewskaya-type theorem in $\mathbb{R}^3$: (1) the loss of regularity in the nonlinear terms; (2) the control of the low horizontal frequency of the solution; (3) the weak damping effect of the parabolic operator $\partial_t - \Delta_\varepsilon$. We will use the semigroup method and the smoothing effect of analyticity to overcome the loss of the α-order derivative in the vertical variable. To control the low horizontal frequency of the solution, we introduce anisotropic Besov spaces. We construct the suitable functional space of the solution to capture the weak damping mechanism of the parabolic operator. Even with these, the special structure of the equation plays an important role in our estimates. We remark that the equation on the vertical component is a linear equation with coefficients depending on the horizontal components and with no loss of regularity in the vertical variable in the pressure term.

Our main result is stated as follows.

Theorem 1.1. *Let $a > 0, s > \frac{1}{2}$. Assume that (α, γ) satisfies*

$$\alpha \in \left(\frac{1}{2}, 1\right), \qquad \frac{2}{1+\alpha} < \gamma \leqslant \frac{1}{\alpha}.$$

Then there exists a constant η such that, for any divergence-free field v_0 satisfying

$$\|e^{a|D_3|^{\gamma\alpha}} v_0\|_{B^{0,s}_{2,\gamma}} + \|e^{a|D_3|^{\gamma\alpha}} v_0\|_{B^{-\alpha,s}_{2,\gamma}} \leqslant \eta$$

for any $\varepsilon \in (0, 1)$, the Navier–Stokes system (NS) with initial data

$$u_0^\varepsilon = \big(\varepsilon^{1-\alpha} v_0^h(x_h, \varepsilon x_3), \varepsilon^{-\alpha} v_0^3(x_h, \varepsilon x_3)\big)$$

has a global smooth solution on $\mathbb{R}^3$.

Remark 1.2. Another important improvement of this paper is that the global well-posedness is established for more wider class of the initial data with Gevrey regularity instead of analytic regularity, since $\gamma\alpha$ could be less than 1.

Let us conclude the introduction by reviewing some other relevant examples of large initial data generating the global solution of 3D Navier–Stokes equations. Chemin and Gallagher [8] construct the first example of periodic initial data which is big in C^{-1}, and strongly oscillating in one direction, which generates a global solution (see [9] for the case of $\mathbb{R}^3$), where the special structure of Navier–Stokes equations was used in their proof. We refer to [13, 2, 11] and references therein for further progress.

2. Structure of the proof

2.1. Reduction to a rescaled problem

We seek the solution of the form

$$u_\varepsilon(t, x) \stackrel{\text{def}}{=} \big(\varepsilon^{1-\alpha} v^h(t, x_h, \varepsilon x_3), \varepsilon^{-\alpha} v^3(t, x_h, \varepsilon x_3)\big).$$

This leads to the following rescaled Navier–Stokes system:

$$(RNS_\varepsilon) \quad \begin{cases} \partial_t v^h - \Delta_h v^h - \varepsilon^2 \partial_3^2 v^h + \varepsilon^{1-\alpha} v \cdot \nabla v^h = -\nabla^h q, \\ \partial_t v^3 - \Delta_h v^3 - \varepsilon^2 \partial_3^2 v^3 + \varepsilon^{1-\alpha} v \cdot \nabla v^3 = -\varepsilon^2 \partial_3 q, \\ \operatorname{div} v = 0, \\ v(0) = v_0(x), \end{cases} \tag{2.1}$$

where $\Delta_h \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2$ and $\nabla_h \stackrel{\text{def}}{=} (\partial_1, \partial_2)$. As there is no boundary, the rescaled pressure q can be computed with the formula

$$-\Delta_\varepsilon q = \varepsilon^{1-\alpha} \operatorname{div}(v \cdot \nabla v), \quad \Delta_\varepsilon = \Delta_h + \varepsilon^2 \partial_3^2. \tag{2.2}$$

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To state our result, we first introduce the Littlewood–Paley decomposition. Choose two nonnegative even functions $\chi,\ \varphi \in \mathcal{S}(\mathbb{R})$ supported respectively in $\mathcal{B} = \{\xi \in \mathbb{R},\ |\xi| \leqslant \frac{4}{3}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R},\ \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j\geqslant 0} \varphi(2^{-j}\xi) = 1 \quad \text{for any } \xi \in \mathbb{R},$$
$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}\setminus\{0\}.$$

We need to introduce two classes of the frequency localization operators. The localization operators Δ_j^v and S_j^v in the vertical direction are defined by

$$\Delta_j^v f = \mathcal{F}^{-1}\big(\varphi(2^{-j}|\xi_3|)\widehat{f}\,\big) \quad \text{for } j \geqslant 0, \quad S_j^v f = \mathcal{F}^{-1}\big(\chi(2^{-j}|\xi_3|)\widehat{f}\,\big) = \sum_{j'\leqslant j-1} \Delta_{j'}^v f,$$
$$\Delta_{-1}^v f = S_0^v f, \quad \Delta_j^v f = 0 \quad \text{for } j \leqslant -2.$$

The localization operators $\dot{\Delta}_j^h$ and S_j^h in the horizontal direction are defined by

$$\dot{\Delta}_j^h f = \mathcal{F}^{-1}\big(\varphi(2^{-j}|\xi_h|)\widehat{f}\big), \quad S_j^h f = \sum_{j'\leqslant j-1} \dot{\Delta}_{j'}^h f \quad \text{for } j \in \mathbb{Z}.$$

Now let us introduce the anisotropic Besov spaces, which are important to control the low horizontal frequency of the function. The role of anisotropic Sobolev or Besov spaces in the study of a Navier–Stokes system with anisotropic vertical viscosity appears in [7, 16].

Definition 2.1. Let $s_1, s_2 \in \mathbb{R}$, $q \in [1, \infty]$. The anisotropic Besov space $B_{2,q}^{s_1,s_2}$ is defined by

$$B_{2,q}^{s_1,s_2} \stackrel{\text{def}}{=} \{f \in S'(\mathbb{R}^3) : \|f\|_{B_{2,q}^{s_1,s_2}} < \infty\},$$

where

$$\|f\|_{B_{2,q}^{s_1,s_2}} \stackrel{\text{def}}{=} \bigg(\sum_{k\in\mathbb{Z}} 2^{qks_1} \sup_{j\in\mathbb{Z}} 2^{qjs_2} \|\Delta_j^v \dot{\Delta}_k^h f\|_{L^2}^q\bigg)^{\frac{1}{q}}.$$

We also need to use Chemin–Lerner-type Besov space $\widetilde{L}^p(0,T; B_{2,q}^{s_1,s_2})$, whose norm is defined by

$$\|u\|_{\widetilde{L}^p(0,T;B_{2,q}^{s_1,s_2})} \stackrel{\text{def}}{=} \bigg(\sum_{k\in\mathbb{Z}} 2^{qks_1} \Big\| \sup_{j\in\mathbb{Z}} 2^{js_2} \|\Delta_j^v \dot{\Delta}_k^h u(t)\|_{L^2} \Big\|_{L^p(0,T)}^q\bigg)^{\frac{1}{q}}.$$

For the sake of simplicity, we will denote $\widetilde{L}^p(0,T; B_{2,q}^{s_1,s_2})$ by $\widetilde{L}_T^p B_{2,q}^{s_1,s_2}$. It is easy to see that $\widetilde{L}_T^p B_{2,q}^{s_1,s_2} = L_T^p B_{2,q}^{s_1,s_2}$ if $p = q$.

