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A UNIFYING APPROACH TO NON-MINIMAL QUASI-STATIONARY DISTRIBUTIONS FOR ONE-DIMENSIONAL DIFFUSIONS

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Abstract

We study convergence to non-minimal quasi-stationary distributions for one-dimensional diffusions. We give a method for reducing the convergence to the tail behavior of the lifetime via a property we call the first hitting uniqueness. We apply the results to Kummer diffusions with negative drift and give a class of initial distributions converging to each non-minimal quasi-stationary distribution.

Keywords: One-dimensional diffusion processes; quasi-stationary distributions; limit theorems; Kummer diffusion processes

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1. Introduction

Let us consider a one-dimensional diffusion $X = (X_t)_{t \geq 0}$ on $I = [0, b)$ or $[0, b]$ $(0 < b \leq \infty)$ killed nowhere and stopped upon hitting 0, and let T_0 denote its first hitting time of 0. A probability distribution ν on $I \setminus \{0\}$ is called a *quasi-stationary distribution* of X when the distribution of X_t with the initial distribution ν conditioned to be away from 0 until time t is time-invariant, that is, the following holds:

$$\mathbb{P}_\nu[X_t \in \mathrm{d}x \mid T_0 > t] = \nu(\mathrm{d}x) \quad (t > 0),$$

where $\mathbb{P}_\nu$ denotes the underlying probability measure of X with its initial distribution ν. For a certain quasi-stationary distribution ν, we study a sufficient condition on an initial distribution μ such that

$$\mu_t(\mathrm{d}x) := \mathbb{P}_\mu[X_t \in \mathrm{d}x \mid T_0 > t] \xrightarrow[t \to \infty]{} \nu(\mathrm{d}x). \tag{1.1}$$

Here and hereafter all convergence of probability distributions is in the sense of weak convergence. In the case where μ is compactly supported, the convergence (1.1) has been studied by many authors (e.g. [8], [10], [14], and [17]), and it has been shown that (1.1) holds under very general conditions and the limit distribution ν does not depend on the choice of a compactly supported μ. The limit measure ν is sometimes called the Yaglom limit or the *minimal quasi-stationary distribution*. On the other hand, for some diffusions there exist infinitely many quasi-stationary distributions. Although it is a natural problem to consider for what initial distributions (1.1) holds for each quasi-stationary distribution ν, there are very few studies considering this problem for non-minimal quasi-stationary distributions. The author

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only knows two papers, Lladser and San Martín [15] and Martínez, Picco, and San Martín [19], whose results we generalize in the present paper.

The present paper has two main results. One is Theorem 3.1, which gives a method for reducing the convergence (1.1) to the tail behavior of T_0. The other is Theorem 5.1, which applies Theorem 3.1 to *Kummer diffusions* with negative drift and derives concrete sufficient conditions for convergence (1.1).

A Kummer diffusion $Y^{(0)} = Y^{(\alpha,\beta)}$ ($\alpha > 0$, $\beta \in \mathbb{R}$) is a diffusion on $[0, \infty)$ stopped upon hitting 0 whose local generator $\mathcal{L}^{(0)} = \mathcal{L}^{(\alpha,\beta)}$ on $(0, \infty)$ is

$$\mathcal{L}^{(0)} = \mathcal{L}^{(\alpha,\beta)} = x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (-\alpha + 1 - \beta x)\frac{\mathrm{d}}{\mathrm{d}x}. \tag{1.2}$$

Note that the process $Y^{(0)} = Y^{(\alpha,\beta)}$ is also called a *radial Ornstein–Uhlenbeck process* in some of the literature (see e.g. [2] and [7]). We write

$$g_\gamma(x) := \mathbb{P}_x\big[\mathrm{e}^{-\gamma T_0^{(0)}}\big] = \int_0^\infty \mathrm{e}^{-\gamma t}\mathbb{P}_x\big[T_0^{(0)} \in \mathrm{d}t\big] \quad (\gamma \geq 0), \tag{1.3}$$

which is the Laplace transform of the first hitting time of 0 for $Y^{(0)} = Y^{(\alpha,\beta)}$. Then g_γ is a γ-eigenfunction for $\mathcal{L}^{(0)}$, i.e. $\mathcal{L}^{(0)}g_\gamma = \gamma g_\gamma$ (see e.g. [22, p. 292]). We define a Kummer diffusion with negative drift $Y^{(\gamma)} = Y^{(\alpha,\beta,\gamma)}$ ($\gamma \geq 0$) as the h-transform of $Y^{(\alpha,\beta)}$ by the function g_γ, that is, the process $Y^{(\alpha,\beta,\gamma)}$ is a diffusion on $[0, \infty)$ stopped at 0 whose local generator on $(0, \infty)$ is

$$\mathcal{L}^{(\gamma)} = \mathcal{L}^{(\alpha,\beta,\gamma)} = \frac{1}{g_\gamma}(\mathcal{L}^{(0)} - \gamma)g_\gamma.$$

If we write

$$\tilde{Y}^{(\alpha,\beta,\gamma)} := \sqrt{2Y^{(\alpha,\beta,\gamma)}},$$

then the local generator $\tilde{\mathcal{L}}^{(\alpha,\beta,\gamma)}$ of $\tilde{Y}^{(\alpha,\beta,\gamma)}$ on $(0, \infty)$ is given by

$$\tilde{\mathcal{L}}^{(\alpha,\beta,\gamma)} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{1-2\alpha}{2x} - \frac{\beta x}{2} + \frac{\tilde{g}'_\gamma}{\tilde{g}_\gamma}\right)\frac{\mathrm{d}}{\mathrm{d}x}, \tag{1.4}$$

where $\tilde{g}_\gamma(x) = \tilde{\mathbb{P}}_x\big[\mathrm{e}^{-\gamma \tilde{T}_0}\big]$ denotes the Laplace transform of the first hitting time of 0 for $\tilde{Y}^{(0)}$ starting from x. When $\alpha = 1/2$ and $\gamma = 0$, the process $\tilde{Y}^{(1/2,\beta,0)}$ is the Ornstein–Uhlenbeck process, and when $\beta = 0$, the process $\tilde{Y}^{(\alpha,0,\gamma)}$ is the Bessel process with negative drift (see e.g. [7]).

Previous studies. We briefly review several previous studies of quasi-stationary distributions for one-dimensional diffusions.

A first remarkable result on quasi-stationary distributions for one-dimensional diffusions was given by Mandl [17]. He treated the case where the right boundary is natural and gave a sufficient condition for the convergence to the minimal quasi-stationary distributions. His condition has been weakened by many authors, e.g. Collet, Martínez, and San Martín [5], Hening and Kolb [8], Kolb and Steinsaltz [10], and Martínez and San Martín [18]. Under certain weak assumptions it is shown that all compactly supported initial distributions imply convergence to the minimal quasi-stationary distribution.

The case where the right boundary is entrance has also been widely studied. Cattiaux *et al.* [3] and Littin [14] showed that in this case there exists a unique quasi-stationary distribution and all compactly supported initial distributions are attracted to the unique quasi-stationary distribution. Takeda [23] generalized their results to symmetric Markov processes with the *tightness property*.

Let us come back to the case where the right boundary is natural. We then have non-minimal quasi-stationary distributions. In the present paper, we let $L^1(I, \nu)$ denote the set of integrable functions on I with respect to the measure ν, and denote $f(x) \sim g(x)$ $(x \to \infty)$ when $\lim_{x\to\infty} f(x)/g(x) = 1$.

First, Martínez, Picco, and San Martín [19] studied Brownian motion with negative drift and showed convergence to non-minimal quasi-stationary distributions under the assumptions on tail behavior of the initial distribution.

Theorem 1.1. ([19, Theorem 1.1].) *Let B_t be a standard Brownian motion and let $\alpha > 0$ and consider the process*

$$X_t = B_t - \alpha t.$$

For an initial distribution μ on $(0, \infty)$, assume $\mu(\mathrm{d}x) = \rho(x)\,\mathrm{d}x$ for some $\rho \in L^1((0, \infty), \mathrm{d}x)$ satisfying

$$\log \rho(x) \sim -(\alpha - \delta)x \quad (x \to \infty)$$

for some $\delta \in (0, \alpha)$. Then we have

$$\mathbb{P}_\mu[X_t \in \mathrm{d}x \mid T_0 > t] \xrightarrow[t\to\infty]{} \nu_\lambda(\mathrm{d}x),$$

with

$$\lambda = (\alpha^2 - \delta^2)/2 \quad \textit{and} \quad \nu_\lambda(\mathrm{d}x) = C_\lambda\, \mathrm{e}^{-\alpha x} \sinh\!\left(x\sqrt{\alpha^2 - 2\lambda}\right) \mathrm{d}x$$

for the normalizing constant C_λ.

Secondly, Lladser and San Martín [15] studied Ornstein–Uhlenbeck processes.

