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A NOTE ON LINEAR RECURSIVE SEQUENCES

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1. Introduction

We consider linear recursive sequences of integers not all zero such that

(1) $$u_{n+k} = a_1 u_{n+k-1} + \cdots + a_k u_n, \qquad \text{for } n = 0, 1, 2, \cdots,$$

where the a_j are rational integers.

If

$$X^k - a_1 X^{k-1} - \cdots - a_{k-1} X - a_k = \prod_{j=1}^{h} (X - \omega_j)^{r_j}$$

is the decomposition of the associated polynomial P, $|\omega_1| \geqq \cdots \geqq |\omega_h|$, it is well-known that u_n is given by

(2) $$u_n = \sum_{j=1}^{h} P_j(n)\omega_j^n,$$

where P_j is a polynomial of degree $< r_j$, with coefficients in $Q(\omega_1, \cdots, \omega_h)$.

We recall first a theorem of Mahler (1969).

THEOREM *A. Suppose that the* u_n *are given by* (1), *where*

$$k = 2,\ a_1^2 + 4a_2 < 0,\ a_2 \leqq -2,\ (a_1, a_2) = 1.$$

Let $\varepsilon > 0$ *be an arbitrary constant. Then, as soon as n is sufficiently large,*

$$|u_n| \geqq |\omega_1|^{(1-\varepsilon)n}.$$

Our aim is to prove the following result.

THEOREM 1. *Let* (u_n) *be a sequence of integers satisfying* (1). *Suppose that P has at most* 3 *roots of greatest modulus and that these roots* $\omega_1, \cdots, \omega_l$ *are* *simple. Then, there exist* n_0 *and* c, *which are calculable, such that, for* $n \geqq n_0$, *we have*

$$|u_n| \geqq |\omega_1|^n n^{-c} \text{ if } v_n = P_1\omega_1^n + \cdots + P_l\omega_l^n \neq 0, \quad l \leqq 3.$$

(*The polynomials* $P_1, \cdots, P_l$ *are constant.*)

It is clear that this result is nearly the best possible. It seems to be difficult to extend this theorem to the general case.

2. A lemma

LEMMA. *Let* x_n *be defined by*

$$x_n = b + b_1 y_1^n + \bar{b}_1 \bar{y}_1^n$$

where b_1, y_1 *are algebraic numbers,* $|y_1| = 1$, $b = 0$ *or* 1. *Then, there exists calculable* n_0 *and* C *such that, for* $n \geqq n_0$, *the following implication holds*

$$x_n \neq 0 \Rightarrow |x_n| \geqq n^{-C}.$$

PROOF. Because of $|x_n| \geqq b - 2|b_1|$, it suffices to consider the case $b \leqq 2|b_1|$. Put

$$b_1 = |b_1|e^{i\psi}, \quad y_1 = e^{i\theta}, \quad |b/b_1| = -2\cos\phi, \ \theta, \phi, \psi \in [-\pi, \pi[.$$

We have

$$|x_n| = 4|b_1|\left|\sin\frac{\psi + n\theta + \phi}{2}\sin\frac{\psi + n\theta - \phi}{2}\right|.$$

The inequality $|x_n| \leqq \eta$ implies

$$\left|\sin\frac{\psi + n\theta + \phi}{2}\sin\frac{\psi + n\theta - \phi}{2}\right| \leqq \frac{\eta}{4|b_1|}. \tag{3}$$

If $\phi \neq 0$, (3) leads to an inequality of the form

$$|n\theta + m\pi \pm \phi + \psi| \leqq c_1\eta, \ m \in \mathbb{Z}, \ |m| \leqq n, \text{ if } \eta < \eta_0,$$

whereas, if $\phi = 0$, it implies

$$|n\theta + m\pi + \psi| \leqq c_2\eta^{\frac{1}{2}}, \text{ if } \eta < \eta_1.$$

In both cases, for $x_n \neq 0$, we get

$$0 < |n\theta + m\pi + \psi \pm \phi| \leqq c\eta^{\frac{1}{2}}, \text{ if } \eta < \eta_2.$$

Here $i\theta$, $i\pi$, $i\phi$, $i\psi$ are values of logartithms of algebraic numbers and the conclusion follows from Baker's theorem (1972):

THEOREM. *Let* $\beta_1, \cdots, \beta_k$ *be fixed algebraic numbers. There exists a calculable constant* C_0, *such that for* $0 < \delta < \frac{1}{2}$, *the inequalities*

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$$0 < \left| b_1 \log \beta_1 + \cdots + b_{k-1} \log \beta_{k-1} - \log \beta_k \right| < \delta^{C_0} e^{-\delta B}$$

have no integer solutions $b_1, \cdots, b_{k-1}$, *with* $\max |b_i| \leqq B$.

Here, for $B > 2C_0$, we choose $\delta = C_0/B$, thus, if $|b_1 \log \beta_1 + \cdots| \neq 0$,

$$|b_1 \log \beta_1 + \cdots| > \left(\frac{C_0}{e}\right)^{C_0} B^{-C_0}.$$

3. Proof of the Theorem

We may write

$$|v_n| = a \left| \omega_1^n x_n \right|,$$

where x_n verifies the hypothesis of the lemma. The conclusion follows at once from the lemma (use (2)).

References

A. Baker (1974), 'A sharpening of the bounds for linear forms in logarithms', *Acta Arith.* **21**, 117–129.

K. Mahler (1966), 'A remark on recursive sequences', *J. Math. Sci.* **1**, 12–17.

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