For the rescaled system (2.1), we prove the following.

Theorem 2.2. *Let $a > 0, s > \frac{1}{2}$. Assume that (α, γ) satisfies*

$$\alpha \in \left(\frac{1}{2}, 1\right), \qquad \frac{2}{1+\alpha} < \gamma \leqslant \frac{1}{\alpha}. \tag{2.3}$$

Then there exists a constant η such that, for any divergence-free field v_0 satisfying

$$\|e^{a|D_3|^{\gamma\alpha}} v_0\|_{B^{0,s}_{2,\gamma}} + \|e^{a|D_3|^{\gamma\alpha}} v_0\|_{B^{-\alpha,s}_{2,\gamma}} \leqslant \eta$$

for any $\varepsilon \in (0, 1)$, (RNS_ε) has a unique global smooth solution.

2.2. Definition of the functional setting

As in [12], the proof relies on exponential decay estimates for the Fourier transform of the solution. Thus, for any locally bounded function Φ on $\mathbb{R}^+ \times \mathbb{R}^3$, and for any function f, continuous in time and compactly supported in Fourier space, we define

$$f_\Phi(t) \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(e^{\Phi(t,\cdot)}\widehat{f}(t, \cdot)\big).$$

Now we introduce two key quantities, which capture the weak damping mechanism of the parabolic operator $\partial_t - \Delta_\varepsilon$. We define the function $\theta(t)$ by

$$\dot{\theta}(t) \stackrel{\text{def}}{=} \varepsilon^{\gamma\alpha}\|v^h_\Phi(t)\|^\gamma_{B^{-\alpha+\frac{2}{\gamma},s}_{2,\gamma}} + \|v^3_\Phi(t)\|^\gamma_{B^{-\alpha+\frac{2}{\gamma},s}_{2,\gamma}} \quad \text{and} \quad \theta(0) = 0, \tag{2.4}$$

and we also define

$$\Psi(t) \stackrel{\text{def}}{=} \|v_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}, \tag{2.5}$$

where $\gamma' = \frac{\gamma}{\gamma-1}$ and

$$\Phi(t, \xi) \stackrel{\text{def}}{=} (a - \lambda\theta(t))\langle\xi_3\rangle^{\gamma\alpha}, \quad \langle\xi_3\rangle = (1 + \xi_3^2)^{\frac{1}{2}}, \tag{2.6}$$

for some λ that will be chosen later on.

Proposition 2.3. *A constant C exists such that, for any positive λ, and for any t satisfying $\theta(t) \leqslant a/\lambda$, we have*

$$\theta(t)^{\frac{1}{\gamma}} \leqslant C\|e^{a|D_3|^{\gamma\alpha}} v_0\|_{B^{-\alpha,s}_{2,\gamma}} + C\Psi(t)\theta(t)^{\frac{1}{\gamma}}.$$

Proposition 2.4. *A constant C exists such that, for any positive λ, and for any t satisfying $\theta(t) \leqslant a/\lambda$, we have*

$$\Psi(t) \leqslant C\|e^{a|D_3|^{\gamma\alpha}} v_0\|_{B^{0,s}_{2,\gamma}} + C\Psi(t)\left(\Psi(t) + \frac{1}{\lambda^{\frac{1}{\gamma}}}\right).$$

The proof of Propositions 2.3 and 2.4 will be presented in §§ 5 and 6 respectively. In the following subsection, we will use Propositions 2.3 and 2.4 to complete the proof of Theorem 2.2.

2.3. Proof of Theorem 2.2

We use a continuation argument. For any η, let us define

$$\mathcal{T}_\lambda \stackrel{\text{def}}{=} \{T : \theta(T)^{\frac{1}{\gamma}} \leqslant 4C\eta, \Psi(T) \leqslant 4C\eta\}.$$

Similar to the argument in [12], $\mathcal{T}_\lambda$ is of the form $[0, T^*)$ for some positive T^*. Hence, it suffices to prove that $T^* = +\infty$. In order to use Propositions 2.3 and 2.4, we need to assume that $\theta(T) \leqslant \frac{a}{\lambda}$, which leads to the condition

$$4C\eta \leqslant \left(\frac{a}{\lambda}\right)^{\frac{1}{\gamma}}.$$

From Propositions 2.3 and 2.4, it follows that, for all $T \in \mathcal{T}_\lambda$,

$$\begin{aligned} \theta(T)^{\frac{1}{\gamma}} &\leqslant C\eta + 16C^2\eta^2, \\ \Psi(T) &\leqslant C\eta + C\eta\left(C\eta + \frac{1}{\lambda^{\frac{1}{\gamma}}}\right). \end{aligned} \tag{2.7}$$

We first choose λ large enough that $\frac{1}{\lambda^{\frac{1}{\gamma}}} \leqslant 1$, and then choose η small enough such that

$$4C\eta \leqslant \min\left(\left(\frac{a}{\lambda}\right)^{\frac{1}{\gamma}}, \frac{1}{2}\right).$$

With this choice of η, we infer from (2.7) that

$$\theta(T)^{\frac{1}{\gamma}} < 4C\eta, \quad \Psi(T) < 4C\eta,$$

which ensures that $T^* = +\infty$, and thus we conclude the proof of Theorem 2.2. □

3. The action of subadditive phases on products

For any function f, we denote by f^+ the inverse Fourier transform of $|\widehat{f}|$. Let us notice that the map $f \mapsto f^+$ preserves the norm of all $B^{s_1,s_2}_{2,q}$ spaces. Throughout this section, Ψ will denote a locally bounded function on $\mathbb{R}^+ \times \mathbb{R}^3$ which satisfies the following inequality:

$$\Psi(t,\xi) \leqslant \Psi(t,\xi-\eta) + \Psi(t,\eta). \tag{3.1}$$

In what follows, we will constantly use Bony's decomposition from [5] that

$$fg = T^v_f g + R^v_f g, \tag{3.2}$$

with

$$T^v_f g = \sum_j S^v_{j-1} f \Delta^v_j g, \quad R_g f = \sum_j S^v_{j+2} f \Delta^v_j g.$$

We also use Bony's decomposition in the horizontal direction,

$$fg = T^h_f g + T^h_f g + R^h(f,g), \tag{3.3}$$

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with

$$T^h_f g = \sum_j S^h_{j-1} f \dot{\Delta}^h_j g, \quad R^h(f,g) = \sum_{|j'-j|\leqslant 1} \dot{\Delta}^h_j f \dot{\Delta}^h_{j'} g.$$

Lemma 3.1 (Bernstein's inequality). *Let $1 \leqslant p \leqslant q \leqslant \infty$. Assume that $f \in L^p(\mathbb{R}^d)$. Then there exists a constant C, independent of f, j, such that*

$$\operatorname{supp}\hat{f} \subset \{|\xi| \leqslant C2^j\} \Rightarrow \|\partial^\alpha f\|_{L^q} \leqslant C2^{j|\alpha|+dj(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p},$$
$$\operatorname{supp}\hat{f} \subset \left\{\frac{1}{C}2^j \leqslant |\xi| \leqslant C2^j\right\} \Rightarrow \|f\|_{L^p} \leqslant C2^{-j|\alpha|} \sup_{|\beta|=|\alpha|} \|\partial^\beta f\|_{L^p}.$$