Theorem 1.2. ([15, Theorem 1.1].) *Let $\alpha > 0$. Let X be the solution of the following SDE:*

$$\mathrm{d}X_t = \mathrm{d}B_t - \alpha X_t\, \mathrm{d}t,$$

where B is a standard Brownian motion. For an initial distribution μ on $(0, \infty)$, assume $\mu(\mathrm{d}x) = \rho(x)\,\mathrm{d}x$ for some $\rho \in L^1((0, \infty), \mathrm{d}x)$ satisfying

$$\rho(x) \sim x^{-2+\delta} \ell(x) \quad (x \to \infty)$$

for some $\delta \in (0, 1)$ and a slowly varying function ℓ at ∞. Then we have

$$\mathbb{P}_\mu[X_t \in \mathrm{d}x \mid T_0 > t] \xrightarrow[t\to\infty]{} \nu_\lambda(\mathrm{d}x)$$

with

$$\lambda = \alpha(1 - \delta) \quad \textit{and} \quad \nu_\lambda(\mathrm{d}x) = C_\lambda \psi_{-\lambda}(x)\, \mathrm{e}^{-\alpha x^2}\, \mathrm{d}x$$

for the normalizing constant C_λ, where $u = \psi_{-\lambda}$ denotes the unique solution for the following differential equation:

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} u - \alpha x \frac{\mathrm{d}}{\mathrm{d}x} u = -\lambda u, \qquad \lim_{x\to 0+} u(x) = 0, \qquad \lim_{x\to 0+} \frac{\mathrm{d}}{\mathrm{d}x} u(x) = 1 \quad (x \in (0, \infty)).$$

We will give a generalization of these two results in Theorem 5.1.

Outline of the paper. The remainder of the present paper is organized as follows. In Section 2 we will recall several known results on one-dimensional diffusions, the quasi-stationary distributions and the spectral theory for second-order ordinary differential operators. In Section 3 we will show one of our main results giving a general condition for convergence to quasi-stationary distributions. In Section 4 we will give the hitting density of Kummer diffusions with negative drift. In Section 5 we will show the second main result, which gives a sufficient condition for convergence to non-minimal quasi-stationary distributions for Kummer diffusions with negative drift.

2. Preliminaries

2.1. Feller's canonical form of second-order differential operators

Let $(X, \mathbb{P}_x)_{x\in I}$ be a one-dimensional diffusion on $I=[0, b)$ or $[0, b]$ $(0<b\leq\infty)$, that is, the process X is a time-homogeneous strong Markov process on I which has a continuous path up to its lifetime. Throughout this paper, we always assume

$$\mathbb{P}_x[T_y<\infty]>0 \quad (x\in I\setminus\{0\},\ \ y\in[0, b)), \tag{2.1}$$

where T_y denotes the first hitting time of y, and assume the point 0 is a trap:

$$X_t=0 \quad \text{for } t\geq T_0.$$

Let us recall Feller's classification of the boundaries (see e.g. Itô [9]). There exist a Radon measure m on $I\setminus\{0\}$ with full support and a strictly increasing continuous function s on $(0, b)$ such that the local generator $\mathcal{L}$ on $(0, b)$ is represented by

$$\mathcal{L}=\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}.$$

We call m *the speed measure* and s *the scale function* of X and we say X is a $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$-diffusion. Let $c=0$ or b and take $d\in(0, b)$. Set

$$I(c)=\int_c^d \mathrm{d}s(x)\int_c^x \mathrm{d}m(y), \qquad J(c)=\int_c^d \mathrm{d}m(x)\int_c^x \mathrm{d}s(y).$$

The boundary c is classified as follows:

$$\text{The boundary is}\quad \begin{cases} \text{regular} & \text{when } I(c)<\infty,\ J(c)<\infty,\\ \text{exit} & \text{when } I(c)=\infty,\ J(c)<\infty,\\ \text{entrance} & \text{when } I(c)<\infty,\ J(c)=\infty,\\ \text{natural} & \text{when } I(c)=\infty,\ J(c)=\infty. \end{cases}$$

Since $\mathbb{P}_x[T_0<\infty]>0$ for every $x>0$, the boundary 0 is necessarily regular or exit, equivalently $J(0)<\infty$. Note that in this case $s(0):=\lim_{x\to 0+} s(x)>-\infty$ holds. We also assume that the boundary b is not exit and that the boundary b is reflecting when it is regular.

Let us consider a diffusion on I whose local generator $\mathcal{L}$ on $(0, b)$ is

$$\mathcal{L}=a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}+c(x)\frac{\mathrm{d}}{\mathrm{d}x} \quad (x\in(0, b))$$

for functions a and c. Assume $a(x) > 0$ ($x \in (0, b)$). Then $\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$, where

$$\mathrm{d}m(x) = \frac{1}{a(x)} \exp\left(\int_d^x \frac{c(y)}{a(y)}\,\mathrm{d}y\right) \mathrm{d}x, \quad \mathrm{d}s(x) = \exp\left(-\int_d^x \frac{c(y)}{a(y)}\,\mathrm{d}y\right) \mathrm{d}x$$

for arbitrary given $d \in (0, b)$.

2.2. Quasi-stationary distributions

Let us summarize known results on quasi-stationary distributions for one-dimensional diffusions and give a necessary and sufficient condition for the existence of quasi-stationary distributions. Let X be a $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$-diffusion on $I = [0, b)$ or $[0, b]$ ($0 < b \leq \infty$). We define a function $u = \psi_\lambda$ as the unique solution of the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}u(x) = \lambda u(x), \quad \lim_{x \to 0+} u(x) = 0, \quad \lim_{x \to 0+} \frac{\mathrm{d}}{\mathrm{d}s}u(x) = 1 \quad (x \in (0, b), \lambda \in \mathbb{R}). \tag{2.2}$$

Note that from the assumption that the boundary 0 is regular or exit, the function ψ_λ always exists. The operator $L = -\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$ defines a non-negative definite self-adjoint operator on

$$L^2(I, \mathrm{d}m) := \left\{ f : I \to \mathbb{R} \mid \int_I |f|^2\,\mathrm{d}m < \infty \right\}.$$

Here we assume the Dirichlet boundary condition at 0 and the Neumann boundary condition at b if the boundary b is regular. We denote the infimum of the spectrum of L by $\lambda_0 \geq 0$.

Let us consider the case where the boundary b is not natural. It is then known that there is a unique quasi-stationary distribution (noting that Takeda [23] showed the corresponding result for general Markov processes with the tightness property).

Proposition 2.1. (See e.g. [14, Lemma 2.2, Theorem 4.1].) *Assume the boundary b is not natural. Then we have*

$$\lambda_0 > 0$$

and the function $\psi_{-\lambda_0}$ is strictly positive on $I \setminus \{0\}$ and integrable with respect to dm*, and there is a unique quasi-stationary distribution given by*

$$\nu_{\lambda_0}(\mathrm{d}x) = \lambda \psi_{-\lambda_0}(x)\,\mathrm{d}m(x), \quad \mathbb{P}_{\nu_{\lambda_0}}[T_0 \in \mathrm{d}t] = \lambda_0\,\mathrm{e}^{-\lambda_0 t}\,\mathrm{d}t.$$

Moreover, for every probability distribution μ on $(0, b)$ *with compact support, we obtain*

$$\mu_t \xrightarrow[t \to \infty]{} \nu_{\lambda_0}.$$

We now assume the boundary b is natural. We now have

$$\mathbb{P}_x[T_b < \infty] = 0 \quad (x \in (0, b)) \tag{2.3}$$

and

$$\frac{s(x) - s(0)}{s(M) - s(0)} = \mathbb{P}_x[T_M < T_0] \quad (0 < x < M < b)$$

(see e.g. Itô [9]). Taking limit $M \to b$, we have from (2.3)

$$\frac{s(x) - s(0)}{s(b) - s(0)} = \mathbb{P}_x[T_0 = \infty].$$

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Hence it follows that

$$\mathbb{P}_x[T_0 < \infty] = 1 \quad \text{for some / any } x > 0 \quad \Leftrightarrow \quad s(b) = \infty.$$

If ν is a quasi-stationary distribution, the distribution $\mathbb{P}_\nu[T_0 \in \mathrm{d}t]$ is exponentially distributed because $\mathbb{P}_\nu[T_0 > t + s \mid T_0 > t] = \mathbb{P}_\nu[X_{t+s} > 0 \mid T_0 > t] = \mathbb{P}_\nu[X_s > 0] = \mathbb{P}_\nu[T_0 > s]$. Then by (2.1) we have $\mathbb{P}_\nu[T_0 = \infty] < 1$, and therefore $\mathbb{P}_\nu[T_0 = \infty] = 0$, which implies $s(b) = \infty$. We recall the following good properties for the function ψ_λ.

Proposition 2.2. ([6, Lemma 6.18].) *Suppose the boundary b is natural and $s(b) = \infty$. Then for $\lambda > 0$ the following hold.*

(i) *For $0 < \lambda \leq \lambda_0$, the function $\psi_{-\lambda}$ is strictly positive on $I \backslash \{0\}$ and*

$$1 = \lambda \int_0^b \psi_{-\lambda}(x)\, \mathrm{d}m(x).$$

(ii) *For $\lambda > \lambda_0$, the function $\psi_{-\lambda}$ changes signs on I.*

Now we state a necessary and sufficient condition for the existence of non-minimal quasi-stationary distributions without proof.

Theorem 2.1. ([6, Theorem 6.34], [12, Theorem 3, Appendix I].) *Suppose the boundary b is natural. Then a non-minimal quasi-stationary distribution exists if and only if*

$$\lambda_0 > 0 \quad \textit{and} \quad s(b) = \infty. \tag{2.4}$$

This condition is equivalent to

$$m(d, b) < \infty \quad \textit{for some } d \in (0, b) \quad \textit{and} \quad \limsup_{x \to b} s(x) m(x, b) < \infty.$$

In this case a probability measure ν is a quasi-stationary distribution if and only if

$$\nu(\mathrm{d}x) = \lambda \psi_{-\lambda}(x)\, \mathrm{d}m(x) \;=: \nu_\lambda(\mathrm{d}x), \quad \mathbb{P}_{\nu_\lambda}[T_0 \in \mathrm{d}t] = \lambda\, \mathrm{e}^{-\lambda t}\, \mathrm{d}t \quad \textit{for some } 0 < \lambda \leq \lambda_0.$$

Here we note that as [6] only dealt with the case where the boundary 0 is regular, the proof also works in the case where the boundary 0 is exit.

