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Equilibrium and termination II: the case of Petri nets

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This paper is concerned with the asymptotic properties of a restricted class of Petri nets equipped with stochastic mass-action semantics. We establish a simple algebraic criterion for the existence of an *equilibrium*, that is to say, an invariant probability that satisfies the *detailed balance* condition familiar from the thermodynamics of reaction networks. We also find that when such a probability exists, it can be described by a free energy function that combines an internal energy term and an entropy term. Under strong additional conditions, we show how the entropy term can be deconstructed using the finer-grained individual-token semantics of Petri nets.

1. Introduction

Markov chains (MC) and differential equations (DE) have been the fundamental means of describing dynamical systems for a very long time. But today, as quantitative modelling efforts try to address decentralised dynamics of increasing complexity and connectedness (Easley and Kleinberg 2010), the direct use of such basic descriptive tools is becoming unwieldy. New modeling situations present a diversity of structures and scales, where the representational challenge is insurmountable without the use of indirect, abstract and structured syntaxes to describe the dynamics of interest (Epstein 1999). Thus, part of the modelling activity has to morph into a sort of domain-specific programming, where MC/DEs play the role of an assembly language, which is best not written out by hand, or even considered extensionally at all.

A case in point, which has drawn considerable attention in the past decade and will be the focus of this paper, is the modelling of what is arguably the complex distributed system *par excellence*, namely the combinatorial processes at work in biomolecular networks (BMN). There is a clear need for higher syntaxes such as Petri nets (Goss and Peccoud 1998) or derivatives of pi-calculus (Regev *et al.* 2001), or, the more recent rule-based propositions (Danos and Laneve 2004; Danos *et al.* 2007; Faeder *et al.* 2009). Rule-based methods, in particular, introduce a whole new level of flexibility by adding the ability to express refined context-independence assumptions about a particular reaction mechanism. As a result, rules succinctly describe entire reaction classes, and they do so in a way that makes the frequently necessary revision of a model a lot easier (Danos 2009). It should be noted that the purpose of creating new modelling languages, and in so doing enriching

the representational apparatus of a particular modelling domain, is not just to make the modelling more agile and far-reaching. As has been long recognised in the context of programming, one can lean on syntactic structures to develop various analyses that would otherwise be impossible. For instance, in our own rule-based framework, which uses the Kappa language, we have developed and adapted methodologies for detecting dead rules (Danos *et al.* 2008b), extracting long-range causal dependencies (Danos *et al.* 2007), and obtaining compressed representations of the models' differential semantics, and thus more accurate approximations of their stochastic behaviour (Harmer *et al.* 2010; Feret *et al.* 2009; Danos *et al.* 2010).

The more structure we can impose on the problem, the easier the analysis. Now, and unlike in the case of man-made programmed distributed systems, when dealing with natural systems, we are not at liberty to impose a structure. To make further progress in the specific modelling and analysis of BMNs, a compelling idea is to borrow structuring features from biophysics, and develop a modelling language where energetic and thermodynamics constraints are put front and centre.

This is not a new idea. There have been interesting attempts at developing syntaxes for reactions that would guarantee thermodynamic consistency by construction (Ederer and Gilles 2008; Ederer and Gilles 2007). Quite recently, these have been successfully extended to restricted forms of rule-based systems (Ollivier *et al.* 2010). This approach is congruent with the fact that knowing whether a computable CTMC admits an energy function is undecidable. This is a result we established using an encoding of the Post correspondence problem in an earlier paper (Danos and Oury 2010). This led to the conclusion that thermodynamic consistency should be ensured by construction rather than verified *ex post*. This parallels the way in which strongly typed languages guarantee termination, which is an otherwise notoriously undecidable problem.

In the present paper, we reconsider the thermodynamic consistency problem (in other words, the existence of a stochastic equilibrium, or, equivalently, of an energy function) for simple and symmetric Petri nets equipped with a stochastic mass-action dynamics (where simple means that no two transitions have the same net balance; and symmetric means every reaction is reversible). We show that in this very restricted class of computable CTMCs, and in contrast with the more expressive Kappa language used in Danos and Oury (2010), deciding the existence of an energy function is possible. The criterion we obtain is of a purely linear algebraic nature. In essence, any reaction invariant (a multiset of reactions with net balance zero) must have zero energy balance. This captures Kolmogorov's criterion (which is also known as Wegscheider's condition in the chemical literature), which states that the product of the CTMC rates along any cycle must equal that of the rates on the reverse cycle. Thanks to the linear structure on the transition graph of a Petri net, the above criterion can be decided by inspecting a finite basis of reaction invariants.

The key ingredient for the algebraisation of the equilibrium problem is the thermodynamic notion of entropy, defined as

$$\Omega(x) = -\sum_{A \in S} \ln x(A)!,$$

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where $x \in \mathbb{N}^S$ is a state of a Petri net – that is, a multiset over a finite set S. This term ensures that the invariant measure, if any, will have finite mass. This unconditional convergence does not happen with non-mass-action semantics, where the jumping rate of the CTMC only depends on the reaction, not the state. Neither does it happen in the more general case of rule-based CTMCs described in the Kappa language, where one is treading a line, which, in Petri net terms, could be described as having an unbounded number of potential species, and there the entropic contribution no longer suffices. In fact, a consequence of our result is that entropy-driven convergence can *only* be disrupted in a Kappa model by an unlimited creation of new species since, otherwise, the dynamics can be faithfully mapped to that of a Petri net (Danos *et al.* 2008a).