Lemma 3.2. *Let $s > \frac{1}{2}, q \in [1,\infty]$, $p, p_1, p_2 \in [1,\infty]$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\sigma_1, \sigma_2 < 1$, and $\sigma_1 + \sigma_2 > 0$. Assume that $a_\Psi \in \widetilde{L}^{p_1}_T B^{\sigma_1,s}_{2,q}$ and $b_\Psi \in \widetilde{L}^{p_2}_T B^{\sigma_2,s}_{2,q}$. Then there holds*

$$\begin{aligned}&\Big\|\sup_{j\in\mathbb{Z}} 2^{js}\big\|\big[\Delta^v_j \dot{\Delta}^h_k (T^v_a b)\big]_\Psi\big\|_{L^2}\Big\|_{L^p_T} + \Big\|\sup_{j\in\mathbb{Z}} 2^{js}\big\|\big[\Delta^v_j \dot{\Delta}^h_k (R^v_a b)\big]_\Psi\big\|_{L^2}\Big\|_{L^p_T}\\ &\quad\leqslant Cc_k 2^{(1-\sigma_1-\sigma_2)k}\|a_\Psi\|_{\widetilde{L}^{p_1}_T B^{\sigma_1,s}_{2,q}}\|b_\Psi\|_{\widetilde{L}^{p_2}_T B^{\sigma_2,s}_{2,q}},\end{aligned}$$

with the sequence $(c_k)_{k\in\mathbb{Z}}$ satisfying $\sum_{k\in\mathbb{Z}} c^q_k \leqslant 1$.

Proof. Let us first prove the case when the function Ψ is identically 0. Below, we only present the proof for $R_a b$; the proof for $T_a b$ is very similar. Using Bony's decomposition (3.3) in the horizontal direction, we write

$$\begin{aligned}\Delta_j \dot{\Delta}^h_k (R^v_a b) &= \sum_{j'} \Delta^v_j \dot{\Delta}^h_k (S^v_{j'+2} a \Delta^v_{j'} b)\\ &= \sum_{j'} \Delta^v_j \dot{\Delta}^h_k \big(T^h_{S^v_{j'+2}a} \Delta^v_{j'} b + T^h_{\Delta^v_{j'}b} S^v_{j'+2} a + R^h(S^v_{j'+2}a, \Delta^v_{j'} b)\big)\\ &\stackrel{\text{def}}{=} A + B + C.\end{aligned}$$

Considering the support of the Fourier transform of $T^h_{S^v_{j'+2}a}\Delta^v_{j'}b$, we find

$$A = \sum_{j'\geqslant j-4} \sum_{|k'-k|\leqslant 4} \Delta^v_j \dot{\Delta}^h_k \big(S^v_{j'+2} S^h_{k'-1} a \Delta^v_{j'} \dot{\Delta}^h_{k'} b\big).$$

Then we get by Lemma 3.1 that

$$\begin{aligned}\|A\|_{L^2} &\leqslant C \sum_{j'\geqslant j-4} \sum_{|k'-k|\leqslant 4} \|S^v_{j'+2} S^h_{k'-1} a \Delta_{j'} \dot{\Delta}^h_{k'} b\|_{L^2}\\ &\leqslant C \sum_{j'\geqslant j-4} \sum_{|k'-k|\leqslant 4} \|S^v_{j'+2} S^h_{k'-1} a\|_{L^\infty} \|\Delta^v_{j'} \dot{\Delta}^h_{k'} b\|_{L^2}.\end{aligned}$$

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We use Lemma 3.1 again to get

$$\begin{aligned}\|S^v_{j'+2}S^h_{k'-1}a\|_{L^\infty} &\leqslant \sum_{j''\leqslant j'+1}\sum_{k''\leqslant k'-2}\|\Delta^v_{j''}\dot{\Delta}^h_{k''}a\|_{L^\infty}\\ &\leqslant C\sum_{j''\leqslant j'+1}\sum_{k''\leqslant k'-2}2^{k''}\|\Delta^v_{j''}\dot{\Delta}^h_{k''}a\|_{L^\infty_{x_3}L^2_{x_h}}\\ &\leqslant C\sum_{j''\leqslant j'+1}\sum_{k''\leqslant k'-2}2^{\frac{j''}{2}}2^{k''}\|\Delta^v_{j''}\dot{\Delta}^h_{k''}a\|_{L^2},\end{aligned}$$

from which and $\sigma_1 < 1$ it follows that

$$\begin{aligned}\Big\|\sup_{j\in\mathbb{Z}}2^{js}\|A\|_{L^2}\Big\|_{L^p_T} &\leqslant C2^{(1-\sigma_1-\sigma_2)k}\|a\|_{\widetilde{L}^{p_1}_T B^{\sigma_1,s}_{2,q}}\|b\|_{\widetilde{L}^{p_2}_T B^{\sigma_2,s}_{2,q}}\sup_{j\in\mathbb{Z}}\sum_{j'\geqslant j-4}\sum_{|k'-k|\leqslant 4}2^{-(j'-j)s}c_{k'}\\ &\leqslant Cc_k2^{(1-\sigma_1-\sigma_2)k}\|a\|_{\widetilde{L}^{p_1}_T B^{\sigma_1,s}_{2,q}}\|b\|_{\widetilde{L}^{p_2}_T B^{\sigma_2,s}_{2,q}}. \end{aligned}\tag{3.4}$$

Similarly, we have

$$B=\sum_{j'\geqslant j-4}\sum_{|k'-k|\leqslant 4}\Delta^v_j\dot{\Delta}^h_k(\Delta^v_{j'}S^h_{k'-1}bS^v_{j'+2}\dot{\Delta}^h_{k'}a).$$

Then we get by Lemma 3.1 and $\sigma_2 < 1$ that

$$\begin{aligned}\Big\|\sup_{j\in\mathbb{Z}}2^{js}\|B\|_{L^2}\Big\|_{L^p_T} &\leqslant C\Big\|\sup_{j\in\mathbb{Z}}2^{js}\sum_{j'\geqslant j-4}\sum_{|k'-k|\leqslant 4}\|\Delta^v_{j'}S^h_{k'-1}b\|_{L^2_{x_3}L^\infty_{x_h}}\|S^v_{j'+2}\dot{\Delta}^h_{k'}a\|_{L^2_{x_h}L^\infty_{x_3}}\Big\|_{L^p_T}\\ &\leqslant Cc_k2^{(1-\sigma_1-\sigma_2)k}\|a\|_{\widetilde{L}^{p_1}_T B^{\sigma_1,s}_{2,q}}\|b\|_{\widetilde{L}^{p_2}_T B^{\sigma_2,s}_{2,q}}. \end{aligned}\tag{3.5}$$

Now let us turn to C. We have

$$C=\sum_{j'\geqslant j-4}\sum_{k',k''\geqslant k-2;|k'-k''|\leqslant 1}\Delta^v_j\dot{\Delta}^h_k(S^v_{j'+2}\dot{\Delta}^h_{k'}a\Delta^v_{j'}\dot{\Delta}^h_{k''}b).$$