For probability distributions on $(0, b)$, we introduce a partial order. For $\mu_1, \mu_2 \in \mathcal{P}(0, \infty)$, we define $\mu_1 \preceq \mu_2$ by

$$\mu_2(0, x] \leq \mu_1(0, x] \quad (x > 0).$$

This order gives a total order for quasi-stationary distributions and, as the following proposition says, the distribution ν_{λ_0} gives the minimal element. This is why we call it the minimal quasi-stationary distribution.

Proposition 2.3. *Suppose the boundary b is natural and (2.4) holds. Then we have*

$$\nu_\lambda \preceq \nu_{\lambda'} \quad (0 < \lambda' \leq \lambda \leq \lambda_0).$$

In particular, the distribution ν_{λ_0} is the minimal one in this order.

Proof. From (2.2) we have

$$\psi_{-\lambda}(x) = s(x) - \lambda \int_0^x \mathrm{d}s(y) \int_0^y \psi_{-\lambda}(z)\, \mathrm{d}m(z) \quad (x > 0, \lambda \in \mathbb{R}).$$

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Hence it follows that

$$\nu_\lambda(0, x] = \lambda \int_0^x \psi_{-\lambda}(y)\, \mathrm{d}m(y) = 1 - \psi_{-\lambda}^+(x) \quad (x > 0,\, 0 < \lambda \leq \lambda_0), \tag{2.5}$$

where $\psi_{-\lambda}^+(x)$ is the right-derivative of $\psi_{-\lambda}$ with respect to the scale function:

$$\psi_{-\lambda}^+(x) := \lim_{h \to 0+} \frac{\psi_{-\lambda}(x+h) - \psi_{-\lambda}(x)}{s(x+h) - s(x)}.$$

Let $0 < \lambda' \leq \lambda \leq \lambda_0$. From (2.5) we have

$$\psi_{-\lambda}^+(x) \leq \psi_{-\lambda'}^+(x) \quad (x > 0)$$

by a similar argument to [6, Lemma 6.11], which yields $\nu_\lambda \preceq \nu_{\lambda'}$. □

2.3. Spectral theory for second-order differential operators

Let us briefly review several results on the spectral theory of second-order differential operators. For the details, see e.g. Coddington and Levinson [4] and Kotani [11].

Set $I = (0, b)$ $(0 < b \leq \infty)$. Let $\mathrm{d}m$ be a Radon measure on I with full support and let $s : I \to (-\infty, \infty)$ be a strictly increasing continuous function. We assume that the boundary 0 is regular or exit, that is,

$$\int_0^d \mathrm{d}m(x) \int_0^x \mathrm{d}s(y) < \infty \quad \text{for some } 0 < d < b,$$

and assume the boundary b is natural, that is,

$$\int_d^b \mathrm{d}m(x) \int_x^b \mathrm{d}s(y) = \infty \quad \text{and} \quad \int_d^b \mathrm{d}s(x) \int_x^b \mathrm{d}m(y) = \infty \quad \text{for some } 0 < d < b.$$

Let $u = \psi_\lambda$ be defined by (2.2). Set

$$g_\lambda(x) = \psi_\lambda(x) \int_x^b \frac{\mathrm{d}s(y)}{\psi_\lambda(y)^2} \quad (\lambda \geq 0).$$

Then the function $u = g_\lambda$ is the unique, non-increasing solution for

$$\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s} u = \lambda u, \qquad \lim_{x \to 0+} u(x) = 1.$$

Define the Green's function

$$G_\lambda(x, y) = G_\lambda(y, x) := \psi_\lambda(x) g_\lambda(y) \quad (0 \leq x \leq y < b,\; \lambda \geq 0).$$

Then there exists a unique Radon measure σ on $[0, \infty)$, which we call the *spectral measure*, such that

$$G_\lambda(x, y) = \int_0^\infty \frac{\psi_{-\xi}(x)\psi_{-\xi}(y)}{\lambda + \xi} \sigma(\mathrm{d}\xi),$$

and the transition density $p(t, x, y)$ with respect to $\mathrm{d}m$ of $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$-diffusion absorbed at 0 is given by

$$p(t, x, y) = \int_0^\infty \mathrm{e}^{-\lambda t} \psi_{-\lambda}(x)\psi_{-\lambda}(y)\sigma(\mathrm{d}\lambda) \quad (t > 0,\, x, y \in I)$$

(see [20] for the details). Note that under the assumptions of Theorem 2.1, the spectral measure has its support on $[\lambda_0, \infty)$.

3. Convergence to quasi-stationary distributions

Let X be a $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$-diffusion on $[0, b)$ $(0 < b \leq \infty)$. For a set I, we denote the set of initial distributions on I by $\mathcal{P}(I)$. For a class $\mathcal{P} \subset \mathcal{P}[0, b)$ of initial distributions, we say that the *first hitting uniqueness* holds on $\mathcal{P}$ if

$$\text{the map } \mathcal{P} \ni \mu \longmapsto \mathbb{P}_\mu[T_0 \in \mathrm{d}t] \text{ is injective.}$$

For the class $\mathcal{P}$, we shall take

$$\mathcal{P}_{\mathrm{exp}} = \{\mu \in \mathcal{P}[0, b) \mid \mathbb{P}_\mu[T_0 \in \mathrm{d}t] = \lambda\, \mathrm{e}^{-\lambda t}\, \mathrm{d}t\ (\lambda > 0)\},$$

the set of initial distributions with exponential hitting probabilities. We refer to Rogers [21] as a general study of the first hitting uniqueness. Provided that the first hitting uniqueness holds on $\mathcal{P}_{\mathrm{exp}}$ and X satisfies the condition of Theorem 2.1, an initial distribution $\mu \in \mathcal{P}[0, b)$ satisfying $\mathbb{P}_\mu[T_0 \in \mathrm{d}t] = \lambda\, \mathrm{e}^{-\lambda t}\, \mathrm{d}t$ for some $0 < \lambda \leq \lambda_0$ must satisfy $\mu = \nu_\lambda$.

One of our main theorems is a general result to reduce the convergence (1.1) to the tail behavior of T_0, provided that the first hitting uniqueness holds on $\mathcal{P}_{\mathrm{exp}}$.

Theorem 3.1. *Let X be a $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$-diffusion on $[0, b)$ $(0 < b \leq \infty)$ and set*

$$\mu_t(\mathrm{d}x) = \mathbb{P}_\mu[X_t \in \mathrm{d}x \mid T_0 > t].$$

Assume the first hitting uniqueness holds on $\mathcal{P}_{\mathrm{exp}}$ and

$$\mathbb{P}_\nu[T_0 \in \mathrm{d}t] = \lambda\, \mathrm{e}^{-\lambda t}\, \mathrm{d}t \quad \textit{for some } \lambda > 0 \textit{ and some } \nu \in \mathcal{P}(0, b).$$

Then, for $\mu \in \mathcal{P}[0, b)$ and $\lambda > 0$, the following are equivalent:

(i) $\lim_{t\to\infty} \dfrac{\mathbb{P}_\mu[T_0 > t + s]}{\mathbb{P}_\mu[T_0 > t]} = \mathrm{e}^{-\lambda s} (s > 0)$,

(ii) $\mathbb{P}_{\mu_t}[T_0 \in \mathrm{d}s] \xrightarrow[t\to\infty]{} \lambda\, \mathrm{e}^{-\lambda s}\, \mathrm{d}s$,

(iii) $\mu_t \xrightarrow[t\to\infty]{} \nu$.

Proof of Theorem 3.1. From the Markov property, we have

$$\mathbb{P}_{\mu_t}[T_0 > s] = \frac{\mathbb{P}_\mu[T_0 > t + s]}{\mathbb{P}_\mu[T_0 > t]} \quad (t, s \geq 0).$$

Now it is obvious that (i) and (ii) are equivalent. In addition, it is not difficult to see that (iii) implies (i).

