Proceedings of the Royal Society of Edinburgh, **142A**, 199–213, *2012*

On a class of Schrödinger systems with critical exponents

Youjun Wang and **Wenming Zou**
Department of Mathematical Sciences, Tsinghua University,
Beijing 100084, People's Republic of China
(yjwang@math.tsinghua.edu.cn; wzou@math.tsinghua.edu.cn)

(MS received 25 September 2010; accepted 31 March 2011)

We consider the elliptic system

$$-\Delta u + \lambda a_1(x)u = \alpha_1 u + \beta v + \frac{2p}{p+q}|u|^{p-2}u|v|^q, \quad x \in \mathbb{R}^N,$$
$$-\Delta v + \lambda a_2(x)v = \beta u + \alpha_2 v + \frac{2q}{p+q}|v|^{q-2}v|u|^p, \quad x \in \mathbb{R}^N,$$

where $N \geqslant 4$, $\lambda > 0$, $\alpha_1, \alpha_2, \beta \in \mathbb{R}$, $p, q > 1$, $p + q = 2^* = 2N/(N-2)$ and $a_1(x)$, $a_2(x) \geqslant 0$ have potential well. By using variational methods and the category theory, we establish the existence of least energy and multiplicity of solutions.

1. Introduction

We are concerned with the following elliptic system

$$\left.\begin{aligned} -\Delta u + \lambda a_1(x)u &= \alpha_1 u + \beta v + \frac{2p}{p+q}|u|^{p-2}u|v|^q, \quad x \in \mathbb{R}^N, \\ -\Delta v + \lambda a_2(x)v &= \beta u + \alpha_2 v + \frac{2q}{p+q}|v|^{q-2}v|u|^p, \quad x \in \mathbb{R}^N, \end{aligned}\right\} \tag{1.1}$$

where $N \geqslant 4$, $\lambda > 0$, $\alpha_1, \alpha_2, \beta \in \mathbb{R}$, $p, q > 1$, $p + q = 2^* = 2N/(N-2)$. For $i = 1, 2$, we assume the potential $a_i(x)$ satisfies the following conditions:

(A_1) $a_i(x) \in C(\mathbb{R}^N, [0, \infty))$; $\Omega_i := \operatorname{int} a_i^{-1}(0)$ is a non-empty bounded set with smooth boundary; $\bar{\Omega}_i := a_i^{-1}(0)$; $\Omega := \Omega_1 \cap \Omega_2$ is a non-empty set;

(A_2) there exists $M_i > 0$ such that the set $F_i := \{x \in \mathbb{R}^N,\ a_i(x) \leqslant M_i\}$ has finite Lebesgue measure.

There exists an extensive literature on the study of elliptic systems with critical nonlinearities on bounded domains (see, for example, [1, 8] and the reference therein). Alves *et al*. [1] generalized the corresponding results in [3] to the following elliptic system:

$$\left.\begin{aligned} -\Delta u &= \alpha_1 u + \beta v + \frac{2p}{p+q}|u|^{p-2}u|v|^q, \quad x \in \Omega, \\ -\Delta v &= \beta u + \alpha_2 v + \frac{2q}{p+q}|v|^{q-2}v|u|^p, \quad x \in \Omega, \end{aligned}\right\} \tag{1.2}$$

https://doi.org/10.1017/S0308210510001526 Published online by Cambridge University Press

$$\left.\begin{array}{lll} u > 0, & v > 0, & x \in \Omega, \\ u = v = 0, & & x \in \partial\Omega. \end{array}\right\} \qquad (1.2\ cont.)$$

They showed that there is at least one solution of (1.2) if $N \geqslant 4$, $\beta \geqslant 0$, $0 < \nu_1 \leqslant \nu_2 < \mu_1(\Omega)$ and $p + q = 2^*$, where ν_1, ν_2 denote the real eigenvalues of the matrix

$$B := \begin{pmatrix} \alpha_1 & \beta \\ \beta & \alpha_2 \end{pmatrix}$$

and satisfy $\nu_1|U|^2 \leqslant (BU, U)_{\mathbb{R}^2} \leqslant \nu_2|U|^2$ for some $U = (u, v)$; $\mu_1(\Omega)$ is the first eigenvalue of the eigenvalue problem $(-\Delta, H_0^1(\Omega))$.

Elliptic systems in unbounded domains with subcritical nonlinearities have received much attention recently (see, for example, [2, 5–7, 9, 10]). The existence and multiplicity of solutions for a coupled elliptic system with subcritical nonlinearities have been proved in [6,9], where the potentials are bounded below by some positive constants. In [5], Costa proved the existence of a non-trivial solution for a class of semilinear elliptic systems under the coercivity of the potentials and a non-quadratic condition on the nonlinearity. We also mention the recent paper [7] in which Furtado *et al*. studied the existence, multiplicity and asymptotic behaviour of solutions for the coupled elliptic system (1.1) with $\alpha_1 = \alpha_2 = \beta = 0$ and $p + q < 2^*$. As far as we know, there are few results for system (1.1) with critical nonlinearities.

The main aim of the present paper is to study the existence and multiplicity of solutions for the elliptic system (1.1) with critical nonlinearities. We do not assume any positive lower bound or coercivity for the potentials. Motivated by [4], we shall show in some sense that problem (1.2) is a limit problem for (1.1) as $\mu_1(\Omega)$ small enough and $\lambda \to \infty$.

In order to state our results, we define by

$$E_i = \left\{ u \in H^1(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} a_i(x) u^2 \, \mathrm{d}x < \infty \right\}$$

the Hilbert space endowed with the norm

$$\|u\|_{E_i} = \left(\|u\|_{H^1}^2 + \int_{\mathbb{R}^N} a_i(x) u^2 \, \mathrm{d}x \right)^{1/2}.$$

Let $E := E_1 \times E_2$ be endowed with the norm

$$\|(u, v)\| = \left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2 + \int_{\mathbb{R}^N} (a_1(x) u^2 + a_2(x) v^2) \, \mathrm{d}x \right)^{1/2},$$

which is clearly equivalent to the norm

$$\|(u, v)\|_\lambda = \left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2 + \lambda \int_{\mathbb{R}^N} (a_1(x) u^2 + a_2(x) v^2) \, \mathrm{d}x \right)^{1/2}$$

for each $\lambda > 0$. We say that $(u, v) \in E$ is a weak solution of system (1.1) if, for any $(\phi, \varphi) \in E$, the following holds:

$$\int_{\mathbb{R}^N} (\nabla u \nabla \phi + \nabla v \nabla \varphi + \lambda a_1(x) u\phi + \lambda a_2(x) v\varphi)\,\mathrm{d}x - \int_{\mathbb{R}^N} (\alpha_1 u\phi + \beta u\varphi + \beta v\phi + \alpha_2 v\varphi)\,\mathrm{d}x - \frac{2p}{p+q}\int_{\mathbb{R}^N} |u|^{p-2}u|v|^q\phi\,\mathrm{d}x - \frac{2q}{p+q}\int_{\mathbb{R}^N} |v|^{q-2}v|u|^p\varphi\,\mathrm{d}x = 0. \tag{1.2}$$

