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Non-Gibbsianness of SRB measures for the natural extension of intermittent systems

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Abstract. For countable-to-one transitive Markov maps, we show that the natural extensions of invariant ergodic weak Gibbs measures absolutely continuous with respect to weak Gibbs conformal measures possess a version of the u-Gibbs property. In particular, if dynamical potentials admit *generalized indifferent periodic points* then the natural extensions exhibit a non-Gibbsian character in statistical mechanics. Our results can be applicable to certain non-hyperbolic number-theoretical transformations of which natural extensions possess unstable (respectively stable) leaves with subexponential expansion (respectively contraction).

0. *Introduction*

In the category of smooth dynamical systems with non-zero Lyapunov exponent, u-Gibbs measures introduced by Pesin and Sinai in [**14**] played important roles in establishing ergodic and further statistical properties [**1, 3, 4**]. Here, u-Gibbs measures are invariant probability measures whose conditional measures along corresponding Pesin unstable leaves are absolutely continuous with respect to Lebesgue measure on the leaves. In particular, hyperbolic u-Gibbs measures are sometimes referred to as Sinai–Ruelle–Bowen (SRB) measures [**2, 4, 13**]. We should notice that in this setting densities of conditional measures with respect to Lebesgue measure on corresponding unstable leaves are strictly positive and continuous (cf. [**1, 4, 10, 11**]). These properties of u-Gibbs measures remind us of the well-known Gibbsian character in statistical mechanics. On the other hand, in the category of piecewise smooth countable-to-one Markov systems (see the definition in §1), even if the systems are not expanding, we can obtain ergodic absolutely continuous invariant weak Gibbs measures (see the definition in §1) with respect to physical reference measures associated to the dynamical potentials. Many such examples come from number theory. We should remark that densities of these invariant measures are typically unbounded at indifferent periodic points with respect to the dynamical potentials (see the definition in §1) and these periodic points typically cause lack of the Gibbs property in the sense of Bowen. Moreover, we can see that some of such non-hyperbolic systems possess a representation of their natural extensions as piecewise diffeomorphisms defined on subsets of compact metric spaces [**21**]. In this paper, we shall restrict our attention to invertible non-hyperbolic dynamical systems $(\overline{T}, \overline{\mu})$ which are the natural extensions of non-invertible absolutely continuous ergodic systems (T, μ). Our purpose is to give a new interpretation of non-Gibbsian character from the point of view of statistical mechanics by showing that $(\overline{T}, \overline{\mu})$ satisfies an analogy of the u-Gibbs property. More specifically, let $\overline{\mu}$ be the natural extension of a weak Gibbs measure μ which is equivalent to a weak Gibbs reference measure. Then the densities of the conditional measures of $\overline{\mu}$ along the unstable direction fail to hold strict positivity when the dynamical potentials admit generalized indifferent periodic points (Theorems 1 and 2). Thus, a non-Gibbsian character which is different from those established for the non-invertible system (T, μ) in [**23, 25**] is observed for the natural extensions $(\overline{T}, \overline{\mu})$ of (T, μ) (cf. [**7, 8, 12, 25**]). Moreover, as we will see in §2, $\overline{\mu}$ do not possess the local product structure (LPS) even if $\overline{\mu}$ is hyperbolic [**2, 13**]. We should recall that there is a common understanding that hyperbolic u-Gibbs measures adapt to describe statistical properties for dynamical systems with sufficiently weak instability of the trajectories [**1, 3, 4, 13**]. In our setting, the weak Gibbsianness of absolutely continuous invariant measures μ is typically caused by subexponential instability of the dynamics which exhibits so-called *intermittency* (see [**19–25**]). In §3 we apply our results to piecewise C^1-smooth non-invertible maps that have piecewise smooth representations of their natural extensions for which expansion rates of unstable leaves and contraction rates of stable leaves are subexponential. These examples are described via number-theoretical algorithms. Both admit periodic points x_0 at which modulus of eigenvalues of the differentials $DT(x_0)$ are just one and these periodic points are indifferent periodic points with respect to the dynamical potential $\phi = -\log|\det DT|$.