We show that (ii) implies (iii). Since $\mathcal{P}[0, b]$, the class of probability measures on the compactification $[0, b]$, is compact under the topology of weak convergence, we can take a sequence $\{t_n\}_n$ which diverges to ∞ such that

$$\mu_{t_n} \xrightarrow[n\to\infty]{} \tilde{\nu} \tag{3.1}$$

for some $\tilde{\nu} \in \mathcal{P}[0, b]$. From (ii), we have

$$\mathbb{P}_{\mu_{t_n}}[T_0 \in \mathrm{d}s] \xrightarrow[n\to\infty]{} \lambda\, \mathrm{e}^{-\lambda s}\, \mathrm{d}s. \tag{3.2}$$

On the other hand, for fixed $t>0$ we have

$$\mathbb{P}_{\mu_{t_n}}[T_0>t]=\int_{[0,b]}\mathbb{P}_x[T_0>t]\mu_{t_n}(\mathrm{d}x),$$

where we understand that

$$\mathbb{P}_x[T_0>t]=\begin{cases}0 & x=0,\\ 1 & x=b.\end{cases}$$

Note that since the boundary b is natural, the function $x\mapsto\mathbb{P}_x[T_0>t]$ is continuous on $[0, b]$. From (3.1) we obtain

$$\lim_{n\to\infty}\mathbb{P}_{\mu_{t_n}}[T_0>t]=\int_{[0,b]}\mathbb{P}_x[T_0>t]\tilde{\nu}(\mathrm{d}x).$$

Then from (3.2) it follows that

$$\int_{[0,b]}\mathbb{P}_x[T_0>t]\tilde{\nu}(\mathrm{d}x)=\mathrm{e}^{-\lambda t}. \tag{3.3}$$

Since

$$\lim_{t\to 0}\mathbb{P}_x[T_0>t]=1\{x>0\},\qquad \lim_{t\to\infty}\mathbb{P}_x[T_0>t]=1\{x=b\}\quad (x\in[0, b]),$$

we have from the dominated convergence theorem and (3.3) that $\tilde{\nu}\{0\}=\tilde{\nu}\{b\}=0$. Therefore $\tilde{\nu}\in\mathcal{P}(0, b)$ and $\mathbb{P}_{\tilde{\nu}}[T_0\in\mathrm{d}s]=\lambda\,\mathrm{e}^{-\lambda s}\,\mathrm{d}s$. Since the first hitting uniqueness holds on $\mathcal{P}_{\mathrm{exp}}$, we have $\tilde{\nu}=\nu$. The limit distribution ν does not depend on the choice of the sequence $\{t_n\}$, and therefore we obtain (iii). □

We give a sufficient condition for Theorem 3.1(i).

Proposition 3.1. *Assume the hitting densities f_x of 0 exist, that is, there exists a non-negative jointly measurable function $f_x(t)$ such that*

$$\mathbb{P}_x[T_0\in\mathrm{d}t]=f_x(t)\,\mathrm{d}t\quad (0<x<b,\ t>0).$$

Let $\mu\in\mathcal{P}(0, b)$ and assume the function

$$f_\mu(t):=\int_0^\infty f_x(t)\mu(\mathrm{d}x)\quad (0<x<b,\ t>0)$$

is differentiable in $t>0$ and

$$-\lim_{t\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}\log f_\mu(t)=\lambda\in(0, \lambda_0]. \tag{3.4}$$

Then we have

$$\lim_{t\to\infty}\frac{\mathbb{P}_\mu[T_0>t+s]}{\mathbb{P}_\mu[T_0>t]}=\mathrm{e}^{-\lambda s}\quad (s>0). \tag{3.5}$$

Proof. Set $g(u)=f_\mu(\log u)$ for $u>1$. From (3.4) we have

$$\lim_{t\to\infty}\frac{tg'(t)}{g(t)}=\lim_{t\to\infty}\frac{\mathrm{e}^t g'(\mathrm{e}^t)}{g(\mathrm{e}^t)}=-\lambda.$$

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Then from [13, Theorem 2], the function g varies regularly at ∞ with exponent $-\lambda$. From L'Hôpital's rule, we have for $u = e^s > 1$

$$\lim_{t\to\infty} \frac{\mathbb{P}_\mu[T_0 > t + \log u]}{\mathbb{P}_\mu[T_0 > t]} = \lim_{t\to\infty} \frac{f_\mu(t + \log u)}{f_\mu(t)} = \lim_{t\to\infty} \frac{g(e^t u)}{g(e^t)} = u^{-\lambda} = e^{-\lambda s}. \qquad \square$$

Remark 3.1. We might expect Proposition 3.1 to be extended, with (3.4) replaced by

$$\log f_\mu(t) \sim -\lambda t \quad (t \to \infty), \tag{3.6}$$

which is weaker than (3.4) by L'Hôpital's rule. In general, however, it does not hold. We give a counterexample that satisfies (3.6) but not (3.5). Let us find a positive function f of the form

$$f(t) = e^{(-\lambda + \varepsilon(t))t},$$

with a function $\varepsilon(t)$ vanishing at ∞ but not satisfying

$$\frac{\int_{t+s}^\infty f(u)\,du}{\int_t^\infty f(u)\,du} \xrightarrow[t\to\infty]{} e^{-\lambda s} \quad (s > 0). \tag{3.7}$$

By the change of variables, we can see that (3.7) is equivalent to the function

$$h(t) := \int_t^\infty u^{-\lambda - 1 + \varepsilon(\log u)}\,du$$

varying regularly with exponent $-\lambda$ at ∞. If the function ε is non-increasing, by the monotone density theorem [1, Theorem 1.7.2] it is equivalent to the slow variation of

$$k(s) = s^{\varepsilon(\log s)} \quad (s > 1).$$

We now set

$$\varepsilon(s) = 2^{-n} \quad (4^n < s \le 4^{n+1}, n \in \mathbb{N}),$$

and then the function ε vanishes at ∞ and

$$\frac{k(e \cdot \exp(4^n))}{k(\exp(4^n))} = \frac{\exp(2^n + 2^{-n})}{\exp(2^{n+1})} = \exp(-2^n + 2^{-n}) \xrightarrow[n\to\infty]{} 0.$$

So the function k does not vary slowly.

We give a sufficient condition for the existence of the hitting densities of 0. For this purpose, we need the following condition on decay of the spectral measure σ of $-\frac{d}{dm}\frac{d}{ds}$:

$$\text{(S)} \qquad \int_0^\infty e^{-\lambda t} \sigma(d\lambda) < \infty \quad (t > 0).$$

A sufficient condition for (S) is as follows.

Proposition 3.2. *Let m be a speed measure and s be a scale function on $(0, b)(0 < b \le \infty)$. Then if $|s(0)| < \infty$ and*

$$m(x, c] \le C(s(x) - s(0))^{-\delta} \quad (0 < x < c)$$

for some $C > 0$, $0 < c < b$ and $0 < \delta < 1$ (in this case, the boundary 0 is automatically regular or exit), the condition (S) holds.

The proof of Proposition 3.2 is given in [25]. The following result by Yano [26] gives existence and a spectral representation of the hitting densities.

Proposition 3.3. ([26, Proposition 2.1].) *Assume* (S) *holds. Then for any* $0 < x < b$ *the distribution of* T_0 *under* $\mathbb{P}_x$ *has density* $f_x(t)$ *on* $(0, \infty)$ *with respect to the Lebesgue measure, that is, the following hold:*

$$\mathbb{P}_x[T_0 \in \mathrm{d}t] = f_x(t)\,\mathrm{d}t \quad (0 < x < b,\ t > 0).$$

The hitting densities have a spectral representation,

$$f_x(t) = \int_0^\infty \mathrm{e}^{-\lambda t}\psi_{-\lambda}(x)\sigma(\mathrm{d}\lambda) \quad (0 < x < b,\ t > 0), \tag{3.8}$$

and have another representation,

$$f_x(t) = \frac{\mathrm{d}}{\mathrm{d}s(y)}p(t, x, y)\bigg|_{y=0} \quad (0 < x < b,\ t > 0).$$

4. Hitting densities of Kummer diffusions with negative drift

Let us give the hitting densities of Kummer diffusions with negative drift.

First we give a speed measure and a scale function for Kummer diffusions with negative drift. Fix $\alpha > 0$ and $\beta \in \mathbb{R}$. From (1.2) we have

$$\mathcal{L}^{(0)} = \mathcal{L}^{(\alpha,\beta)} = x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (-\alpha + 1 - \beta x)\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}m^{(0)}}\frac{\mathrm{d}}{\mathrm{d}s^{(0)}}$$

with

$$\mathrm{d}m^{(0)}(x) := \mathrm{d}m^{(\alpha,\beta)}(x) = x^{-\alpha}\,\mathrm{e}^{-\beta x}\,\mathrm{d}x, \quad \mathrm{d}s^{(0)}(x) := \mathrm{d}s^{(\alpha,\beta)}(x) = x^{\alpha-1}\,\mathrm{e}^{\beta x}\,\mathrm{d}x. \tag{4.1}$$

In addition, for $\gamma \geq 0$, we have

$$\mathcal{L}^{(\gamma)} = \mathcal{L}^{(\alpha,\beta,\gamma)} = x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(-\alpha + 1 - \beta x + \frac{xg'_\gamma(x)}{g_\gamma(x)}\right)\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}m^{(\gamma)}}\frac{\mathrm{d}}{\mathrm{d}s^{(\gamma)}},$$

with

$$\mathrm{d}m^{(\gamma)} = g_\gamma^2\,\mathrm{d}m^{(0)} \quad \mathrm{d}s^{(\gamma)} = g_\gamma^{-2}\,\mathrm{d}s^{(0)},$$

where g_γ is the function given in (1.3). Note that since $g_\gamma(0) = 1$, the classification of the boundary 0 for $\mathcal{L}^{(\gamma)}$ does not depend on $\gamma \geq 0$. The boundary ∞ for $\mathcal{L}^{(\gamma)}$ is always natural, which we will see in Proposition 4.1. We also have

$$\mathcal{L}^{(\gamma)} = \mathcal{L}^{(0)} + \frac{xg'_\gamma}{g_\gamma}\frac{\mathrm{d}}{\mathrm{d}x},$$

and since

$$\tilde{g}_\gamma(x) = g_\gamma(x^2/2),$$

it follows that

$$\tilde{\mathcal{L}}^{(\alpha,\beta,\gamma)} = \tilde{\mathcal{L}}^{(\alpha,\beta,0)} + \frac{\tilde{g}'_\gamma}{\tilde{g}_\gamma}\frac{\mathrm{d}}{\mathrm{d}x},$$

which implies (1.4).