We shall search for the critical points of the functional

$$J_\lambda(u, v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + \lambda a_1(x)u^2 + \lambda a_2(x)v^2)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} (\alpha_1 u^2 + 2\beta uv + \alpha_2 v^2)\,\mathrm{d}x - \frac{2}{p+q}\int_{\mathbb{R}^N} |u|^p|v|^q\,\mathrm{d}x$$

on the space E. Clearly, $J_\lambda \in C^1(E, \mathbb{R})$. Define

$$S_{p,q} := \inf_{(u,v)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)\setminus(0,0)} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)\,\mathrm{d}x \left(\int_{\mathbb{R}^N} |u|^p|v|^q\,\mathrm{d}x\right)^{2^*/2}. \tag{1.3}$$

By [1, theorem 5], we get

$$S_{p,q} = \left[\left(\frac{p}{q}\right)^{q/2^*} + \left(\frac{p}{q}\right)^{-p/2^*}\right] S, \tag{1.4}$$

where S is the best constant of the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

Set $\mu_1 := \min\{\mu_1(\Omega_1), \mu_1(\Omega_2)\}$. Our main results are the following.

THEOREM 1.1. *Assume (A_1) and (A_2) hold and $\beta \geqslant 0$. Then, for $0 < \nu_1 \leqslant \nu_2 < \mu_1$, there exists $\lambda_1 > 0$ such that for $\lambda > \lambda_1$ the problem (1.1) has a least energy solution.*

THEOREM 1.2. *Assume (A_1) and (A_2) hold and $\beta \geqslant 0$. Then for $0 < \nu_1 \leqslant \nu_2 < \mu_1$, every sequence of solutions (u_n, v_n) of (1.1) such that $\lambda_n \to \infty$ with*

$$J_{\lambda_n}(u_n, v_n) \to c_{\lambda_n} < \frac{2}{N}\left(\frac{S_{p,q}}{2}\right)^{N/2}$$

concentrates at a solution of the following elliptic system in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$:

$$-\Delta u = \alpha_1 u + \beta v + \frac{2p}{p+q}|u|^{p-2}u|v|^q, \quad x \in \Omega_1,$$
$$-\Delta v = \beta u + \alpha_2 v + \frac{2q}{p+q}|v|^{q-2}v|u|^p, \quad x \in \Omega_2,$$
$$u \in H_0^1(\Omega_1), \qquad v \in H_0^1(\Omega_2).$$

https://doi.org/10.1017/S0308210510001526 Published online by Cambridge University Press

THEOREM 1.3. *Assume (A_1) and (A_2) hold and $\beta \geqslant 0$. Then for $0 < \nu_2 (< \mu_1)$ small enough, there exist $\lambda_2 > 0$ and*

$$0 < c_\lambda < \frac{2}{N}\left(\frac{S_{p,q}}{2}\right)^{N/2}$$

such that if $\lambda > \lambda_2$, (1.1) has at least $\operatorname{cat}(\Omega)$ *solutions with $J_\lambda \leqslant c_\lambda$.*

Let $a_1(x) = a_2(x) = a(x)$, $\alpha_1 = \alpha_2 = \mu$, $\beta = 0$, $p = q = 2^*/2$, $u = v$. Then system (1.1) reduces to the Schrödinger equation:

$$-\Delta u + \lambda a(x) u = \mu u + |u|^{2^*-2}u, \quad x \in \mathbb{R}^N. \tag{1.5}$$

In [4], Clapp *et al*. established the existence and multiplicity of positive solutions for (1.5) which localize near the potential well for μ small and λ large under conditions (A_1) and (A_2). To obtain the multiplicity of solutions, by using the concentration-compactness principle for elliptic systems and the category theory we prove that problem (1.1) has at least $\operatorname{cat}(\Omega)$ solutions, where $\operatorname{cat}(\Omega)$ denotes the Lyusternik–Schnirelmann category of $\bar{\Omega}$ in itself [12].

The paper is organized in the following way. In § 2 we present some technical results which will be used throughout the work. We prove theorems 1.1 and 1.2 in § 3. Finally, § 4 is devoted to the proof of theorem 1.3.

2. Preliminaries

LEMMA 2.1. *Let $\lambda_n \geqslant 1$ and $(u_n, v_n) \in E$ be such that $\lambda_n \to \infty$ and $\|(u_n, v_n)\|^2_{\lambda_n} < K$. Then there exists a $(u, v) \in H^1_0(\Omega_1) \times H^1_0(\Omega_2)$ such that, up to a subsequence, $(u_n, v_n) \rightharpoonup (u, v)$ in E and $(u_n, v_n) \to (u, v)$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$.*

Proof. The proof is similar to that of [4, lemma 4]. For convenience, we give a sketch here. Since $\lambda_n \geqslant 1$, we have $\|(u_n, v_n)\|^2 \leqslant \|(u_n, v_n)\|^2_{\lambda_n} < K$. Then, we may assume that $(u_n, v_n) \rightharpoonup (u, v)$ in E and $(u_n, v_n) \to (u, v)$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N) \times L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Set $C_m = \{x \in \mathbb{R}^N : |x| \leqslant m,\ a_1(x) \geqslant 1/m\}$. For every $m \in \mathbb{N}$, we have

$$\int_{C_m} u_n^2 \,\mathrm{d}x \leqslant m \int_{C_m} a_1(x) u_n^2 \,\mathrm{d}x \leqslant \frac{mK}{\lambda_n} \to 0$$

as $n \to \infty$. Similarly, set $C_j = \{x \in \mathbb{R}^N : |x| \leqslant j,\ a_2(x) \geqslant 1/j\}$. For every $j \in \mathbb{N}$, we have

