Canad. J. Math. Vol. **60** (1), 2008 pp. 64–87

Classification of Linear Weighted Graphs Up to Blowing-Up and Blowing-Down

Daniel Daigle

Abstract. We classify linear weighted graphs up to the blowing-up and blowing-down operations which are relevant for the study of algebraic surfaces.

The word *graph* in this paper means a finite undirected graph such that no edge connects a vertex to itself and at most one edge joins any given pair of vertices. A *weighted graph* is a graph in which each vertex is assigned an integer (called its weight). Two operations are performed on weighted graphs: the blowing-up and its inverse, the blowing-down. Two weighted graphs are said to be equivalent if one can be obtained from the other by means of a finite sequence of blowings-up and blowings-down (see Definitions 1.1–1.2). These weighted graphs and operations are well known to geometers who study algebraic surfaces. Many problems in the geometry of surfaces can be formulated in graph-theoretic terms and solving these sometimes requires elaborate graph-theoretic considerations. This gives rise to a variety of questions about weighted graphs, all in connection with the equivalence relation generated by blowing-up and blowing-down.

The first four sections of the present paper classify linear chains up to equivalence, where by a *linear chain* we mean a weighted graph of the form:

$$\overset{x_1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{x_2}{\bullet}\!\!-\!\!\!-\ \cdots\ -\!\!\!-\!\!\overset{x_q}{\bullet} \qquad (x_i \in \mathbb{Z}).$$

In particular Theorem 3.2 shows that each equivalence class of linear chains contains a canonical form, unique up to an operation which we call *transposition*; and Corollary 3.4 states that two linear chains are equivalent if and only if they have the same invariants. The fourth section introduces the notion of *prime class* and uses it to express the classification in a very simple form (see Corollary 4.4).

Although this paper is motivated by the theory of algebraic surfaces, we make only one brief incursion into geometry: Section 5 recalls the geometric interpretation of weighted graphs and blowing-up, and characterizes the linear chains which occur in the context of algebraic surfaces. The rest of the paper is pure graph theory, and the results of Section 5 are not used in other parts of the paper.

The last two sections are concerned with the problem of listing all minimal weighted graphs equivalent to a given linear chain: the preliminary technical results

Received by the editors August 9, 2004.
The author's research was supported by a grant from NSERC Canada.
AMS subject classification: Primary: 14J26; secondary: 14E07, 14R05, 05C99.
Keywords: weighted graph, dual graph, blowing-up, algebraic surface.

https://doi.org/10.4153/CJM-2008-003-2 Published online by Cambridge University Press

are gathered in Section 6 and the conclusions are given in Section 7. We give a general recursive solution and, in some simple cases, an explicit solution. Incidentally, the cases that we are able to describe explicitely are precisely those which are relevant for algebraic surfaces.

This paper is essentially a subset of our unpublished [1], with some improvements and clarifications. Note that [1] is more general, as it classifies weighted forests; however the classification of linear chains—arguably the most important case—is given in [1] after a relatively long route, and this is one of the reasons for writing the present paper. As we wanted this paper to be self-contained we avoided replacing proofs by references to [1], the only exception being Lemma 6.6: including that proof here would have required the addition of a substantial amount of material (some version of Section 6 of [1]); that material is needed for other purposes in [1], but not here.

Remarks Papers [6,7] classify weighted forests up to an equivalence relation weaker than the one considered here (the relation is generated by blowing-up, blowing-down and other operations which are not allowed here). Result 3.2.1 of [8] classifies linear chains but, again, this is relative to a weak equivalence relation. Paper [9] uses the same equivalence relation as we do, but classifies a set of weighted trees which does not contain all linear chains.

Proposition 3.2 of [8] almost[1] implies the fact (Lemma 2.26) that each linear chain is equivalent to at least one canonical chain. As we realized *a posteriori,* there is even some similarity between the cited result and our method for proving Lemma 2.26.

1 Weighted Graphs

See the introduction for the definition of weighted graph. If $\mathcal{G}$ is a weighted graph, $\text{Vtx}(\mathcal{G})$ is its vertex set. If $v \in \text{Vtx}(\mathcal{G})$ then $w(v, \mathcal{G})$ denotes the weight of v in $\mathcal{G}$; $\deg(v, \mathcal{G})$ denotes the degree of v in $\mathcal{G}$, that is, the number of neighbors of v.

Definition 1.1 We define three types of *blowing-up* of a weighted graph $\mathcal{G}$.

(i) If v is a vertex of $\mathcal{G}$ then the *blowing-up of* $\mathcal{G}$ *at* v is the weighted graph $\mathcal{G}'$ obtained from $\mathcal{G}$ by adding one vertex e of weight -1, adding one edge joining e to v, and decreasing the weight of v by 1. (This process is called a blowing-up "at a vertex".)

(ii) If $\varepsilon = \{v_1, v_2\}$ is an edge of $\mathcal{G}$ (so v_1, v_2 are distinct vertices of $\mathcal{G}$), then the *blowing-up of* $\mathcal{G}$ *at* ε is the weighted graph $\mathcal{G}'$ obtained from $\mathcal{G}$ by adding one vertex e of weight -1, deleting the edge $\varepsilon = \{v_1, v_2\}$, adding the two edges $\{v_1, e\}$ and $\{e, v_2\}$, and decreasing the weights of v_1 and v_2 by 1. (This is called a blowing-up "at an edge", or a "subdivisional blowing-up".)

(iii) The *free blowing-up of* $\mathcal{G}$ is the weighted graph $\mathcal{G}'$ obtained by taking the disjoint union of $\mathcal{G}$ and of a vertex e of weight -1.

In each of the above three cases, we call e the vertex *created* by the blowing-up. If $\mathcal{G}'$ is a blowing-up of $\mathcal{G}$, then there is a natural way to identify $\text{Vtx}(\mathcal{G})$ with a subset of $\text{Vtx}(\mathcal{G}')$ (whose complement is $\{e\}$). It is understood that whenever a blowing-up is

[1] One also needs Lemma 2.20 for the proof.

performed, such an injective map $\mathrm{Vtx}(\mathcal{G}) \hookrightarrow \mathrm{Vtx}(\mathcal{G}')$ is chosen. If $\mathcal{G}'$ is a blowing-up of $\mathcal{G}$ and $\mathcal{G}''$ is a weighted graph isomorphic to $\mathcal{G}'$, then $\mathcal{G}''$ is a blowing-up of $\mathcal{G}$.

Definitions 1.2

(i) A vertex e of a weighted graph $\mathcal{G}'$ is said to be *contractible* if the following three conditions hold:

(a) e has weight -1;
(b) e has at most two neighbors;
(c) if v_1 and v_2 are distinct neighbors of e, then v_1, v_2 are not neighbors of each other.

If e is a contractible vertex of $\mathcal{G}'$ then $\mathcal{G}'$ is the blowing-up of some weighted graph $\mathcal{G}$ in such a way that e is the vertex created by this process. Up to isomorphism of weighted graphs, $\mathcal{G}$ is uniquely determined by $\mathcal{G}'$ and e. We say that $\mathcal{G}$ is obtained by *blowing-down* $\mathcal{G}'$ at e. The blowing-down is the inverse operation of the blowing-up.

(ii) A weighted graph is *minimal* if it does not have a contractible vertex.

(iii) Two weighted graphs $\mathcal{G}$ and $\mathcal{H}$ are *equivalent* (notation: $\mathcal{G} \sim \mathcal{H}$) if one can be obtained from the other by a finite sequence of blowings-up and blowings-down.

Definition 1.3 Given a weighted graph $\mathcal{G}$, consider the real vector space V with basis $\mathrm{Vtx}(\mathcal{G})$ and define a symmetric bilinear form $B_{\mathcal{G}} \colon V \times V \to \mathbb{R}$ by:

$$B_{\mathcal{G}}(u, v) = \begin{cases} w(u, \mathcal{G}) & \text{if } u = v \in \mathrm{Vtx}(\mathcal{G}), \\ 1 & \text{if } u, v \in \mathrm{Vtx}(\mathcal{G}) \text{ are distinct and joined by an edge}, \\ 0 & \text{if } u, v \in \mathrm{Vtx}(\mathcal{G}) \text{ are distinct and not joined by an edge}. \end{cases}$$

One calls $B_{\mathcal{G}}$ the *intersection form* of $\mathcal{G}$. Then define the natural number $\|\mathcal{G}\| = \max_W \dim W$, where W runs in the set of subspaces of V satisfying

$$\forall_{x \in W}\, B_{\mathcal{G}}(x, x) \geq 0.$$

Note that $\|\mathcal{G}\| = 0$ if and only if $B_{\mathcal{G}}$ is negative definite, in which case we say that $\mathcal{G}$ is negative definite.

Lemma 1.4 *For weighted graphs $\mathcal{G}$ and $\mathcal{G}'$, $\mathcal{G} \sim \mathcal{G}' \implies \|\mathcal{G}\| = \|\mathcal{G}'\|$.*

Proof See, for instance, [8, 1.14]. ∎

Definition 1.5 Consider a weighted graph $\mathcal{G}$ and its intersection form $B_{\mathcal{G}}$: $V \times V \to \mathbb{R}$ (see Definition 1.3). Let $v_1, \dots, v_n$ be the distinct vertices of $\mathcal{G}$ (enumerated in any order) and let M be the $n \times n$ matrix representing $B_{\mathcal{G}}$ with respect to the basis $(v_1, \dots, v_n)$ of V. That is, $M_{ii} = w(v_i, \mathcal{G})$ and if $i \neq j$, $M_{ij} = 1$ (resp. 0) if v_i, v_j are neighbors (resp. are not neighbors) in $\mathcal{G}$. Note that $\det(-M)$ is independent of

https://doi.org/10.4153/CJM-2008-003-2 Published online by Cambridge University Press

the choice of an ordering for $\mathrm{Vtx}(\mathcal{G})$. One defines the *determinant* of the weighted graph $\mathcal{G}$ by:

$$\det(\mathcal{G}) = \det(-M).$$

Note that $\det(\mathcal{G}) \in \mathbb{Z}$. By convention, the empty graph has determinant 1.

The following is well known, and easily verified.

Lemma 1.6 *For weighted graphs $\mathcal{G}$ and $\mathcal{G}'$, $\mathcal{G} \sim \mathcal{G}' \implies \det(\mathcal{G}) = \det(\mathcal{G}')$.*

Remark Without the minus sign in $\det(-M)$, Lemma 1.6 would only be true up to sign.

2 Finite Sequences of Integers

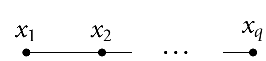
This section classifies finite sequences of integers up to the equivalence relation defined in Definition 2.4 (which of course mimics equivalence of linear chains). From this, it will be very easy to derive, in the next section, a classification of linear chains.

The material up to Corollary 2.14 is well known when stated for linear chains. The main results of the section are Theorem 2.28 and Corollary 2.29.