We summarize several results on the hitting densities for Kummer diffusions with negative drift. Note that from (4.1) and Proposition 3.2, the condition (S) holds for $\frac{\mathrm{d}}{\mathrm{d}m^{(0)}}\frac{\mathrm{d}}{\mathrm{d}s^{(0)}}$.

Theorem 4.1. *For the process* $Y^{(\alpha,\beta,\gamma)}$ $(\alpha>0,\ \beta\in\mathbb{R},\ \gamma\geq 0)$*, the hitting densities* $f_x^{(\gamma)}$ *of* 0 *and the spectral measure* $\sigma^{(\gamma)}$ *for* $\mathcal{L}^{(\gamma)}$ *are given by*

$$f_x^{(\gamma)}(t)=\frac{\mathrm{e}^{-\gamma t}}{g_\gamma(x)}f_x^{(0)}(t)\quad(0<x<\infty,\ t>0) \tag{4.2}$$

and

$$\sigma^{(\gamma)}(\mathrm{d}\lambda)=\sigma^{(0)}(\mathrm{d}(\lambda-\gamma)), \tag{4.3}$$

where

$$f_x^{(0)}(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}x^\alpha t^{-\alpha-1}\,\mathrm{e}^{-x/t} & (\beta=0),\\[2ex] \dfrac{x^\alpha\,\mathrm{e}^{\beta t}}{\Gamma(\alpha)}\left(\dfrac{\beta\,\mathrm{e}^{-\beta t}}{1-\mathrm{e}^{-\beta t}}\right)^{1+\alpha}\exp\left(\dfrac{-x\beta\,\mathrm{e}^{-\beta t}}{1-\mathrm{e}^{-\beta t}}\right) & (\beta\neq 0),\end{cases} \tag{4.4}$$

and

$$\sigma^{(0)}(\mathrm{d}\lambda)=\begin{cases}\beta^{\alpha+1}\displaystyle\sum_{n=0}^{\infty}\frac{(\alpha)_{n+1}}{n!\Gamma(\alpha)}\delta_{\beta(n+\alpha)}(\mathrm{d}\lambda) & (\beta>0),\\[2ex] \dfrac{1}{\Gamma(\alpha)^2}\lambda^\alpha\,\mathrm{d}\lambda & (\beta=0),\\[2ex] (-\beta)^{\alpha+1}\displaystyle\sum_{n=0}^{\infty}\frac{(\alpha)_{n+1}}{n!\Gamma(\alpha)}\delta_{(-\beta)(n+1)}(\mathrm{d}\lambda) & (\beta<0),\end{cases} \tag{4.5}$$

where $(a)_k$ $(a\in\mathbb{R},\ k\in\mathbb{N})$ *is a Pochhammer symbol,*

$$(a)_k=a(a+1)\cdots(a+k-1).$$

In particular, we have

$$\lambda_0^{(\gamma)}=\begin{cases}\alpha\beta+\gamma & (\beta>0),\\ \gamma & (\beta=0),\\ -\beta+\gamma & (\beta<0).\end{cases}$$

Remark 4.1. From [16, Section 3.7], for example, we have

$$g_\gamma(x)=\begin{cases}\dfrac{1}{2^{\alpha-1}\Gamma(\alpha)}(2\sqrt{\gamma x})^\alpha K_\alpha(2\sqrt{\gamma x}) & (\beta=0),\\[2ex] \dfrac{\Gamma(\alpha+\gamma/\beta)}{\Gamma(\alpha)}(\beta x)^\alpha U(\alpha+\gamma/\beta,\alpha+1;\beta x) & (\beta>0),\\[2ex] \dfrac{\Gamma(1-\gamma/\beta)}{\Gamma(\alpha)}(-\beta x)^\alpha\,\mathrm{e}^{\beta x}U(1-\gamma/\beta,\alpha+1;-\beta x) & (\beta<0),\end{cases} \tag{4.6}$$

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where K_α denotes the modified Bessel function of the second kind (see e.g. [16, Section 3.1]) and U denotes the Tricomi confluent hypergeometric function

$$U(a, b; x) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-sx} s^{a-1} (1+s)^{b-a-1} \, \mathrm{d}s \quad (a > 0, \ b \in \mathbb{R}, \ x > 0).$$

Note that

$$K_\alpha(x) \sim 2^{\alpha-1}\Gamma(\alpha)x^{-\alpha}, \quad U(a, b; x) \sim \frac{\Gamma(b-1)}{\Gamma(a)} x^{-b+1} \quad (x \to +0, \ a > 0, \ b > 1)$$

and

$$K_\alpha(x) \sim \sqrt{\frac{\pi}{2x}} \, \mathrm{e}^{-x}, \quad U(a, b; x) \sim x^{-a} \quad (x \to +\infty, \ a > 0) \tag{4.7}$$

(see e.g. [16, Section 3.14.1]).

Although Theorem 4.1 can be easily shown by compiling some known results, we give a proof for completeness.

Proof of Theorem 4.1. First we show (4.2) and (4.4). We denote the transition probability of $Y^{(\gamma)} = Y^{(\alpha,\beta,\gamma)}$ by

$$\mathbb{P}_x\big[Y_t^{(\gamma)} \in \mathrm{d}y\big] = p^{(\gamma)}(t, x, y) \, \mathrm{d}m^{(\gamma)}(y).$$

Then we have

$$p^{(\gamma)}(t, x, y) = \mathrm{e}^{-\gamma t} \frac{p^{(0)}(t, x, y)}{g_\gamma(x) g_\gamma(y)}$$

(see e.g. [24, p. 172]), where we write $\mathbb{P}_x$ for the underlying probability measure for $Y^{(\gamma)}$ starting from x. From [2, Appendix 1], the transition density $p^{(0)}(t, x, y)$ is given by

$$p^{(0)}(t, x, y) = \begin{cases} \dfrac{1}{t}(xy)^{\alpha/2} \, \mathrm{e}^{-(x+y)/t} I_\alpha\left(\dfrac{2\sqrt{xy}}{t}\right) & (\beta = 0), \\[2ex] \dfrac{\beta \, \mathrm{e}^{-\alpha\beta t/2}}{1 - \mathrm{e}^{-\beta t}} (xy)^{\alpha/2} \exp\left(-\dfrac{(x+y)\beta \, \mathrm{e}^{-\beta t}}{1 - \mathrm{e}^{-\beta t}}\right) I_\alpha\left(\dfrac{2\sqrt{xy}\beta \, \mathrm{e}^{-\beta t/2}}{1 - \mathrm{e}^{-\beta t}}\right) & (\beta \neq 0), \end{cases}$$

where the function I_ν is the modified Bessel function of the first kind:

$$I_\nu(x) = \sum_{n=0}^\infty \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{\nu+2n} \quad (\nu \in \mathbb{R}, \ x \in \mathbb{R}).$$

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We now have

$$\begin{aligned}
\mathbb{P}_x\big[T_0^{(\gamma)} > t\big] &= \int_0^b p^{(\gamma)}(t, x, y)\, \mathrm{d}m^{(\gamma)}(y) \\
&= \frac{\mathrm{e}^{-\gamma t}}{g_\gamma(x)} \int_0^b p^{(0)}(t, x, y) g_\gamma(y)\, \mathrm{d}m^{(0)}(y) \\
&= \frac{\mathrm{e}^{-\gamma t}}{g_\gamma(x)} \int_0^b p^{(0)}(t, x, y)\, \mathrm{d}m^{(0)}(y) \int_0^\infty \mathrm{e}^{-\gamma u} f_y^{(0)}(u)\, \mathrm{d}u \\
&= \frac{\mathrm{e}^{-\gamma t}}{g_\gamma(x)} \int_0^\infty \mathrm{e}^{-\gamma u}\, \mathrm{d}u \int_0^b p^{(0)}(t, x, y) f_y^{(0)}(u)\, \mathrm{d}m^{(0)}(y) \\
&= \frac{\mathrm{e}^{-\gamma t}}{g_\gamma(x)} \int_0^\infty \mathrm{e}^{-\gamma u} f_x^{(0)}(u + t)\, \mathrm{d}u \\
&= \frac{1}{g_\gamma(x)} \int_t^\infty \mathrm{e}^{-\gamma u} f_x^{(0)}(u)\, \mathrm{d}u.
\end{aligned}$$

This shows (4.2). Then from Proposition 3.3 we obtain (4.4).