The paper is organized as follows. In §1, we collect definitions and fundamental results related to the non-Gibbsian character of weak Gibbs measures for piecewise C^0-invertible transitive finite-range-structure (FRS) Markov systems. The main results are stated in §2 and proofs are postponed until §4. In §3, we apply our results to the complex continued fraction and the inhomogeneous Diophantine algorithm.

1. *Preliminaries*

Let $(T, X, Q = \{X_i\}_{i\in I})$ be a *piecewise C^0-invertible system*, i.e., X is a compact metric space with metric d, $T : X \to X$ is a non-invertible map which is not necessarily continuous, and $Q = \{X_i\}_{i\in I}$ is a countable disjoint generating partition of X such that for each $i \in I$ with $\operatorname{int} X_i \neq \emptyset$, $T|_{\operatorname{int} X_i} : \operatorname{int} X_i \to T(\operatorname{int} X_i)$ is a homeomorphism and $(T|_{\operatorname{int} X_i})^{-1}$ extends to a homeomorphism ψ_i on $\operatorname{cl}(T(\operatorname{int} X_i))$. Let $\underline{i} = (i_1 \dots i_n) \in I^n$ satisfy $\operatorname{int}(X_{i_1} \cap T^{-1}X_{i_2} \cap \cdots \cap T^{-(n-1)}X_{i_n}) \neq \emptyset$. Then, we define $X_{\underline{i}} := X_{i_1} \cap T^{-1}X_{i_2} \cap \cdots \cap T^{-(n-1)}X_{i_n}$ which is called a *cylinder of rank n*. Suppose that $\mathcal{U} = \{\operatorname{int}(T^n X_{i_1\dots i_n}) : \forall X_{i_1\dots i_n}, \forall n > 0\}$ consists of finitely many open subsets $U_1, U_2, \dots, U_N$ of X (*finite range structure*). If (T, X, Q) satisfies the Markov property (i.e., $\operatorname{int} X_i \cap \operatorname{int} TX_j \neq \emptyset$ implies $\operatorname{int} TX_j \supset \operatorname{int} X_i$), then $\mathcal{U} = \{\operatorname{int}(TX_i) : \forall i \in I\}$ and

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we say that (T, X, Q) is an FRS Markov system. Let $\phi : X \to \mathbb{R}$ be a potential of *weak bounded variation* (WBV), i.e., there exists a sequence of positive numbers $\{C_n\}$ satisfying $\lim_{n\to\infty}(1/n)\log C_n = 0$ and for all $n \geq 1$, and all $X_{i_1\ldots i_n} \in \bigvee_{i=0}^{n-1} T^{-i}Q$,

$$\frac{\sup_{x\in X_{i_1\ldots i_n}} \exp\left(\sum_{i=0}^{n-1}\phi(T^i x)\right)}{\inf_{x\in X_{i_1\ldots i_n}} \exp\left(\sum_{i=0}^{n-1}\phi(T^i x)\right)} \leq C_n.$$

For a potential ϕ of WBV and for each $n \geq 1$ we define a partition function $Z_n(\phi)$ by

$$Z_n(\phi) := \sum_{\underline{i}:|\underline{i}|=n,\mathrm{int}(TX_{i_n})\supset \mathrm{int}\, X_{i_1}} \exp\left[\sum_{h=0}^{n-1}\phi T^h(x(\underline{i}))\right],$$

where $x(\underline{i})$ is the unique point satisfying $\psi_{\underline{i}}x(\underline{i}) = x(\underline{i}) \in \mathrm{cl}(\mathrm{int}\, X_{\underline{i}})$. Then there exists the limit $P_{\mathrm{top}}(T, \phi) := \lim_{n\to\infty}(1/n)\log Z_n(\phi) \in (-\infty, \infty]$ under the following transitivity condition (see [**23**]).

