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ON THE EXPONENTIAL DIOPHANTINE EQUATION $(m^2+1)^x + (cm^2-1)^y = (am)^z$

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Abstract

Let m, a, c be positive integers with $a \equiv 3, 5 \pmod 8$. We show that when $1 + c = a^2$, the exponential Diophantine equation $(m^2+1)^x + (cm^2-1)^y = (am)^z$ has only the positive integer solution $(x, y, z) = (1, 1, 2)$ under the condition $m \equiv \pm 1 \pmod a$, except for the case $(m, a, c) = (1, 3, 8)$, where there are only two solutions: $(x, y, z) = (1, 1, 2)$, $(5, 2, 4)$. In particular, when $a = 3$, the equation $(m^2+1)^x + (8m^2-1)^y = (3m)^z$ has only the positive integer solution $(x, y, z) = (1, 1, 2)$, except if $m = 1$. The proof is based on elementary methods and Baker's method.

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1. Introduction

Let a, b, c be fixed relatively prime positive integers greater than one. The exponential Diophantine equation

$$a^x + b^y = c^z \tag{1.1}$$

in positive integers x, y, z has been studied by a number of authors. In 1956, Sierpiński [S] considered the case of $(a, b, c) = (3, 4, 5)$, and showed that $(x, y, z) = (2, 2, 2)$ is the only solution. Jeśmanowicz [J] conjectured that if a, b, c are Pythagorean numbers, that is, positive integers satisfying $a^2 + b^2 = c^2$, then (1.1) has only the solution $(x, y, z) = (2, 2, 2)$. As an analogue of Jeśmanowicz's conjecture, the second author proposed that if a, b, c, p, q, r are fixed positive integers satisfying $a^p + b^q = c^r$ with $a, b, c, p, q, r \geq 2$ and $\gcd(a, b) = 1$, then, apart from a handful of exceptions, (1.1) has only the solution $(x, y, z) = (p, q, r)$. This conjecture has been proved to be true in many special cases (see [CD, Le, M1, M2, T1, T2]), but is still unsolved in general.

In the other direction, many of the recent works on (1.1) concern the case where two of a, b and c are congruent to ± 1 modulo a (relatively) large divisor of the other one.

For example, see [HT1, HT2, HY, MT, T3]. In this paper, we consider the exponential Diophantine equation

$$(m^2+1)^x + (cm^2-1)^y = (am)^z \tag{1.2}$$

with m a positive integer. Our main result is the following theorem.

Theorem 1.1. *Let a be a positive integer with $a \equiv 3,\ 5 \pmod 8$. Let c be a positive integer with $1 + c = a^2$. Suppose that $m \equiv \pm 1 \pmod a$. Then (1.2) has only the positive integer solution $(x, y, z) = (1, 1, 2)$, except for the case $(m, a, c) = (1, 3, 8)$, where the equation $2^x + 7^y = 3^z$ has only the positive integer solution $(x, y, z) = (1, 1, 2),\ (5, 2, 4)$.*

In particular, for $a = 3$, we can completely solve (1.2) without any assumption on m. The proof is based on applying a result on linear forms in p-adic logarithms due to Bugeaud [Bu] to (1.2) with $m \equiv 0 \pmod 3$.

Corollary 1.2. *Let m be a positive integer. Then the equation*

$$(m^2+1)^x + (8m^2-1)^y = (3m)^z$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$, except for the case $m = 1$, where the equation $2^x + 7^y = 3^z$ has only the positive integer solutions $(x, y, z) = (1, 1, 2),\ (5, 2, 4)$.

2. Preliminaries

In order to obtain an upper bound for a solution of Pillai's equation $C^z - B^y = A$, we need a lower bound for linear forms in two logarithms. Now we introduce some notation. Let α_1 and α_2 be real algebraic numbers with $|\alpha_1| \ge 1$ and $|\alpha_2| \ge 1$. We consider the linear form

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

where b_1 and b_2 are positive integers. As usual, the *logarithmic height* of an algebraic number α of degree n is defined as

$$h(\alpha) = \frac{1}{n}\Big(\log |a_0| + \sum_{j=1}^{n} \log \max\{1, |\alpha^{(j)}|\}\Big),$$

where a_0 is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$) and $(\alpha^{(j)})_{1\le j\le n}$ are the conjugates of α. Let A_1 and A_2 be real numbers greater than one with

$$\log A_i \ge \max\Big\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\Big\},$$

for $i \in \{1, 2\}$, where D is the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2)$ over $\mathbb{Q}$. Define

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

We choose to use a result due to Laurent [La, Corollary 2] with $m = 10$ and $C_2 = 25.2$.