From [24, p. 173] we have (4.3). We show (4.5). First we consider the case $\beta > 0$. By some computation, we can check that

$$\psi_\lambda(x) = \frac{1}{\alpha} x^\alpha M(\lambda/\beta + \alpha,\, 1 + \alpha;\, \beta x) \quad (x > 0,\ \lambda \in \mathbb{R}), \tag{4.8}$$

where the function M is Kummer's confluent hypergeometric function:

$$M(a, b; x) = \sum_{n=0}^\infty \frac{(a)_n x^n}{(b)_n n!} \quad (a, b \in \mathbb{R},\ x \in \mathbb{R}).$$

We consider the values of λ for which the function ψ_λ is square-integrable. We may assume $\lambda < 0$. Since the asymptotic behavior of the function M is given by

$$M(a, b; x) \sim \frac{\Gamma(b)}{\Gamma(a)} x^{a-b}\, \mathrm{e}^x \quad (x \to \infty)$$

for $a \neq 0, -1, -2, \ldots$ (see e.g. [16, p. 289]), the function ψ_λ is not square-integrable with respect to $\mathrm{d}m$ when $\lambda/\beta + \alpha \neq 0, -1, -2, \ldots$. When $\lambda/\beta + \alpha = 0, -1, -2, \ldots$, the function ψ_λ is a polynomial and obviously square-integrable with respect to $\mathrm{d}m$. Note that

$$M(-n, 1 + \alpha; \beta x) = \frac{n!}{(1 + \alpha)_n} L_n^{(\alpha)}(\beta x),$$

where $L_n^{(\alpha)}(x)$ is the nth Laguerre polynomial of parameter α, that is,

$$L_n^{(\alpha)}(x) = \mathrm{e}^x \frac{x^{-\alpha}}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (\mathrm{e}^{-x} x^{n+\alpha}) \quad (n \in \mathbb{N})$$

(see e.g. [16, p. 241]). Since the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_n$ comprise an orthogonal basis of $L^2((0, \infty), x^\alpha\, \mathrm{e}^{-x}\, \mathrm{d}x)$, the functions $\{\psi_{-\beta(\alpha+n)}(x)\}$ are an orthogonal basis on

$L^2((0,\infty), x^{-\alpha}\,\mathrm{e}^{-\beta x}\,\mathrm{d}x)$. Hence the spectral measure only has the point spectrum, and the support of σ is $\{\beta(\alpha+n),\ n\geq 0\}$. Since we have

$$\int_0^\infty L_i^{(\alpha)}(x)L_j^{(\alpha)}(x)x^\alpha\,\mathrm{e}^{-x}\,\mathrm{d}x=\delta_{ij}\frac{\Gamma(i+\alpha+1)}{i!}\quad (i,j\in\mathbb{N})$$

(see e.g. [16, p. 241]), it follows that

$$\begin{aligned}\int_0^\infty \psi_{-\beta(\alpha+n)}(x)^2\,\mathrm{d}m(x)&=\frac{(n!)^2}{\alpha^2\beta^{\alpha+1}\{(1+\alpha)_n\}^2}\int_0^\infty L_n^{(\alpha)}(x)^2x^\alpha\,\mathrm{e}^{-x}\,\mathrm{d}x\\&=\frac{n!\Gamma(\alpha)}{\beta^{\alpha+1}(\alpha)_{n+1}}.\end{aligned}$$

Hence we obtain

$$\sigma\{\beta(n+\alpha)\}=\frac{\beta^{\alpha+1}(\alpha)_{n+1}}{n!\Gamma(\alpha)}\quad (n\geq 0).$$

Next we show the case $\beta<0$. Let us consider the map

$$L^2\big((0,\infty),\mathrm{d}m^{(\alpha,-\beta)}\big)\ni f\longmapsto \mathrm{e}^{\beta x}f\in L^2\big((0,\infty),\mathrm{d}m^{(\alpha,\beta)}\big).\tag{4.9}$$

Obviously this map is unitary. Moreover, since we have

$$\mathcal{L}^{(\alpha,\beta)}\big(\mathrm{e}^{\beta x}\psi_\lambda^{(\alpha,-\beta)}(x)\big)=(\lambda-\beta(\alpha-1))\big(\mathrm{e}^{\beta x}\psi_\lambda^{(\alpha,-\beta)}(x)\big)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s^{(\alpha,\beta)}}\big(\mathrm{e}^{\beta x}\psi_\lambda^{(\alpha,-\beta)}(x)\big)=\beta x^{1-\alpha}\psi_\lambda^{(\alpha,-\beta)}(x)+\mathrm{e}^{\beta x}\frac{\mathrm{d}}{\mathrm{d}s^{(\alpha,-\beta)}}\psi_\lambda^{(\alpha,-\beta)}(x),$$

we can see from (4.8) that

$$\psi_\lambda^{(\alpha,\beta)}(x)=\mathrm{e}^{\beta x}\psi_{\lambda+\beta(\alpha-1)}^{(\alpha,-\beta)},$$

where we denote the function defined in (2.2) for $\mathcal{L}^{(\alpha,\beta)}$ by $\psi_\lambda^{(\alpha,\beta)}$. Then, from the unitarity of the map (4.9) and the argument for the case $\beta>0$, the functions $\big\{\psi_{-\beta(n+1)}^{(\alpha,\beta)},\ n\geq 0\big\}$ comprise the orthogonal basis of $L^2((0,\infty),\mathrm{d}m^{(\alpha,\beta)})$ and therefore we obtain (4.5) for $\beta<0$.

Finally, we show the case $\beta=0$. Note that we can see from some computation that

$$\psi_\lambda(x)=\Gamma(\alpha)\left(\frac{x}{\lambda}\right)^{\alpha/2}I_\alpha(2\sqrt{\lambda x})\quad (x>0,\ \lambda\in\mathbb{R}).$$

From (3.8) and (4.4) we have

$$\int_0^\infty \mathrm{e}^{-\lambda t}\psi_{-\lambda}(x)\sigma^{(0)}(\mathrm{d}\lambda)=\frac{1}{\Gamma(\alpha)}x^\alpha t^{-\alpha-1}\,\mathrm{e}^{-x/t}.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^\nu I_\nu(x))=x^\nu I_{\nu-1}(x),\qquad I_\nu(x)\sim\frac{\mathrm{e}^x}{\sqrt{2\pi x}}\quad (\nu\in\mathbb{R},\ x\to\infty)$$

(see e.g. [16, p. 67, p. 139]), we can see that

$$\int_0^\infty \mathrm{e}^{-\lambda t}\left|\frac{\mathrm{d}}{\mathrm{d}x}\psi_{-\lambda}(x)\right|\sigma^{(0)}(\mathrm{d}\lambda)<\infty\quad (x>0).$$

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Thus we have

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\lambda t}\sigma^{(0)}(\mathrm{d}\lambda) &= \frac{\mathrm{d}}{\mathrm{d}s(x)}\int_0^\infty \mathrm{e}^{-\lambda t}\psi_{-\lambda}(x)\sigma^{(0)}(\mathrm{d}\lambda)\bigg|_{x=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}s(x)}\frac{1}{\Gamma(\alpha)}x^\alpha t^{-\alpha-1}\,\mathrm{e}^{-x/t}\bigg|_{x=0}\\ &= \frac{\alpha t^{-\alpha-1}}{\Gamma(\alpha)}.\end{aligned}$$

From the uniqueness of the Laplace transform, we obtain (4.5). □

We give the classification of the boundary ∞ for $\mathcal{L}^{(\gamma)}$.

Proposition 4.1. *For* $\alpha > 0,\ \beta \in \mathbb{R},\ \gamma \geq 0$*, the boundary* ∞ *for* $\mathcal{L}^{(\gamma)}$ *is natural.*

Proof. Let $\beta > 0$. From (4.6) and (4.7) we have

$$\begin{aligned}s^{(\gamma)}(x) - s^{(\gamma)}(1) &= \int_1^x y^{\alpha-1}\,\mathrm{e}^{\beta y}\frac{\mathrm{d}y}{g_\gamma^2(y)}\\ &\asymp \int_1^x y^{\alpha+2\gamma/\beta-1}\,\mathrm{e}^{\beta y}\,\mathrm{d}y \xrightarrow[x\to\infty]{} \infty,\end{aligned}$$

where $f_1 \asymp f_2$ means that there exists a constant $c > 0$ such that $(1/c)f_1(x) \leq f_2(x) \leq cf_1(x)$ for large $x > 0$. Note that from L'Hôpital's rule, it holds for $\delta \in \mathbb{R}$ that

$$\int_x^\infty y^\delta\,\mathrm{e}^{-\beta y}\,\mathrm{d}y \sim \frac{1}{\beta}x^\delta\,\mathrm{e}^{-\beta x} \quad (x \to \infty).$$

We have

$$\begin{aligned}\int_1^\infty \mathrm{d}s^{(\gamma)}(x)\int_x^\infty \mathrm{d}m^{(\gamma)}(y) &\asymp \int_1^\infty x^{\alpha+2\gamma/\beta-1}\,\mathrm{e}^{-\beta x}\,\mathrm{d}x\int_x^\infty y^{-\alpha-2\gamma/\beta}\,\mathrm{e}^{-\beta y}\,\mathrm{d}y\\ &\asymp \int_1^\infty \frac{\mathrm{d}x}{x}\\ &= \infty.\end{aligned}$$

Thus the boundary ∞ is natural. We can show the cases of $\beta = 0$ and $\beta < 0$ by a similar argument and hence we omit them. □

5. Convergence to non-minimal quasi-stationary distributions for Kummer diffusions with negative drift

Let us apply Theorem 3.1 to Kummer diffusions with negative drift, and give a sufficient condition on initial distributions under which the conditional process converges to each non-minimal quasi-stationary distribution specified.

We classify $Y^{(\gamma)} = Y^{(\alpha,\beta,\gamma)}$ $(\alpha > 0,\ \beta \in \mathbb{R},\ \gamma \geq 0)$ into the following five cases by β and γ:

$$\begin{array}{lll} \text{Case 1} & \beta = 0, & \gamma > 0,\\ \text{Case 2} & \beta > 0, & \gamma \geq 0,\\ \text{Case 3} & \beta < 0, & \gamma > 0,\\ \text{Case 1}' & \beta = 0, & \gamma = 0,\\ \text{Case 3}' & \beta < 0, & \gamma = 0. \end{array} \tag{5.1}$$

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We give a necessary and sufficient condition for Kummer diffusions with negative drift to satisfy the condition of Theorem 2.1.