Notation 2.1 If E is a set, then $E^* = \bigcup_{n=0}^{\infty} E^n$ denotes the set of finite sequences in E, including the empty sequence $\varnothing \in E^*$. We write A^- for the *reversal* of $A \in E^*$, *i.e.,* if $A = (a_1, \dots, a_n)$, then $A^- = (a_n, \dots, a_1)$. We shall consider $\mathbb{Z}^*$ and $\mathcal{N}^*$, where

$$\mathcal{N} = \{x \in \mathbb{Z} \mid x < -1\}.$$

Definition 2.2 A *linear chain* is a weighted tree in which every vertex has degree at most two. An *admissible chain* is a linear chain in which every weight is strictly less than -1. The empty graph is an admissible chain. Given an element $X = (x_1, \dots, x_q)$ of $\mathbb{Z}^*$, the linear chain



is denoted $[x_1, \dots, x_q]$ or $[X]$. So we distinguish between the graph $[X]$ and the sequence X and we note that $[X] = [X^-]$.

Notation 2.3 For each $i \in \{1, \dots, r\}$, let A_i be either an integer or an element of $\mathbb{Z}^*$. We write $(A_1, \dots, A_r)$ for the concatenation of $A_1, \dots, A_r$, that is, $(A_1, \dots, A_r) \in \mathbb{Z}^*$ is a single sequence. Also, we will use superscripts to indicate repetitions. For instance, if $A = (0^3, -5, -1) \in \mathbb{Z}^*$ and $B = (-2^3, 3, -2) \in \mathbb{Z}^*$, then

$$\begin{aligned}(A, -2, B) &= (0^3, -5, -1, -2, -2^3, 3, -2)\\ &= (0, 0, 0, -5, -1, -2, -2, -2, -2, 3, -2).\end{aligned}$$

Superscripts occurring in sequences (or linear chains) should always be interpreted in this way, never as exponents.

https://doi.org/10.4153/CJM-2008-003-2 Published online by Cambridge University Press

Definition 2.4 If $X = (x_1, \dots, x_n) \in \mathbb{Z}^*$ and $X \neq \varnothing$, then any one of the following sequences $X' \in \mathbb{Z}^*$ is called a *blowing-up* of X:

- $X' = (-1, x_1 - 1, x_2, \dots, x_n)$;
- $X' = (x_1, \dots, x_{i-1}, x_i - 1, -1, x_{i+1} - 1, x_{i+2}, \dots, x_n)$ (where $1 \le i < n$);
- $X' = (x_1, \dots, x_{n-1}, x_n - 1, -1)$.

Moreover, we regard the one-term sequence (-1) as a blowing-up of the empty sequence $\varnothing$. If X' is a blowing-up of X, we also say that X is a blowing-down of X'. Two elements of $\mathbb{Z}^*$ are said to be *equivalent* if one can be obtained from the other by a finite sequence of blowings-up and blowings-down. This defines an equivalence relation "$\sim$" on the set $\mathbb{Z}^*$ and we write $\mathbb{Z}^*/{\sim}$ for the set of equivalence classes. We also consider the partial order relation "$\le$" on the set $\mathbb{Z}^*$ which is generated by the condition

$$X \le X' \text{ whenever } X' \text{ is a blowing-up of } X.$$

Thus a minimal element of $\mathbb{Z}^*$ is a sequence which cannot be blown-down, *i.e.*, an element of $(\mathbb{Z} \setminus \{-1\})^*$.

The exact relation between equivalence of sequences (Definition 2.4) and equivalence of linear chains (Definitions 1.2) is given by Lemma 2.6, below. But first we need to point out the following.

Lemma 2.5 *Let $\mathcal{L}$ and $\mathcal{L}'$ be equivalent linear chains. Then there exists a sequence of blowings-up and blowings-down which transforms $\mathcal{L}$ into $\mathcal{L}'$ and which has the additional property that every graph which occurs in the sequence is itself a linear chain.*

This fact is obtained in [1] as an immediate consequence of a more general result (see [1, 3.3]). However, Lemma 2.5 is rather trivial and we leave it without proof.

Lemma 2.6 *Given $X, Y \in \mathbb{Z}^*$,*

(i) $X \sim Y \iff X^- \sim Y^-$;
(ii) $[X] \sim [Y] \iff X \sim Y$ *or* $X \sim Y^-$.

Proof The only nontrivial claim is implication "$\Rightarrow$" in assertion (ii); for proving this implication we may, by Lemma 2.5, restrict ourselves to the case where $[Y]$ is obtained from $[X]$ by blowing-up once; then it is clear that $X \sim Y$ or $X \sim Y^-$. ∎

Refer to Definitions 1.3 and 1.5 for the following.

Definition 2.7 Given $X \in \mathbb{Z}^*$, we define $\det(X) = \det([X])$ and $\|X\| = \|[X]\|$.

Lemma 2.8 *If $X, Y \in \mathbb{Z}^*$ satisfy $X \sim Y$, then* $\det(X) = \det(Y)$ *and* $\|X\| = \|Y\|$.

Proof Follows from Lemma 2.6, Lemma 1.4 and Lemma 1.6. ∎

By Lemma 2.8 we may define $\det(\mathcal{C})$ and $\|\mathcal{C}\|$ for any equivalence class $\mathcal{C} \in \mathbb{Z}^*/{\sim}$ (the definitions are the obvious ones).

https://doi.org/10.4153/CJM-2008-003-2 Published online by Cambridge University Press

Notation 2.9 Given $X = (x_1, \dots, x_n) \in \mathbb{Z}^*$, define:

$$\det_i(X) = \begin{cases} \det(x_{i+1}, \dots, x_n) & \text{if } 0 \le i < n, \\ 1 & \text{if } i = n, \\ 0 & \text{if } i > n; \end{cases}$$

$$\det_*(X) = \begin{cases} \det(x_2, \dots, x_{n-1}) & \text{if } n > 2, \\ 1 & \text{if } n = 2, \\ 0 & \text{if } n < 2. \end{cases}$$

In particular, note that $\det_0(X) = \det(X)$. The sequence X determines the ordered pair $\operatorname{Sub}(X) = (\det_1(X), \det_1(X^-))$ which is an element of the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}$. This gives in particular $\operatorname{Sub}(\varnothing) = (0, 0)$ and if $a \in \mathbb{Z}$, $\operatorname{Sub}((a)) = (1, 1)$. Finally, let $d = \det(X)$ and define the pair

$$\overline{\operatorname{Sub}}(X) = (\pi(\det_1(X)), \pi(\det_1(X^-))) \in \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z},$$

where $\pi \colon \mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ is the canonical epimorphism and where we regard $\mathbb{Z}/d\mathbb{Z}\times\mathbb{Z}/d\mathbb{Z}$ as a $\mathbb{Z}$-module.

Lemmas 2.10–2.13 and Corollary 2.14 are, in one form or another, contained in [4]. We include the proofs for the reader's convenience.

Lemma 2.10 *If* $X = (x_1, \dots, x_n) \in \mathbb{Z}^*$ *and* $n \ge 1$, *then*

$$\det_i(X) = (-x_{i+1}) \det_{i+1}(X) - \det_{i+2}(X) \quad (0 \le i < n).$$

In particular, $\det X = (-x_1) \det_1(X) - \det_2(X)$.

Proof Let $v_1, \dots, v_n$ be the vertices of $\mathcal{G} = [X] = [x_1, \dots, x_n]$, where the labelling is such that $w(v_i, \mathcal{G}) = x_i$ and $\{v_i, v_{i+1}\}$ is an edge for every i. Let M be as in 1.5. The Laplace expansion of $\det(-M)$ along the first row gives $\det_0(X) = (-x_1) \det_1(X) - \det_2(X)$. Applying this formula to $(x_{i+1}, \dots, x_n)$ gives the desired result. ∎

Lemma 2.11 *The assignment* $X \mapsto (\det(X), \det_1(X))$ *is a well-defined bijection:*

$$\mathcal{N}^* \longrightarrow \{(r_0, r_1) \in \mathbb{N}^2 \mid 0 \le r_1 < r_0 \text{ and } \gcd(r_0, r_1) = 1\}.$$

Proof Consider $X = (-q_1, \dots, -q_n) \in \mathcal{N}^*$, where $q_i \ge 2$ for all i. Define $r_i = \det_i(X)$; then Lemma 2.10 gives

$$r_0 = q_1 r_1 - r_2, \tag{2.1}$$

$$r_1 = q_2 r_2 - r_3, \tag{2.2}$$

$$\vdots$$

$$r_{n-2} = q_{n-1} r_{n-1} - r_n, \tag{2.3}$$

$$r_{n-1} = q_n r_n - r_{n+1}, \tag{2.4}$$

https://doi.org/10.4153/CJM-2008-003-2 Published online by Cambridge University Press

where $r_{n+1} = 0$ and $r_n = 1$ by definition of $\det_i(X)$. Then (2.4) gives $r_{n-1} > r_n$ and by descending induction we get

$$0 = r_{n+1} < r_n < r_{n-1} < \cdots < r_1 < r_0 = \det(X). \tag{2.5}$$

Thus $0 \leq r_1 < r_0$ and $\gcd(r_0, r_1) = 1$. Moreover, we may interpret (2.1), (2.2), (2.3), and (2.4) together with (2.5) as the "outer" Euclidean algorithm of the pair (r_0, r_1), which shows that the sequence $(q_1, \dots, q_n)$, and hence X, is completely determined by $(\det(X), \det_1(X))$. Bijectivity follows from this remark. ■

Lemma 2.12 *If $X \in \mathbb{Z}^*$, $d = \det(X)$ and $(x, y) = \mathrm{Sub}(X)$, then $xy \equiv 1 \pmod{d}$.*

Proof The result holds trivially when $X = \varnothing$. For $X \neq \varnothing$, we prove

$$xy = 1 + d \det_*(X). \tag{2.6}$$

Write $X = (x_1, \dots, x_n)$. We leave the cases $n = 1, 2$ to the reader. Assume that $n > 2$ and that (2.6) holds for the shorter sequence $(x_2, \dots, x_n)$, *i.e.*, we are assuming that

$$\det_2(X) \det_*(X) = 1 + x\delta, \tag{2.7}$$

where $\delta = \det_*(x_2, \dots, x_n)$. We obtain $d = -x_1 x - \det_2(X)$ by Lemma 2.10, so

$$\det_2(X) = -x_1 x - d. \tag{2.8}$$

Applying Lemma 2.10 to $(x_1, \dots, x_{n-1})$ gives $y = -x_1 \det_*(X) - \delta$ and hence

$$\delta = -x_1 \det_*(X) - y. \tag{2.9}$$

Substituting (2.8) and (2.9) in (2.7) yields the desired conclusion (2.6). ■

Lemma 2.13 *Suppose that $A, B \in \mathbb{Z}^*$ satisfy $A \sim B$ and let $d = \det(A) = \det(B)$. Then there exists $(x, y) \in \mathbb{Z}^2$ such that*

$$\mathrm{Sub}(A) = \mathrm{Sub}(B) + d(x, y). \tag{2.10}$$

Proof Note that $\det(A) = \det(B)$ by Lemma 2.8. Since $A \sim B$, performing a certain sequence of blowings-up and blowings-down on A produces B; if the same sequence of operations is performed on $(0, A)$, then (obviously) we obtain (x, B) for some $x \in \mathbb{Z}$, which shows that $(0, A) \sim (x, B)$. By the same argument, $(A, 0) \sim (B, y)$ for some $y \in \mathbb{Z}$. By Lemma 2.10, we have $\det(0, A) = -\det_1(A)$ and $\det(x, B) = -xd - \det_1(B)$; since $(0, A) \sim (x, B)$ implies $\det(0, A) = \det(x, B)$, we obtain $\det_1(A) = \det_1(B) + dx$. Similarly, we have $(0, A^-) = (A, 0)^- \sim (B, y)^- = (y, B^-)$, so $\det(0, A^-) = \det(y, B^-)$ and consequently $\det_1(A^-) = \det_1(B^-) + dy$. So (x, y) satisfies (2.10). ■

Corollary 2.14 *If $A, B \in \mathbb{Z}^*$ and $A \sim B$, then $\overline{\mathrm{Sub}}(A) = \overline{\mathrm{Sub}}(B)$.*

Proof This is an obvious consequence of Lemma 2.13. ∎

We shall now develop a classification of sequences up to equivalence. Sequences of the form $(0^{2i}, A)$ (see Notation 2.3) play an important role in that classification.