Hence, we have by Lemma 3.1 and $\sigma_1+\sigma_2>0$ that

$$\begin{aligned}\Big\|\sup_{j\in\mathbb{Z}}2^{js}\|C\|_{L^2}\Big\|_{L^p_T} &\leqslant C\Big\|\sup_{j\in\mathbb{Z}}2^{js}\sum_{j'\geqslant j-4}\sum_{k',k''\geqslant k-2;|k'-k''|\leqslant 1}2^k\|S^v_{j'+2}\dot{\Delta}^h_{k'}a\Delta^v_{j'}\dot{\Delta}^h_{k''}b\|_{L^2_{x_3}L^1_{x_h}}\Big\|_{L^p_T}\\ &\leqslant C\Big\|\sup_{j\in\mathbb{Z}}2^{js}\sum_{j'\geqslant j-4}\sum_{k',k''\geqslant k-2;|k'-k''|\leqslant 1}2^k\|S^v_{j'+2}\dot{\Delta}^h_{k'}a\|_{L^\infty_{x_3}L^2_{x_h}}\|\Delta^v_{j'}\dot{\Delta}^h_{k''}b\|_{L^2}\Big\|_{L^p_T}\\ &\leqslant C2^{(1-\sigma_1-\sigma_2)k}\|a\|_{\widetilde{L}^{p_1}_T B^{\sigma_1,s}_{2,q}}\|b\|_{\widetilde{L}^{p_2}_T B^{\sigma_2,s}_{2,q}}\sum_{k'\geqslant k-2}2^{-(\sigma_1+\sigma_2)(k'-k)}c_{k'}\\ &\leqslant Cc_k2^{(1-\sigma_1-\sigma_2)k}\|a\|_{\widetilde{L}^{p_1}_T B^{\sigma_1,s}_{2,q}}\|b\|_{\widetilde{L}^{p_2}_T B^{\sigma_2,s}_{2,q}}. \end{aligned}\tag{3.6}$$

Summing up (3.4)–(3.6) yields that

$$\Big\|\sup_{j\in\mathbb{Z}}2^{js}\|\Delta^v_j\dot{\Delta}^h_k(R_ab)\|_{L^2}\Big\|_{L^p_T}\leqslant Cc_k2^{(1-\sigma_1-\sigma_2)k}\|a\|_{\widetilde{L}^{p_1}_T B^{\sigma_1,s}_{2,q}}\|b\|_{\widetilde{L}^{p_2}_T B^{\sigma_2,s}_{2,q}}.$$

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The lemma is proved in the case when the function Ψ is identically 0. In order to deal with the general case, we only need to notice the fact that

$$\left|\mathcal{F}\left[\Delta_j \dot{\Delta}_k^h (R_a b)\right]_\Psi(\xi)\right| \leqslant \mathcal{F}\left[\Delta_j \dot{\Delta}_k^h (R_{a_\Psi^+} b_\Psi^+)\right](\xi).$$

The proof of Lemma 3.2 is finished. □

Remark 3.3. From the proof of Lemma 3.2, it is easy to see that

$$\begin{aligned}\left\|\left[\Delta_j^v \dot{\Delta}_k^h(ab)\right]_\Psi\right\|_{L^2} &\leqslant C2^k 2^{-js}\|a_\Psi\|_{B_{2,q}^{\sigma_1,s}} \sum_{k'\geqslant k-4} 2^{-k'\sigma_1} \sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j^v \dot{\Delta}_{k'}^h b_\Psi\|_{L^2}\\ &\quad + C2^{-k\sigma_1}2^{-js}\|a_\Psi\|_{B_{2,q}^{\sigma_1,s}} \sum_{k'\leqslant k+4} 2^{k'} \sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j^v \dot{\Delta}_{k'}^h b_\Psi\|_{L^2}\end{aligned}$$

for $\sigma_1 < 1$ and $s > \frac{1}{2}$.

The following lemma is a direct consequence of Lemma 3.2.

Lemma 3.4. *Let $s > \frac{1}{2}, q \in [1,\infty],\ p, p_1, p_2 \in [1,\infty]$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Assume that $a_\Psi \in \widetilde{L}_T^{p_1} B_{2,q}^{\sigma_1,s}$ and $b_\Psi \in \widetilde{L}_T^{p_2} B_{2,q}^{\sigma_2,s}$ with $\sigma_1, \sigma_2 < 1$ and $\sigma_1 + \sigma_2 > 0$. Then we have*

$$\|(ab)_\Psi\|_{\widetilde{L}_T^p B_{2,q}^{\sigma_1+\sigma_2-1,s}} \leqslant C\|a_\Psi\|_{\widetilde{L}_T^{p_1} B_{2,q}^{\sigma_1,s}} \|b_\Psi\|_{\widetilde{L}_T^{p_2} B_{2,q}^{\sigma_2,s}}.$$

4. The action of the phase Φ on the heat operator

Let $\Phi(t,\xi)$ be given by (2.6). In this section, we will study the action of the multiplier $e^{\Phi(t,D)}$ on the heat operator $e^{t\Delta_\varepsilon}$.

Lemma 4.1. *Let $q_1, q_2, r \in [1,\infty], s_1, s_2 \in \mathbb{R}$. Then there holds*

$$\left\|\left(e^{t\Delta_\varepsilon} v_0\right)_\Phi\right\|_{\widetilde{L}_T^r B_{2,q}^{s_1,s_2}} \leqslant C\|e^{a|D_3|^{\gamma\alpha}} v_0\|_{B_{2,q}^{s_1-\frac{2}{r},s_2}}.$$

Proof. Thanks to the definition of $\widetilde{L}_T^r B_{2,q}^{s_1,s_2}$ space, we have

$$\begin{aligned}\left\|\left(e^{t\Delta_\varepsilon} v_0\right)_\Phi\right\|_{\widetilde{L}_T^r B_{2,q}^{s_1,s_2}} &\leqslant \left(\sum_{k\in\mathbb{Z}} 2^{qks_1}\|e^{-ct2^{2k}}\|_{L_t^r}^q \sup_{j\in\mathbb{Z}} 2^{qjs_2}\|\Delta_j^v \dot{\Delta}_k^h e^{a|D_3|^{\gamma\alpha}} v_0\|_{L^2}^q\right)^{\frac{1}{q}}\\ &\leqslant C\left(\sum_{k\in\mathbb{Z}} 2^{qk(s_1-\frac{2}{r})} \sup_{j\in\mathbb{Z}} 2^{qjs_2}\|\Delta_j^v \dot{\Delta}_k^h e^{a|D_3|^{\gamma\alpha}} v_0\|_{L^2}^q\right)^{\frac{1}{q}}\\ &= C\|e^{a|D_3|^{\gamma\alpha}} v_0^3\|_{B_{2,q}^{s_1-\frac{2}{r},s_2}}.\end{aligned}$$

The proof of Lemma 4.1 is finished. □

We define E_ε by

$$E_\varepsilon f(t) \stackrel{\text{def}}{=} \int_0^t e^{(t-\tau)\Delta_\varepsilon} f(\tau)d\tau.$$

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Lemma 4.2. *Let* $q, r_1, r_2 \in [1,\infty], r_2 \leqslant r_1, s_1, s_2 \in \mathbb{R}$. *Then, for any* $f \in \widetilde{L}^{r_2}(0,T; B^{s_1-\sigma,s_2}_{2,q})$ *with* $\sigma = 2(1+\frac{1}{r_1}-\frac{1}{r_2})$, *we have*

$$\big\|\big(E_\varepsilon f\big)_\Phi\big\|_{\widetilde{L}^{r_1}_T B^{s_1,s_2}_{2,q}} \leqslant C\|f_\Phi\|_{\widetilde{L}^{r_2}_T B^{s_1-\sigma,s_2}_{2,q}}.$$