Proposition 5.1. *For $\mathcal{L}^{(\alpha,\beta,\gamma)}$ $(\alpha > 0,\ \beta \in \mathbb{R},\ \gamma \geq 0)$, the condition of Theorem 2.1 holds if and only if one of Cases 1–3 in (5.1) holds.*

Proof. Let $\beta > 0$. Obviously $m^{(\gamma)}(1, \infty) < \infty$ and $s^{(\gamma)}(\infty) = \infty$. From (4.7) we have

$$m^{(\gamma)}(x, \infty)\big(s^{(\gamma)}(x) - s^{(\gamma)}(1)\big) \asymp \big(x^{-\alpha-2\gamma/\beta}\,\mathrm{e}^{-\beta x}\big)\big(x^{\alpha+2\gamma/\beta-1}\,\mathrm{e}^{\beta x}\big)$$
$$\asymp 1/x \xrightarrow[x\to\infty]{} 0.$$

Let $\beta = 0$. We can easily check $s^{(\gamma)}(\infty) = \infty$ for $\gamma \geq 0$ and

$$\lim_{x\to\infty} m^{(0)}(x, \infty)\big(s^{(0)}(x) - s^{(0)}(1)\big) = \infty.$$

For $\gamma > 0$, from (4.7) we have

$$m^{(\gamma)}(x, \infty)\big(s^{(\gamma)}(x) - s^{(\gamma)}(1)\big) \asymp \mathrm{e}^{-4\sqrt{\gamma x}} \cdot \mathrm{e}^{4\sqrt{\gamma x}} = 1.$$

Let $\beta < 0$. From (4.1) we obtain $s^{(0)}(\infty) < \infty$. For $\gamma > 0$, we have from (4.6)

$$s^{(\gamma)}(x) - s^{(\gamma)}(1) \asymp \int_1^x y^{-\alpha-\gamma/\beta}\,\mathrm{e}^{-\beta y}\,\mathrm{d}y$$
$$\asymp x^{1-\alpha-2\gamma/\beta}\,\mathrm{e}^{-\beta x} \xrightarrow[x\to\infty]{} \infty.$$

Similarly, we can show $m^{(\gamma)}(1, x) \asymp x^{-2+\alpha+2\gamma/\beta}\,\mathrm{e}^{\beta x}$ and thus $m^{(\gamma)}(1, \infty) < \infty$. Then we have

$$m^{(\gamma)}(x, \infty)\big(s^{(\gamma)}(x) - s^{(\gamma)}(1)\big) \asymp 1/x \xrightarrow[x\to\infty]{} 0. \qquad \square$$

The following is another main result of the present paper. For Kummer diffusions with negative drift, it gives a sufficient condition for an initial distribution under which the conditioned distribution converges to a non-minimal quasi-stationary distribution.

Theorem 5.1. *Let $X = Y^{(\gamma)} = Y^{(\alpha,\beta,\gamma)}$ $(\alpha > 0,\ \beta \in \mathbb{R},\ \gamma \geq 0)$ satisfying one of Cases 1–3 in (5.1) and let $\mu \in \mathcal{P}(0, \infty)$. Then the following hold.*

(i) *If Case 1 holds and $\mu(\mathrm{d}x) = \rho(x)\,\mathrm{d}x$ for some $\rho \in L^1((0, \infty), \mathrm{d}x)$ and*

$$\log \rho(x) \sim (\delta - 2\sqrt{\gamma})\sqrt{x} \quad (x \to \infty)$$

for some $0 < \delta < 2\sqrt{\gamma}$, then we have

$$\mu_t \xrightarrow[t\to\infty]{} \nu_\lambda$$

with $\lambda = \gamma - \delta^2/4 \in \big(0, \lambda_0^{(\gamma)}\big)$, where $\lambda_0^{(\gamma)} = \gamma > 0$ is the spectral bottom.

(ii) *If Case 2 holds and*

$$\mu(x, \infty) \sim x^{-\alpha-\gamma/\beta+\delta}\,\ell(x) \quad (x \to \infty) \tag{5.2}$$

for some $0 < \delta < \alpha + \gamma/\beta$ and some slowly varying function ℓ at ∞, then we have

$$\mu_t \xrightarrow[t\to\infty]{} \nu_\lambda$$

with $\lambda = \beta(\alpha - \delta) + \gamma \in \big(0, \lambda_0^{(\gamma)}\big)$, where $\lambda_0^{(\gamma)} = \alpha\beta + \gamma > 0$ is the spectral bottom.

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(iii) *If Case 3 holds and*

$$\mu(x, \infty) \sim x^{-1+\gamma/\beta+\delta}\ell(x) \quad (x \to \infty)$$

for some $0 < \delta < 1 - \gamma/\beta$ *and some slowly varying function* ℓ *at* ∞*, then we have*

$$\mu_t \xrightarrow[t\to\infty]{} \nu_\lambda$$

with $\lambda = -\beta(1-\delta) + \gamma \in \big(0, \lambda_0^{(\gamma)}\big)$*, where* $\lambda_0^{(\gamma)} = -\beta + \gamma > 0$ *is the spectral bottom.*

The proof of Theorem 5.1 will be given after several preparatory results.

Remark 5.1.

- When $\alpha = 1/2$, $\beta = 0$ and $\gamma > 0$, the process $\sqrt{2Y^{(1/2,0,\gamma)}}$ is a Brownian motion with negative drift $-\sqrt{2\gamma}t$. Hence Theorem 5.1(i) gives a generalization of Theorem 1.1.
- In Theorem 5.1(ii), if $\mu(\mathrm{d}x) = \rho(x)\,\mathrm{d}x$ for $\rho \in L^1((0, \infty), \mathrm{d}x)$ and

$$\rho(x) \sim x^{-\alpha-\gamma/\beta+\delta-1}\ell(x) \quad (x \to \infty)$$

for a slowly varying function ℓ, then (5.2) holds from Karamata's theorem [1, Proposition 1.5.8]. Hence Theorem 5.1(ii) is an extension of Theorem 1.2.

For the process $Y^{(\alpha,\beta,\gamma)}$, the first hitting uniqueness holds on $\mathcal{P}(0, \infty)$. We show this fact in more general settings as follows.

Theorem 5.2. *Let* X *be a* $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$*-diffusion on* $[0, b)(0 < b \le \infty)$ *and* $s(b) = \infty$*. Suppose the hitting densities* $f_x(t)$ *of* 0 *have the following form:*

$$f_x(t) = u(x)w(t)\,\mathrm{e}^{-v(x)y(t)} \quad (0 < x < b,\ t > 0) \tag{5.3}$$

for some strictly positive functions $u(x)$ *and* $v(x)$ *on* $(0, b)$ *and some strictly positive function* $w(t)$ *and* $y(t)$ *on* $(0, \infty)$*. In addition, suppose* v *is strictly increasing continuous and* $y(0, \infty) = (0, \infty)$*. Then the first hitting uniqueness holds on* $\mathcal{P}(0, \infty)$*.*

Proof. Suppose μ_1 and $\mu_2 \in \mathcal{P}(I)$ satisfy

$$\mathbb{P}_{\mu_1}[T_0 \in \mathrm{d}t] = \mathbb{P}_{\mu_2}[T_0 \in \mathrm{d}t]$$

and set $\mu = \mu_1 - \mu_2$. We have

$$\int_0^b f_x(t)\mu(\mathrm{d}x) = 0 \quad (t > 0). \tag{5.4}$$

Note that from the continuity of $f_x(t)/w(t)$ with respect to t, the equality (5.4) holds for every $t > 0$. From (5.3) and by a change of variables, we have

$$0 = \int_{v(0)}^{v(b)} u\big(v^{-1}(x)\big)\,\mathrm{e}^{-xy(t)}\mu\big(v^{-1}(\mathrm{d}x)\big).$$

Since $y(0, \infty) = (0, \infty)$, from the uniqueness of the Laplace transform we obtain

$$u(x)\mu(\mathrm{d}x) = 0 \quad \text{on } (0, b).$$

Since $u(x) > 0$, we obtain the desired result. □

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Now we go on to the proof of Theorem 5.1. For the proof of Theorem 5.1(i) we need the following lemma, which enables us to cut off the integral region for the asymptotic behavior of the Laplace transform.

Lemma 5.1. *Let* $f:(0,\infty)\to[0,\infty)$ *and assume*

$$\log f(x)\sim\delta\sqrt{x}\quad(x\to\infty)\tag{5.5}$$

for $\delta>0$ *and*

$$\int_0^\infty \mathrm{e}^{-x/t}f(x)\,\mathrm{d}x<\infty\quad(t>0).$$

Then we have

$$\log\int_0^\infty \mathrm{e}^{-x/t}f(x)\,\mathrm{d}x\sim\frac{\delta^2}{4}t,$$

and for every $\varepsilon>0$

$$\int_0^\infty \mathrm{e}^{-x/t}f(x)\,\mathrm{d}x\sim\int_{(\delta^2/4-\varepsilon)t^2}^{(\delta^2/4+\varepsilon)t^2}\mathrm{e}^{-x/t}f(x)\,\mathrm{d}x\quad(t\to\infty).$$

Proof. Since we have

$$\lim_{t\to\infty}\int_0^1 \mathrm{e}^{-x/t}f(x)\,\mathrm{d}x<\infty\quad\text{and}\quad\lim_{t\to\infty}\int_1^\infty \mathrm{e}^{-x/t}f(x)\,\mathrm{d}x=\infty,$$

we may assume without loss of generality that $f(x)=0$ for $0<x<1$. It is enough to show that

$$\lim_{t\to\infty}\frac{\int_1^{(\delta^2/4-\varepsilon)t}\mathrm{e}^{-x/t}f(x)\,\mathrm{d}x}{\int_1^\infty \mathrm{e}^{-x/t}f(x)\,\mathrm{d}x}=0\tag{5.6}$$

and

$$\lim_{t\to\infty}\frac{\int_{(\delta^2/4+\varepsilon)t}^\infty \mathrm{e}^{-x/t}f(x)\,\mathrm{d}x}{\int_1^\infty \mathrm{e}^{-x/t}f(x)\,\mathrm{d}x}=0.\tag{5.7}$$