Lemma 2.15 *Let $i \in \mathbb{N}$ and $A \in \mathbb{Z}^*$.*

(i) $\det(0^{2i}, A) = (-1)^i \det(A)$;
(ii) $\operatorname{Sub}(0^{2i}, A) = (-1)^i \operatorname{Sub}(A)$.

Proof We may assume that $i > 0$; then Lemma 2.10 gives

$$\det(0^{2i}, A) = 0 \det_1(0^{2i}, A) - \det_2(0^{2i}, A) = -\det(0^{2i-2}, A),$$

and assertion (i) follows by induction. We also have:

$$\text{(2.11)} \quad \det_1(0^{2i}, A) = \det(0^{2i-2}, 0, A) \overset{(1)}{=} (-1)^{i-1} \det(0, A) \overset{2.10}{=} (-1)^{i-1}(0 \det(A) - \det_1(A)) = (-1)^i \det_1(A)$$

so, to prove (ii), there remains only to show that

$$\text{(2.12)} \qquad \det_1((0^{2i}, A)^-) = (-1)^i \det_1(A^-).$$

If $A = \varnothing$ then (2.12) reads $\det(0^{2i-1}) = 0$, which is true by assertion (i). So we may assume that $A = (a_1, \dots, a_n)$ with $n \geq 1$, in which case

$$\det_1\left((0^{2i}, A)^-\right) = \det(a_{n-1}, \dots, a_1, 0^{2i}) = \det(0^{2i}, a_1, \dots, a_{n-1}) \overset{(1)}{=} (-1)^i \det(a_1, \dots, a_{n-1}) = (-1)^i \det(a_{n-1}, \dots, a_1) = (-1)^i \det_1(A^-).$$

So (2.12) holds and assertion (ii) follows from (2.11) and (2.12). ∎

Lemma 2.16 *If $i \in \mathbb{N}$ and $A \in \mathbb{Z}^*$, then $\|(0^{2i}, A)\| = i + \|A\|$.*

Proof This is an exercise in diagonalization. It suffices to prove that $\|(0, 0, A)\| = 1 + \|A\|$ for every $A \in \mathbb{Z}^*$. This is obvious if $A = \varnothing$, so assume that $A \neq \varnothing$ and write $A = (a_1, \dots, a_n)$. Consider the linear chain

$$\mathcal{L} = [0, 0, A] = \underset{u_1}{\overset{0}{\bullet}} \!-\!\!-\! \underset{u_2}{\overset{0}{\bullet}} \!-\!\!-\! \underset{v_1}{\overset{a_1}{\bullet}} \!-\!\!-\! \cdots \!-\!\!-\! \underset{v_n}{\overset{a_n}{\bullet}}$$

and let V be the real vector space with basis $\operatorname{Vtx}(\mathcal{L})$. Then the matrix representing $B_{\mathcal{L}}$ with respect to the basis $(u_1, u_2, v_1 - u_1, v_2, \dots, v_n)$ of V is:

$$\text{(2.13)} \qquad \left(\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & M & \\ 0 & 0 & & & \end{array}\right)$$

where M is the $n \times n$ matrix given by $M_{ii} = a_i$, $M_{ij} = 1$ if $|i - j| = 1$ and $M_{ij} = 0$ if $|i - j| > 1$, that is, M is the matrix representing the intersection form of the linear chain $[A]$. Now $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ can be diagonalized to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and we conclude that a diagonal matrix congruent to (2.13) has $1 + \|A\|$ nonnegative entries on its main diagonal, *i.e.*, $\|\mathcal{L}\| = 1 + \|A\|$. ■

Lemma 2.17 *Let $a, b, x \in \mathbb{Z}$ and $A, B \in \mathbb{Z}^*$. Then*

$$(A, a, 0, b, B) \sim (A, a - x, 0, b + x, B) \sim (A, 0, 0, a + b, B).$$

Proof $(A, a, 0, b, B) \sim (A, a - 1, -1, -1, b, B) \sim (A, a - 1, 0, b + 1, B)$, from which the result follows. ■

Lemma 2.18 *Let $n \in \mathbb{N}$ and $A, B, C \in \mathbb{Z}^*$. Then $(A, B, 0^{2n}, C) \sim (A, 0^{2n}, B, C)$.*

Proof If $A, C \in \mathbb{Z}^*$ and $b \in \mathbb{Z}$ then by Lemma 2.17

$$(A, b, 0, 0, C) \sim (A, b - b, 0, 0 + b, C) = (A, 0, 0, b, C),$$

from which the result follows. ■

Lemma 2.19 *Let $n \in \mathbb{N}$, $x, y \in \mathbb{Z}$ and $A \in \mathbb{Z}^*$. Then $(0^{2n+1}, x, A) \sim (0^{2n+1}, y, A)$.*

Proof We first consider the case $n = 0$: $(0, x, A) \sim (-1, -1, x - 1, A) \sim (0, x - 1, A)$, from which we deduce $(0, x, A) \sim (0, y, A)$. Now the general case:

$$(0^{2n+1}, x, A) \overset{2.18}{\sim} (0, x, 0^{2n}, A) \overset{(n=0)}{\sim} (0, y, 0^{2n}, A) \overset{2.18}{\sim} (0^{2n+1}, y, A). \qquad ■$$

Lemma 2.20 *Let $n \in \mathbb{N}$ and $A, B \in \mathbb{Z}^*$. Then*

$$A \sim B \implies (0^{2n}, A) \sim (0^{2n}, B).$$

Remark We will see in Corollary 2.30 that the converse of Lemma 2.20 is also true.

Proof of Lemma 2.20 We may assume that $n \geq 1$. If $A \sim B$, then performing a certain sequence of blowings-up and blowings-down on A produces B; if the same sequence of operations is performed on $(0^{2n}, A) = (0^{2n-1}, 0, A)$, then we obtain $(0^{2n-1}, x, B)$ for some $x \in \mathbb{Z}$, *i.e.*, only the rightmost zero in 0^{2n} is affected. So

$$(0^{2n}, A) \sim (0^{2n-1}, x, B) \overset{2.19}{\sim} (0^{2n-1}, 0, B) = (0^{2n}, B). \qquad ■$$

Definition 2.21 Let $B = (b_1, \ldots, b_n) \in \mathbb{Z}^*$.

(i) Given $x \in \mathbb{Z}$, define ${}_xB = (0, x, B) = (0, x, b_1, \ldots, b_n) \in \mathbb{Z}^*$ and $B_x = (B, x, 0) = (b_1, \ldots, b_n, x, 0) \in \mathbb{Z}^*$.

https://doi.org/10.4153/CJM-2008-003-2 Published online by Cambridge University Press

(ii) Suppose that $B \neq \varnothing$. Given $i \in \{1, \dots, n\}$ and $x, y \in \mathbb{Z}$ such that $x + y = b_i$, define $B_{(i;x,y)} = (b_1, \dots, b_{i-1}, x, 0, y, b_{i+1}, \dots, b_n) \in \mathbb{Z}^*$.

Definition 2.22 Given a minimal element $M = (m_1, \dots, m_k)$ of $\mathbb{Z}^*$, let $M^{\oplus}$ be the set of sequences $Z \in \mathbb{Z}^*$ which can be constructed in one of the following ways.

(i) Pick $x \in \mathbb{Z}$ and let Z be the unique minimal sequence such that $Z \le {}_xM$.
(ii) Pick $x \in \mathbb{Z}$ and let Z be the unique minimal sequence such that $Z \le M_x$.
(iii) Assuming that $M \neq \varnothing$, pick $j \in \{1, \dots, k\}$ and $x, y \in \mathbb{Z}$ such that $x + y = m_j$ and let Z be the unique minimal sequence such that $Z \le M_{(j;x,y)}$.
(iv) Pick $M' = (\mu_1, \dots, \mu_\ell)$ such that $M' \ge M$ and exactly one term μ_j is equal to -1; pick $x, y \in \mathbb{Z} \setminus \{-1\}$ such that $x + y = -1$ and let $Z = M'_{(j;x,y)}$.

Note that each element of $M^{\oplus}$ is a minimal element of $\mathbb{Z}^*$.

Lemma 2.23 *If M is a minimal element of $\mathbb{Z}^*$ and $Z \in M^{\oplus}$, then $Z \sim (0, 0, M)$. Moreover,* $\det Z = -\det M$ *and* $\|Z\| = \|M\| + 1$.

Proof By definition of $M^{\oplus}$, one of the following holds:

$$Z \le {}_xM, \quad Z \le M_x, \quad Z \le M_{(j;x,y)} \quad \text{or} \quad Z = M'_{(j;x,y)} \text{ where } M' \sim M.$$

Consequently, one of the following holds:

$$Z \sim {}_xM, \quad Z \sim M_x \quad \text{or} \quad Z \sim M'_{(j;x,y)} \text{ where } M' \sim M.$$

By Lemma 2.19, ${}_xM = (0, x, M) \sim (0, 0, M)$. Since $X \sim Y$ implies $X^- \sim Y^-$, we also have $M_x = ({}_x(M^-))^- \sim (0, 0, M^-)^- = (M, 0, 0) \sim (0, 0, M)$ by Lemma 2.18.