Unlike the literature on non-deterministic PNs, where algebraic invariants play a fundamental role (Pedersen 2008; Chaouiya 2007), the literature on stochastic Petri nets seems more oriented towards rich and scalable modelling environments than towards analysing asymptotic properties of infinite state systems (Marsan 1990). In the finite case, the theory of continuous-time Markov chains is already largely computational (Norris 1998), and since the size of the state space permits, such systems are amenable to automated verification techniques (see, for example, Desharnais and Panangaden (2003)). The chemistry literature contains similar arguments using Lyapounov potentials for the existence of an equilibrium for reversible Petri nets equipped with a differential mass-action semantics (Schuster and Schuster 1989); and there are also numerous proofs of our result for the simple case of finite-state systems (Yang *et al.* 2008; Kimura *et al.* 2007). However, none seem to cover the specific stochastic infinite-state scenario we propose here.

1.1. *Outline*

We begin in Section 2 by recalling those properties of continuous-time Markov chains we will need. In Section 3, we give a very brief review of Petri nets, narrowing down to the specific class of simple and symmetric nets. In Section 4, we establish a criterion for the existence of an equilibrium for this class, with a thermodynamic aside on entropy in Section 5. Section 6 concludes the paper with a discussion of the limitations we had to impose on the notion of a Petri net to obtain the result, as well as potential avenues for further research.

2. Continuous-time Markov chains

In this section we recall the basics of continuous-time Markov chain that we will need. We suppose we are given, once and for all, a countable state space X. We write

$$\Delta_X = \{(x, x) \mid x \in X\} \subseteq X^2$$

for the diagonal of X.

An exponential timer with parameter $\lambda > 0$ is a random variable T on $[0, +\infty)$ defined by $p(T > t) = \exp(-\lambda t)$ for $t \geqslant 0$. If we pick finitely many independent timers T_i with

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parameters λ_i, it is easy to prove that

$$p(T_i = \min T_j, T_i < t) = \frac{\lambda_i}{\sum_j \lambda_j} \cdot (1 - e^{-(\sum_j \lambda_j)t}) \to_{t \to 0^+} \lambda_i t. \tag{1}$$

Definition 2.1. A rate function on X is a map $q : X^2 \setminus \Delta_X \to \mathbb{R}_+$ such that for x in X, $\{y \mid q(x, y) > 0\}$ is finite.

Given a rate function q, we define a binary relation on X, written $|q|$, as $(x, y) \in |q|$ if and only if $q(x, y) > 0$. We say $|q|$ is the *transition graph* or the support of q. We can think of $|q|$ as a directed graph on X, and of q as weighting edges of $|q|$ with positive reals. As, according to our definition, $q(x, x)$ is not defined, $|q|$ is irreflexive. We also assume that $|q|$ is image-finite or equivalently has finite out-degree. We say that q is *symmetric* if $|q|$ is, in other words, if $q(x, y) = 0 \Rightarrow q(y, x) = 0$.

Given a rate function, we can generate a random sequence with values in $X \times [0, +\infty)$ in the following inductive way. Supposing we are at state x, we draw independently for each jump from x an exponential time $T(y)$ with parameter $q(x, y)$. Then we set:

— the next state to be the (almost surely unique) y such that $T(y) = \min T(y')$; and
— the time increment to be $T(y)$.

Image-finiteness guarantees that there are only finitely many timers to choose from. If $q(x, y) > 0$, it follows from (1) that the jump to y has a probability proportional to $q(x, y)$. If $q(x, y) = 0$, the chain never jumps from x to y.

This simulation protocol defines a probability to be in state y at time t, starting in state x at time 0, which we will denote by $P(t, x, y)$, and this is known as a continuous-time Markov chain (CTMC).

Technically, we have to assume more from q to prevent Zenonian explosions, where a process accumulates infinitely many jumps in finite time (Norris 1998). We do not need to get into these delicate questions as the actual CTMCs we are interested in will be sufficiently regular. Indeed, our rate functions will come from a certain class (to be defined in detail in the next section) of simple and symmetric Petri nets (which is why we need a countable state space), and will therefore have a homogeneous structure – with a bounded out-degree (though rates themselves will be unbounded).

Definition 2.2. Given a rate function q on X, we say a probability p on X is:

— *invariant* if for x in X, $t \geqslant 0$, $p(x) = \sum_y p(y)P(t, y, x)$;
— an *equilibrium* if for distinct x, y in X, we have $p(x)q(x, y) = p(y)q(y, x)$.

Note that the equilibrium condition breaks naturally into two conditions having different natures:

$$p(x)q(x, y) = p(y)q(y, x) \tag{E1}$$

$$\sum_x p(x) = p(X) = 1 \qquad \text{(convergence)}. \tag{E2}$$

Condition (E1) is a purely algebraic condition for which there is a simple criterion (Kolmogorov's or Wegscheider's, see the introduction). It is known as *reversibility* in the

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probabilistic literature, and *detailed balance* in the chemical one. On the other hand, condition (E2) is a convergence condition, which only comes into play because the state space X is infinite.