Let

$$h(x)=\frac{\log(x^2f(x))}{\sqrt{x}}-\delta\quad(x>1).$$

Then from (5.5) we have $\lim_{x\to\infty}h(x)=0$. It follows that

$$\int_1^\infty \mathrm{e}^{-x/t}f(x)\,\mathrm{d}x=\int_1^\infty \mathrm{e}^{-\varphi_t(x)}\frac{\mathrm{d}x}{x^2},$$

where

$$\varphi_t(x)=x/t-(\delta+h(x))\sqrt{x}.$$

Note that

$$\varphi_t(x)=\frac{1}{t}\left(\sqrt{x}-\frac{\delta+h(x)}{2}t\right)^2-\frac{(\delta+h(x))^2}{4}t.$$

Let $\theta:=\delta/2-\sqrt{\delta^2/4-\varepsilon}>0$ and take $R>1$ so that

$$|h(x)|<\theta\quad\text{and}\quad\frac{2\delta|h(x)|+h(x)^2}{4}<\theta^2/8\quad(x>R).$$

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Then for $R < x < (\delta^2/4 - \varepsilon)t^2$ it follows that

$$\frac{\delta + h(x)}{2}t - \sqrt{x} > \frac{\delta + h(x)}{2}t - t\sqrt{\delta^2/4 - \varepsilon} > \frac{\theta}{2}t$$

and thus

$$\varphi_t(x) \geq (\theta^2/8 - \delta^2/4)t.$$

Then it follows that

$$\int_R^{(\delta^2/4-\varepsilon)t^2} \mathrm{e}^{-\varphi_t(x)} \frac{\mathrm{d}x}{x^2} \leq \mathrm{e}^{(\delta^2/4-\theta^2/8)t} \int_R^{(\delta^2/4-\varepsilon)t^2} \frac{\mathrm{d}x}{x^2}$$
$$\leq \mathrm{e}^{(\delta^2/4-\theta^2/8)t}.$$

To show (5.6), it is enough to show

$$\log \int_1^\infty \mathrm{e}^{-x/t} f(x)\,\mathrm{d}x \sim \frac{\delta^2}{4}t \quad (t \to \infty). \tag{5.8}$$

From [1, Theorem 4.12.10(ii)], we have

$$\log \int_0^x f(y)\,\mathrm{d}y \sim \delta\sqrt{x} \quad (x \to \infty).$$

From Kohlbecker's Tauberian Theorem [1, Theorem 4.12.1], we therefore obtain (5.8). We can show (5.7) by a similar argument. □

Now we proceed to the proof of Theorem 5.1.

Proof of Theorem 5.1. First we show (i). From Proposition 3.1 and Theorem 4.1, it is enough to show that

$$\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \log \int_0^\infty \mathrm{e}^{-x/t} \frac{x^{\alpha/2}}{K_\alpha(2\sqrt{\gamma x})} \mu(\mathrm{d}x) = \delta^2/4.$$

From (4.7) we have

$$\log \tilde{\rho}(x) := \log \frac{x^{\alpha/2}\rho(x)}{K_\alpha(2\sqrt{\gamma x})} \sim \delta\sqrt{x} \quad (x \to \infty). \tag{5.9}$$

Take $\varepsilon > 0$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \int_0^\infty \mathrm{e}^{-x/t} \frac{x^{\alpha/2}}{K_\alpha(2\sqrt{\gamma x})} \mu(\mathrm{d}x) = \frac{\int_0^\infty \mathrm{e}^{-x/t} x\tilde{\rho}(x)\,\mathrm{d}x}{t^2 \int_0^\infty \mathrm{e}^{-x/t}\tilde{\rho}(x)\,\mathrm{d}x},$$

we have from (5.9) and Lemma 5.1

$$\frac{\int_0^\infty \mathrm{e}^{-x/t} x\tilde{\rho}(x)\,\mathrm{d}x}{t^2 \int_0^\infty \mathrm{e}^{-x/t}\tilde{\rho}(x)\,\mathrm{d}x} \sim \frac{\int_{(\delta^2/4-\varepsilon)t^2}^{(\delta^2/4+\varepsilon)t^2} \mathrm{e}^{-x/t} x\tilde{\rho}(x)\,\mathrm{d}x}{t^2 \int_{(\delta^2/4-\varepsilon)t^2}^{(\delta^2/4+\varepsilon)t^2} \mathrm{e}^{-x/t}\tilde{\rho}(x)\,\mathrm{d}x},$$

and obviously we have

$$\int_{(\delta^2/4-\varepsilon)t^2}^{(\delta^2/4+\varepsilon)t^2} \mathrm{e}^{-x/t} x\tilde{\rho}(x)\,\mathrm{d}x \lesseqgtr (\delta^2/4 \pm \varepsilon)t^2 \int_{(\delta^2/4-\varepsilon)t^2}^{(\delta^2/4+\varepsilon)t^2} \mathrm{e}^{-x/t}\tilde{\rho}(x)\,\mathrm{d}x.$$

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Since $\varepsilon > 0$ can be arbitrarily small, we obtain

$$\frac{\int_0^\infty \mathrm{e}^{-x/t} x\tilde{\rho}(x)\,\mathrm{d}x}{t^2 \int_0^\infty \mathrm{e}^{-x/t}\tilde{\rho}(x)\,\mathrm{d}x} \xrightarrow{t\to\infty} \delta^2/4.$$

Next we show (ii). From the proof of Proposition 3.1, it is enough to show that the function $f_\mu(\log t)$ varies regularly at ∞ with exponent $-\lambda$. From Theorem 4.1, we have

$$f_\mu(\log t) = \frac{1}{\Gamma(\alpha)} t^{\beta-\gamma} h(t)^{1+\alpha} \int_0^\infty \frac{x^\alpha}{g_\gamma(x)}\, \mathrm{e}^{-h(t)x}\mu(\mathrm{d}x),$$

where

$$h(t) = \frac{\beta}{t^\beta - 1} \quad (t > 1).$$

The inverse function h^{-1} of h is given by

$$h^{-1}(s) = \left(1 + \frac{\beta}{s}\right)^{1/\beta} \quad (s > 0).$$

Note that the function $h^{-1}(s)$ varies regularly at $s = 0$ with exponent $-1/\beta$. By considering the function $f(\log h^{-1}(s))$, it follows that the function $f_\mu(\log t)$ varies regularly at $t = \infty$ with exponent $-\lambda$ if and only if the function

$$\int_0^\infty \frac{x^\alpha}{g_\gamma(x)}\, \mathrm{e}^{-sx}\mu(\mathrm{d}x)$$

varies regularly at $s = 0$ with exponent $-\alpha - (\gamma - \lambda)/\beta$. From Karamata's Tauberian Theorem [1, Theorem 1.7.1], it is equivalent to the function

$$\int_0^x \frac{y^\alpha}{g_\gamma(y)}\mu(\mathrm{d}y)$$

varying regularly at $x = \infty$ with exponent $\alpha + (\gamma - \lambda)/\beta$. Then from (4.7) and [1, Theorem 1.6.4], it is equivalent to the function $\mu(x, \infty)$ varying regularly at $x = \infty$ with exponent $-\lambda/\beta$, and therefore we obtain (ii).

Finally, we show (iii). The proof of this case is quite similar to that of (ii). From Theorem 4.1, we have

$$f_\mu(\log t) = \frac{1}{\Gamma(\alpha)} t^{\beta-\gamma} h(t)^{1+\alpha} \int_0^\infty \frac{x^\alpha}{g_\gamma(x)}\, \mathrm{e}^{-h(t)x}\mu(\mathrm{d}x).$$

Note that for $\beta < 0$ we obtain $\lim_{t\to\infty} h(t) = -\beta$. Then the function $f_\mu(\log t)$ varies regularly at $t = \infty$ with exponent $-\lambda$ if and only if the function

$$\int_0^\infty \frac{x^\alpha}{g_\gamma(x)}\, \mathrm{e}^{-h(t)x}\mu(\mathrm{d}x) = \frac{(-\beta)^{-\alpha}\Gamma(\alpha)}{\Gamma(1-\gamma/\beta)} \int_0^\infty \frac{\mathrm{e}^{-(h(t)+\beta)x}}{U(1-\gamma/\beta, \alpha+1;\ -\beta x)}\mu(\mathrm{d}x) \tag{5.10}$$

varies regularly at $t = \infty$ with exponent $-\lambda - \beta + \gamma$. Note that the function $h^{-1}(s)$ varies regularly at $s = -\beta + 0$ with exponent $-1/\beta$. Thus, by denoting $u = s + \beta$, the regular variation at $t = \infty$ of (5.10) with exponent $-\lambda - \beta + \gamma$ is equivalent to that at $u = 0$ of

$$\int_0^\infty \frac{\mathrm{e}^{-ux}}{U(1-\gamma/\beta, \alpha+1; -\beta x)}\mu(\mathrm{d}x)$$

with exponent $1 + (\lambda - \gamma)/\beta$. Using (4.7), the rest of the proof can be made by the same argument in (ii) and hence we omit it. The proof is complete. □

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