Let $M' = (b_1, \dots, b_m)$ be any nonempty sequence equivalent to M and let $j \in \{1, \dots, m\}$ and $x, y \in \mathbb{Z}$ be such that $x + y = b_j$; then

$$\begin{aligned} &M'_{(j;x,y)} \\ &\quad = (b_1, \dots, b_{j-1}, x, 0, y, b_{j+1}, \dots, b_m) \overset{2.17}{\sim} (b_1, \dots, b_{j-1}, 0, 0, x + y, b_{j+1}, \dots, b_m) \\ &\quad = (b_1, \dots, b_{j-1}, 0, 0, b_j, b_{j+1}, \dots, b_m) \overset{2.18}{\sim} (0, 0, M') \overset{2.20}{\sim} (0, 0, M). \end{aligned}$$

Thus $Z \sim (0, 0, M)$ whenever $Z \in M^{\oplus}$. By Lemma 2.15 and Lemma 2.16 we get $\det Z = -\det M$ and $\|Z\| = \|M\| + 1$. ■

Proposition 2.24 *Let Z be a minimal element of $\mathbb{Z}^*$ such that $\|Z\| > 0$ and $Z \neq (0)$. Then $Z \in M^{\oplus}$ for some minimal element M of $\mathbb{Z}^*$.*

Proof Assume that $Z = (z_1, \dots, z_n)$ is minimal, $\|Z\| > 0$ and $Z \neq (0)$. In particular, $\|Z\| > 0$ implies that $z_i \ge -1$ for some i; so by minimality of Z there exists i such that $z_i \ge 0$. If $z_i = 0$ for some i, we distinguish three cases:

(i) If $z_1 = 0$, then since $Z \neq (0)$, we have $Z = (0, x, M) = {}_xM$ for some $M \in \mathbb{Z}^*$ and $x \in \mathbb{Z}$; then M is minimal and $Z \in M^{\oplus}$.

(ii) If $z_n = 0$, then, similarly, $Z = M_x$ for some $M \in \mathbb{Z}^*$ and $x \in \mathbb{Z}$; then M is minimal and $Z \in M^{\oplus}$.

(iii) If $z_i = 0$ for some i such that $1 < i < n$, then

$$Z = (z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_n) = B_{(i-1; z_{i-1}, z_{i+1})},$$

where $B = (z_1, \dots, z_{i-2}, z_{i-1} + z_{i+1}, z_{i+2}, \dots, z_n)$. If B is minimal, then $Z \in M^{\oplus}$ where $M = B$. If B is not minimal, then its $(i-1)$-th term $(z_{i-1} + z_{i+1})$ is the only one which is equal to -1; we have $B \geq M$ for some minimal M, then $Z \in M^{\oplus}$.

From now on, assume that $z_j \neq 0$ for all $j \in \{1, \dots, n\}$. Then $z_i > 0$ for some i and we have four cases.

(iv) If $Z = (p)$ where $p > 0$, then $Z \leq (0, -1, -2^{p-1}) = {}_{-1}M$ where $M = (-2^{p-1})$ is minimal; then $Z \in M^{\oplus}$.

(v) If $z_1 > 0$ and $n > 1$, then $Z \leq (0, -1, -2^{z_1-1}, z_2 - 1, z_3, \dots, z_n) = {}_{-1}M$ where $M = (-2^{z_1-1}, z_2 - 1, z_3, \dots, z_n)$ is minimal; then $Z \in M^{\oplus}$.

(vi) If $z_n > 0$ and $n > 1$, then $Z \leq (z_1, \dots, z_{n-2}, z_{n-1} - 1, -2^{z_n-1}, -1, 0) = M_{-1}$ where $M = (z_1, \dots, z_{n-2}, z_{n-1} - 1, -2^{z_n-1})$ is minimal; then $Z \in M^{\oplus}$.

(vii) If $z_i > 0$ and $1 < i < n$, then

$$Z \leq (z_1, \dots, z_{i-1}, 0, -1, -2^{z_i-1}, z_{i+1} - 1, z_{i+2}, \dots, z_n) = M_{(i-1; z_{i-1}, -1)},$$

where $M = (z_1, \dots, z_{i-2}, z_{i-1} - 1, -2^{z_i-1}, z_{i+1} - 1, z_{i+2}, \dots, z_n)$ is minimal; then $Z \in M^{\oplus}$. ∎

Definition 2.25 An element C of $\mathbb{Z}^*$ is a *canonical sequence* if it has the form $C = (0^r, A)$, where $r \in \mathbb{N}$, $A \in \mathcal{N}^*$ and if $A \neq \varnothing$ then r is even.

We now proceed to show that each element of $\mathbb{Z}^*$ is equivalent to a unique canonical sequence. The proof consists of Lemmas 2.26 and 2.27, below.

Lemma 2.26 *Every element of $\mathbb{Z}^*$ is equivalent to a canonical sequence.*

Proof It suffices to show that every minimal element Z of $\mathbb{Z}^*$ is equivalent to a canonical sequence. We proceed by induction on $\|Z\|$. If $\|Z\| = 0$, then $Z \in \mathcal{N}^*$, so Z itself is canonical. If $\|Z\| > 0$ then, by Proposition 2.24, either $Z = (0)$ or $Z \in M^{\oplus}$ for some minimal element M of $\mathbb{Z}^*$. In the first case, Z is canonical and we are done. In the second case, Lemma 2.23 gives $\|M\| < \|Z\|$ so we may assume by induction that M is equivalent to a canonical sequence C; then $Z \sim (0, 0, M) \sim (0, 0, C)$ by Lemma 2.23 and Lemma 2.20, and clearly $(0, 0, C)$ is canonical. ∎

Lemma 2.27 *Let $L \in \mathbb{Z}^*$, let $n = \|L\|$ and let d be the absolute value of* $\det(L)$.

If $(0^r, A)$ (where $r \in \mathbb{N}$ and $A \in \mathcal{N}^*$) is a canonical sequence equivalent to L, then

(i) *if $d = 0$, then $r = 2n - 1$ and $A = \varnothing$;*

(ii) *if $d \neq 0$, then $r = 2n$ and A is the unique element of $\mathcal{N}^*$ which satisfies*

$$\det(A) = d \quad \textit{and} \quad \overline{\mathrm{Sub}}(A) = (-1)^n\, \overline{\mathrm{Sub}}(L).$$

In particular, r and A are uniquely determined by L.

Proof The claim that r and A are uniquely determined by L is obvious in case (i), and follows from Lemma 2.11 in case (ii). Consider any canonical sequence $(0^r, A)$ equivalent to L; we have $r \in \mathbb{N}$, $A \in \mathcal{N}^*$, and if $A \neq \varnothing$, then r is even. To prove (i) and (ii), it suffices to show

(i′) If r is odd, then $d = 0$ and $r = 2n - 1$.
(ii′) If r is even, then $\det(A) = d \neq 0$, $r = 2n$ and $\overline{\mathrm{Sub}}(A) = (-1)^n \overline{\mathrm{Sub}}(L)$.

If r is odd, then $A = \varnothing$; writing $r = 2i + 1$, we get $\pm d = \det(L) = \det(0^{2i+1}) = \det(0^{2i}, 0) = (-1)^i \det(0) = 0$ by Lemma 2.15 and

$$n = \|L\| = \|(0^{2i+1})\| = i + \|(0)\| = i + 1$$

by Lemma 2.16. This proves (i′).

If r is even, then Lemma 2.16 gives $n = \|L\| = \|(0^r, A)\| = \frac{r}{2} + \|A\| = \frac{r}{2}$, so $r = 2n$. Then Lemma 2.15 gives $\pm d = \det(L) = \det(0^{2n}, A) = (-1)^n \det(A)$; since $\det(A) > 0$ by Lemma 2.11, we obtain $\det(A) = d \neq 0$. Since $(0^{2n}, A) \sim L$, Lemma 2.13 implies that there exist $(u, v) \in \mathbb{Z}^2$ such that $\mathrm{Sub}(0^{2n}, A) = \mathrm{Sub}(L) + d(u, v)$. On the other hand, Lemma 2.15 gives $\mathrm{Sub}(A) = (-1)^n \mathrm{Sub}(0^{2n}, A)$, so

$$\mathrm{Sub}(A) = (-1)^n(\mathrm{Sub}(L) + d(u, v)).$$

It follows that $\overline{\mathrm{Sub}}(A) = (-1)^n \overline{\mathrm{Sub}}(L)$ and that (ii′) is true. ■

As an immediate consequence of Lemmas 2.26 and 2.27, we obtain the following fundamental result.

Theorem 2.28 *Each element of $\mathbb{Z}^*$ is equivalent to a unique canonical sequence.*

Corollary 2.29 *For $L, L' \in \mathbb{Z}^*$, the following are equivalent:*

(i) $L \sim L'$;
(ii) $\|L\| = \|L'\|$, $\det(L) = \det(L')$ *and* $\overline{\mathrm{Sub}}(L) = \overline{\mathrm{Sub}}(L')$.

Proof (i) implies (ii) by Lemma 2.8 and Corollary 2.14, and (ii) implies (i) by Lemma 2.27. ■

Remark Note the following consequence of Lemma 2.27:

If $L \in \mathbb{Z}^*$ and $\det(L) = 0$ then $L \sim (0^{2i+1})$ for some $i \in \mathbb{N}$.

Remark One can state some variants of Corollary 2.29, for instance:

- *Suppose that $L, L' \in \mathbb{Z}^*$ satisfy $\det(L) = 0 = \det(L')$. Then*

$$L \sim L' \iff \|L\| = \|L'\|.$$

https://doi.org/10.4153/CJM-2008-003-2 Published online by Cambridge University Press

- *Suppose that $L, L' \in \mathbb{Z}^*$ satisfy $\det(L) = d = \det(L')$ and $\|L\| = \|L'\|$. Then*
$$L \sim L' \iff \det_1(L) \equiv \det_1(L') \pmod{d}.$$

We may now prove the converse of Lemma 2.20.

Corollary 2.30 *Let $n \in \mathbb{N}$ and $A, B \in \mathbb{Z}^*$. Then*
$$A \sim B \iff (0^{2n}, A) \sim (0^{2n}, B).$$

Proof Implication "$\Rightarrow$" is Lemma 2.20. Conversely, if $(0^{2n}, A) \sim (0^{2n}, B)$, then
$$\|A\| \overset{2.16}{=} -n + \|(0^{2n}, A)\| \overset{2.29}{=} -n + \|(0^{2n}, B)\| \overset{2.16}{=} \|B\|,$$
$$\det A \overset{2.15}{=} (-1)^n \det(0^{2n}, A) \overset{2.29}{=} (-1)^n \det(0^{2n}, B) \overset{2.15}{=} \det B,$$
$$\overline{\mathrm{Sub}}(A) \overset{2.15}{=} (-1)^n \,\overline{\mathrm{Sub}}(0^{2n}, A) \overset{2.29}{=} (-1)^n \,\overline{\mathrm{Sub}}(0^{2n}, B) \overset{2.15}{=} \overline{\mathrm{Sub}}(B),$$
so we obtain $A \sim B$ by Corollary 2.29. ■

Definition 2.31 Let $C = (0^r, A) \in \mathbb{Z}^*$ be a canonical sequence (where $r \in \mathbb{N}$ and $A \in \mathcal{N}^*$). The *transpose* C^t of C is defined by $C^t = (0^r, A^-)$. Note that C^t is a canonical sequence.