(Transitivity). $\mathrm{int}\, X = \bigcup_{k=1}^{N} U_k$ and for all $l \in \{1, 2, \ldots, N\}$, there exists $0 < s_l < \infty$ such that for each $k \in \{1, 2, \ldots, N\}$, U_k contains an interior of a cylinder $X^{(k,l)}(s_l)$ of rank s_l such that $T^{s_l}(\mathrm{int}\, X^{(k,l)}(s_l)) = U_l$.

Suppose that there exists $B_1 \subset X$ consisting of *full cylinders* X_b (i.e., $T\,\mathrm{cl}(\mathrm{int}\, X_b) = X$). Define the stopping time over B_1, $R : X \to \mathbb{N}\cup\{\infty\}$ by $R(x) = \inf\{n \geq 0 : T^n x \in B_1\}+1$ and for each $n > 1$, define inductively:

$$B_n := \{x \in X \mid R(x) = n\}, \quad D_n := \{x \in X \mid R(x) > n\} \quad \left(= \bigcap_{m=0}^{n-1} T^{-m}B_1^c\right).$$

Now we define Schweiger's jump transformation [**16**], $T^* : \bigcup_{n=1}^{\infty} B_n \to X$ by $T^*x = T^{R(x)}x$. Let $I^* := \bigcup_{n\geq 1}\{(i_1 \ldots i_n) \in I^n \mid X_{i_1\ldots i_n} \subseteq B_n\}$. If T^* is uniformly expanding in the following sense,

$$\sup_{\underline{i}\in I^*}\sup_{x,y\in X} d(\psi_{\underline{i}}(x), \psi_{\underline{i}}(y))/d(x, y) < 1,$$

then we say that (T, X, Q) is *locally uniformly expanding* with respect to B_1. Define $X^* := X\setminus\bigcup_{m=0}^{\infty} T^{*-m}\left(\bigcap_{n\geq 0}\{R(x) > n\}\right)$ and $Q^* := \{X_{\underline{i}}\}_{\underline{i}\in I^*}$. Then, (T^*, X^*, Q^*) is a piecewise C^0-invertible Markov system with FRS, where Q^* consists of full cylinders and X^* is not necessarily compact. For a function $\phi : X \to \mathbb{R}$, if there exists $\theta > 0$ such that (1): for all $n \geq 0$, and all $X_{i_1\ldots i_n} \subset B_n$, there exists $0 < L_\phi(i_1 \ldots i_n) < \infty$ satisfying

$$\sup_{x,y\in X}\frac{|\phi(\psi_{i_1\ldots i_n}(x)) - \phi(\psi_{i_1\ldots i_n}(y))|}{d(x, y)^\theta} \leq L_\phi(i_1 \ldots i_n)$$

and

$$L_\phi(\infty) := \sup_{n>0}\sup_{X_{i_1\ldots i_n}\subset\mathcal{B}_n}\sum_{j=1}^{n} L_\phi(i_j \ldots i_n) < \infty,$$

then we say that ϕ satisfies the *local bounded distortion* (LBD) with respect to B_1. We define an induced potential $\phi^* : \bigcup_{n=1}^{\infty} B_n \to \mathbb{R}$ associated to T^* by $\phi^*(x) := \sum_{h=0}^{R(x)-1}\phi T^h(x)$. Then, under the locally uniformly expanding property the LBD property

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implies equi-Hölder continuity of a family $\{\phi^* \circ \psi_{\underline{i}}\}_{\underline{i}\in I^*}$ so that ϕ^* is a potential of summable variation with respect to (T^*, X^*, Q^*). A Borel probability measure ν on X is called an *f-conformal measure* if $d(\nu T)|_{X_i}/d\nu|_{X_i} = f|_{X_i}$ (for all $i \in I$) and $\nu\big(\bigcup_{i\in I} \partial X_i\big) = 0$. The following fundamental result was established in [**23**] (cf. [**19**]).