$$\int_{C_j} v_n^2 \,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Therefore, $u = 0$ for almost every (a.e.) $x \in \mathbb{R}^N \setminus \Omega_1$ and $v = 0$ for a.e. $x \in \mathbb{R}^N \setminus \Omega_2$. Since $\partial\Omega_1$ and $\partial\Omega_2$ are smooth, $(u, v) \in H^1_0(\Omega_1) \times H^1_0(\Omega_2)$. On the other hand,

$$\int_{F_1^{\mathrm{c}}} u_n^2 \,\mathrm{d}x \leqslant \frac{1}{\lambda_n M_1} \int_{F_1^{\mathrm{c}}} \lambda_n a_1(x) u_n^2 \,\mathrm{d}x \leqslant \frac{K}{\lambda_n M} \to 0 \tag{2.1}$$

https://doi.org/10.1017/S0308210510001526 Published online by Cambridge University Press

as $n \to \infty$. Setting $B_R = \{x \in \mathbb{R}^N : |x| \leqslant R\}$ and choosing $r \in (1, N/(N-2))$, by (A_2) we have

$$\int_{F_1 \cap B_R^c} (u_n - u)^2 \, dx \leqslant |u_n - u|_{2r}^2 |\{x \in \mathbb{R}^N : x \in B_R^c \cap F_1\}|^{1/r'}$$
$$\leqslant c\|(u_n - u, v_n - v)\|^2 |\{x \in \mathbb{R}^N : x \in B_R^c \cap F_1\}|^{1/r'} \to 0 \tag{2.2}$$

as $R \to \infty$, where $r' = r/(r-1)$, $B_R^c = \mathbb{R}^N \setminus B_R$. Therefore, combining (2.1) and (2.2) and noting that $u_n \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we have $u_n \to u$ in $L^2(\mathbb{R}^N)$. Similarly, we obtain $v_n \to v$ in $L^2(\mathbb{R}^N)$. □

For $i = 1, 2$, set $a_{\lambda,i} := \inf \sigma(-\Delta + \lambda a_i(x))$, the infimum of the spectrum of the self-adjoint operator $-\Delta + \lambda a_i(x)$. Observe that

$$0 \leqslant a_{\lambda,i} = \inf \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda a_i(x) u^2) \, dx : u \in H^1(\mathbb{R}^N), \ \int_{\mathbb{R}^N} u^2 \, dx = 1 \right\}.$$

Define the functional

$$T_\lambda(u, v) := \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + \lambda a_1(x) u^2 + \lambda a_2(x) v^2) \, dx - \int_{\mathbb{R}^N} (\alpha_1 u^2 + 2\beta uv + \alpha_2 v^2) \, dx.$$

LEMMA 2.2. *For any $\sigma \in (0, \mu_1)$, there exists $\lambda_1 > 0$ such that*

$$a_{\lambda,i} \geqslant \tfrac{1}{2}(\sigma + \mu_1)$$

for $\lambda \geqslant \lambda_1$. Moreover,

$$\delta \|(u, v)\|_\lambda^2 \leqslant \int_{\mathbb{R}^N} [|\nabla u|^2 + |\nabla v|^2 + (\lambda a_1(x) - \sigma) u^2 + (\lambda a_2(x) - \sigma) v^2] \, dx \tag{2.3}$$

for $(u, v) \in E$ and $\lambda \geqslant \lambda_1$, where $\delta = (\mu_1 - \sigma)/(\mu_1 + \sigma + 2)$. In particular, if $0 < \nu_1 \leqslant \nu_2 < \mu_1$,

$$\delta \|(u, v)\|_\lambda^2 \leqslant T_\lambda(u, v) \tag{2.4}$$

for $(u, v) \in E$ and $\lambda \geqslant \lambda_1$.

Proof. Assume by contradiction that there exists a sequence $\lambda_n \to \infty$ such that $a_{\lambda_n,1} < \frac{1}{2}(\sigma + \mu_1)$ for all n and $a_{\lambda_n,1} \to \kappa \leqslant \frac{1}{2}(\sigma + \mu_1)$. We assume that $u_n \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |u_n| \, dx = 1$$

and

$$\int_{\mathbb{R}^N} [|\nabla u_n|^2 + (\lambda_n a_1(x) - a_{\lambda_n,1}) u_n^2] \, dx \to 0.$$

https://doi.org/10.1017/S0308210510001526 Published online by Cambridge University Press

Then, for large n, there is $C > 0$ such that

$$\|u_n\|^2_{\lambda_n} = \int_{\mathbb{R}^N} [|\nabla u_n|^2 + (\lambda_n a_1(x) - a_{\lambda_n,1})u_n^2]\,dx + \int_{\mathbb{R}^N} (1 + a_{\lambda_n,1})u_n^2\,dx$$
$$\leqslant C + 1 + \tfrac{1}{2}(\sigma + \mu_1). \tag{2.5}$$

By the proof of lemma 2.1, there exists $u \in H^1_0(\Omega_1)$ such $u_n \rightharpoonup u$ in E and $u_n \to u$ in $L^2(\mathbb{R}^N)$. Therefore,

$$\int_{\Omega_1} |u|\,dx = 1$$

and

$$\int_{\Omega_1} (|\nabla u|^2 - \kappa u^2)\,dx \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 - a_{\lambda_n,1}u_n^2)\,dx$$
$$\leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} [|\nabla u_n|^2 + (\lambda_n a_1(x) - a_{\lambda_n,1})u_n^2]\,dx \to 0.$$

Hence,

$$\int_{\Omega_1} |\nabla u|^2\,dx \leqslant \kappa \leqslant \tfrac{1}{2}(\sigma + \mu_1) < \mu_1 \leqslant \mu_1(\Omega_1),$$

which is a contradiction since

$$\mu_1(\Omega_1) \leqslant \int_{\Omega_1} |\nabla u|^2\,dx.$$

Similarly, we can prove $a_{\lambda,2} \geqslant \frac{1}{2}(\sigma + \mu_1)$. □

In order to find least energy solutions, we shall use the Nehari manifold

$$\mathcal{N}_\lambda = \{(u, v) \in E \setminus (0, 0)\colon J'_\lambda(u, v)(u, v) = 0\}.$$

It is clear that $\mathcal{N}_\lambda$ is radially diffeomorphic to the set

$$\mathcal{V}_\lambda := \biggl\{(u, v) \in E\colon \int_{\mathbb{R}^N} |u|^p|v|^q\,dx = 1\biggr\},$$

where the diffeomorphism is given by

$$\mathcal{V}_\lambda \to \mathcal{N}_\lambda\colon (u, v) \to 2^{(2-N)/4} T_\lambda^{(N-2)/4}(u, v)(u, v).$$