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PROPOSITION 2.1 (Laurent [La]). *Let Λ be given as above, with $\alpha_1 > 1$ and $\alpha_2 > 1$. Suppose that α_1 and α_2 are multiplicatively independent. Then*

$$\log|\Lambda| \geq -25.2\, D^4\Big(\max\Big\{\log b' + 0.38, \frac{10}{D}\Big\}\Big)^2 \log A_1 \log A_2.$$

Next, we shall quote a result on linear forms in p-adic logarithms due to Bugeaud [Bu]. Here we consider the case where $y_1 = y_2 = 1$ in the notation of [Bu, page 375].

Let p be an odd prime. Let a_1 and a_2 be nonzero integers prime to p. Let g be the least positive integer such that

$$\mathrm{ord}_p(a_1^g - 1) \geq 1, \quad \mathrm{ord}_p(a_2^g - 1) \geq 1,$$

where we denote the p-adic valuation by $\mathrm{ord}_p(\cdot)$. Assume that there exists a real number E such that

$$\frac{1}{p-1} < E \leq \mathrm{ord}_p(a_1^g - 1).$$

We consider the integer

$$\Lambda = a_1^{b_1} - a_2^{b_2},$$

where b_1 and b_2 are positive integers. We let A_1 and A_2 be real numbers greater than one with

$$\log A_i \geq \max\{\log|a_i|, E\log p\} \quad i = 1, 2,$$

and we put $b' = b_1/\log A_2 + b_2/\log A_1$.

PROPOSITION 2.2 (Bugeaud [Bu]). *With the above notation, if a_1 and a_2 are multiplicatively independent, then we have the upper estimate*

$$\mathrm{ord}_p(\Lambda) \leq \frac{36.1g}{E^3(\log p)^4}(\max\{\log b' + \log(E\log p) + 0.4, 6E\log p, 5\})^2 \log A_1 \log A_2.$$

3. Proof of Theorem 1.1

3.1. The case $m = 1$. We first show that when $m = 1$, (1.2) has only the positive integer solution $(x, y, z) = (1, 1, 2)$, except for the case $(a, c) = (3, 8)$.

LEMMA 3.1. *Let a be a positive integer with $a \equiv 3, 5 \pmod 8$. The equation*

$$2^x + (a^2 - 2)^y = a^z \tag{3.1}$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$ except for the case $a = 3$, where the equation $2^x + 7^y = 3^z$ has only the positive integer solutions $(x, y, z) = (1, 1, 2), (5, 2, 4)$.

REMARK 3.2. In 1958, Nagell [N2] showed that the equation

$$2^x + 7^y = 3^z$$

has only the positive integer solutions $(x, y, z) = (1, 1, 2), (5, 2, 4)$.

PROOF. We use the following proposition to show our assertion.

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Proposition 3.3.

(i) (Bennett [Be]) *Let a and b be integers with* $a, b \geq 2$. *Then the equation*

$$a^x - b^y = 2$$

has at most one solution in positive integers x and y.

(ii) (Nagell [N1]) *The equation*

$$x^2 + 4 = y^n$$

has only the positive integer solution $(x, y, n) = (11, 5, 3)$ *with y odd and* $n \geq 3$.

Let (x, y, z) be a solution of (3.1).

Case 1: $x = 1$. It follows from (i) in Proposition 3.3 that

$$a^z - (a^2 - 2)^y = 2$$

has only the positive integer solution $y = 1$, $z = 2$.

Case 2: $x = 2$. If $a \equiv 3 \pmod 8$, then, from (3.1),

$$1 = \left(\frac{-1}{a}\right)\left(\frac{-2}{a}\right)^y = (-1) \cdot 1 = -1,$$

where $(\cdot/\cdot)$ denotes the Jacobi symbol. This is impossible.

If $a \equiv 5 \pmod 8$, then

$$4 + 7^y \equiv 5^z \pmod 8.$$

Hence y is even and z is odd. It follows from (ii) in Proposition 3.3 that

$$((a^2 - 2)^{y/2})^2 + 4 = a^z$$

has no solutions y, z.