Let $\beta \in [0,1]$ *with* $\beta < \sigma$. *Then, for any* $f \in \widetilde{L}^{r_2}(0,T; B^{s_1-\sigma+1,s_2}_{2,q})$, *we have*

$$\big\|\big(E_\varepsilon\big((\varepsilon\partial_3)^\beta f\big)\big)_\Phi\big\|_{\widetilde{L}^{r_1}_T B^{s_1,s_2}_{2,q}} \leqslant C\|f_\Phi\|_{\widetilde{L}^{r_2}_T B^{s_1-\sigma+\beta,s_2}_{2,q}}.$$

Proof. Applying the Fourier multiplier $e^{\Phi(t,D)}\Delta^v_j\dot{\Delta}^h_k$ to $E_\varepsilon f$ gives

$$e^{\Phi(t,D)}\Delta^v_j\dot{\Delta}^h_k E_\varepsilon f = \int_0^t e^{(t-\tau)\Delta_\varepsilon+\Phi(t,D)-\Phi(\tau,D)}\Delta^v_j\dot{\Delta}^h_k f_\Phi(\tau)d\tau.$$

Due to the definition of Φ, we have

$$\Phi(t,\xi)-\Phi(\tau,\xi) = -\lambda\langle\xi_3\rangle^{\gamma\alpha}\int_\tau^t \dot{\theta}(t')dt'.$$

Then we take the Fourier transform to get

$$\mathcal{F}\Big(e^{\Phi(t,D)}\Delta^v_j\dot{\Delta}^h_k E_\varepsilon f\Big) = \int_0^t e^{-(t-\tau)|\xi|^2_\varepsilon-\langle\xi_3\rangle^{\gamma\alpha}\int_\tau^t\dot{\theta}(t')dt'}\mathcal{F}\big[\Delta^v_j\dot{\Delta}^h_k f\big]_\Phi(\tau)d\tau, \tag{4.1}$$

where $|\xi|^2_\varepsilon \stackrel{\text{def}}{=} |\xi_h|^2+\varepsilon^2|\xi_3|^2$. We take the L^2 norm to obtain

$$\big\|\big[\Delta^v_j\dot{\Delta}^h_k E_\varepsilon f\big]_\Phi\big\|_{L^2} \leqslant \int_0^t e^{-c2^{2k}(t-\tau)}\|\Delta^v_j\dot{\Delta}^h_k f_\Phi(\tau)\|_{L^2}d\tau,$$

from which and Young's inequality we infer that

$$\Big\|\sup_{j\in\mathbb{Z}} 2^{js_2}\big\|\big[\Delta^v_j\dot{\Delta}^h_k E_\varepsilon f\big]_\Phi\big\|_{L^2}\Big\|_{L^{r_1}_T} \leqslant C2^{-k\sigma}\Big\|\sup_{j\in\mathbb{Z}} 2^{js_2}\|\Delta^v_j\dot{\Delta}^h_k f_\Phi\|_{L^2}\Big\|_{L^{r_2}_T}.$$

This gives the first inequality of Lemma 4.2.

Noticing that

$$|\varepsilon\xi_3|^\beta e^{-t\varepsilon^2|\xi_3|^2} \leqslant Ct^{-\frac{\beta}{2}},$$

we infer from (4.1) that

$$\big\|\big[\Delta^v_j\dot{\Delta}^h_k E_\varepsilon(\varepsilon\partial_3)f\big]_\Phi\big\|_{L^2} \leqslant C\int_0^t (t-\tau)^{-\frac{\beta}{2}}e^{-c2^{2k}(t-\tau)}\|\Delta^v_j\dot{\Delta}^h_k f_\Phi(\tau)\|_{L^2}d\tau.$$

Due to $\beta < \sigma$, we have

$$\int_0^t t^{-\frac{\beta}{\sigma}}e^{-ct2^{2k}}dt \leqslant Ct^{2(-1+\frac{\beta}{\sigma})k}.$$

Then, applying Young's inequality gives

$$\Big\|\sup_{j\in\mathbb{Z}} 2^{js_2}\big\|\big[\Delta^v_j\dot{\Delta}^h_k E_\varepsilon(\varepsilon\partial_3)f\big]_\Phi\big\|_{L^2}\Big\|_{L^{r_1}_T} \leqslant C2^{-k(\sigma-\beta)}\Big\|\sup_{j\in\mathbb{Z}} 2^{js_2}\|\Delta^v_j\dot{\Delta}^h_k f_\Phi\|_{L^2}\Big\|_{L^{r_2}_T}.$$

This gives the second inequality of Lemma 4.2. □

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Lemma 4.3. *Let $a(D)$ be a Fourier multiplier such that $|a(\xi)| \leqslant C|\xi_3|$. We have*

$$\varepsilon^{1-\alpha}\big\|\big[E_\varepsilon(a(D)(v^3 f))\big]_\Phi\big\|_{\widetilde{L}_t^{\gamma'}B_{2,\gamma}^{\frac{2}{\gamma'},s}} \leqslant \frac{C}{\lambda^{\frac{1}{\gamma}}}\|f_\Phi\|_{\widetilde{L}_t^{\gamma'}B_{2,\gamma}^{\frac{2}{\gamma'},s}},$$

$$\varepsilon^{1-\alpha}\big\|\big[E_\varepsilon(a(D)(v^3 \nabla_h f))\big]_\Phi\big\|_{\widetilde{L}_t^{\gamma'}B_{2,\gamma}^{\frac{2}{\gamma'}-1,s}} \leqslant \frac{C}{\lambda^{\frac{1}{\gamma}}}\|f_\Phi\|_{\widetilde{L}_t^{\gamma'}B_{2,\gamma}^{\frac{2}{\gamma'},s}}.$$

Here (s, α, γ) satisfy the conditions in Theorem 2.2.

Proof. Noticing that

$$|\varepsilon\xi_3|^{1-\alpha}e^{-t\varepsilon^2|\xi_3|^2} \leqslant Ct^{-\frac{1-\alpha}{2}},$$

we infer from (4.1) that

$$\begin{aligned}
&\varepsilon^{1-\alpha}\big\|\big[\Delta_j^v\dot{\Delta}_k^h E_\varepsilon(a(D)(v^3 f))\big]_\Phi\big\|_{L^2}\\
&\quad\leqslant C2^{j\alpha}\int_0^t (t-\tau)^{-\frac{1-\alpha}{2}}e^{-c2^{2k}(t-\tau)-c\lambda 2^{j\gamma\alpha}\int_\tau^t\dot{\theta}(t')dt'}\|\Delta_j^v\dot{\Delta}_k^h(v^3 f)_\Phi\|_{L^2}d\tau.
\end{aligned}$$