PROPOSITION 1. *Let (T, X, Q) be a piecewise C^0-invertible transitive FRS Markov system. Let ϕ be a potential of WBV with $P_{\text{top}}(T, \phi) < \infty$. Suppose that there exists an* $\exp[P_{\text{top}}(T, \phi) - \phi]$*-conformal measure ν on X. Assume further that there exists $B_1 \subset X$ consisting of full cylinders with respect to which (T, X, Q) is locally uniformly expanding and ϕ satisfies the LBD. If $\int_{X^*} R\, d\nu < \infty$, then there exists a T-invariant probability measure μ equivalent to ν which is exact.*

Let $\mathcal{F}$ be the σ-algebra of Borel subsets of X. We define an operator $\mathcal{L}_\phi$ associated to the potential ϕ by

$$\mathcal{L}_\phi f(x) = \sum_{i\in I} \exp[\phi\psi_i(x)] f\psi_i(x) 1_{\text{cl}(T(\text{int}\, X_i))}(x),$$

whenever the series converges for $f : X \to \mathbb{R}$. Define the dual $\mathcal{L}^*_\phi$ of $\mathcal{L}_\phi$ by $(\mathcal{L}^*_\phi m)(f) = \int_X \mathcal{L}_\phi f\, dm$ for all probability measures m on $(X, \mathcal{F})$.

Definition. [**19–25**] A probability measure ν on $(X, \mathcal{F})$ is called a *weak Gibbs measure* for a function ϕ with a constant P if there exists a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n\to\infty}(1/n)\log K_n = 0$ such that for all $n \geq 1$, for all $X_{i_1\ldots i_n} \in \bigvee_{h=0}^{n-1} T^{-h}Q$, ν-a.e. $x \in X_{i_1\ldots i_n}$,

$$K_n^{-1} \leq \frac{\nu(X_{i_1\ldots i_n})}{\exp\big(\sum_{j=0}^{n-1}\phi T^j(x) + nP\big)} \leq K_n.$$

We note that ν obtained in Proposition 1 satisfies $\mathcal{L}^*_{\phi - P_{\text{top}}(T,\phi)}\nu = \nu$ and ν is a weak Gibbs measure for ϕ with $-P_{\text{top}}(T, \phi)$ [**19, 23**]. We define for each $n \geq 1$ $\nu_n := \nu T^{-n} (n \geq 1)$. Then by non-singularity of ν, the Radon–Nikodyn derivative $d\nu_n/d\nu$ exists ν-a.e., and we can write $d\nu_n/d\nu = \mathcal{L}^n_{\phi - P_{\text{top}}(T,\phi)}1$. For each $n \geq 1$ we define $\hat{\phi}_n := \log(\mathcal{L}^n_\phi 1/\mathcal{L}^n_\phi 1 \circ T) + \phi - P_{\text{top}}(T, \phi)$. Then, we have the following facts.

LEMMA 1. *$\hat{\phi}_n$ satisfies the WBV property and ν_n is a weak Gibbs measure for $\hat{\phi}_n$ with constant* 0.

PROPOSITION 2. *$\nu_n := \nu T^{-n} \to \mu$ weakly as $n \to \infty$ and $\mathcal{L}^*_{\hat{\phi}}\mu = \mu$ for $\hat{\phi} :=$* $\log(h/h \circ T) + \phi - P_{\text{top}}(T, \phi)$*, where $h := d\mu/d\nu$. In particular, if $\hat{\phi}$ is of WBV, then μ is a weak Gibbs measure for $\hat{\phi}$.*