An equilibrium is a particular kind of invariant probability. To see this, let us return to (1), which says that an exponential timer of parameter λ fires within $[0, \tau]$ with a probability that tends to $\lambda\tau$ as $\tau \to 0^+$. Using this first-order expansion for a small time τ, we get

$$P(\tau, y, x) \sim \tau q(y, x)$$

if $x \neq y$, and therefore

$$P(\tau, x, x) \sim 1 - \sum_{y \neq x} \tau q(x, y).$$

Hence, the invariant distribution equation given above can be rewritten as:

$$p(x) \sum_{y \neq x} q(x, y) = \sum_{y \neq x} p(y) q(y, x). \tag{S}$$

Condition (S) expresses the fact that p is a probabilistic state of the system such that the compound rate at which we leave x (the left-hand side) is the same as the rate at which we enter x (the right-hand side). By contrast, (E1) expresses the fact that p is a state such that the rate at which we jump from x to y is the same as the rate at which we jump from y to x. It is clear that (S) is implied by (E1).

From the classic theory of CTMCs (Norris 1998), we know that if q's transition graph is strongly connected (in other words q is irreducible) and p is a probability invariant under q, then:

— p is unique;

— q is recurrent, meaning the probability to return to any x is 1;

— q is positive-recurrent, meaning the mean return time to any x is finite; and, importantly,

— for $x, y \in X$, we have $P(t, x, y) \to p(y)$ when $t \to +\infty$.

In words, wherever we start in X, the dynamics will invariably lead the system to p – this is a strong property, and reminiscent of both termination and confluence. All of the above holds for an invariant probability, and, *a fortiori*, for one that satisfies the stronger detailed balance condition (E1). (This raises the question of whether having an invariant probability is decidable for Petri nets; which is a question we will not attempt to answer here.)

We will call the set of states where an equilibrium p is not zero its *support*. It is easy to see that the sub-graph induced by the support of p in $|q|$ is symmetric (and thus strongly connected), even if $|q|$ is not, and terminal (that is, it has no outgoing edges). It follows that all equilibria are convex combinations of equilibria localised to such symmetric terminal components of $|q|$.

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2.1. *Examples*

We will now give a couple of examples to illustrate the definitions so far:

(1) Take $X = \mathbb{N} \setminus \{0\}$ and $q(x, x+1) = 1$; this is a walk on X where only the times are random, the sequence of states being inflexibly determined. It is connected, but clearly neither symmetric nor strongly connected. To make the transition graph strongly connected, we can add 'resets', for example, we can set $q(2^k, 1) = 1$ for powers of 2, $k > 0$. The resulting graph is still not symmetric, and we may also observe that with this specific choice, resets are so sparse that the mean return time to 1 is actually infinite, so the system has no invariant probability. This could not happen with a finite transition graph.

(2) Take $X = \mathbb{N}$ and $q(x, x+1) = \alpha \cdot q(x+1, x)$ with $\alpha > 0$. This (biased) random walk on $\mathbb{N}$ is symmetric and (hence) strongly connected. We see that (E1) has geometric solutions verifying $p(x+1)/p(x) = \alpha$, and therefore (E2) only holds if $\alpha < 1$.

2.2. *A thermodynamic aside: energy*

The condition (E1) of detailed balance refers to some notion of thermodynamic consistency. Indeed, the equilibrium problem can be reformulated as the existence of a function $H : X \to \mathbb{R}$ such that:

$$\ln(q(y, x)/q(x, y)) = H(y) - H(x) \tag{F1}$$

$$\sum_x \exp(-H(x)) \leqslant \infty. \tag{F2}$$

We can choose to describe an equilibrium either as a probability p or by means of a real-valued map as above. The correspondence between the two definitions is given by Boltzmann's relation:

$$p(x) = e^{-H(x)} / \sum_x e^{-H(x)}. \tag{2}$$

The map H is called an *energy* function and is defined up to an additive constant (which reflects the convention that a probability should sum up to 1).

We will return to this thermodynamic formulation when we discuss our main result at the end of the paper.

3. Petri nets

In this section we start with a quick review of Petri nets, with an eye on:

(1) narrowing down our question to a subclass of Petri nets for which we can derive a criterion;

(2) describing the reaction invariants which correspond to loops in the state space, and which we will use to decide (E1).

The terminology of Petri nets reflects the history of the subject, and we will mostly use the chemical side of the dictionary, for example, talking about reactions rather than transitions, and species rather than places.

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3.1. *Basic definition*

A Petri net consists of:

— two disjoint finite sets S (species), R (reactions); and
— an input and an output function $i, o : R \to S \to \mathbb{N}$.

The idea is that $i(r)(A)$ is the number of tokens of species A that reaction r consumes, while $o(r)(A)$ is the number it produces. The state space is $S \to \mathbb{N} \simeq \mathbb{N}^S$, which can be viewed equivalently as: multisets over S; integer vectors of size $|S|$; or integer-valued maps defined on S. States form a subset of the real vector space $\mathbb{R}^S$ of dimension $|S|$; they are closed under linear combination with integer coefficients, and can be added, subtracted and compared. We write $\leqslant$ for the pointwise order on finite maps.

The data above allows us to define a labelled transition system on the state space, that is to say, a family of binary relations on $\mathbb{N}^S$ labelled in R, written $x \to_r y$. Specifically, a transition labelled by $r \in R$ (also known as a firing of r) is defined by

$$x \to_r x + o(r) - i(r) \text{ if } i(r) \leqslant x$$

where the condition $i(r) \leqslant x$ expresses the fact that r needs a minimal number of inputs to fire.