Lemma 2.32 *Let $X \in \mathbb{Z}^*$. If C is the unique canonical sequence equivalent to X, then C^t is the unique canonical sequence equivalent to X^-.*

Proof Since $X^- \sim C^-$, it suffices to show that $C^- \sim C^t$. Write $C = (0^r, A)$ with $A \in \mathcal{N}^*$. If r is odd, then $A = \varnothing$ and the result holds trivially. Assume that r is even. Then
$$C^- = (0^r, A)^- = (A^-, 0^r) \overset{2.18}{\sim} (0^r, A^-) = C^t.$$
■

3 Classification of Linear Chains

This section reformulates Theorem 2.28 and Corollary 2.29 in terms of linear chains.

Definition 3.1 By a *canonical chain*, we mean a linear chain of the form $[L]$ where $L \in \mathbb{Z}^*$ is a canonical sequence. The *transpose* $\mathcal{L}^t$ of a canonical chain $\mathcal{L}$ is defined by $\mathcal{L}^t = [L^t]$ where $L \in \mathbb{Z}^*$ is a canonical sequence satisfying $\mathcal{L} = [L]$ and where L^t was defined in Definition 2.31. Note that $\mathcal{L}^t$ is a canonical chain.

Remark The linear chain $\mathcal{L}^t$ is well defined even when L is not uniquely determined by $\mathcal{L}$. Indeed, if L and L' are distinct canonical sequences such that $[L] = \mathcal{L} = [L']$, then $L \in \mathcal{N}^*$ and $L' = L^- = L^t$, so $[L^t] = [L^-] = \mathcal{L}$ and $[(L')^t] = [L] = \mathcal{L}$, so $\mathcal{L}^t$ is well defined and equal to $\mathcal{L}$.

Concretely, a linear chain is canonical if it is $[0^r]$ with r odd or if it has the form

$$(3.1)\qquad \underbrace{\overset{0}{\bullet}\!\!-\!\cdots\!-\!\!\overset{0}{\bullet}}_{r\text{ vertices}}\!-\!\overset{a_1}{\bullet}\!-\!\cdots\!-\!\overset{a_n}{\bullet} \qquad (r \geq 0 \text{ is even, } n \geq 0 \text{ and } \forall_i\ a_i \leq -2).$$

The transpose of $[0^r]$ is the same graph $[0^r]$ and the transpose of (3.1) is:

$$\underbrace{\overset{0}{\bullet}\!\!-\!\cdots\!-\!\!\overset{0}{\bullet}}_{r\text{ vertices}}\!-\!\overset{a_n}{\bullet}\!-\!\cdots\!-\!\overset{a_1}{\bullet}.$$

As a corollary to the classification of sequences, we obtain the following.

Theorem 3.2 *Every linear chain is equivalent to a canonical chain. Moreover, if $\mathcal{L}$ and $\mathcal{L}'$ are canonical chains, then*

$$\mathcal{L} \sim \mathcal{L}' \iff \mathcal{L}' \in \left\{\mathcal{L}, \mathcal{L}^t\right\}.$$

Proof In view of Lemma 2.6, this is a corollary to Theorem 2.28 and Lemma 2.32. ■

For the next result, we need the following.

Definition 3.3 Let $\mathcal{L}$ be a linear chain. Define a subset $\operatorname{Sub}(\mathcal{L})$ of $\mathbb{Z}$ as follows: choose $L \in \mathbb{Z}^*$ such that $\mathcal{L} = [L]$, let $(x, y) = \operatorname{Sub}(L) \in \mathbb{Z} \times \mathbb{Z}$ and set

$$\operatorname{Sub}(\mathcal{L}) = \{x, y\}.$$

We also define the subset $\overline{\operatorname{Sub}}(\mathcal{L})$ of $\mathbb{Z}/d\mathbb{Z}$, where $d = \det(\mathcal{L})$, by taking the image of $\operatorname{Sub}(\mathcal{L})$ via the canonical epimorphism $\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$.

Corollary 3.4 *For linear chains $\mathcal{L}$ and $\mathcal{L}'$, the following are equivalent.*

(i) $\mathcal{L} \sim \mathcal{L}'$;
(ii) $\|\mathcal{L}\| = \|\mathcal{L}'\|$, $\det \mathcal{L} = \det \mathcal{L}'$ *and* $\overline{\operatorname{Sub}}(\mathcal{L}) \cap \overline{\operatorname{Sub}}(\mathcal{L}') \neq \varnothing$;
(iii) $\|\mathcal{L}\| = \|\mathcal{L}'\|$, $\det \mathcal{L} = \det \mathcal{L}'$ *and* $\overline{\operatorname{Sub}}(\mathcal{L}) = \overline{\operatorname{Sub}}(\mathcal{L}')$.

Proof Follows immediately from Corollary 2.29. Note that the condition $\overline{\operatorname{Sub}}(\mathcal{L}) \cap \overline{\operatorname{Sub}}(\mathcal{L}') \neq \varnothing$ is equivalent to $\overline{\operatorname{Sub}}(\mathcal{L}) = \overline{\operatorname{Sub}}(\mathcal{L}')$ by Lemma 2.12. ■

4 Prime Classes of Sequences

Let $\mathbb{Z}^*/\sim$ denote the set of equivalence classes of sequences. Given $\mathcal{C} \in \mathbb{Z}^*/\sim$, let $\min \mathcal{C} = \{M \in \mathcal{C} \mid M \text{ is a minimal element of } \mathbb{Z}^*\}$ denote the set of minimal elements of $\mathcal{C}$ (see Definition 2.4 for the notion of minimal sequence).

Recall that if $M \in \mathbb{Z}^*$ is a minimal sequence, then $M^\oplus \subset \mathbb{Z}^*$ is a nonempty set of minimal sequences (see Definition 2.22).

Definition 4.1 For each $\mathcal{C} \in \mathbb{Z}^*/\sim$ we define an element $\mathcal{C}^\oplus$ of $\mathbb{Z}^*/\sim$ in two ways:

https://doi.org/10.4153/CJM-2008-003-2 Published online by Cambridge University Press

(i) Pick a minimal element M of $\mathcal{C}$, pick $X \in M^{\oplus}$ and let $\mathcal{C}^{\oplus}$ be the class of X.
(ii) Pick any $X \in \mathcal{C}$ and let $\mathcal{C}^{\oplus}$ be the class of $(0, 0, X)$.

The two definitions are equivalent (and logically sound) by Lemma 2.23 and Corollary 2.30. Corollary 2.30 also gives:

$$\mathcal{C} \longmapsto \mathcal{C}^{\oplus} \text{is an injective map from } \mathbb{Z}^*/\sim \text{ to itself.} \tag{4.1}$$

Note that $\|\mathcal{C}^{\oplus}\| = 1 + \|\mathcal{C}\|$ by Lemma 2.16. We call $\mathcal{C}^{\oplus}$ the *successor* of $\mathcal{C}$. If $\mathcal{C} = \mathcal{C}_1^{\oplus}$ for some $\mathcal{C}_1$, then $\mathcal{C}_1$ is unique by (4.1); in this case we say that $\mathcal{C}$ has a predecessor and we call $\mathcal{C}_1$ the *predecessor* of $\mathcal{C}$.

Remark The symbol $U^{\oplus}$ has two meanings, depending on the nature of U.

(i) If $U \in \mathbb{Z}^*$ is a minimal sequence, then $U^{\oplus}$ is the set of minimal sequences defined in Definition 2.22;
(ii) if $U \in \mathbb{Z}^*/\sim$ is an equivalence class of sequences, then $U^{\oplus} \in \mathbb{Z}^*/\sim$ is another equivalence class of sequences, as defined in Definition 4.1.

Lemma 4.2 *For an element $\mathcal{C}$ of $\mathbb{Z}^*/\sim$, the following are equivalent:*

(i) $\min \mathcal{C}$ *is a singleton;*
(ii) $\min \mathcal{C}$ *is a finite set;*
(iii) *the canonical element of $\mathcal{C}$ is either (0) or an element of $\mathcal{N}^*$;*
(iv) *$\mathcal{C}$ does not have a predecessor.*

Proof Note that $\neg$(ii) $\Rightarrow$ $\neg$(i) is trivial; we prove $\neg$(i) $\Rightarrow$ $\neg$(iv) $\Rightarrow$ $\neg$(iii) $\Rightarrow$ $\neg$(ii).

If $\|\mathcal{C}\| = 0$ then the canonical element of $\mathcal{C}$ is a sequence $X \in \mathcal{N}^*$; clearly, X is then the unique minimal element of $\mathcal{C}$, so the condition $\|\mathcal{C}\| = 0$ implies (i).

Hence, if (i) is false then $\|\mathcal{C}\| > 0$; since $\min \mathcal{C}$ has more than one element, we may pick a minimal $X \in \mathcal{C}$ such that $X \neq (0)$; then Proposition 2.24 gives $X \in M^{\oplus}$ for some minimal element M of $\mathbb{Z}^*$. Thus (iv) is false.

If (iv) is false, then we may consider the canonical element C of the predecessor of $\mathcal{C}$; then $(0, 0, C)$ is the canonical element of $\mathcal{C}$, so (iii) is false.

If (iii) is false, then the canonical element $(0^r, A)$ of $\mathcal{C}$ (where $r \in \mathbb{N}$ and $A \in \mathcal{N}^*$) satisfies $r \geq 2$. By Lemma 2.19, $(0, x, 0^{r-2}, A) \in \min \mathcal{C}$ for every $x \in \mathbb{Z} \setminus \{-1\}$, so (ii) is false. ∎

Definition 4.3 A *prime class* is an element $\mathcal{C}$ of $\mathbb{Z}^*/\sim$ which satisfies conditions (i)–(iv) of Lemma 4.2.

Remark All prime classes are known explicitly, by condition (iii) of Lemma 4.2.

We now give a remarkably simple formulation of the classification of linear chains. Given $\mathcal{C} \in \mathbb{Z}^*/\sim$ and $n \in \mathbb{N}$, let $\mathcal{C}^{\oplus n} \in \mathbb{Z}^*/\sim$ be the equivalence class of $(0^{2n}, X)$, where X is an arbitrary element of $\mathcal{C}$. Thus $\mathcal{C}^{\oplus 0} = \mathcal{C}$, $\mathcal{C}^{\oplus 1} = \mathcal{C}^{\oplus}$, $\mathcal{C}^{\oplus 2} = (\mathcal{C}^{\oplus})^{\oplus}$, etc. Then we note that Theorem 2.28 implies the following.

https://doi.org/10.4153/CJM-2008-003-2 Published online by Cambridge University Press

Corollary 4.4 *If $\mathcal{P} \subset \mathbb{Z}^*/\!\sim$ denotes the set of prime classes, then the map*

$$\begin{array}{ccc} \mathcal{P} \times \mathbb{N} & \longrightarrow & \mathbb{Z}^*/\!\sim \\ (\mathcal{C}, n) & \longmapsto & \mathcal{C}^{\oplus n} \end{array}$$

is bijective.