For $u \in \mathcal{N}_\lambda$, we have

$$J_\lambda(u, v) = \frac{T_\lambda(u, v)}{N}.$$

Therefore,

$$c_\lambda := \inf_{(u,v)\in\mathcal{N}_\lambda} J_\lambda(u, v) = \frac{2^{1-N/2}}{N} \inf_{(u,v)\in\mathcal{V}_\lambda} T_\lambda^{N/2}(u, v).$$

For any bounded domain $\mathcal{D} \subset \mathbb{R}^N$, we consider the functional

$$J_{\mathcal{D}}(u,v) = \frac{1}{2}\int_{\mathcal{D}} (|\nabla u|^2 + |\nabla v|^2)\,\mathrm{d}x - \frac{1}{2}\int_{\mathcal{D}} (\alpha_1 u^2 + \beta uv + 2\alpha_2 v^2)\,\mathrm{d}x - \frac{2}{p+q}\int_{\mathcal{D}} |u|^p|v|^q\,\mathrm{d}x$$

for $(u,v) \in H_0^1(\mathcal{D}) \times H_0^1(\mathcal{D})$. Similar to the above arguments, its Nehari manifold

$$\mathcal{N}_{\mathcal{D}} := \{(u,v) \in H_0^1(\mathcal{D}) \times H_0^1(\mathcal{D}) \setminus (0,0) \colon J'_{\mathcal{D}}(u,v)(u,v) = 0\}$$

is radially diffeomorphic to the set

$$\mathcal{V}_{\mathcal{D}} := \left\{(u,v) \in H_0^1(\mathcal{D}) \times H_0^1(\mathcal{D}) \colon \int_{\mathcal{D}} |u|^p|v|^q\,\mathrm{d}x = 1\right\}.$$

Set

$$c_{\mathcal{D}} := \inf_{(u,v)\in\mathcal{N}_{\mathcal{D}}} J_{\mathcal{D}}(u,v) = \frac{2^{1-N/2}}{N} \inf_{(u,v)\in\mathcal{V}_{\mathcal{D}}} T_{\mathcal{D}}^{N/2}(u,v),$$

where

$$T_{\mathcal{D}} := \int_{\mathcal{D}} (|\nabla u|^2 + |\nabla v|^2)\,\mathrm{d}x - \int_{\mathcal{D}} (\alpha_1 u^2 + \beta uv + \alpha_2 v^2)\,\mathrm{d}x.$$

LEMMA 2.3. *Assume that* $\beta \geqslant 0$, $0 < \nu_1 \leqslant \nu_2 < \mu_1$ *and* $\lambda \geqslant \lambda_1$. *Then*

$$\frac{2}{N}\left(\frac{\delta S_{p,q}}{2}\right)^{N/2} \leqslant c_\lambda < c_\Omega < \frac{2}{N}\left(\frac{S_{p,q}}{2}\right)^{N/2}.$$

Proof. By (2.4), $\delta\|(u,v)\|_{H^1}^2 \leqslant \delta\|(u,v)\|_\lambda^2 \leqslant T_\lambda(u,v)$. Thus, taking infima over $(u,v) \in \mathcal{V}_\lambda$ we get

$$\frac{2}{N}\left(\frac{\delta S_{p,q}}{2}\right)^{N/2} \leqslant c_\lambda.$$

Since $\mathcal{V}_\Omega \subset \mathcal{V}_\lambda$ and for any $(u,v) \in \mathcal{V}_\Omega$, $T_\lambda(u,v) = T_\Omega(u,v)$, it follows that $c_\lambda \leqslant c_\Omega$. By [1, lemma 4], we know for $\beta \geqslant 0$, $0 < \nu_1 \leqslant \nu_2 < \mu_1(\Omega)$, that

$$c_\Omega < \frac{2}{N}\left(\frac{S_{p,q}}{2}\right)^{N/2}$$

and c_Ω is obtained by some $(u,v) > 0$. Finally, if $c_\lambda = c_\Omega$, then c_λ would also be achieved at a (u,v) that vanishes outside $\Omega \times \Omega$, contradicting the maximum principle. □

LEMMA 2.4. *If* $\nu_2 \to 0$, *then*

$$c_\Omega \to \frac{2}{N}\left(\frac{S_{p,q}}{2}\right)^{N/2}.$$

Proof. By the definition of c_Ω, we take $\{(u_n,v_n)\} \subset \mathcal{V}_\Omega$ such that

$$\frac{2^{1-N/2}}{N} T_\Omega^{N/2}(u_n,v_n) \to c_\Omega.$$

https://doi.org/10.1017/S0308210510001526 Published online by Cambridge University Press

Then, by the Poincaré inequality, we get

$$\int_{\Omega}(|\nabla u_n|^2+|\nabla v_n|^2)\,\mathrm{d}x \leqslant C$$

for ν_2 small enough. Therefore, by lemma 2.3,

$$\begin{aligned}
S_{p,q} &\leqslant \lim_{n\to\infty}\int_{\Omega}(|\nabla u_n|^2+|\nabla v_n|^2)\,\mathrm{d}x \\
&\leqslant \lim_{n\to\infty}\left(T_{\Omega}(u_n,v_n)+\int_{\Omega}(\alpha_1 u_n^2+2\beta u_n v_n+\alpha_2 v_n^2)\,\mathrm{d}x\right) \\
&\leqslant \lim_{n\to\infty} T_{\Omega}(u_n,v_n)+\nu_2 C \\
&= 2^{1-2/N}(Nc_{\Omega})^{2/N}+\nu_2 C \\
&< S_{p,q}+\nu_2 C.
\end{aligned}$$

So, for $\nu_2\to 0$, we obtain

$$c_{\Omega}\to \frac{2}{N}\left(\frac{S_{p,q}}{2}\right)^{N/2}.$$

□

3. Proof of theorems 1.1 and 1.2

In this section, for $M := \min\{M_1, M_2\}$, we take λ_1 large such that $\lambda M-\nu_2\geqslant 0$ for all $\lambda\geqslant\lambda_1$.