We will write $r \cdot x$ for $x + o(r) - i(r)$, that is, the new state resulting from applying r to x.

The transition system we have just defined is *monotonic* in the sense that

$$x \to_r y \Rightarrow x + x' \to_r y + x'$$

for any $x' \in \mathbb{N}^S$. More tokens never inhibit a transition. Extensions of Petri nets, including reactions where the *absence* of a species can be required for a reaction to fire, make the entire framework a lot more complex.

3.2. *Linear structure*

Using the linear structure of the state space, we can represent the input and output functions as matrices indexed on $S \times R$ with coefficients in $\mathbb{N}$. This gives the so-called *stoichiometric* matrix C as $C = o - i$.

The column vector $C_r \in \mathbb{Z}^S$ represents the net effect of applying r, or the jump/translation in the state space that results from applying r. This jump does not depend on the state it is applied to (and on the manner in which it is applied – unlike in the case of rules (Danos and Oury 2010)); what does depend on the source state is the application condition $x \geqslant i(r)$, the fullfilment of which will determine whether the jump is possible (or enabled) from x.

More generally, a vector $y \in \mathbb{N}^R$ represents a linear combination of reactions, and $Cy \in \mathbb{Z}^S$ is its total resulting effect. In general, y will be realised by countably many trajectories in the underlying state space $\mathbb{N}^S$. Specifically, any trajectory whose sequence of reactions $\tilde{y}$ enumerate the multiset y will have to have enough inputs at each step for the next reaction to proceed. By monotony, any y with integer coefficients can be realised, provided we start with a large enough state.

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A *reaction invariant*, or R-invariant, is a $y \in \mathbb{N}^R$ such that $Cy = 0$. Equivalently, a multiset of reactions y such that, wherever realisable, the underlying trajectory is a loop in $\mathbb{N}^S$. Such invariants can be thought of as loop schemes.

There is also the dual notion of a *species invariant*, or S-invariant, namely, a $u \in \mathbb{Z}^R$ in the null-space of C^t the transpose of C; equivalently, us such that $\langle u, Cy\rangle = \langle C^t u, y\rangle = 0$, that is to say, the linear form $\langle u, \_\rangle$ is a (linear) invariant of the trajectories.

We will use later the following basic linear algebraic fact. If C is a (stoichiometric) matrix, then $ker(C)^\perp = Im(C^t)$. To see this, suppose $\kappa = C^t\epsilon$ for some $\epsilon \in \mathbb{R}^S$, and pick ϕ in $ker(C)$. We have $\langle \kappa, \phi\rangle = \langle C^t\epsilon, \phi\rangle = \langle \epsilon, C\phi\rangle = 0$. Hence $Im(C^t) \subseteq ker(C)^\perp$, and since both subspaces have the same dimension, this inclusion is in fact an equality.

Note that C^t might not be injective (equivalently the associated system might have non-trivial S-invariants). But if we pick ϵ_1, ϵ_2 such that $C^t\epsilon_1 = C^t\epsilon_2$, then the linear forms $\langle \epsilon_1, \_\rangle$, $\langle \epsilon_2, \_\rangle$ are equal when restricted to $Im(C)$; in other words, $\epsilon_1 - \epsilon_2$ is an S-invariant. In the following, this will imply that any choice of ϵ amounts to the same.

We will restrict ourselves to Petri nets that are simple and symmetric, as defined below.

Definition 3.1. Let N be a net, with reactions R and stoichiometric matrix C. Then N is said to be:

— *simple* if there are no identical columns in C;
— *symmetric* if every reaction $r \in R$ has an inverse reaction $r^\star \in R$ with $i(r^\star) = o(r)$ and $i(r) = o(r^\star)$.

The first condition says that no two reactions have identical jumps; the second says that every reaction is reversible. Note that if N is symmetric and simple, then there is a unique $r^\star$ inverse to r, so in this case we can unambiguously refer to *the* reaction inverse to r. Note also that r can be inverse to itself when $i(r) = o(r)$, that is, when r has no effect on the state.

A net can always be symmetrised by adding an inverse where it lacks one. However, it is not clear how a given net can be 'simplicised' naturally; we could select among identical columns, but that would be arbitrary.

3.3. *Examples*

In practice, we will present a Petri net as a list of reactions – here are 3 examples:

$$\rightarrow A,\ A \rightarrow 2A \tag{3}$$

$$S + I \rightarrow 2I,\ S \leftarrow I \tag{4}$$

$$\rightarrow A,\ A \rightarrow . \tag{5}$$

(5) and (4) are simple, but (3) is not since both transitions have the same net balance. (5) is symmetric; and (4) would have a symmetric underlying transition graph if we were to add $S \rightarrow I$, but it would no longer be simple, neither would it be symmetric in the intensional sense we have used here. This illustrates the difference between the extensional notion of a symmetric transition graph and the more restrictive and intensional one we have defined and will be using for Petri nets.

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Continuing with example (4), we see that there is a reaction invariant $y^t = (1\,1)$, which can be realised by a loop provided we start from a state x where $x(I) > 1$. It also has a species invariant, $x^t = (1\,1)$, expressing the fact that $x(S) + x(I)$ is invariant under any transition (the actual value depending on the initial state).

3.4. *Mass-action semantics*

We turn now to the quantitative aspects of Petri nets.