5 Geometric Weighted Graphs

We recall the classical notion of the dual graph of a divisor on an algebraic surface. Then we characterize the linear chains which can arise as dual graphs.

Definition 5.1 Let S be a nonsingular projective algebraic surface (over some algebraically closed field). By an *SNC-divisor* of S we mean a reduced effective divisor of S, say $D = \sum_{i=1}^{n} C_i$ where $C_1, \dots, C_n$ are distinct irreducible curves on S, satisfying the following conditions:

- each C_i is a nonsingular curve;
- if $i \neq j$ then the intersection number $C_i \cdot C_j$ is 0 or 1;
- if i, j, k are distinct then $C_i \cap C_j \cap C_k = \varnothing$.

Given an SNC-divisor $D = \sum_{i=1}^{n} C_i$ of S, one defines a weighted graph $\mathcal{G}(D, S)$ by stipulating the following:

- the vertices of $\mathcal{G}(D, S)$ are $C_1, \dots, C_n$;
- distinct vertices C_i, C_j are joined by an edge if and only if $C_i \cap C_j \neq \varnothing$;
- the weight of the vertex C_i is the self-intersection number C_i^2 of the curve C_i.

The weighted graph $\mathcal{G}(D, S)$ is called *the dual graph of D in S*.

Remark Let D be an SNC-divisor of a nonsingular projective surface S, let $\pi\colon S' \to S$ be the blowing-up of S at a point $P \in S$ and let D' be the unique SNC-divisor of S' whose support is equal to $\pi^{-1}\big(\{P\} \cup \operatorname{supp}(D)\big)$. Then $\mathcal{G}(D', S')$ is a blowing-up of $\mathcal{G}(D, S)$. The exceptional curve $E = \pi^{-1}(P)$ is an irreducible component of D' and hence is a vertex of $\mathcal{G}(D', S')$; in the terminology of Definition 1.1, E is in fact the vertex which is created by the blowing-up of $\mathcal{G}(D, S)$. Moreover, if we write $D = \sum_{i=1}^{n} C_i$, then

- if P belongs to exactly one irreducible component C_i of D, then $\mathcal{G}(D', S')$ is the blowing-up of $\mathcal{G}(D, S)$ at the vertex C_i;
- if P belongs to C_i and C_j where $i \neq j$ (so $C_i \cap C_j = \{P\}$), then $\mathcal{G}(D', S')$ is the blowing-up of $\mathcal{G}(D, S)$ at the edge $\{C_i, C_j\}$;
- if $P \notin \operatorname{supp}(D)$, then $\mathcal{G}(D', S')$ is the free blowing-up of $\mathcal{G}(D, S)$.

Remark If D' is an SNC-divisor of a nonsingular projective surface S' and E is a contractible vertex of $\mathcal{G}(D', S')$, then we may blow-down the graph $\mathcal{G}(D', S')$ at the vertex E, and this graph-theoretic operation can be realized geometrically if and only if the curve E is rational. Indeed, if E is a rational curve then it can be shrunk to a smooth point; what we mean by this is that there exists a nonsingular projective

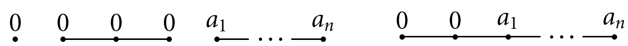
https://doi.org/10.4153/CJM-2008-003-2 Published online by Cambridge University Press

surface S such that the blowing-up of S at a suitable point $P \in S$ is S' and the exceptional curve is E; if $\pi\colon S' \to S$ is this blowing-up morphism, then there is a unique SNC-divisor D of S satisfying $\pi(\operatorname{supp} D') = \{P\} \cup \operatorname{supp} D$. Then $\mathcal{G}(D,S)$ is the blowing-down of $\mathcal{G}(D',S')$ at the vertex E.

Definition 5.2 A weighted graph $\mathcal{G}$ is said to be *geometric* if it is isomorphic to $\mathcal{G}(D,S)$ for some pair (D,S), where:

- S is a smooth projective algebraic surface over an algebraically closed field;
- D is an SNC-divisor of S and every irreducible component of D is a rational curve.

Proposition 5.4, below, states (in particular) that a linear chain is geometric if and only if it is equivalent to one of the following:

(5.1) 

where in the last two graphs n is any nonnegative integer and $a_1, \dots, a_n$ are any integers satisfying $a_i \le -2$ for all i. This claim, and more generally the fact that the first three conditions of Proposition 5.4 are equivalent, was at least partially known prior to this work (compare, for instance, [8, 3.2.4]), but we do not know a suitable reference so we shall give a proof. The main novelty in Proposition 5.4 is the observation that conditions (i) and (iv) are equivalent, which can be paraphrased as follows:

> *The prime classes and their immediate successors give exactly the set of geometric linear chains.*

Also note that the weighted graphs pictured in (5.1) are canonical chains, by Definition 3.1. So, by Theorem 3.2, we immediately know when two such chains are equivalent.

The following is needed for proving Proposition 5.4.

Lemma 5.3 *Let $\mathcal{G}$ be a geometric weighted graph.*

(i) $\|\mathcal{G}\| \le 1$ *or* $\det(\mathcal{G}) = 0$.
(ii) *If $\mathcal{G}' \sim \mathcal{G}$ then $\mathcal{G}'$ is geometric.*
(iii) *Every induced subgraph of $\mathcal{G}$ is geometric.*
(iv) *Let $\mathcal{G}'$ be a weighted graph with the same underlying graph as $\mathcal{G}$ and such that $w(v,\mathcal{G}') \le w(v,\mathcal{G})$ holds for every vertex v. Then $\mathcal{G}'$ is geometric.*

Note that a subgraph G' of a graph G is *induced* if every edge of G which has its two endpoints in G' is an edge of G'. Lemma 5.3 is well known. (The first assertion is a consequence of the Hodge index theorem, see, for instance, [8]; (ii) and (iii) are trivial; and (iv) follows from (ii) and (iii).)

Proposition 5.4 *For a linear chain $\mathcal{L}$, the following conditions are equivalent:*

(i) *$\mathcal{L}$ is geometric;*
(ii) $\|\mathcal{L}\| \le 1$ *or* $\mathcal{L} \sim [0,0,0]$;
(iii) *$\mathcal{L}$ is equivalent to either* $[0]$, $[0,0,0]$, $[A]$, *or* $[0,0,A]$ *(for some $A \in \mathcal{N}^*$);*

https://doi.org/10.4153/CJM-2008-003-2 Published online by Cambridge University Press

(iv) *Let $X \in \mathbb{Z}^*$ be such that $\mathcal{L} = [X]$; then the equivalence class of X is either a prime class or the successor of a prime class.*

Proof By Lemma 4.2, (iii) is equivalent to (iv); we prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Suppose that $\mathcal{L}$ is geometric and that $\det(\mathcal{L}) = 0$. By Lemma 2.27, $\mathcal{L} \sim [0^{2n+1}]$ for some $n \in \mathbb{N}$; by parts (ii) and (iii) of Lemma 5.3, it follows that $[0^{2n+1}]$ is geometric and then that $[0^{2n}]$ is geometric. We have $\det[0^{2n}] \neq 0$ and $\|[0^{2n}]\| = n$ by Lemmas 2.15 and 2.16, so $n \leq 1$ by part (i) of Lemma 5.3. Thus

If $\mathcal{L}$ is geometric and $\det \mathcal{L} = 0$ *then $\mathcal{L}$ is equivalent to* $[0]$ *or* $[0, 0, 0]$.

The fact that (i) and (ii) follows from this and part (i) of Lemma 5.3.

Consider a canonical linear chain $[0^r, A]$ equivalent to $\mathcal{L}$ (with $r \in \mathbb{N}$, $A \in \mathcal{N}^*$ and r is even if $A \neq \varnothing$). If (ii) holds, then $r < 4$, so $[0^r, A]$ is one of the chains displayed in assertion (iii). So (ii) implies (iii).

To show that (iii) implies (i), we have to check that each of $[0]$, $[0, 0, 0]$, $[A]$, and $[0, 0, A]$ (where $A \in \mathcal{N}^*$) is geometric; by part (iii) of Lemma 5.3, it suffices to prove that $[0, 0, 0]$ and $[0, 0, A]$ are geometric, where we may assume that $A \neq \varnothing$. Considering a pair of lines in $\mathbb{P}^2$ shows that $[1, 1]$ is geometric; so $[0, 0, 0] \sim [1, 1]$ is geometric. Let $n \geq 1$ be such that $A = (a_1, \dots, a_n)$. If $n = 1$, then $[0, 0, A]$ is geometric by applying part (iv) of Lemma 5.3 to $[0, 0, A]$ and $[0, 0, 0]$; if $n > 1$, then $[0, 0, -1, -2^{n-2}, -1] \sim [0, 0, 0]$ is geometric and, by part (iv) of Lemma 5.3 applied to $[0, 0, A]$ and $[0, 0, -1, -2^{n-2}, -1]$, $[0, 0, A]$ is geometric. ■

6 Description of Certain Sets of Sequences

Given a minimal element M of $\mathbb{Z}^*$, a set $M^{\oplus} \subset \mathbb{Z}^*$ of minimal sequences was defined in Definition 2.22. This notion was used in proving the classification results of Section 2 and in discussing the concepts of prime class and successor in Section 4. The aim of this section is to solve the following two problems.

Problem 1 *Given a minimal element M of $\mathbb{Z}^*$, describe the set $M^{\oplus}$.*

Actually, the problem that really interests us is:

Problem 2 *List all minimal weighted graphs equivalent to a given linear chain.*

The latter problem will be addressed in Section 7, where we will in fact reduce Problem 2 to Problem 1.

Remark In the present section and the next one we prove some mathematical results and then claim that those results solve certain problems. Such claims are useful for psychological reasons, but the extent to which the results are indeed satisfactory solutions to the problems is partly a matter of interpretation, because the problems are stated in imprecise terms. What do we mean by "describing" the set $M^{\oplus}$, or by "listing" all minimal weighted graphs equivalent to a given one?

Definition 6.1 Let $Z, Z' \in \mathbb{Z}^*$. We say that Z can be $(+,-)$-*contracted* to Z' (resp. $(-,+)$-*contracted*, $(+,+)$-*contracted*) if there exists a sequence of blowings-down which transforms Z into Z' and such that no blowing-down is performed at the leftmost (resp. rightmost, leftmost or rightmost) term of a sequence.

For instance, let $Z = (0,-3,-1,-2)$; then Z can be $(+,-)$-contracted to $Z' = (1)$, but Z cannot be $(-,+)$-contracted or $(+,+)$-contracted to Z'.