LEMMA 3.1. *Assume that* $0<\nu_1\leqslant\nu_2<\mu_1$ *and* $\lambda\geqslant\lambda_1$. *Then* J_λ *satisfies the Palais–Smale* $(PS)_c$ *condition for all*

$$c<\frac{2}{N}\left(\frac{S_{p,q}}{2}\right)^{N/2}.$$

Proof. Assume that $\{(u_n,v_n)\}$ is a $(\mathrm{PS})_c$ sequence. Since

$$J_\lambda(u_n,v_n)-\frac{J'_\lambda(u_n,v_n)(u_n,v_n)}{2^*}=\frac{1}{N}T_\lambda(u_n,v_n) \tag{3.1}$$

and

$$J_\lambda(u_n,v_n)-\frac{J'_\lambda(u_n,v_n)(u_n,v_n)}{2}=\frac{2}{N}\int_{\mathbb{R}^N}|u_n|^p|v_n|^q\,\mathrm{d}x, \tag{3.2}$$

then, combining (3.1) and (3.2), we get

$$\lim_{n\to\infty}T_\lambda(u_n,v_n)=2\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p|v_n|^q\,\mathrm{d}x=Nc. \tag{3.3}$$

By (2.4) and (3.3), $\delta\|(u_n,v_n)\|_\lambda^2\leqslant T_\lambda(u_n,v_n)=Nc+o(1)$. So, $\{(u_n,v_n)\}$ is bounded in E. We may assume that $(u_n,v_n)\rightharpoonup(u,v)$ in E, $(u_n,v_n)\to(u,v)$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)\times L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $(u_n,v_n)\to(u,v)$ almost everywhere $x\in\mathbb{R}^N$. A standard argument shows that (u,v) is a weak solution of (1.1).

https://doi.org/10.1017/S0308210510001526 Published online by Cambridge University Press

Let $\omega_n = u_n - u$, $\psi_n = v_n - v$. Since $|u_n - tu|^{p-2}(u_n - tu)|v_n - tv|^q$ is uniformly bounded in $L^{2^*/(2^*-1)}(\mathbb{R}^N)$ for $t \in [0,1]$ and $|u_n - tu|^{p-2}(u_n - tu)|v_n - tv|^q \to (1-t)^{2^*-1}|u|^{p-1}|v|^q$ almost everywhere $(t,x) \in [0,1] \times \mathbb{R}^N$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} (|u_n|^p|v_n|^q - |\omega_n|^p|\psi_n|^q)\,\mathrm{d}x
&= -\int_{\mathbb{R}^N}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(|u_n - tu|^p|v_n - tv|^q)\,\mathrm{d}t\,\mathrm{d}x \\
&= p\int_{\mathbb{R}^N}\int_0^1 u|u_n - tu|^{p-2}(u_n - tu)|v_n - tv|^q\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + q\int_{\mathbb{R}^N}\int_0^1 v|v_n - tv|^{q-2}(v_n - tv)|u_n - tu|^q\,\mathrm{d}t\,\mathrm{d}x \\
&\to p\int_{\mathbb{R}^N}\int_0^1 (1-t)^{2^*-1}|u|^p|v|^q\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + q\int_{\mathbb{R}^N}\int_0^1\int_{\mathbb{R}^N} (1-t)^{2^*-1}|v|^q|u|^p\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} |u|^p|v|^q\,\mathrm{d}x.
\end{aligned} \tag{3.4}$$

Since $J'_\lambda(u_n, v_n)(u_n, v_n) \to 0$, $J'_\lambda(u,v)(u,v) = 0$ and by the Brézis–Lieb lemma, we have

$$J'_\lambda(\omega_n, \psi_n)(\omega_n, \psi_n) = T_\lambda(\omega_n, \psi_n) - 2\int_{\mathbb{R}^N} |\omega_n|^p|\psi_n|^q\,\mathrm{d}x \to 0. \tag{3.5}$$

Thus, by (3.5), we may assume

$$T_\lambda(\omega_n, \psi_n) \to b, \qquad \int_{\mathbb{R}^N} |\omega_n|^p|\psi_n|^q\,\mathrm{d}x \to \tfrac{1}{2}b \leqslant Nc < 2(\tfrac{1}{2}S_{p,q})^{N/2}.$$

By (2.2) and note that $(u_n, v_n) \to (u,v)$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N) \times L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we have

$$\int_{F_1} \omega_n^2\,\mathrm{d}x \to 0, \qquad \int_{F_2} \psi_n^2\,\mathrm{d}x \to 0.$$

Therefore,

$$\begin{aligned}
S_{p,q}(\tfrac{1}{2}b)^{2/2^*} &= S_{p,q}\lim_{n\to\infty}\biggl(\int_{\mathbb{R}^N} |\omega_n|^p|\psi_n|^q\,\mathrm{d}x\biggr)^{2/2^*} \\
&\leqslant \lim_{n\to\infty}\int_{\mathbb{R}^N} (|\nabla\omega_n|^2 + |\nabla\psi_n|^2)\,\mathrm{d}x \\
&\leqslant \lim_{n\to\infty}\biggl[\int_{\mathbb{R}^N} (|\nabla\omega_n|^2 + |\nabla\psi_n|^2)\,\mathrm{d}x \\
&\qquad + \int_{F_1^c} (\lambda a_1 - \nu_2)\omega_n^2\,\mathrm{d}x + \int_{F_2^c} (\lambda a_2 - \nu_2)\psi_n^2\,\mathrm{d}x\biggr]
\end{aligned}$$

https://doi.org/10.1017/S0308210510001526 Published online by Cambridge University Press

$$\leqslant \lim_{n\to\infty} T_\lambda(\omega_n,\psi_n) + \nu_2 \int_{F_1} \omega_n^2 \,\mathrm{d}x + \nu_2 \int_{F_2} \psi_n^2 \,\mathrm{d}x$$
$$= b. \tag{3.6}$$

If $b \neq 0$, we get $b \geqslant 2(\frac{1}{2}S_{p,q})^{N/2}$, which contradicts the fact that $b < 2(\frac{1}{2}S_{p,q})^{N/2}$. Thus, $b = 0$. Hence, $(\omega_n,\psi_n) \to (0,0)$ in E. □

Proof of theorem 1.1. Let $\{(u_n,v_n)\} \subset \mathcal{N}_\lambda$ be a minimizing sequence for J_λ, By Ekeland's variational principle [12], we may assume that it is a Palais–Smale sequence. By lemmas 2.3 and 2.4, there exists a subsequence of $\{(u_n,v_n)\}$ that converges to a least energy solution (u,v) of problem (1.1). □