PROPOSITION 3. (cf. [**20, 22**]) *Suppose that there exists $\{E_k\}_{k\geq 1}$, and $\{F_k\}_{k\geq 1}$ such that for all $x, y \in \{R \leq k\} \cap T^n X_{i_1\ldots i_n}$, $d(\psi_{i_1\ldots i_n}x, \psi_{i_1\ldots i_n}y) \leq E_k d(x, y)$ and*

$$\left|1 - \exp\left[\sum_{h=0}^{n-1}\{\phi T^h(\psi_{i_1\ldots i_n}(x)) - \phi T^h(\psi_{i_1\ldots i_n}(y))\}\right]\right| \leq F_k d(x, y)^\theta.$$

Then the following hold:

(i) *$\mathcal{L}^n_\phi 1$ converges to h uniformly on compact subsets of $X \setminus \bigcap_{n\geq 0}\{R > n\}$;*

(ii) *$\hat{\phi}_n$ converges to $\hat{\phi}$ uniformly on compact subsets of $X \setminus \bigcap_{n\geq 0}\{R > n\}$.*

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Definition. x_0 is called a *generalized indifferent periodic point* with period q with respect to ϕ if $P_{\text{top}}(T,\phi) = (1/q)\sum_{h=0}^{q-1}\phi T^h(x_0)$.

As we will see in §3, we can apply all the above results to the complex continued fraction and the Inhomogeneous Diophantine algorithm, both of which admit indifferent periodic points with respect to $\phi = -\log|\det DT|$ of WBV and $\bigcap_{n\geq 0}\{R > n\}$ consists of these periodic points. For these examples, the normalized Lebesgue measure ν is an $\exp[-\phi]$-conformal measure and $\nu(\{R > n\})$ decay polynomially fast.

The next result was established in [**20**] and [**22**].

LEMMA 2. *Let x_0 be a generalized indifferent periodic point with respect to ϕ. Then the following hold.*

(i) $$\sup_{x,y\in X_{i_1\ldots i_n}(x_0)} \frac{\exp\big[\sum_{h=0}^{n-1}\phi T^h(x)\big]}{\exp\big[\sum_{h=0}^{n-1}\phi T^h(y)\big]} \to \infty (n\to\infty) \quad \textit{and} \quad x_0 \in \bigcap_{n\geq 0}\{R > n\}$$

*(see [**22**, Lemma 6.1]).*

(ii) *$h := d\mu/d\nu$ is unbounded at x_0.*

If h is away from zero and infinity, then μ satisfies the weak Gibbs property for both $\hat{\phi}$ and ϕ. As such a case, we can consider a parabolic rational map T on the Riemann sphere of degree ≥ 2 and a Hölder potential ϕ defined on the Julia set $J(T)$ satisfying $P(\phi) > \sup_{x\in J(T)}\phi(x)$ so that there is no generalized indifferent periodic point with respect to ϕ. It is known that there exists a finite generating Markov partition of $J(T)$. Furthermore, there exists a Borel probability measure ν on $J(T)$ with $\mathcal{L}^*_{\phi-P(\phi)}\nu = \nu$ [**6**] and a nowhere-vanishing bounded Hölder function h with $\mathcal{L}_{\phi-P(\phi)}h = h$ [**5**]. Hence, we see that $\hat{\phi}$ satisfies the WBV property. On the other hand, it follows from (ii) in Lemma 2 that $\hat{\phi}$ fails to hold the WBV property under the existence of indifferent periodic points with respect to ϕ, and μ cannot be a weak Gibbs measure for $\hat{\phi}$. However, μ can be still a weak Gibbs measure for ϕ under certain conditions. Indeed, we can show in §3 that the weak Gibbs property of μ for the potential $\phi = -\log|\det DT|$ is valid for the complex continued fraction and the inhomogeneous Diophantine algorithm although $\hat{\phi}$ fails to hold the WBV property because of singularities of h at indifferent periodic points with respect to ϕ. We recall two different characterizations of non-Gibbsianness of the weak Gibbs measures μ for ϕ with generalized indifferent periodic points x_0 established in [**23**] and [**25**]. More specifically, non-Gibbsianness in the sense of Bowen, established in [**23**], is a direct consequence of the existence of a generalized indifferent periodic point. On the other hand, we observed in [**25**] that conditional measures of μ in finite boxes (cylinders) with boundary conditions outside the boxes could be described in terms of $\hat{\phi}$ as follows:

$$\mu\bigg(X_{i_1\ldots i_n}\,\bigg|\,\sigma\bigg(\bigvee_{h=n}^{\infty}T^{-h}Q\bigg)\bigg)(x) = \exp\bigg[\sum_{k=0}^{n-1}\hat{\phi}T^k\psi_{i_1\ldots i_n}\bigg]\circ T^n(x),$$

where $\sigma(\bigvee_{h=n}^{\infty}T^{-h}Q)$ denotes the sigma-algebra generated by $\bigvee_{h=n}^{\infty}T^{-h}Q$. This formula allows us to show non-Gibbsianness in the following sense: the conditional measures of μ fail to hold positivity because of (ii) in Lemma 2. The new characterization

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of non-Gibbsianness that we shall establish in §2 shows that densities of conditional measures of the natural extension of μ along an unstable direction fail to hold strict positivity and this is a direct consequence of the existence of a generalized indifferent periodic point.

2. *Main results*

We recall Rohlin's construction of an invertible extension of a T-invariant ergodic probability measure μ. Let Σ_- be the set of all semi-infinite sequences of $I^{\mathbb{N}}, \omega = (\omega_{-1}, \omega_{-2}, \ldots, \omega_{-n}, \ldots)$ such that there exists a sequence of points in X, $\{x_{-n}\}_{n\geq 0}$ satisfying $Tx_{-n} = x_{-n+1}$ and $x_{-n} \in X_{\omega_{-n}}$ for all $n \geq 1$. Let $\overline{X}$ be a subset of $X \times \Sigma_-$ composed of all pairs (x, ω) and define $\overline{T} : \overline{X} \to \overline{X}$ by $\overline{T}(x, \omega) = (Tx, \omega_0\omega)$ where $x \in X_{\omega_0}$ and $\omega_0\omega = (\omega_0, \omega_{-1}, \omega_{-2}, \ldots)$. It is known that the projection π_+ onto X commutes with the map T and $\overline{T}$, i.e., $\pi_+\overline{T} = T\pi_+$, and that there exists a unique ergodic invariant measure $\overline{\mu}$ on $\overline{X}$ whose image by π_+ is μ, i.e., $\overline{\mu}\pi_+^{-1} = \mu$ (see [**15**]). For each $i \in I$ we define a cylinder set $[i]$ on Σ_- by $[i] := \{(\omega_{-1}, \omega_{-2}, \ldots) \in \Sigma_- \mid \omega_{-1} = i\}$. Then, $Q_- := \{[i]\}_{i\in I}$ is a countable disjoint partition of Σ_-. Let $\sigma : \Sigma_- \to \Sigma_-$ be a (left) shift, (i.e., $\sigma(\omega_{-1}, \omega_{-2}, \ldots) = (\omega_{-2}, \omega_{-3}, \ldots)$). Then, for each $i \in I$, $\sigma|_{[i]}$ is a homeomorphism onto its image $\sigma[i] = \bigcup_{j\in I: TX_j \supset X_i}[j]$. Thus, (σ, Σ_-, Q_-) is a piecewise invertible Markov system, where Σ_- is non-compact. The system (σ, Σ_-, Q_-) is called a *dual system* with respect to (T, X, Q). We can easily see that

$$\overline{X} := \bigcup_{i\in I} X_i \times \sigma[i] \left(= \bigcup_{i\in I} TX_i \times [i] \right),$$

$$\overline{T}(x, \omega) = (T|_{X_i}x, i\omega) \quad \text{if } (x, \omega) \in X_i \times \sigma[i],$$

and

$$\overline{T}^{-1}(x, \omega) = (\psi_{\omega_{-1}}x, \sigma\omega) \quad \text{if } (x, \omega) \in TX_{\omega_{-1}} \times [\omega_{-1}].$$