Definition 6.2 We define two subsets of $\mathbb{Z}^*$.

$$\mathcal{M} = \{(x_1,\dots,x_n) \in \mathbb{Z}^* \mid n \geq 1,\ x_1 \neq -1 \text{ and } \forall_{i>1}\, x_i \leq -2\},$$
$$\mathcal{M}^- = \{X^- \mid X \in \mathcal{M}\}$$

and, given $\alpha, \beta \in \mathbb{Z}$, four subsets of $\mathbb{Z}^* \times \mathbb{Z}^*$:

$$E = \{(X,Y) \in \mathcal{N}^* \times \mathcal{N}^* \mid (X,-1,Y) \sim \varnothing\}$$
$${}^{\alpha}E = \{(X,Y) \in \mathcal{M} \times \mathcal{N}^* \mid (X,-1,Y) \text{ can be } (+,-)\text{-contracted to } (\alpha)\}$$
$$E^{\alpha} = \{(X,Y) \in \mathcal{N}^* \times \mathcal{M}^- \mid (X,-1,Y) \text{ can be } (-,+)\text{-contracted to } (\alpha)\}$$
$${}^{\alpha}E^{\beta} = \{(X,Y) \in \mathcal{M} \times \mathcal{M}^- \mid (X,-1,Y) \text{ can be } (+,+)\text{-contracted to } (\alpha,\beta)\}.$$

Note that in the above, (X,Y) is an ordered pair of sequences (*i.e.,* we do not concatenate X and Y), whereas $(X,-1,Y)$ is the sequence obtained by concatenating X, -1 and Y.

The first step in solving Problem 1 is to describe the four subsets E, ${}^{\alpha}E$, E^{α} and ${}^{\alpha}E^{\beta}$ of $\mathbb{Z}^* \times \mathbb{Z}^*$, for any choice of $\alpha, \beta \in \mathbb{Z}$. This is achieved by Lemma 6.6, below.

If $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the least integer n such that $x \leq n$.

Lemma 6.3 *$X \mapsto (\det X, \det_1 X)$ is a bijection from $\mathcal{M}$ to*

$$\mathcal{S} := \{(r_0, r_1) \in \mathbb{Z}^2 \mid r_1 > 0,\ \gcd(r_0,r_1) = 1 \text{ and } \lceil \tfrac{r_0}{r_1} \rceil \neq 1\}$$

and $X \mapsto (\det(X^-), \det_1(X^-))$ is a bijection from $\mathcal{M}^-$ to $\mathcal{S}$.

Proof It is well known that $\gcd(\det(X), \det_1(X)) = 1$ holds for every $X \in \mathbb{Z}^*$. Consider an element $X = (-q, N)$ of $\mathbb{Z}^* \setminus \{\varnothing\}$, where $q \in \mathbb{Z}$ and $N \in \mathbb{Z}^*$. By Lemma 2.10,

$$\det(X) = q\det(N) - \det_1(N).$$

If $N \in \mathcal{N}^*$, then by Lemma 2.11 we have $0 \leq \det_1(N) < \det(N)$, so $q = \lceil \frac{\det X}{\det N} \rceil$; thus $\mathcal{M}$ is mapped into the set $\mathcal{S}$. If (r_0, r_1) belongs to the set $\mathcal{S}$, there is a unique pair $(q, r_2) \in \mathbb{Z}^2$ such that $r_0 = qr_1 - r_2$ and $0 \leq r_2 < r_1$; by Lemma 2.11, a unique $N \in \mathcal{N}^*$ satisfies $\det(N) = r_1$ and $\det_1(N) = r_2$; then $(-q, N) \in \mathcal{M}$ and this defines a map from the set $\mathcal{S}$ to $\mathcal{M}$. It is clear that the two maps are inverse of each other, so the first assertion is proved. The second assertion follows from the first. ∎

https://doi.org/10.4153/CJM-2008-003-2 Published online by Cambridge University Press

Elaborating the above proof yields the following fact, which gives a concrete description of the inverse of the bijection $X \mapsto (\det X, \det_1 X)$ given in Lemma 6.3.

Lemma 6.4 *If (r_0, r_1) belongs to the set $\mathcal{S}$, (cf. Lemma 6.3), then the unique $X \in \mathcal{M}$ satisfying $(\det X, \det_1 X) = (r_0, r_1)$ is the sequence $X = (-q_1, \dots, -q_k)$ determined by the "outer" Euclidean algorithm of the pair (r_0, r_1):*

$$\begin{aligned} r_0 &= q_1 r_1 - r_2, \\ &\ \ \vdots \\ r_{k-2} &= q_{k-1} r_{k-1} - r_k, \\ r_{k-1} &= q_k r_k - 0, \end{aligned}$$

where $q_i, r_i \in \mathbb{Z}$ and $r_1 > \cdots > r_k = 1$.

Definition 6.5 Given $\alpha, \beta \in \mathbb{Z}$, define the following subsets of $\mathbb{N}^3$.

$$\begin{aligned} P &= \{(n,p,c) \in \mathbb{N}^3 \mid 1 \le p \le c \text{ and } \gcd(c,p) = 1\}, \\ {}^{\alpha}P = P^{\alpha} &= \{(n,p,c) \in \mathbb{N}^3 \mid 1 \le p \le c,\ \gcd(p,c) = 1 \text{ and } \lceil \tfrac{c}{nc+p} \rceil \ne \alpha + 1\}, \\ {}^{\alpha}P^{\beta} &= \{(n,p,c) \in \mathbb{N}^3 \mid 1 \le p \le c,\ \gcd(p,c) = 1,\ \lceil \tfrac{c}{nc+p} \rceil \ne \alpha + 1 \text{ and } n \ne \beta\}, \end{aligned}$$

and define four maps

(i) $f \colon P \to E$, $(n,p,c) \mapsto (X,Y)$,
(ii) ${}^{\alpha}f \colon {}^{\alpha}P \to {}^{\alpha}E$, $(n,p,c) \mapsto (X,Y)$,
(iii) $f^{\alpha} \colon P^{\alpha} \to E^{\alpha}$, $(n,p,c) \mapsto (X,Y)$,
(iv) ${}^{\alpha}f^{\beta} \colon {}^{\alpha}P^{\beta} \to {}^{\alpha}E^{\beta}$, $(n,p,c) \mapsto (X,Y)$,

by declaring in each case that (X,Y) is the unique pair of sequences satisfying the following:

(i′) $(X,Y) \in \mathcal{N}^* \times \mathcal{N}^*$ and

$$\begin{aligned} \det(X) &= nc + p, & \det(Y^-) &= c, \\ \det_1(X) &\equiv -c \pmod{nc+p}, & \det_1(Y^-) &= c - p. \end{aligned}$$

(ii′) $(X,Y) \in \mathcal{M} \times \mathcal{N}^*$ and

$$\begin{aligned} \det(X) &= c - \alpha(nc+p), & \det(Y^-) &= c, \\ \det_1(X) &= nc + p, & \det_1(Y^-) &= c - p. \end{aligned}$$

(iii′) $(X,Y) \in \mathcal{N}^* \times \mathcal{M}^-$ and

$$\begin{aligned} \det(X) &= c, & \det(Y^-) &= c - \alpha(nc+p), \\ \det_1(X) &= c - p, & \det_1(Y^-) &= nc + p. \end{aligned}$$

(iv′) $(X, Y) \in \mathcal{M} \times \mathcal{M}^-$ and

$$\det(X) = c - \alpha(nc + p), \qquad \det(Y^-) = (n - \beta)c + p,$$
$$\det_1(X) = nc + p, \qquad \det_1(Y^-) = c.$$

Lemma 6.6 *The maps f, ${}^\alpha f$, f^α and ${}^\alpha f^\beta$ are well defined and bijective.*

For the proof of Lemma 6.6, refer to [1, Lemma 7.4.1].

Example 6.7 To describe ${}^{-2}E^{-3}$, we first note that

$$\begin{aligned} {}^{-2}P^{-3} &= \{(n, p, c) \in \mathbb{N}^3 \mid 1 \leq p \leq c,\ \gcd(p, c) = 1,\ \lceil \tfrac{c}{nc+p} \rceil \neq -1 \text{ and } n \neq -3\} \\ &= \{(n, p, c) \in \mathbb{N}^3 \mid 1 \leq p \leq c,\ \gcd(p, c) = 1\}. \end{aligned}$$

Then the desired description of ${}^{-2}E^{-3}$ is given by the bijection ${}^{-2}f^{-3}\colon {}^{-2}P^{-3} \to {}^{-2}E^{-3}$, where by definition ${}^{-2}f^{-3}(n, p, c)$ is the unique element (X, Y) of $\mathcal{M} \times \mathcal{M}^-$ satisfying

$$\det(X) = c + 2(nc + p), \qquad \det(Y^-) = (n + 3)c + p,$$
$$\det_1(X) = nc + p, \qquad \det_1(Y^-) = c.$$

For any given element (n, p, c) of ${}^{-2}P^{-3}$, the actual sequences X and Y can be obtained (if desired) from these equalities via the outer Euclidean algorithm; see Lemma 6.4.

Remark The description of E, ${}^\alpha E$, E^α and ${}^\alpha E^\beta$ given by Lemma 6.6 is explicit to some extent, but not fully explicit. However we think that knowing the determinants of sequences is often more useful than knowing the sequences themselves, so the present form of Lemma 6.6 is probably more useful than would be a fully explicit description. Similar comments apply to Propositions 6.8 and 6.9, below.

The next two results solve Problem 1 (Proposition 6.8 solves the case where M is the empty sequence and Proposition 6.9 solves all other cases). Note that the solution is expressed in terms of the sets E, ${}^\alpha E$, E^α and ${}^\alpha E^\beta$, which are described by Lemma 6.6.

Proposition 6.8 *The elements of $\varnothing^\oplus$ are:*

(i) (1),
(ii) $(0, x)$ *where* $x \in \mathbb{Z} \setminus \{-1\}$,
(iii) $(x, 0)$ *where* $x \in \mathbb{Z} \setminus \{-1\}$,
(iv) $(X, x, 0, -1 - x, Y)$, *where* $x \in \mathbb{Z} \setminus \{-1, 0\}$ *and* $(X, Y) \in E$.