Proof of theorem 1.2. Let $\{(u_n,v_n)\}$ be a sequence of solutions of problem (1.1) such that $\beta \geqslant 0$, $0 < \nu_1 \leqslant \nu_2 < \mu_1$, $\lambda_n \to \infty$ and

$$J_{\lambda_n}(u_n,v_n) \to c < \frac{2}{N}\left(\frac{S_{p,q}}{2}\right)^{N/2}.$$

By lemmas 2.1 and 2.2, there exists a $(u,v) \in H_0^1(\Omega_1)\times H_0^1(\Omega_2)$ such that $(u_n,v_n) \rightharpoonup (u,v)$ in E and $(u_n,v_n) \to (u,v)$ in $L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$. Therefore, for any $(\phi,\varphi) \in H_0^1(\Omega_1)\times H_0^1(\Omega_2)$,

$$\int_{\mathbb{R}^N} (\nabla u_n \nabla\phi + \nabla v_n \nabla\varphi)\,\mathrm{d}x - \int_{\mathbb{R}^N} (\alpha_1 u_n\phi + \beta u_n\varphi + \beta v_n\phi + \alpha_2 v_n\varphi)\,\mathrm{d}x$$
$$= \frac{2p}{p+q}\int_{\mathbb{R}^N} |u_n|^{p-2}u_n|v_n|^q\phi\,\mathrm{d}x + \frac{2q}{p+q}\int_{\mathbb{R}^N} |v_n|^{q-2}v_n|u_n|^p\varphi\,\mathrm{d}x. \tag{3.7}$$

Letting $n\to\infty$ in (3.7), we get

$$\int_{\mathbb{R}^N} (\nabla u \nabla\phi + \nabla v \nabla\varphi)\,\mathrm{d}x - \int_{\mathbb{R}^N} (\alpha_1 u\phi + \beta u\varphi + \beta v\phi + \alpha_2 v\varphi)\,\mathrm{d}x$$
$$= \frac{2p}{p+q}\int_{\mathbb{R}^N} |u|^{p-2}u|v|^q\phi\,\mathrm{d}x + \frac{2q}{p+q}\int_{\mathbb{R}^N} |v|^{q-2}v|u|^p\varphi\,\mathrm{d}x. \tag{3.8}$$

Thus, (u,v) is a solution of (1.2). To show $(u_n,v_n) \to (u,v)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, we set $\omega_n = u_n - u$ and $\psi_n = v_n - v$. Then, since $(u_n,v_n) \in \mathcal{N}_\lambda$ and (u,v) satisfies (3.8), analogously to the arguments in lemma 2.4, we have

$$T_\lambda(\omega_n,\psi_n) - 2\int_{\mathbb{R}^N} |\omega_n|^p|\psi_n|^q\,\mathrm{d}x = o(1).$$

We claim that

$$\int_{\mathbb{R}^N} |\omega_n|^p|\psi_n|^q\,\mathrm{d}x \to 0.$$

Assume by contradiction that

$$\int_{\mathbb{R}^N} |\omega_n|^p|\psi_n|^q\,\mathrm{d}x \to a > 0.$$

https://doi.org/10.1017/S0308210510001526 Published online by Cambridge University Press

Then,

$$S_{p,q}\bigg(\int_{\mathbb{R}^N}|\omega_n|^p|\psi_n|^q\,\mathrm{d}x\bigg)^{2/2^*} \leqslant \int_{\mathbb{R}^N}(|\nabla\omega_n|^2+|\nabla\psi_n|^2)\,\mathrm{d}x$$
$$\leqslant T_{\lambda_n}(\omega_n,\psi_n)+o(1)$$
$$= 2\int_{\mathbb{R}^N}|\omega_n|^p|\psi_n|^q\,\mathrm{d}x. \tag{3.9}$$

It follows from (3.4) that

$$S_{p,q}\leqslant 2\bigg(\int_{\mathbb{R}^N}|\omega_n|^p|\psi_n|^q\,\mathrm{d}x\bigg)^{1-2/2^*}+o(1)\leqslant 2\bigg(\int_{\mathbb{R}^N}|u_n|^p|v_n|^q\,\mathrm{d}x\bigg)^{1-2/2^*}+o(1).$$

So,

$$\bigg(\frac{S_{p,q}}{2}\bigg)^{N/2}\leqslant\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p|v_n|^q\,\mathrm{d}x=\frac{Nc}{2}<\bigg(\frac{S_{p,q}}{2}\bigg)^{N/2},$$

which is a contradiction. Therefore,

$$\int_{\mathbb{R}^N}|\omega_n|^p|\psi_n|^q\,\mathrm{d}x\to 0 \quad\text{and}\quad T_{\lambda_n}(\omega_n,\psi_n)\to 0.$$

Thus, we get

$$\lim_{n\to\infty}T_{\lambda_n}(u_n,v_n)=T_{\Omega}(u,v). \tag{3.10}$$

Since $(u_n,v_n)=(\omega_n,\psi_n)$ in $\mathbb{R}^N\setminus\Omega_1\times\mathbb{R}^N\setminus\Omega_2$, $a_1(x)=0$ in Ω_1 and $a_2(x)=0$ in Ω_2, we have

$$\int_{\mathbb{R}^N}a_1(x)u_n^2\,\mathrm{d}x\leqslant\int_{\mathbb{R}^N}\lambda_n a_1(x)u_n^2\,\mathrm{d}x=\int_{\mathbb{R}^N}\lambda_n a_1(x)\omega_n^2\,\mathrm{d}x\leqslant T_{\lambda_n}(\omega_n,\psi_n)\to 0.$$

Similarly, we get

$$\int_{\mathbb{R}^N}a_2(x)v_n^2\,\mathrm{d}x\to 0.$$

Thus, by (3.10), $(u_n,v_n)\to(u,v)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. □

4. Proof of theorem 1.3

PROPOSITION 4.1. *Assume $p+q=2^*$. Let $\{(u_n,v_n)\}\subset D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$ be a sequence such that*

$$\begin{aligned}(u_n,v_n)&\rightharpoonup(u,v) && \text{in } D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N),\\ (u_n,v_n)&\to(u,v) && \text{almost everywhere on } \mathbb{R}^N,\\ |\nabla(u_n-u)|^2+|\nabla(v_n-v)|^2&\rightharpoonup\mu && \text{in } \mathcal{M}(\mathbb{R}^N),\\ |u_n|^p|v_n|^q&\rightharpoonup\nu && \text{in } \mathcal{M}(\mathbb{R}^N),\end{aligned}$$

https://doi.org/10.1017/S0308210510001526 Published online by Cambridge University Press

where $\mathcal{M}(\mathbb{R}^N)$ *denotes the space of finite measures on* $\mathbb{R}^N$. *Define*

$$\mu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} (|\nabla u_n|^2 + |\nabla v_n|^2)\,\mathrm{d}x,$$

$$\nu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^p |v_n|^q\,\mathrm{d}x.$$