PNs have a countably infinite state space and finitely many reactions – so they will generate countably infinite transition graphs with finite degree, which fits the definition of rate function given in Section 2. We still need to assign a rate to each transition.

Suppose x, z are multisets over S, we define:

— the number of symmetries of any enumeration of x

$$x! := \prod_{A \in S} x(A)!;$$

— the total number of elements of x

$$|x| := \sum_{A \in S} x(A);$$

— when $z \leqslant x$, the number of matches for z in x

$$[z\,;x] := \prod_{A \in S} [z(A)\,;x(A)] := \prod_{A \in S} x(A)!/(x(A) - z(A))! = x!/(x - z)!. \tag{6}$$

Definition 3.2. Given a simple PN on S, R and a *rate constant* map $k : R \to \mathbb{R}^+ \setminus \{0\}$, we define the jumping rate as

$$x \to_r x + o(r) - i(r) \quad \text{with rate} \quad \tau(x, r) = k(r) \cdot [i(r)\,;x].$$

By the application condition, $i(r) \leqslant x$, so $[i(r)\,;x]$ is well defined.

Note that if the PN were not simple, the above definition would induce a compound rate function

$$q(x, y) = \textstyle\sum_{\{r \mid r \cdot x = y\}} \tau(x, r).$$

As we will see, this would create a problem when it comes to constructing equilibria, which is why we restrict to simple PNs. As a consequence, the set of reactions leading from x to y is either empty or a singleton $\{r\}$, and we can write $q(x, r \cdot x) = \tau(x, r)$ and $q(x, y) = 0$ otherwise.

Another point worth making is that we have chosen to use $[a\,;b] = b!/(b - a)!$ the number of injections of a in b to count matches, but sometimes, $\binom{b}{a} = [a\,;b]/a!$, that is, the number of subsets, is used instead. The difference between the two conventions is independent of the state (that is, it is static), and can (therefore) be entirely hidden in the rate constant $k(r)$. The convention we follow is noticeably more natural when considering the rule-based extension of PNs and their refinement theory (Danos *et al.* 2008a).

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3.5. *Comparison with constant rate semantics*

The semantics above will be called the *mass-action* semantics.

It is important to contrast it with another semantics, which associates to a jump $x \to_r x + o(r) - i(r)$ a flat rate $\tau(x, r) = k(r)$. These nets are often referred to as stochastic Petri nets with marking-independent rates (Marsan 1990), since the rate of a jump only depends on the reaction r and not on the state x to which the reaction applies.

The difference between the two semantics has drastic consequences on their long-term behaviours. To see this, let us reconsider example (5), a birth-and-death process $\to_k A \to_d$ where rate constants are indicated as subscripts (which we recall are both > 0). If we follow the non-mass-action definition, (E1) becomes $p(x)k = dp(x + 1)$, so $p(x)$ is a geometric sequence, and will verify (E2) if and only if $\mu := k/d < 1$. With the mass-action definition, the condition (E1) reads

$$p(x)k = d(x + 1)p(x + 1),$$

and it is easy to verify that the solution is a Poisson distribution,

$$p(x) = \exp(-\mu)\, \mu^x / x!,$$

which therefore converges independently of the value of μ. This unconditional convergence is a general phenomenon and so is the occurrence of the Poisson distribution; somewhat paradoxically, mass action will turn out to be a mathematically simpler semantics when it comes to deciding equilibria.

3.6. *A lemma*

We will say that a Petri net N with a rate constant map $k : T \to \mathbb{R}_+$ is *sisma*, if it is simple, symmetric (as in Definition 3.1), and has mass-action semantics.

Lemma 3.3. Let N be sisma. Then for x in $\mathbb{N}^S$, $r \in R$ and $y = r \cdot x$, we have

$$q(y, x)/q(x, y) \;=\; k(r^\star)/k(r) \cdot y!/x!.$$

Proof. Because N is simple, r is the only reaction that brings x to y, so, by the definition of mass action, the rate of the jump from x to y is

$$q(x, y) \;=\; k(r) \cdot x!/(x - i(r))!.$$

Because N is symmetric, $r^\star$ brings y back to x, so $q(y, x) > 0$, and again, because N is simple, $r^\star$ is the only reaction that does this, so the rate of the reverse jump is

$$q(y, x) \;=\; k(r^\star) \cdot y!/(y - o(r))!$$

where we have used $i(r^\star) = o(r)$, which holds by the definition of $r^\star$.

The conclusion then follows since $y - o(r) = x - i(r)$. □

4. Equilibrium

Everything is now in place for us to present our main result.

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Theorem 4.1. Let N be sisma with stochiometric matrix C, and define N's transitional energy vector $\kappa \in \mathbb{R}^R$ as $\kappa(r) = \ln(k(r^\star)/k(r))$ for $r \in R$. Then N has a nowhere zero equilibrium if and only if $\kappa \in ker(C)^\perp$.

Proof.