Case 3: $x \geq 3$. Taking (3.1) modulo 8 implies that

$$7^y \equiv 3^z,\ 5^z \pmod 8,$$

so y and z are even, say $y = 2Y$ and $z = 2Z$. Thus

$$(a^Z + (a^2 - 2)^Y)(a^Z - (a^2 - 2)^Y) = 2^x,$$

so

$$a^Z + (a^2 - 2)^Y = 2^{x-1} \quad \text{and} \quad a^Z - (a^2 - 2)^Y = 2.$$

Adding these yields

$$2^{x-2} + 1 = a^Z.$$

If $x = 3, 4$, then the above equation has no solutions. Indeed,

$$a^2 = (a^2 - 2) + 2 \leq (a^2 - 2)^Y + 2 = a^Z = 2^{x-2} + 1 \leq 5,$$

which is impossible. If $x \geq 5$, then $1 \equiv a^Z \pmod 8$. Since $a \equiv 3, 5 \pmod 8$, we see that Z is even, say $Z = 2Z_1$. Then

$$(a^{Z_1} + 1)(a^{Z_1} - 1) = 2^{x-2},$$

so

$$a^{Z_1} + 1 = 2^{x-3} \quad \text{and} \quad a^{Z_1} - 1 = 2.$$

We therefore obtain $a = 3$, $Z_1 = 1$ and so $x = 5$, $Y = 1$. □

3.2. The case $m \geq 2$. Let (x, y, z) be a solution of (1.2). By Lemma 3.1, we may suppose that $m \geq 2$. We examine parities of x, y, z. Using $a \equiv 3, 5 \pmod 8$ and $m \equiv \pm 1 \pmod a$, we show the following lemma.

LEMMA 3.4. *If (x, y, z) is a solution of (1.2), then both x and y are odd, and z is even.*

PROOF. Let (x, y, z) be a solution of (1.2). Suppose that our conditions are all satisfied.

Now it follows from $1 + c = a^2$ that $cm^2 - 1 = (a^2 - 1)m^2 - 1 > am$. Hence $z \geq 2$ from (1.2). Taking (1.2) modulo m^2 implies that $1 + (-1)^y \equiv 0 \pmod{m^2}$. Since $m \geq 2$, we see that y is odd. In view of $1 + c = a^2$ and $m \equiv \pm 1 \pmod a$, (1.2) leads to

$$2^x + (c-1)^y \equiv 2^x - 2^y \equiv 0 \pmod a,$$

so $(2/a)^x = (2/a)^y$. Since $(2/a) = -1$, from $a \equiv 3, 5 \pmod 8$, we have $x \equiv y \pmod 2$. Therefore, the fact that y is odd implies that x is odd.

We first show that $(m/(cm^2 - 1)) = 1$ and $(a/(cm^2 - 1)) = -1$. Note that $cm^2 - 1 \equiv -1 \pmod 8$. Write $m = 2^\alpha t$ with $\alpha \geq 0$ and t odd. Then

$$\left(\frac{m}{cm^2-1}\right) = \left(\frac{2}{cm^2-1}\right)^\alpha \left(\frac{t}{cm^2-1}\right) = 1 \cdot \left(\frac{t}{cm^2-1}\right) = \left(\frac{t}{cm^2-1}\right) = 1.$$

If $a \equiv 3 \pmod 8$, then

$$\left(\frac{a}{cm^2-1}\right) = -\left(\frac{cm^2-1}{a}\right) = -\left(\frac{c-1}{a}\right) = -\left(\frac{-2}{a}\right) = (-1) \cdot 1 = -1.$$

If $a \equiv 5 \pmod 8$, then

$$\left(\frac{a}{cm^2-1}\right) = \left(\frac{cm^2-1}{a}\right) = \left(\frac{c-1}{a}\right) = \left(\frac{-2}{a}\right) = -1.$$

Therefore,

$$\left(\frac{am}{cm^2-1}\right) = \left(\frac{a}{cm^2-1}\right)\left(\frac{m}{cm^2-1}\right) = (-1) \cdot 1 = -1.$$

Since $1 + c = a^2$,

$$\left(\frac{m^2+1}{cm^2-1}\right) = \left(\frac{m^2+cm^2}{cm^2-1}\right) = \left(\frac{a^2m^2}{cm^2-1}\right) = 1.$$

In view of these, we conclude that z is even from (1.2). □

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We can easily show that if m is even, then (1.2) has only the positive integer solution $(x, y, z) = (1, 1, 2)$ by taking (1.2) modulo m^3.