We define a set function μ_- on the cylinders of the dual system (σ, Σ_-, Q_-) as follows: $\mu_-([\omega_{-1}\ldots\omega_{-n}]) = \mu(X_{\omega_{-n}\ldots\omega_{-1}})$, where $[\omega_{-1}\ldots\omega_{-n}] := [\omega_{-1}] \cap \sigma^{-1}[\omega_{-2}]\ldots\sigma^{-(n-1)}[\omega_{-n}]$, and μ_- can be extended to a sigma additive set function which is σ-invariant (see [**16**, pp. 157–158]). Since $(\overline{T}^{-1}, \overline{\mu})$ is the natural extension of (σ, μ_-), μ_- is ergodic, too. Σ_- provides the usual symbolic metric and if $(T, X, Q = \{X_i\}_{i\in I})$ is a finite-to-one system then Σ_- is a compact metric space. We define $\overline{X_i} = \overline{X} \cap \pi_+^{-1}X_i \big(= \bigcup_{X_i\subset TX_j}(X_i \times [j])\big)$ and $\overline{Q} := \{\overline{X_i}\}_{i\in I}$. We further define $\xi := \bigvee_{k=1}^{\infty} \overline{T}^k\overline{Q}$ and $\eta := \overline{Q} \vee \xi$. Then it is easy to see that $\overline{T}^{-1}\eta \geq \eta$ and so $\bigvee_{n=-\infty}^{\infty} \overline{T}^{-n}\eta$ is the partition into points. Let $\xi(x, \omega)$ and $\eta(x, \omega)$ be the elements of ξ and η containing $(x, \omega) \in \overline{X}$ respectively and let $X_i(x) \in Q$ be the cylinder containing x. Then, $\xi(x, \omega) = TX_{\omega_{-1}} \times \{\omega\}$ and $\eta(x, \omega) = X_i(x) \times \{\omega\}$, where $\omega = (\omega_{-1}, \omega_{-2}, \ldots) \in \Sigma_-$ and (ω_{-1}, i) is T-admissible. Let ϕ be a potential of WBV with $P_{\text{top}}(T, \phi) < \infty$ and let $\{C_n\}_{n\geq 1}$ be the WBV sequence for ϕ. For each $(x, \omega) \in \overline{X}$ we define a function $\Phi_{(x,\omega)} : \xi(x, \omega) \to \mathbb{R}$ by: for all $(x', \omega) \in \xi(x, \omega)$

$$\Phi_{(x,\omega)}(x', \omega) := \limsup_{n\to\infty} \exp\left[\sum_{h=1}^{n}(-\phi)(\pi_+ \circ \overline{T}^{-h}(x, \omega)) - \sum_{h=1}^{n}(-\phi)(\pi_+ \circ \overline{T}^{-h}(x', \omega))\right].$$

Now we come to state our main theorem.

THEOREM 1. *Suppose that all conditions in Proposition 1 are satisfied. Assume further that the T-invariant probability measure μ is a weak Gibbs measure for ϕ with a weak Gibbs sequence $\{C_n\}_{n\geq 1}$ and ν satisfies $H_\nu(Q) < \infty$ and $\phi \in L^1(\nu)$. If $\sum_{n=0}^{\infty} \nu(\{x \in X \mid R(x) > n\}) \log C_n < \infty$, then the conditional measure of the natural extension $\overline{\mu}$ of μ with respect to the partition η is absolutely continuous with respect to the conformal measure ν on each fibre of η. In particular, if ϕ admits a generalized indifferent periodic point, then the densities of these conditional measures fail to hold strict positivity.*

The next result easily follows from Theorem 1 (cf. [**10, 18**]).