($\Rightarrow$) (This is the easy direction.) We pick ϕ in $ker(C)$. Since C has values in $\mathbb{Z}$, we can suppose without loss of generality that ϕ has coefficients in $\mathbb{Z}$. In fact, we can suppose ϕ has coefficients in $\mathbb{N}$, since N being symmetric, we can always replace a negative coefficient on r (say) with a positive one on $r^\star$. Next, pick a state x_0 sufficiently large for ϕ to be realisable as a cycle γ_ϕ at x_0 – this is always possible by monotony. By (E1), for all jumps (x, y) in γ_ϕ, we have

$$p(x)q(x, y) = p(y)q(y, x).$$

Since p is nowhere zero, this can be rewritten as

$$p(x)/p(y) = q(y, x)/q(x, y),$$

which implies

$$\prod_{\gamma_\phi} q(y, x)/q(x, y) = \prod_{\gamma_\phi} p(x)/p(y) = 1,$$

or, equivalently (by taking a log),

$$0 \;=\; \sum_{(x,y)\in\gamma_\phi} \ln(p(x)/p(y)) \;=\; \sum_{(x,y)\in\gamma_\phi} \ln(q(y, x)/q(x, y)).$$

We can now evaluate $q(y, x)/q(x, y)$ using Lemma 3.3, and obtain

$$\begin{aligned} 0 &= \sum_r \kappa(r)\phi(r) + \sum_{(x,y)\in\gamma_\phi} \ln(y!/x!) \\ &= \langle \kappa, \phi \rangle + \sum_{(x,y)\in\gamma_\phi} \ln(y!/x!) \\ &= \langle \kappa, \phi \rangle, \end{aligned}$$

since the terms $\ln(y!/x!)$ add up to zero along any cycle.

($\Leftarrow$) Because N is symmetric, the transition graph of q is a disjoint sum of symmetric components.

Pick a component D, together with a distinguished state z_0, and for each z in D choose a path γ_z from z_0 to z.

Up to the choice of $p(z_0)$, the following uniquely defines a function p on D:

$$\begin{aligned} \ln(p(z_0)/p(z)) \;&:=\; \textstyle\sum_{(x,y)\in\gamma_z} \ln(q(y, x)/q(x, y)) \\ &=\; \langle \kappa, \tilde{\gamma}_z \rangle + \ln(z!/z_0!) \end{aligned}$$

where $\tilde{\gamma}_z$ is the R-vector to which γ_z projects.

We can rewrite the above as

$$\ln p(z) + \ln z! \;=\; \ln p(z_0) + \ln z_0! - \langle \kappa, \tilde{\gamma}_z \rangle.$$

This expression for $p(z)$ does not depend on the choice of γ_z since $\langle \kappa, \tilde{\gamma}_z \rangle$ is constant over all paths from z_0 to z. This is because we assume that $\langle \kappa, \tilde{\gamma} \rangle = 0$ for any cycle γ. For the same reason, this assignment verifies (E1).

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We also have to take care of (E2).

Using an earlier remark in Section 3.2, we know there is an $\epsilon \in \mathbb{R}^S$ such that $\langle \kappa, \tilde{\gamma}_z \rangle = \langle \epsilon, C\tilde{\gamma}_z \rangle$, and since $C\tilde{\gamma}_z = z - z_0$, we can rewrite the above as

$$\ln p(z) + \ln z! + \langle \epsilon, z \rangle \;=\; \ln p(z_0) + \ln z_0! + \langle \epsilon, z_0 \rangle. \tag{7}$$

From equation (7), we have

$$\textstyle\sum_{z \in D} p(z) \;\propto\; \sum_{z \in D} e^{-\langle \epsilon, z \rangle}/z! \;\leqslant\; \sum_{z \in \mathbb{N}^S} e^{-\langle \epsilon, z \rangle}/z!.$$

Writing $\mu_A = e^{-\epsilon(A)}$, we have

$$\textstyle\sum_{z \in \mathbb{N}^S} e^{-\langle \epsilon, z \rangle}/z! \;=\; \sum_{z \in \mathbb{N}^S} \prod_{A \in S} {\mu_A}^{z(A)}/z(A)! \;=\; \prod_{A \in S} e^{\mu_A} < \infty.$$

Hence (E2) also holds. Since we can repeat the construction for each component, we can define a nowhere zero equilibrium. □

Note that in the (E2) part of the argument, the choice of ϵ such that $C^t\epsilon = \kappa$ does not matter. If we had picked another ϵ', it would follow from (7) that

$$\ln(p'(z)/p(z)) = \langle \epsilon' - \epsilon, z_0 - z \rangle = 0$$

because $\epsilon' - \epsilon$ is an S-invariant (Section 3.2) and z_0, z are connected in the transition graph.

We can conclude immediately that whether a sisma Petri net N has a nowhere zero equilibrium is decidable. We can equally use the theorem to choose a κ in the solution space $ker(C)^{\perp}$. This solution space has dimension $d = |R| - dim(ker(C))$, which means we can fix d of the $|R|$ transitional energies to arbitrary values and fill in the rest uniquely according to the constraint. In particular, there is always the trivial choice $\kappa = \mathbf{0}$, corresponding to $k(r) = k(r^\star)$ for all r in R.

Yang *et al.* (2008) explores, in the case of a finite state space, how the solution space can be further constrained by experimental evidence. Our criterion shows that this is possible in general for a countable state space.

It is clear from the proof that, provided κ is the solution space, we can define a unique equilibrium with support any of the components of q; all other equilibria can then be obtained by convex combinations of such minimal ones.

It is also clear that the convergence part of the proof (E2) makes great use of the specific mass-action shape of the rates. This does not come as a surprise as we saw earlier with the birth-and-death example (Section 3.5) that the constant rate semantics does not always yield a convergent solution to (E1). Moreover, as foreseen in Section 3.5, the equilibrium solution must be a Poisson distribution when q is irreducible.