LEMMA 3.5. *If m is even, then (1.2) has only the positive integer solution $(x, y, z) = (1, 1, 2)$.*

PROOF. If $z \leq 2$, then we obtain $(x, y, z) = (1, 1, 2)$ from (1.2). Hence we may suppose that $z \geq 3$. It follows from Lemma 3.4 that x and y are odd.

Taking (1.2) modulo m^3 implies that

$$1 + m^2x - 1 + cm^2y \equiv 0 \pmod{m^3},$$

so

$$x + cy \equiv 0 \pmod{m},$$

which is impossible, since x is odd, c is even and m is even. This completes the proof of Lemma 3.5. □

LEMMA 3.6. *In (1.2), if m is odd then $x = 1$.*

PROOF. From Lemma 3.4, it follows that y is odd and z is even. Suppose that $x \geq 2$. Taking (1.2) modulo 4 implies that

$$3^y \equiv (am)^z \equiv 1 \pmod{4}.$$

This implies that y is even, which contradicts the fact that y is odd. We therefore obtain $x = 1$. □

3.3. Pillai's equation $C^z - B^y = A$. From Lemmas 3.4 and 3.6, it follows that $x = 1$ and y is odd. If $y = 1$, then we obtain $z = 2$ from (1.2). From now on, we may suppose that $y \geq 3$. Hence our theorem is reduced to solving Pillai's equation

$$C^z - B^y = A \tag{3.2}$$

with $y \geq 3$, where $A = m^2 + 1$, $B = cm^2 - 1$ and $C = am$.

We now want to obtain a lower bound for y.

LEMMA 3.7. *In (3.2), $y \geq (m^2 - 1)/c$.*

PROOF. Since $y \geq 3$, (3.2) yields

$$(am)^z = m^2 + 1 + (cm^2 - 1)^y \geq m^2 + 1 + (cm^2 - 1)^3 > (am)^3.$$

Hence $z \geq 4$. Taking (3.2) modulo m^4 implies that

$$m^2 + 1 - 1 + cm^2y \equiv 0 \pmod{m^4},$$

so $1 + cy \equiv 0 \pmod{m^2}$. Hence we obtain our assertion. □

We next want to obtain an upper bound for y.

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LEMMA 3.8. *In (3.2), $y < 2521 \log C$.*

PROOF. From (3.2), we now consider the following linear form in two logarithms:

$$\Lambda = z \log C - y \log B \quad (>0).$$

Using the inequality $\log(1+t) < t$ for $t > 0$,

$$0 < \Lambda = \log\left(\frac{C^z}{B^y}\right) = \log\left(1 + \frac{A}{B^y}\right) < \frac{A}{B^y}. \tag{3.3}$$

Hence

$$\log \Lambda < \log A - y \log B. \tag{3.4}$$

On the other hand, we use Proposition 2.1 to obtain a lower bound for Λ. It follows from Proposition 2.1 that

$$\log \Lambda \geq -25.2(\max\{\log b' + 0.38, 10\})^2 (\log B)(\log C), \tag{3.5}$$

where $b' = y/\log C + z/\log B$.

We note that $B^{y+1} > C^z$. Indeed,

$$\begin{aligned} B^{y+1} - C^z &= B(C^z - A) - C^z = (B-1)C^z - AB \\ &\geq (cm^2 - 2)(1+c)m^2 - (m^2+1)(cm^2 - 1) > 0. \end{aligned}$$

Hence $b' < (2y+1)/\log C$.