Thanks to Remark 3.3, we have

$$\begin{aligned}
\|\Delta_j^v\dot{\Delta}_k^h(v^3 f)_\Phi\|_{L^2} &\leqslant C2^k 2^{-js}\|v_\Phi^3\|_{B_{2,\gamma}^{-\alpha+\frac{2}{\gamma},s}}\sum_{k'\geqslant k-4}2^{k'(\alpha-\frac{2}{\gamma})}\sup_{j\in\mathbb{Z}}2^{js}\|\Delta_j^v\dot{\Delta}_{k'}^h f_\Phi\|_{L^2}\\
&\quad + C2^{k(\alpha-\frac{2}{\gamma})}2^{-js}\|v_\Phi^3\|_{B_{2,\gamma}^{-\alpha+\frac{2}{\gamma},s}}\sum_{k'\leqslant k+4}2^{k'}\sup_{j\in\mathbb{Z}}2^{js}\|\Delta_j^v\dot{\Delta}_{k'}^h f_\Phi\|_{L^2}.
\end{aligned}$$

Consequently, we obtain

$$\varepsilon^{1-\alpha}\big\|\big[\Delta_j^v\dot{\Delta}_k^h E_\varepsilon(a(D)(v^3 f))\big]_\Phi\big\|_{L^2} \leqslant C2^{-js}2^{-\frac{2}{\gamma'}k}\big(G_{j,k}^1(t)+G_{j,k}^2(t)\big), \tag{4.2}$$

where $G_{j,k}^i(i=1,2)$ is given by

$$\begin{aligned}
G_{j,k}^1(t) &= 2^{j\alpha}2^{k(3-\frac{2}{\gamma})}\int_0^t (t-\tau)^{-\frac{1-\alpha}{2}}e^{-c2^{2k}(t-\tau)-c\lambda 2^{j\gamma\alpha}\int_\tau^t\dot{\theta}(t')dt'}\|v_\Phi^3(\tau)\|_{B_{2,\gamma}^{-\alpha+\frac{2}{\gamma},s}}\\
&\quad\times\sum_{k'\geqslant k-4}2^{k'(\alpha-\frac{2}{\gamma})}\sup_{j\in\mathbb{Z}}2^{js}\|\Delta_j^v\dot{\Delta}_{k'}^h f_\Phi(\tau)\|_{L^2}d\tau,\\
G_{j,k}^2(t) &= 2^{j\alpha}2^{k(2+\alpha-\frac{4}{\gamma})}\int_0^t (t-\tau)^{-\frac{1-\alpha}{2}}e^{-c2^{2k}(t-\tau)-c\lambda 2^{j\gamma\alpha}\int_\tau^t\dot{\theta}(t')dt'}\|v_\Phi^3(\tau)\|_{B_{2,\gamma}^{-\alpha+\frac{2}{\gamma},s}}\\
&\quad\times\sum_{k'\leqslant k+4}2^{k'}\sup_{j\in\mathbb{Z}}2^{js}\|\Delta_j^v\dot{\Delta}_{k'}^h f_\Phi(\tau)\|_{L^2}d\tau.
\end{aligned}$$

We get by the Hölder inequality that

$$\begin{aligned}
G_{j,k}^1(t) &\leqslant 2^{j\alpha}2^{k(3-\frac{2}{\gamma})}\left(\int_0^t e^{-c\lambda 2^{j\gamma\alpha}\int_\tau^t\dot{\theta}(t')dt'}\dot{\theta}(\tau)d\tau\right)^{\frac{1}{\gamma}}\\
&\quad\times\left(\int_0^t (t-\tau)^{-\frac{(1-\alpha)\gamma'}{2}}e^{-c2^{2k}(t-\tau)}\left(\sum_{k'\geqslant k-4}2^{k'(\alpha-\frac{2}{\gamma})}\sup_{j\in\mathbb{Z}}2^{js}\|\Delta_j^v\dot{\Delta}_{k'}^h f_\Phi(\tau)\|_{L^2}\right)^{\gamma'}d\tau\right)^{\frac{1}{\gamma'}},
\end{aligned}$$

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from which we infer that

$$\sum_{k\in\mathbb{Z}}\|\sup_{j\in\mathbb{Z}} G^1_{j,k}\|^{\gamma}_{L^{\gamma'}_t} \leqslant \frac{C}{\lambda}\sum_{k\in\mathbb{Z}} 2^{(2-\alpha)\gamma k}\bigg(\int_0^t\bigg(\sum_{k'\geqslant k-4} 2^{k'(\alpha-\frac{2}{\gamma})}\sup_{j\in\mathbb{Z}} 2^{js}\|\Delta^v_j\dot{\Delta}^h_{k'}f_\Phi(\tau)\|_{L^2}\bigg)^{\gamma'}d\tau\bigg)^{\gamma-1}$$
$$\leqslant \frac{C}{\lambda}\sum_{k\in\mathbb{Z}} 2^{(2-\alpha)\gamma k}\bigg(\sum_{k'\geqslant k-4} 2^{k'(\alpha-\frac{2}{\gamma})}\Big\|\sup_{j\in Z} 2^{js}\|\Delta^v_j\dot{\Delta}^h_{k'}f_\Phi\|_{L^2}\Big\|_{L^{\gamma'}_t}\bigg)^{\gamma}$$
$$\leqslant \frac{C}{\lambda}\|f_\Phi\|^{\gamma}_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}. \tag{4.3}$$

Here, we used the fact that, due to $\gamma > \frac{2}{1+\alpha}$,

$$\frac{(1-\alpha)\gamma'}{2} < \frac{1-\alpha}{2}\frac{2}{1-\alpha} = 1.$$

Similarly, we have

$$\sum_{k\in\mathbb{Z}}\|\sup_{j\in\mathbb{Z}} G^2_{j,k}\|^{\gamma}_{L^{\gamma'}_t} \leqslant \frac{C}{\lambda}\sum_{k\in\mathbb{Z}} 2^{(\gamma-2)k}\bigg(\int_0^t\bigg(\sum_{k'\leqslant k+4} 2^{k'}\sup_{j\in\mathbb{Z}} 2^{js}\|\Delta^v_j\dot{\Delta}^h_{k'}f_\Phi(\tau)\|_{L^2}\bigg)^{\gamma'}d\tau\bigg)^{\gamma}$$
$$\leqslant \frac{C}{\lambda}\sum_{k\in\mathbb{Z}} 2^{(\gamma-2)k}\bigg(\sum_{k'\leqslant k+4} 2^{k'}\sup_{j\in\mathbb{Z}} 2^{js}\|\Delta^v_j\dot{\Delta}^h_{k'}f_\Phi\|_{L^{\gamma'}_t L^2}\bigg)^{\gamma}$$
$$\leqslant \frac{C}{\lambda}\|f_\Phi\|^{\gamma}_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}. \tag{4.4}$$

Combining (4.3) and (4.4) with (4.2), we get

$$\varepsilon^{1-\alpha}\bigg(\sum_{k\in\mathbb{Z}} 2^{2(\gamma-1)k}\Big\|\sup_{j\in\mathbb{Z}} 2^{js}\|[\Delta^v_j\dot{\Delta}^h_k E_\varepsilon(a(D)(v^3 f))]_\Phi\|_{L^2}\Big\|^{\gamma}_{L^{\gamma'}_t}\bigg)^{\frac{1}{\gamma}}$$
$$\leqslant C\bigg(\sum_{k\in\mathbb{Z}}\|\sup_{j\in\mathbb{Z}} G^1_{j,k}\|^{\gamma}_{L^{\gamma'}_t} + \sum_{k\in\mathbb{Z}}\|\sup_{j\in\mathbb{Z}} G^1_{j,k}\|^{\gamma}_{L^{\gamma'}_t}\bigg)^{\frac{1}{\gamma}}$$
$$\leqslant \frac{C}{\lambda^{\frac{1}{\gamma}}}\|f_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}.$$

This gives the first inequality of Lemma 4.3. Since the proof of the second inequality is similar, we omit the details here. □

5. Classical analytical-type estimates

This section is devoted to the proof of Proposition 2.3. In this part, we do not need to use any regularizing effect from the analyticity, but only the fact that $e^{\Phi(t,\xi_3)}$ is a sublinear function.