Corollary 4.2. Let N be sisma and irreducible. Then its invariant probability, if it exists, is unique and is a product of Poisson distributions.

Proof. To see this, observe that from (7), we get for $z \in \mathbb{N}^S$

$$p(z) \;\propto\; \prod_{A \in S} \mu_A^{z(A)}/z(A)! \tag{8}$$

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with $\mu_A = e^{-\epsilon(A)}$. Now if N's transition graph is (strongly) connected, this says exactly that $p(z)$ is an S-indexed product of Poisson distributions with parameters μ_A. □

Hence, a thermodynamically consistent and irreducible sisma net N is equivalent in the long term to a juxtaposition of independent birth-and-death processes with parameter ratios μ_A. All correlations between As and Bs are transient. If N is not irreducible, the formula above is still correct, but the normalisation factor for $p(z)$ depends on the component of z, so while there can now be long-term correlations, they will have to be entirely encoded in the qualitative reachability properties of N. This intriguing lack of expressivity in the asymptotic behaviour of (irreducible) sisma nets (which one might call their 'normal forms' to borrow from the terminology of rewriting systems) can be traced back to the fact that the transitional energies κ can be 'tokenised' as $C^t\epsilon$ when N satisfies (E1).

4.1. *A stronger version?*

The result only decides the existence of an equilibrium that is everywhere non-zero (or, equivalently, whether it is possible, for every connected component of N's transition graph to define an equilibrium with support this component). However, what if we are interested in deciding whether N has an equilibrium with support the component D of some specific initial state z_0? Admittedly, this is a rather gratuitous question since if such a situation is not already covered by our result, the 'physics' of our Petri net is consistent for some specific inputs only. Nevertheless, it seems fairly subtle.

Returning to the second part of the proof, it can be seen that to build an equilibrium on D, we need κ to be orthogonal only to those reaction invariants ϕ that are realised by a loop γ_ϕ *in* the distinguished component D. So the refined statement becomes that N has a solution to (E1) with support D if and only if κ is orthogonal to those. The problem is that we no longer know how to conclude to (E2), and even if we could prove it always holds, it is unclear how to decide the partial orthogonality property above.

True, it is easy to decide whether any given ϕ is realised in D. First, ϕ is realisable in D if and only if one of the $|\phi|!$ orderings ϕ_σ of ϕ is. For each ϕ_σ, with $A \in S$, we can compute the minimal number of As needed to complete $\phi_\sigma(A)$, say $x(A)$; and ϕ_σ is realisable in D if and only if there exists a z reachable from z_0 such that $z \geqslant x$ (in PN terms, whether x is *coverable* starting from z_0). This is decidable: for example, using coverability trees *à la* Karp–Miller (Karp and Miller 1969).

But checking realisability within D on a generating family of R-invariants (in $\mathbb{N}^R$) is no longer enough since non-realisability is *not* stable under linear combination (with integer coefficients).

Consider the following two 'co-operating' (ordered) R-invariants:

$$\phi_1 = A_1 \to B_1,\ \mathbf{B_2} + B_1 \to \mathbf{B_2} + C_1,\ C_1 \to A_1$$
$$\phi_2 = A_2 \to B_2,\ \mathbf{C_1} + B_2 \to \mathbf{C_1} + C_2,\ C_2 \to A_2.$$

ϕ_1 needs an A_1 and a B_2; ϕ_2 needs an A_2 and a C_1.

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If we start from $z_0 = A_1 + A_2$, none of the loops go through. Nevertheless, $\phi_1 + \phi_2$ can be realised since ϕ_1, ϕ_2 can exchange intermediates $\mathbf{B_2}$ and $\mathbf{C_1}$:

$$A_1 + A_2 \to_2 A_1 + B_2 \to_1 B_1 + B_2 \to_1 C_1 + B_2$$
$$C_1 + B_2 \to_2 C_1 + C_2 \to_1 A_1 + C_2 \to_2 A_1 + A_2.$$

Whether it is possible to work around this additional complication and find a finite description of the loops realisable in an arbitrary component of a Petri net remains to be seen.

5. A thermodynamic aside: entropy

It is instructive to rephrase our main result in terms of energy, and to build a different intuition for it.

We have proved that a sisma Petri net N with transitional energies κ has an equilibrium if and only if $\kappa \in ker(C)^{\perp}$, where we, essentially, use equation (7) to define the equilibrium. Equivalently, using the correspondence explained in equation (2) in Section 2.2, the equilibrium can be described as the following (free) energy assignment:

$$F(x) = \langle \epsilon, x \rangle + \ln x! \tag{9}$$

where ϵ is such that $\kappa = C^t \epsilon$ (recall that we write $x!$ for $\prod_A x(A)!$). Note that the choice of a particular representative of ϵ will only change the above definition by an additive constant per connected component.

It can be seen that $F(x)$ decomposes as $E(x) - \Omega(x)$ with:

— $E(x) = \langle \epsilon, x \rangle$, which can be interpreted as the internal energy of x, or the energy of creating x;

— $\Omega(x) = -\ln x!$, which is a combinatorial symmetry discount term that can be interpreted as the entropy of creating x (see the discussion below).