Then it follows that

$$\|\nu\|^{2/2^*} \leqslant S_{p,q}^{-1}\|\mu\|, \qquad \nu_\infty^{2/2^*} \leqslant S_{p,q}^{-1}\mu_\infty,$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2)\,\mathrm{d}x = \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)\,\mathrm{d}x + \|\mu\| + \mu_\infty,$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p |v_n|^q\,\mathrm{d}x = \int_{\mathbb{R}^N} |u|^p |v|^q\,\mathrm{d}x + \|\nu\| + \nu_\infty.$$

Moreover, if $(u, v) = 0$ *and* $\|\nu\|^{2/2^*} = S_{p,q}^{-1}\|\mu\|$, *then* ν *and* μ *are concentrated at a single point.*

Proof. By the definition of (1.3), the proof can follow step by step from the proof of [12, lemma 1.40]. We refer to the detailed proof in [8]. □

For $r > 0$ small enough, we consider the sets

$$\Omega_{2r}^+ = \{x \in \mathbb{R}^N\colon \operatorname{dist}(x, \Omega) < 2r\}$$

and

$$\Omega_r^- = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > r\}.$$

Define the map

$$\gamma(u, v) := \int_\Omega |u|^p |v|^q x\,\mathrm{d}x.$$

Assume that $B_r \subset \Omega$, Then

$$c_\Omega < c_{B_r} < \frac{2}{N}\left(\frac{S_{p,q}}{2}\right)^{N/2}.$$

We have the following result.

Lemma 4.2. *For* $\nu_2 \to 0$, $\lambda \geqslant \lambda_2$ *and all* $(u, v) \in \mathcal{V}_\Omega$ *with*

$$T_\Omega(u, v) \leqslant 2^{1-2/N}(Nc_{B_r})^{2/N} < S_{p,q}, \quad \gamma(u, v) \in \Omega_r^+.$$

Proof. Assume that, by contradiction, for $\nu_2 \to 0$, there exists $\{(u_n, v_n)\} \subset \mathcal{V}_\Omega$ with $T_\Omega(u_n, v_n) \leqslant 2^{1-2/N}(Nc_{B_r})^{2/N}$ such that $\gamma(u_n, v_n) \notin \Omega_r^+$. It is easy to see that

$$\int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2)\,\mathrm{d}x$$

is bounded. Moreover,

$$\begin{aligned} S_{p,q} &\leqslant \lim_{n\to\infty} \int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2)\,\mathrm{d}x \\ &\leqslant \lim_{n\to\infty} \left(T_\Omega(u_n, v_n) + \int_\Omega (\alpha_1 u_n^2 + 2\beta u_n v_n + \alpha_2 v_n^2)\,\mathrm{d}x \right) \\ &\leqslant 2^{1-2/N}(Nc_{B_r})^{2/N} + \nu_2 C < S_{p,q} + \nu_2 C. \end{aligned} \tag{4.1}$$

Up to a subsequence, we assume that

$$\begin{aligned} (u_n, v_n) &\rightharpoonup (u, v) \text{ in } D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N), \\ (u_n, v_n) &\to (u, v) \text{ almost everywhere on } \Omega, \\ |\nabla(u_n - u)|^2 + |\nabla(v_n - v)|^2 &\rightharpoonup \mu \qquad \text{in } \mathcal{M}(\mathbb{R}^N), \\ |u_n|^p |v_n|^q &\rightharpoonup \nu \qquad \text{in } \mathcal{M}(\mathbb{R}^N). \end{aligned}$$

Since Ω is bounded, by proposition 4.1 and (4.1), for ν_2 small enough, we have

$$S_{p,q} = \int_\Omega (|\nabla u|^2 + |\nabla v|^2)\,\mathrm{d}x + \|\mu\|, \quad 1 = \int_\Omega |u|^p |v|^q\,\mathrm{d}x + \|\nu\|$$

and

$$\|\nu\|^{2/2^*} \leqslant S_{p,q}^{-1} \|\mu\|, \quad \left(\int_\Omega |u|^p |v|^q\,\mathrm{d}x \right)^{2/2^*} \leqslant S^{-1} \int_\Omega (|\nabla u|^2 + |\nabla v|^2)\,\mathrm{d}x.$$

It follows that

$$\int_\Omega |u|^p |v|^q\,\mathrm{d}x$$

and $\|\nu\|$ are equal either to 0 or to 1. Since S is never attained in any bounded domain, by (1.4), $S_{p,q}$ is also never attained in any bounded domain. So, $(u, v) = (0, 0)$. We then deduce from proposition 4.1 that ν is concentrated at a single point $y \in \bar{\Omega}$ and

$$\gamma(u_n, v_n) \to \int_\Omega x\,\mathrm{d}\nu = y \in \bar{\Omega},$$

a contradiction. □

Choose $R > 0$ such that $\bar{\Omega} \in B_R$ and define

$$\gamma_0(u, v) := \int_\Omega |u|^p |v|^q \xi(|x|) x\,\mathrm{d}x, \tag{4.2}$$

where $\xi(t) = 1$ if $0 \leqslant t \leqslant R$ and $\xi(t) = R/t$ if $t \geqslant R$.