First of all, we consider the estimate of the vertical part. Applying the Duhamel formula to the second equation of (2.1) gives

$$v^3(t) = e^{t\Delta_\varepsilon}v^3_0 - \varepsilon^{1-\alpha}E_\varepsilon(v\cdot\nabla v^3) - \varepsilon^2 E_\varepsilon(\partial_3 q). \tag{5.1}$$

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From Lemmas 4.1 and 4.2, it follows that

$$\|v^3_\Phi\|_{L^\gamma_t B^{-\alpha+\frac{2}{\gamma},s}_{2,\gamma}} \leqslant C\|e^{a|D_3|^{\gamma\alpha}} v^3_0\|_{B^{-\alpha,s}_{2,\gamma}} + C\|\varepsilon^{1-\alpha}(v\cdot\nabla v^3)_\Phi + \varepsilon^2\partial_3 q_\Phi\|_{\widetilde{L}^1_t B^{-\alpha,s}_{2,\gamma}}. \tag{5.2}$$

Since $\operatorname{div} v = 0$, we write

$$v\cdot\nabla v^3 = v^h\cdot\nabla_h v^3 - v^3\operatorname{div}_h v^h = \nabla_h\cdot(v^h v^3) - 2v^3\operatorname{div}_h v^h.$$

Due to (2.3), we get by Lemma 3.4 that

$$\begin{aligned}&\|\nabla_h\cdot(v^h v^3)_\Phi\|_{\widetilde{L}^1_t B^{-\alpha,s}_{2,\gamma}} + \|(v^3\operatorname{div}_h v^h)_\Phi\|_{\widetilde{L}^1_t B^{-\alpha,s}_{2,\gamma}}\\ &\leqslant C\|v^3_\Phi\|_{L^\gamma_t B^{-\alpha+\frac{2}{\gamma},s}_{2,\gamma}}\|v^h_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}.\end{aligned}$$

We next turn to the pressure term in (5.2). By (2.2) and since $\operatorname{div} v = 0$, the pressure q can be written as

$$\begin{aligned}q = \varepsilon^{1-\alpha}\sum_{\ell,m=1,2}(-\Delta_\varepsilon)^{-1}\partial_\ell\partial_m(v^\ell v^m) + 2\varepsilon^{1-\alpha}\sum_{\ell=1,2}(-\Delta_\varepsilon)^{-1}\partial_\ell\partial_3(v^\ell v^3)\\ - \varepsilon^{1-\alpha}(-\Delta_\varepsilon)^{-1}\partial_3(v^3\operatorname{div}_h v^h) \overset{\text{def}}{=} q^1 + q^2 + q^3.\end{aligned}$$

Using the fact that the Fourier multiplier $\nabla^2_\varepsilon(-\Delta_\varepsilon)^{-1}$ is bounded in $B^{s_1,s_2}_{2,q}$ and Lemma 3.4, we infer that

$$\begin{aligned}\|\varepsilon^2\partial_3 q^1_\Phi\|_{\widetilde{L}^1_t B^{-\alpha,s}_{2,\gamma}} &\leqslant C\varepsilon^{2-\alpha}\|(v^h v^h)_\Phi\|_{\widetilde{L}^1_t B^{1-\alpha,s}_{2,\gamma}}\\ &\leqslant C\varepsilon^{2-\alpha}\|v^h_\Phi\|_{L^\gamma_t B^{-\alpha+\frac{2}{\gamma},s}_{2,\gamma}}\|v^h_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}},\end{aligned}$$

and

$$\begin{aligned}\|\varepsilon^2\partial_3 q^2_\Phi\|_{\widetilde{L}^1_t B^{-\alpha,s}_{2,\gamma}} + \|\varepsilon^2\partial_3 q^3_\Phi\|_{\widetilde{L}^1_t B^{-\alpha,s}_{2,\gamma}} &\leqslant C\varepsilon^{1-\alpha}\|(v^3 v^h)_\Phi\|_{\widetilde{L}^1_t B^{1-\alpha,s}_{2,\gamma}} + C\varepsilon^{1-\alpha}\|(v^3\operatorname{div}_h v^h)_\Phi\|_{\widetilde{L}^1_t B^{-\alpha,s}_{2,\gamma}}\\ &\leqslant C\varepsilon^{1-\alpha}\|v^3_\Phi\|_{L^\gamma_t B^{-\alpha+\frac{2}{\gamma},s}_{2,\gamma}}\|v^h_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}.\end{aligned}$$

Plugging the above estimates into (5.2), and noting that $\alpha < 1$, yields that

$$\|v^3\|_{L^{\frac{1}{\alpha}}_t B^{-\alpha+\frac{2}{\gamma},s}_{2,\gamma}} \leqslant C\|e^{a|D_3|^{\gamma\alpha}} v^3_0\|_{B^{-\alpha,s}_{2,\gamma}} + C\Psi(t)\theta(t)^{\frac{1}{\gamma}}. \tag{5.3}$$

Next, we consider the estimate of the horizontal part. Applying the Duhamel formula to the first equation of (2.1) gives

$$v^h(t) = e^{t\Delta_\varepsilon}v^h_0 - \varepsilon^{1-\alpha}E_\varepsilon(v\cdot\nabla v^h) - E_\varepsilon(\nabla^h q). \tag{5.4}$$

Since $\operatorname{div} v = 0$, we write

$$v\cdot\nabla v^h = \nabla_h\cdot(v^h\otimes v^h) + \partial_3(v^3 v^h).$$

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From Lemmas 4.1 and 4.2, it follows that

$$\varepsilon^\alpha \|v^h_\Phi\|_{L^\gamma_t B^{-\alpha+\frac{2}{\gamma},s}_{2,\gamma}} \leqslant C\|e^{a|D_3|^{\gamma\alpha}} v^h_0\|_{B^{-\alpha,s}_{2,\gamma}} + C\varepsilon\|\nabla^h \cdot (v^h \otimes v^h)_\Phi\|_{\widetilde{L}^1_t B^{-\alpha,s}_{2,\gamma}} + C\|(v^3 v^h)_\Phi\|_{\widetilde{L}^1_t B^{1-\alpha,s}_{2,\gamma}} + C\varepsilon^\alpha \|\nabla_h q_\Phi\|_{\widetilde{L}^1_t B^{-\alpha,s}_{2,\gamma}}.$$

We get by Lemma 3.4 that

$$\varepsilon\|\nabla^h \cdot (v^h \otimes v^h)_\Phi\|_{\widetilde{L}^1_t B^{-\alpha,s}_{2,\gamma}} + \|(v^3 v^h)_\Phi\|_{\widetilde{L}^1_t B^{1-\alpha,s}_{2,\gamma}} \leqslant C\big(\varepsilon\|v^h_\Phi\|_{L^\gamma_t B^{-\alpha+\frac{2}{\gamma},s}_{2,\gamma}} + \|v^3_\Phi\|_{L^\gamma_t B^{-\alpha+\frac{2}{\gamma},s}_{2,\gamma}}\big)\|v^h_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}.$$