Proposition 6.9 *If $M = (m_1, \dots, m_k) \neq \varnothing$ is a minimal element of $\mathbb{Z}^*$, the elements of $M^\oplus$ are:*

(1) (a) $(0, x, m_1, \dots, m_k)$, *for all* $x \in \mathbb{Z} \setminus \{-1\}$,
(b) *the unique minimal sequence obtained by blowing-down* $(0, -1, m_1, \dots, m_k)$*;*

(2) (a) $(m_1, \dots, m_k, x, 0)$, *for all* $x \in \mathbb{Z} \setminus \{-1\}$,
(b) *the unique minimal sequence obtained by blowing-down* $(m_1, \dots, m_k, -1, 0)$*;*

(3) *For each* $j \in \{1, \dots, k\}$,
(a) $(m_1, \dots, m_{j-1}, x, 0, m_j - x, m_{j+1}, \dots, m_k)$, *for all* $x \in \mathbb{Z} \setminus \{-1, m_j + 1\}$,
(b) *the unique minimal sequence obtained by blowing-down*

$$(m_1, \dots, m_{j-1}, -1, 0, m_j + 1, m_{j+1}, \dots, m_k),$$

(c) *the unique minimal sequence obtained by blowing-down*

$$(m_1, \dots, m_{j-1}, m_j + 1, 0, -1, m_{j+1}, \dots, m_k);$$

(4) (a) $(X, x, 0, -1 - x, Y, m_2, \dots, m_k)$, *for all* $x \in \mathbb{Z} \setminus \{-1, 0\}$ *and all* $(X, Y) \in E^{m_1}$,
(b) $(m_1, \dots, m_{i-1}, X, x, 0, -1 - x, Y, m_{i+2}, \dots, m_k)$, *for all* $x \in \mathbb{Z} \setminus \{-1, 0\}$, *all* $(X, Y) \in {}^{m_i}E^{m_{i+1}}$ *and all i such that* $1 \le i < k$,
(c) $(m_1, \dots, m_{k-1}, X, x, 0, -1-x, Y)$, *for all* $x \in \mathbb{Z} \setminus \{-1, 0\}$ *and all* $(X, Y) \in {}^{m_k}E$.

Proof of Propositions 6.8 and 6.9 Both results follow immediately from Definitions 2.22 (of $M^\oplus$) and 6.2 (of E, ${}^\alpha E$, E^α and ${}^\alpha E^\beta$). ∎

Example 6.10 Let $M = (-2, -3)$. By 6.9, the elements of $M^\oplus$ are

(1a)	$(0, x, -2, -3)$, for all $x \in \mathbb{Z} \setminus \{-1\}$,
(1b)	$(2, -2)$,
(2a)	$(-2, -3, x, 0)$, for all $x \in \mathbb{Z} \setminus \{-1\}$,
(2b)	$(-2, -2, 1)$,
$(3_{j=1}$ a)	$(x, 0, -2 - x, -3)$, for all $x \in \mathbb{Z} \setminus \{-1\}$,
$(3_{j=1}$ b,c)	$(2, -2)$,
$(3_{j=2}$ a)	$(-2, x, 0, -3 - x)$, for all $x \in \mathbb{Z} \setminus \{-1, -2\}$,
$(3_{j=2}$ b)	$(2, -2)$,
$(3_{j=2}$ c)	$(-2, -2, 1)$,
(4a)	$(X, x, 0, -1 - x, Y, -3)$ for all $x \in \mathbb{Z} \setminus \{-1, 0\}$ and $(X, Y) \in E^{-2}$,
(4b)	$(X, x, 0, -1 - x, Y)$ for all $x \in \mathbb{Z} \setminus \{-1, 0\}$ and $(X, Y) \in {}^{-2}E^{-3}$,
(4c)	$(-2, X, x, 0, -1 - x, Y)$ for all $x \in \mathbb{Z} \setminus \{-1, 0\}$ and $(X, Y) \in {}^{-3}E$.

7 Minimal Sequences and Minimal Linear Chains

We are interested in the second problem stated in Section 6, which we repeat for the reader's convenience:

Problem 2 *List all minimal weighted graphs equivalent to a given linear chain.*

It is well known that any minimal weighted graph equivalent to a linear chain is itself a linear chain. Consequently, if $X \in \mathbb{Z}^*$, and if $\mathcal{X} \subset \mathbb{Z}^*$ is the set of minimal sequences equivalent to X, then $\{[Y] \mid Y \in \mathcal{X}\}$ is the set of minimal weighted graphs equivalent to $[X]$. So the above problem reduces to:

Problem 3 *Given $X \in \mathbb{Z}^*$, list all minimal sequences equivalent to X.*

Apparently, very little is known about these problems. One notable exception is [5], which can be interpreted as solving Problem 3 for $X = (1)$. This section is a modest contribution to solving Problem 3; in particular, result 7.1 gives a recursive solution.

The notations $\mathbb{Z}^*/\!\sim$ and $\min(\mathcal{C})$ are defined before Definition 4.1.

Proposition 7.1 *If $\mathcal{C} \in \mathbb{Z}^*/\!\sim$ then,* $\min\left(\mathcal{C}^\oplus\right) = \bigcup_{M \in \min \mathcal{C}} M^\oplus.$

Proof The inclusion "$\supseteq$" is trivial by definition 4.1 of $\mathcal{C}^\oplus$. Consider $Z \in \min(\mathcal{C}^\oplus)$. Since $\mathcal{C}^\oplus$ has a predecessor we have $(0) \notin \mathcal{C}^\oplus$ by 4.2 and hence $Z \neq (0)$; we also have $\|Z\| = \|\mathcal{C}^\oplus\| = 1 + \|\mathcal{C}\| > 0$, so Proposition 2.24 gives $Z \in M^\oplus$ for some minimal element M of $\mathbb{Z}^*$. We have $M \in \mathcal{C}$ by uniqueness of the predecessor of $\mathcal{C}^\oplus$, so $Z \in \cup_{M \in \min \mathcal{C}} M^\oplus$. ∎

Together with Propositions 6.8 and 6.9, and keeping in mind Corollary 4.4, this gives substantial information about Problem 3. Note in particular that Proposition 7.1 immediately implies the following.

Corollary 7.2 *Suppose that $\mathcal{C} \in \mathbb{Z}^*/\!\sim$ is the successor of a prime class $\mathcal{C}_*$, and let M be the unique minimal element of $\mathcal{C}_*$ (see Lemma 4.2). Then* $\min \mathcal{C} = M^\oplus$.

Example 7.3 Let $\mathcal{C}$ denote the equivalence class of the sequence (1). Then $\mathcal{C} = \mathcal{C}_\varnothing^\oplus$, where $\mathcal{C}_\varnothing$ is the equivalence class of the empty sequence $\varnothing$. Since $\mathcal{C}_\varnothing$ is a prime class and its unique minimal element is $\varnothing$, we have $\min \mathcal{C} = \varnothing^\oplus$ by Corollary 7.2 so, by Proposition 6.8, the minimal elements of $\mathcal{C}$ are:

- (1),
- $(0, x)$ where $x \in \mathbb{Z} \setminus \{-1\}$,
- $(x, 0)$ where $x \in \mathbb{Z} \setminus \{-1\}$,
- $(X, x, 0, -1-x, Y)$, where $x \in \mathbb{Z} \setminus \{-1, 0\}$ and $(X, Y) \in E$.

See Lemma 6.6 for a description of E.

Remark The result contained in 7.3 first appeared in [5] (with a different formulation) and was later reproved by several authors.

Example 7.4 Let $\mathcal{C} \in \mathbb{Z}^*/\!\sim$ be the equivalence class of $(0, 0, 0)$. Then $\mathcal{C} = \mathcal{C}_0^\oplus$, where $\mathcal{C}_0$ is the equivalence class of the sequence (0). By Lemma 4.2, $\mathcal{C}_0$ is a prime class and its unique minimal element is (0); so Corollary 7.2 gives $\min \mathcal{C} = (0)^\oplus$ and, by Proposition 6.9, the complete list of minimal elements of $\mathcal{C}$ is:

- $(1, 1)$,
- $(0, x, 0)$ where $x \in \mathbb{Z} \setminus \{-1\}$,
- $(x, 0, -x)$ where $x \in \mathbb{Z} \setminus \{1, -1\}$
- $(X, x, 0, -1 - x, Y)$, where $x \in \mathbb{Z} \setminus \{-1, 0\}$ and $(X, Y) \in E^0 \cup {}^0E$.

See Lemma 6.6 for a description of E^0 and ${}^0E$.

By Section 6, we know how to describe $M^\oplus$ for any given $M \in \min \mathbb{Z}^*$. So by Corollary 7.2:

> *We can list the minimal elements of any class $\mathcal{C} \in \mathbb{Z}^*/\sim$ which is either a prime class or the successor of a prime class.*

In other words, we can solve Problems 2 and 3 exactly in the cases which are relevant for the study of algebraic surfaces (see Section 5). For the other cases, one would have to describe $M^\oplus$ for infinitely many M, and we do not know how to do that. To illustrate this point, consider the following:

Example 7.5 Let $\mathcal{C} \in \mathbb{Z}^*/\sim$ be the equivalence class of $(0, 0, 1)$. Then $\mathcal{C} = \mathcal{C}_1^\oplus$, where $\mathcal{C}_1$ is the equivalence class of (1). In view of the description of $\min(\mathcal{C}_1)$ given in Example 7.3, Proposition 7.1 tells us that $\min(\mathcal{C})$ is the following union:

$$(1)^\oplus \cup \bigcup_{x \in \mathbb{Z} \setminus \{-1\}} (0, x)^\oplus \cup \bigcup_{x \in \mathbb{Z} \setminus \{-1\}} (x, 0)^\oplus \cup \bigcup_{\substack{x \in \mathbb{Z} \setminus \{-1,0\} \\ (X,Y) \in E}} (X, x, 0, -1 - x, Y)^\oplus.$$

Here we do not know how to describe the last union, even though we can describe $(X, x, 0, -1 - x, Y)^\oplus$ for any given choice of $x \in \mathbb{Z} \setminus \{-1, 0\}$ and $(X, Y) \in E$.

References

[1] D. Daigle, *Classification of weighted graphs up to blowing-up and blowing-down.* electronic publication (arXiv:math.AG/0305029), 2003.

[2] D. Daigle and P. Russell, *Affine rulings of normal rational surfaces.* Osaka J. Math. **38**(2001), no. 1, 37–100.

[3] ———, *On log $\mathbb{Q}$-homology planes and weighted projective planes.* Canad. J. Math. **56**(2004), 1145–1189.

[4] F. Hirzebruch, *Über vierdimensionale Riemannsche Flächen mehrdeutiger Funktionen von zwei komplexen Veränderlichen.* Math. Ann. **126** (1953), 1–22.

[5] J. Morrow, *Minimal normal compactifications of $\mathbb{C}^2$.* Rice Univ. Studies **59**(1973), 97–111.

[6] W. Neumann, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves.* Trans. Amer. Math. Soc. **268**(1981), no. 2, 299–344.

[7] ———, *On bilinear forms represented by trees.* Bull. Austral. Math. Soc. **40**(1989), no. 2, 303–321.

[8] K. P. Russell, *Some formal aspects of the theorems of Mumford-Ramanujam.* In: Algebra, Arithmetic and Geometry. Tata Inst. Fund. Res. Stud. Math. 16, Tata Inst. Fund. Res., Bombay, 2002, pp. 557–584.

[9] A. R. Shastri, *Divisors with Finite Local Fundamental Group on a Surface.* In: Algebraic Geometry. Proceedings of Symposia in Pure Mathematics 46, American Mathematical Society, Providence, RI, 1987, pp. 467–481.

Department of Mathematics and Statistics
University of Ottawa
Ottawa, ON
K1N 6N5
e-mail: ddaigle@uottawa.ca