Put $M = y/\log C$. Combining (3.4) and (3.5) leads to

$$y \log B < \log A + 25.2\left(\max\left\{\log\left(2M + \frac{1}{\log C}\right) + 0.38, 10\right\}\right)^2 (\log B)(\log C),$$

so

$$M < 1 + 25.2\left(\max\left\{\log\left(2M + \frac{1}{2}\right) + 0.38, 10\right\}\right)^2,$$

since $\log C = \log(am) \geq \log 9 > 2$ and $A < B$. We therefore obtain $M < 2521$. This completes the proof of Lemma 3.8. □

We are now in a position to prove Theorem 1.1. Recall that $A = m^2 + 1$, $B = (a^2 - 1)m^2 - 1$ and $C = am$. Since $A + B = C^2$ and z is even, (3.2) can be written as

$$(C^2)^Z - B^y = C^2 - B$$

with $z = 2Z$. Then $y \geq Z$. If $y = Z$, then we obtain $y = Z = 1$. If $y > Z$, then we consider a 'gap' between the trivial solution $(y, Z) = (1, 1)$ and (possibly) another solution (y, Z), and show that making the 'gap' small leads to a contradiction. (See Bennett [Be, page 901] and Terai [T2, page 21] for a 'gap principle' for solutions of Pillai's equation.) Since $C^{2Z} > B^y$, it follows from Lemma 3.8 that

$$1 \leq y - Z < y - \frac{\log B}{\log C^2}\, y = \frac{\log(C^2/B)}{2 \log C}\, y < \frac{2521}{2} \log\left(\frac{C^2}{B}\right).$$

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By definition of B and C,

$$\frac{C^2}{B} = \frac{a^2m^2}{(a^2-1)m^2-1} = \frac{1}{1-(m^2+1)/a^2m^2}.$$

Therefore $\alpha := 1-(e^{2/2521})^{-1} < (m^2+1)/a^2m^2$. Since $m \geq 3$, this yields

$$a^2 < \frac{1}{\alpha}\Big(1+\frac{1}{m^2}\Big) \leq \frac{1}{\alpha}\Big(1+\frac{1}{9}\Big) = 1401.111.$$

Consequently, $a \leq 37$.

It follows from Lemmas 3.7 and 3.8, together with $a \leq 37$, that

$$m^2 - 1 < 2521(a^2-1)\log(am) \leq 3448728\log(37m).$$

Hence $m \leq 6538$.

From (3.3), we have the inequality

$$\left|\frac{\log B}{\log C} - \frac{z}{y}\right| < \frac{A}{yB^y \log C},$$

which implies that $|\log B/\log C - z/y| < 1/2y^2$, since $y \geq 3$. Thus z/y is a convergent in the simple continued fraction expansion to $\log B/\log C$.

On the other hand, if p_r/q_r is the rth such convergent, then

$$\left|\frac{\log B}{\log C} - \frac{p_r}{q_r}\right| > \frac{1}{(a_{r+1}+2)q_r^2},$$

where a_{r+1} is the $(r+1)$th partial quotient to $\log B/\log C$ (see, for example, Khinchin [K]). Put $z/y = p_r/q_r$. Note that $q_r \leq y$. It follows, then, that

$$a_{r+1} > \frac{B^y \log C}{Ay} - 2 \geq \frac{B^{q_r}\log C}{Aq_r} - 2. \tag{3.6}$$

Finally, we checked by Magma [BC] that for each $a \leq 37$ with $a \equiv 3,\ 5 \pmod 8$, (3.6) does not hold for any r with $q_r < 2521\log(am)$ in the range $3 \leq m \leq 6538$. This completes the proof of Theorem 1.1. □

4. Proof of Corollary 1.2

Let m be a positive integer. Let (x, y, z) be a positive solution of the Diophantine equation

$$(m^2+1)^x + (8m^2-1)^y = (3m)^z. \tag{4.1}$$

By Theorem 1.1, we may assume $m \equiv 0 \pmod 3$. Similarly to the proof of Lemma 3.4, we can show that y is odd. Here, we apply Proposition 2.2. For this we set $p := 3$, $a_1 := m^2+1$, $a_2 := 1-8m^2$, $b_1 := x$, $b_2 := y$, $\Lambda := (m^2+1)^x - (1-8m^2)^y$. Then we