Lemma 4.3. *There exists $\lambda_2 > \lambda_1$ such that for $\nu_2 \to 0$ and all $(u, v) \in \mathcal{V}_\lambda$ with $T_\lambda(u, v) \leqslant 2^{1-2/N}(Nc_{B_r})^{2/N} < S_{p,q}$, $\gamma_0(u, v) \in \Omega_{2r}^+$.*

Proof. Assume, by contradiction, that for ν_2 small there is a sequence (u_n, v_n) such that $(u_n, v_n) \in \mathcal{V}_{\lambda_n}$, $\lambda_n \to \infty$, $T_{\lambda_n}(u_n, v_n) \leqslant 2^{1-2/N}(Nc_{B_r})^{2/N}$ and $\gamma_0(u_n, v_n) \notin \Omega_{2r}^+$. By lemma 2.1, $(u_n, v_n) \rightharpoonup (u, v)$ in E and $(u_n, v_n) \to (u, v)$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$

https://doi.org/10.1017/S0308210510001526 Published online by Cambridge University Press

for some $(u,v) \in H_0^1(\Omega_1) \times H_0^1(\Omega_2)$. Let $\omega_n = u_n - u$ and $\psi_n = v_n - v$. Analogously to the arguments of the proof of lemma 3.1, we get

$$\int_{\mathbb{R}^N} |u_n|^p |v_n|^q \,\mathrm{d}x \to \int_{\mathbb{R}^N} |u|^p |v|^q \,\mathrm{d}x.$$

Therefore, $\gamma_0(u_n, v_n) \to \gamma(u,v)$. However, since

$$T_\Omega(u,v) \leqslant \lim_{n\to\infty} T_{\lambda_n}(u_n, v_n) \leqslant 2^{1-2/N}(Nc_{B_r})^{2/N},$$

it follows from lemma 4.2 that $\gamma(u,v) \in \Omega_r^+$. This contradicts $\gamma_0(u_n,v_n) \notin \Omega_{2r}^+$. □

To prove theorem 1.3, we need the following result [4].

Proposition 4.4. *Let $I\colon M \to \mathbb{R}$ be an even C^1-functional on a complete symmetric $C^{1,1}$-submanifold $M \subset V \setminus \{0\}$ of some Banach space V and set $I^b := \{z \in M\colon I(z) \leqslant b\}$. Assume that I is bounded below and satisfies (PS)$_c$ for all $c \leqslant b$. Further, assume that there are maps*

$$X \xrightarrow{\iota} I^b \xrightarrow{\gamma} Y,$$

whose composition $\gamma \cdot \iota$ is a homotopy equivalence, and that $\gamma(z) = \gamma(-z)$ for all $z \in M \cap I^b$. Then I has at least $\operatorname{cat}(X)$ pairs $\{z,-z\}$ of critical points with $I(z) = I(-z) \leqslant b$.

Now, we begin to prove theorem 1.3.

Proof of theorem 1.3. Set

$$\tilde{T}_\lambda := \frac{2^{1-N/2}}{N} T_\lambda^{N/2}, \qquad \tilde{T}_\Omega := \frac{2^{1-N/2}}{N} T_\Omega^{N/2}.$$

Let (u_r, v_r) be a non-negative radial minimizer of $\tilde{T}_{B_r}$ on $\mathcal{V}_{B_r}$ with $\tilde{T}_{B_r} = c_{B_r}$ [11]. Define

$$\iota\colon \Omega_r^- \to \mathcal{V}_\lambda \cap \tilde{T}_\lambda^{c_{B_r}}, \qquad \iota(x) = (u_r(\cdot - x), v_r(\cdot - x)),$$

and $\gamma_0\colon \mathcal{V}_\lambda \cap \tilde{T}_\lambda^{c_{B_r}} \to \Omega_{2r}^+$ as in (4.2). By lemma 4.3, γ_0 is well defined. Since $\iota(x) = (0,0)$ in $\mathbb{R}^N \setminus \Omega$ for every $x \in \Omega_r^-$, it follows that $\iota(x) \in \mathcal{V}_\lambda$ and that $\tilde{T}_\lambda(\iota(x)) = \tilde{T}_{B_r}(\iota(x)) = c_{B_r}$. Moreover, $\tilde{T}_\lambda(-u,-v) = \tilde{T}_\lambda(u,v)$ and $\gamma_0(-u,-v) = \gamma_0(u,v)$ for $(u,v) \in E \setminus \{(0,0)\}$. By [1, lemma 4], we get

$$c_{B_r} < \frac{2}{N}\left(\frac{S_{p,q}}{2}\right)^{N/2}.$$

So, by the expression of c_λ and lemma 2.4, $\tilde{T}_\lambda$ satisfies (PS)$_c$ for all $c_\lambda \leqslant c_{B_r}$. It follows from lemma 4.3 and proposition 4.4 that $\tilde{T}_\lambda$ has at least $\operatorname{cat}(\Omega)$ critical points, that is, J_λ has at least $\operatorname{cat}(\Omega)$ critical points with

$$J_\lambda < \frac{2}{N}\left(\frac{S_{p,q}}{2}\right)^{N/2}.$$

□

Acknowledgements

This research was supported by the NSFC (Grant nos 10871109 and 11025106) and the China Postdoctoral Science Foundation (Grant no. 20100470175).

References

1 C. O. Alves, D. C. de Morais and M. A. S. Souto. On systems of elliptic equations involving subcritical or critical Sobolev exponents. *Nonlin. Analysis TMA* **42** (2000), 771–787.

2 A. Ambrosetti and E. Colorado. Standing waves of some coupled nonlinear Schrödinger equations. *J. Lond. Math. Soc.* **75** (2007), 67–82.

3 H. Brézis and L. Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent. *Commun. Pure Appl. Math.* **36** (1983), 437–478.

4 M. Clapp and Y. H. Ding. Positive solutions of a Schrödinger equation with critical nonlinearity. *Z. Angew. Math. Phys.* **55** (2004), 592–605.

5 D. G. Costa. On a class of elliptic systems in $\mathbb{R}^N$. *Electron. J. Diff. Eqns* **7** (1994), 1–14.

6 M. F. Furtado, L. A. Maia and E. A. B. Silva. Solutions for a resonant elliptic system with coupling in $\mathbb{R}^N$. *Commun. PDEs* **27** (2002), 1515–1536.

7 M. F. Furtado, E. A. B. Silva and M. S. Xavier. Multiplicity and concentration of solutions for elliptic systems with vanishing potentials. *J. Diff. Eqns* **249** (2010), 2377–2396.

8 P. G. Han. The effect of the domain topology on the number of positive solutions of an elliptic system involving critical Sobolev exponents. *Houston J. Math.* **32** (2006), 1241–1257.

9 L. A. Maia and E. A. B. Silva. On a class of coupled elliptic systems in $\mathbb{R}^N$. *Nonlin. Diff. Eqns Applic.* **14** (2007), 303–313.

10 L. A. Maia, E. Montefusco and B. Pellacci. Positive solutions for a weakly coupled nonlinear Schrödinger system. *J. Diff. Eqns* **229** (2006), 743–67.

11 W. C. Troy. Symmetry properties in systems of semilinear elliptic equations. *J. Diff. Eqns* **42** (1981), 400–413.

12 W. Willem. *Minimax theorems* (Birkhäuser, 1996).

(*Issued 17 February 2012*)