For the pressure term, we can proceed as above to get

$$\varepsilon^\alpha \|\nabla_h q_\Phi\|_{\widetilde{L}^1_t B^{-\alpha,s}_{2,\gamma}} \leqslant C\big(\varepsilon\|v^h_\Phi\|_{L^\gamma_t B^{-\alpha+\frac{2}{\gamma},s}_{2,\gamma}} + \|v^3_\Phi\|_{L^\gamma_t B^{-\alpha+\frac{2}{\gamma},s}_{2,\gamma}}\big)\|v^h_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}.$$

Summing up the above estimates yields that

$$\varepsilon^\alpha \|v^h\|_{L^\gamma_t B^{-\alpha+\frac{2}{\gamma},s}_{2,\gamma}} \leqslant C\|e^{a|D_3|^{\gamma\alpha}} v^h_0\|_{B^{-\alpha,s}_{2,\gamma}} + C\Psi(t)\theta(t)^{\frac{1}{\gamma}}. \tag{5.5}$$

Hence, Proposition 2.3 follows from (5.3) and (5.5). □

6. Regularizing effect due to analyticity

This section is devoted to the proof of Proposition 2.4. Here we need to use the regularizing effect from the analyticity.

Let us first consider the estimates of the horizontal part. Due to (5.4), we write

$$v^h(t) = e^{t\Delta_\varepsilon} v^h_0 - \varepsilon^{1-\alpha} E_\varepsilon \nabla^h \cdot (v^h \otimes v^h) - \varepsilon^{1-\alpha} E_\varepsilon \partial_3 (v^3 v^h) + E_\varepsilon(\nabla^h q).$$

First of all, we have by Lemma 4.1 that

$$\|e^{\Phi(t,D)} e^{t\Delta_\varepsilon} v^h_0\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} \leqslant C\|e^{a|D_3|^{\gamma\alpha}} v^h_0\|_{B^{0,s}_{2,\gamma}}. \tag{6.1}$$

We get by Lemmas 4.2 and 3.4 that

$$\begin{aligned} &\|[E_\varepsilon \nabla^h \cdot (v^h \otimes v^h)]_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} \\ &\quad \leqslant C\|(v^h \otimes v^h)_\Phi\|_{\widetilde{L}^{\frac{\gamma'}{2}}_t B^{\frac{4}{\gamma'}-1,s}_{2,\gamma}} \leqslant C\|v^h_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} \|v^h_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}. \end{aligned} \tag{6.2}$$

And we infer from Lemma 4.3 that

$$\varepsilon^{1-\alpha}\|[E_\varepsilon \partial_3 (v^3 v^h)]_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} \leqslant \frac{C}{\lambda^{\frac{1}{\gamma}}}\|v^h\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}. \tag{6.3}$$

We next turn to the estimates of the pressure term. Recall that in § 5 we rewrote the pressure q as

$$q = q^1 + q^2 + q^3.$$

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Using the fact that the Fourier multiplier $\nabla_\varepsilon^2(-\Delta_\varepsilon)^{-1}$ is bounded in $B^{s_1,s_2}_{2,q}$, we get by (6.2) that

$$\|[E_\varepsilon \nabla^h q^1]_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} \leqslant C\|v^h_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} \|v^h_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}, \tag{6.4}$$

and, similar to (6.3), we have

$$\|[E_\varepsilon \nabla^h q^2]_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} \leqslant \frac{C}{\lambda^{\frac{1}{\gamma}}}\|v^h\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}. \tag{6.5}$$

For q^3, we use Lemma 4.3 to get

$$\|[E_\varepsilon \nabla^h q^3]_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} \leqslant C\varepsilon^{1-\alpha}\|[E_\varepsilon \partial_3(v^3 \mathrm{div}_h v^h)]_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{1-2\alpha,s}_{2,\gamma}} \tag{6.6}$$

$$\leqslant \frac{C}{\lambda^{\frac{1}{\gamma}}}\|v^h\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}. \tag{6.7}$$

Summing up (6.1)–(6.6) yields that

$$\|v^h\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} \leqslant C\|e^{a|D_3|^{\alpha\gamma}} v^h_0\|_{B^{0,\frac{1}{2}}_{2,\gamma}} + C\Psi(t)\left(\frac{1}{\lambda^{\frac{1}{\gamma}}} + \Psi(t)\right). \tag{6.8}$$

We next consider the estimates of the vertical part. Thanks to (5.1) and Lemma 4.1, we get

$$\|v^3_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} \leqslant C\|e^{a|D_3|^{\gamma\alpha}} v^3_0\|_{B^{0,s}_{2,\gamma}} + C\varepsilon^{1-\alpha}\|[E_\varepsilon(v\cdot\nabla v^3)]_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}$$

$$+ C\varepsilon^2\|[E_\varepsilon(\partial_3 q)]_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}. \tag{6.9}$$

Using Lemmas 4.2 and 3.4, we have

$$\|[E_\varepsilon \nabla^h\cdot(v^h v^3)]_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} \leqslant C\|(v^3 v^h)_\Phi\|_{\widetilde{L}^{\frac{\gamma'}{2}}_t B^{\frac{4}{\gamma'}-1,s}_{2,\gamma}}$$

$$\leqslant C\|v^h_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}\|v^3_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}. \tag{6.10}$$

And by Lemma 4.3 we get

$$\varepsilon^{1-\alpha}\|[E_\varepsilon \partial_3(v^3 v^3)]_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} \leqslant \frac{C}{\lambda^{\frac{1}{\gamma}}}\|v^3\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}. \tag{6.11}$$

We have by Lemmas 4.2, 4.3 and 3.4 that

$$\varepsilon^2\|[E_\varepsilon(\partial_3 q)]_\Phi\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} \leqslant C\|(v^h v^h)_\Phi\|_{\widetilde{L}^{\frac{\gamma'}{2}}_t B^{\frac{4}{\gamma'}-1,s}_{2,\gamma}} + C\|(v^h v^3)_\Phi\|_{\widetilde{L}^{\frac{\gamma'}{2}}_t B^{\frac{4}{\gamma'}-1,s}_{2,\gamma}} + \frac{C}{\lambda^{\frac{1}{\gamma}}}\|v^3\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}}$$

$$\leqslant C\|v_\Phi\|^2_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}} + \frac{C}{\lambda^{\frac{1}{\gamma}}}\|v^3\|_{\widetilde{L}^{\gamma'}_t B^{\frac{2}{\gamma'},s}_{2,\gamma}},$$

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which along with (6.9)–(6.11) gives

$$\|v^3\|_{\widetilde{L}_t^{\gamma'} B_{2,\gamma}^{\frac{2}{\gamma'},s}} \leqslant C\|e^{a|D_3|^{\gamma\alpha}} v_0^3\|_{B_{2,\gamma}^{0,s}} + C\Psi(t)\left(\Psi(t) + \frac{1}{\lambda^{\frac{1}{\gamma}}}\right). \tag{6.12}$$

Then Proposition 2.4 can be deduced from (6.8) and (6.12). □

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