Since $p(x) \propto e^{-F(x)}$, those states x that minimise $E(x)$ and maximise $\Omega(x)$ will be favoured by the invariant probability. Minimising the E-term is easy: if A has the lowest ϵ_A, then x should have only As. Maximising the Ω-term is also easy: x should have a low $x!$, which means x should be as uniformly distributed among species as possible. To see this, observe that $|x|!/x!$ is a multinomial coefficient, which, for a fixed value of $|x|$, gets its highest values when x partitions $|x|$ as evenly as possible. This second term, unlike the first one, favours disorder. To minimise both at once is complicated as the two goals are clearly in contention.

Note that we can rescale κ by any factor λ, without leaving $ker(C)^{\perp}$, and obtain a rescaled equilibrium:

$$H(x) = \lambda E(x) - \Omega(x) = \lambda(E(x) - \lambda^{-1}\Omega(x)),$$

which leads us to interpret $\|\epsilon\|^{-1}$ as a formal notion of temperature arbitrating the competition between the internal energy and entropic terms.

To return to the entropy term, supposing $|x| := \sum_A x(A)$ is a constant n (that is, the total number of tokens per component is constant), we can explain this term as follows.

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Consider the set of words S^n, or, equivalently, the set of maps $n \to S$. Permutations of n have an action on this set. Write Π for the (canonical) projection Π from words to multisets.

By the orbit-stabiliser relation, for any word w, we have

$$n! = sym(w) \times orb(w).$$

On the other hand,

$$sym(w) = \prod_{A \in S} \Pi(w)(A)!$$

since $sym(w)$ is the set of name-preserving permutations of w.

Hence, the cardinality of the inverse image of x is

$$|\Pi^{-1}(x)| = (\sum_A x(A))! / \prod_A x(A)! = |x|! / \prod_A x(A)! = |x|!/x!. \tag{10}$$

Taking logs, we get

$$\ln |\Pi^{-1}(x)| = \ln |x|! + \Omega(x),$$

and we see that the equilibrium F is the image of E under the quotient induced by Π, that is to say,

$$F(x) = E(w) + \ln |\Pi^{-1}(x)|$$

up to an additive constant $\ln n!$, for any $w \in \Pi^{-1}(x)$.

This hints at the existence of a concrete counterpart of N, acting on words (which is reminiscent of the *individual-token semantics* of Petri nets (Bruni *et al.* 1999)) for which Π would be a functional stochastic bisimulation. It is not too difficult to carry out this programme explicitly for the simple case where there is a constant number of tokens. In so doing, the entropic term is seen as a side-effect of changing the semantics and switching to the more abstract collective one, as described by the projection Π. It is however unclear at the time of writing whether this microscopic rendition of entropy can be freed from the simplifying assumption of an invariant number of tokens.

6. Conclusion

We have established a computable criterion for a simple and symmetric Petri net equipped with a mass-action semantics to admit an equilibrium probability. This result, put alongside our earlier result in Danos and Oury (2010) where we proved that the same problem is undecidable for general computable Markov chains, sheds some light on the decidability boundary. There are many possible questions to pursue from this point.

We have mentioned already the decision of the weaker property of having an invariant probability, as well as the more foundational question of the extent to which we can deconstruct entropy terms by constructing covers, as sketched at the end of the previous section. Ideally, this would be done in some axiomatic framework to give maximal generality, perhaps following the leads in Bruni *et al.* (1999) or the more encompassing notion of adhesive categories (Lack and Sobociński 2005; Lack and Sobociński 2004; Ehrig *et al.* 2004).

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Another question is how our result relates to the MC/DE divide. If a Petri net has an invariant probability, we would expect its deterministic DE approximation to be defined at all times (Darling and Norris 2008). For instance, the reaction $2A \to 3A$ will give the mass-action differential equation $x' = x^2$, which is a Ricati equation with explosive solutions $x(t) = 1/(x(0) - kt)$. As soon as we add the reverse reaction $2A \leftarrow 3A$, the new differential equation $x' = x^2 - x^3$ has positive solutions defined at all times. The extent to which convergence in the MC world carries over to differential limits should be investigated. A related question is the relation to Feinberg's chemical reaction theory, which is also partly based on algebraic conditions (the so-called deficiency space of a reaction system) to ensure multistationarity of the differential semantics (Feinberg 1987). Certainly, a probabilistic equilibrium is adverse to multistationarity, and the link should be made precise. Anderson *et al.* (2010) might help shed some light on this issue.

Another direction to pursue, which we mentioned in the introduction and which also prompted this investigation in the first place, is to bring the experience gained here to bear on the study of the thermodynamic consistency of the larger class of rule-based models of BMNs. As we said earlier, there is no hope of getting an all-encompassing criterion of computational significance because the question is undecidable. However, it is possible to envision the synthesis of specific classes of consistent rule sets by using local energy terms to constrain the allowed rules. This is an exciting question, which we have already started investigating in trying to extend the energy-based modelling techniques of Ollivier *et al.* (2010).

Finally, we have dealt here with what might be called a quantitative termination question. It is pleasing, if unsurprising, to see thermodynamics inviting itself into the conceptual apparatus. It seems that a thermodynamics-based quantitative rewriting theory could renew in an interesting way the classical questions of reachability (for a recent example of such an investigation in Kappa-related formalisms, see, for example, Delzanno *et al.* (2009)), confluence and termination that have been its traditional concerns (for a recent example in concurrency theory, see, for example, Bacci *et al.* (2011)).

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