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may take $g = 1$, $E = 2$, $A_1 = m^2 + 1$, $A_2 := 8m^2 - 1$. Hence

$$2z \le \frac{36.1}{8(\log 3)^4}(\max\{\log b' + \log(2\log 3) + 0.4, 12\log 3\})^2 \log(m^2+1)\log(8m^2-1),$$

where $b' := x/\log(8m^2 - 1) + y/\log(m^2 + 1)$. Suppose $z \ge 4$. We will observe that this leads to a contradiction. Taking (4.1) modulo m^4, we find $x + 8y \equiv 0 \pmod{m^2}$. In particular, we see $M := \max\{x, y\} \ge m^2/9$. Therefore, since $z \ge M$ and $b' \le M/\log m$,

$$2M \le \frac{36.1}{8(\log 3)^4}\Big(\max\Big\{\log\Big(\frac{M}{\log m}\Big) + \log(2\log 3) + 0.4, 12\log 3\Big\}\Big)^2 \times \log(m^2+1)\log(8m^2-1). \tag{4.2}$$

If $m \ge 3450$, then

$$2M \le \frac{36.1}{8(\log 3)^4}\Big(\log\Big(\frac{M}{\log m}\Big) + \log(2\log 3) + 0.4\Big)^2 \log(m^2+1)\log(8m^2-1).$$

Since $m^2 \le 9M$, the above inequality gives

$$2M \le 3.1(\log M - \log(\log 3450) + 1.19)^2 \log(9M+1)\log(72M-1).$$

We therefore obtain $M \le 22486$, which contradicts the fact that $M \ge m^2/9 \ge 1322500$.

If $m < 3450$, then inequality (4.2) gives

$$2M \le \frac{649.8}{(\log 3)^2}\log(m^2+1)\log(8m^2-1).$$

This implies $m \le 693$. Hence all x, y and z are also bounded. It is not hard to verify by Magma [BC] that there is no (m, x, y, z) under consideration satisfying (4.1). We conclude $z \le 3$. In this case, one can easily show that $(x, y, z) = (1, 1, 2)$. This completes the proof of Corollary 1.2. □

5. Concluding remarks

In Theorem 1.1, when $1 + c = a^2$ with a odd, we considered the equation

$$(m^2+1)^x + (cm^2-1)^y = (am)^z.$$

The proof is based on the properties that $m^2 + 1 \equiv 2 \pmod 8$ with m odd and $cm^2 - 1 \equiv -1 \pmod 8$.

On the other hand, we cannot apply our method used in the proof of Theorem 1.1 to the equation

$$(cm^2+1)^x + (m^2-1)^y = (am)^z, \tag{5.1}$$

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since $cm^2 + 1 \equiv 1 \pmod 8$ and $m^2 - 1 \equiv 0 \pmod 8$ with m odd. But it would be interesting to make the remark that (5.1) has (at least) two solutions $(x, y, z) = (1, 1, 2)$ and $(2, 3, 4)$ for m and a satisfying

$$m^2 - 2a^2 = 1.$$

We can easily verify this. There are infinitely many m, a, since m, a are solutions of the Pell equation $t^2 - 2u^2 = 1$. Similarly, when $2 + c = a^2$, the equation

$$(cm^2 + 1)^x + (2m^2 - 1)^y = (am)^z \tag{5.2}$$

also has (at least) two solutions $(x, y, z) = (1, 1, 2)$ and $(2, 3, 4)$ for m and a satisfying

$$a^2 - 2m^2 = -1.$$

There are infinitely many m, a, since m, a are solutions of the Pell equation $t^2 - 2u^2 = -1$. Note that $\gcd(cm^2 + 1,\ m^2 - 1) = \gcd(cm^2 + 1,\ 2m^2 - 1) = a^2\ (>1)$ in (5.1) and (5.2). These give new (nontrivial) 'counterexamples' to the generalised Jeśmanowicz' conjecture (Terai's conjecture), which states that if a, b, c, p, q, r are fixed positive integers satisfying $a^p + b^q = c^r$ with $\min\{a, b, c, p, q, r\} \geq 2$ and $\gcd(a, b) = 1$, then the Diophantine equation

$$a^x + b^y = c^z$$

has only the positive integer solution $(x, y, z) = (p, q, r)$, except for $(a, b, c) = (2, 7, 3)$ and $(a, b, c) = (2,\ 2^k - 1,\ 2^k + 1)$, where k is a positive integer with $k \geq 2$. (See [M1, M2, T1, T2].) So far, the known 'counterexamples' with $\gcd(a, b) \neq 1$ have been the following:

$$2^n + 2^n = 2^{n+1} \quad (\text{with } n \geq 1),$$
$$3 + 6 = 3^2, \quad 3^3 + 6^3 = 3^5.$$

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