COROLLARY 1. *Conditional measures of $\overline{\mu}$ with respect to η are all absolutely continuous with respect to $\overline{\mu}$ and (T, μ) is weak Bernoulli.*

Remark 1. As long as $\nu(\{x \in X \mid R(x) > n\})$ decays polynomially fast and C_n is a polynomial sequence, the condition $\sum_{n=0}^{\infty} \nu(\{x \in X \mid R(x) > n\}) \log C_n < \infty$ is automatically satisfied. Such a polynomial behaviour is typically caused by polynomical instability of trajectories near indifferent periodic points for intermittent systems (see §3).

The next lemma plays an important role in proving Theorem 1.

KEY LEMMA. *For $\overline{\mu}$-a.e. $(x, \omega) \in \overline{X}$ the conditional measures $\overline{\mu}_{(x,\omega)}$ of $\overline{\mu}$ with respect to the partition η are given by*

$$\overline{\mu}_{(x,\omega)}(E) = \frac{\int_{E\cap\eta(x,\omega)} \Phi_{(x,\omega)}(x', \omega)\, d\nu|_{\eta(x,\omega)}(x', \omega)}{\int_{\eta(x,\omega)} \Phi_{(x,\omega)}(x', \omega)\, d\nu|_{\eta(x,\omega)}(x', \omega)}$$

for any measurable subset E of $\overline{X}$. (Here $d\nu$ denotes the conformal measure on each element of η so that $d\nu|_{\eta(x,\omega)}(x', \omega) = d\nu|_{X_i(x)}(x')$ for $\eta(x, \omega) = X_i(x) \times \{\omega\}$.)

For the proof of the key lemma, we need the next four lemmas. We define

$$F_1 := \{(x, \omega) \in \overline{X} \mid X_{\omega_{-1}} \subset X\backslash B_1\}, \quad A_1 := \{(x, \omega) \in \overline{X} \mid X_{\omega_{-1}} \subset B_1\}$$

and inductively define for $n \geq 1$,

$$F_n := \{(x, \omega) \in F_{n-1} \mid X_{\omega_{-n}} \subset X\backslash B_1\}, \quad A_n := \{(x, \omega) \in F_{n-1} \mid X_{\omega_{-n}} \subset B_1\}.$$

LEMMA 3. (cf. [**18**]) $\overline{\mu}(F_n) = \mu(\{x \in X \mid R(x) > n\})$ *and* $\overline{X} = \bigcup_{n\geq 1} A_n$ ($\overline{\mu}$ mod 0).

LEMMA 4. *There exists a positive sequence $\{H_n\}_{n\geq 1}$ with $\lim_{n\to\infty}(1/n)\log H_n = 0$ satisfying for $\overline{\mu}$-a.e. $(x, \omega) \in A_n$ and for all $(x', \omega) \in \eta(x, \omega) \cap A_n$,*

$$H_n^{-1} \leq \frac{\Phi_{(x,\omega)}(x', \omega)}{\int_{\eta(x,\omega)} \Phi_{(x,\omega)}(x'', \omega)\, d\nu|_{\eta(x,\omega)}(x'', \omega)} \leq H_n.$$

Denote $g(x, \omega) := \int_{\eta(x,\omega)} \Phi_{(x,\omega)}(x', \omega)\, d\nu|_{\eta(x,\omega)}(x', \omega)$.

LEMMA 5. *For all $n \geq 1$,*

$$\frac{\int_{[\overline{T}^{-n}\eta](x,\omega)} \Phi_{(x,\omega)}(x', \omega)\, d\nu|_{\eta(x,\omega)}(x', \omega)}{g(x, \omega)} = \frac{g \circ \overline{T}^n(x, \omega)}{g(x, \omega)} \sum_{j=0}^{n-1} (\phi - P_{\rm top}(T, \phi))T^j(x).$$

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LEMMA 6. *For all* $n \geq 1$, $\log\big(\int_{[\overline{T}^{-n}\eta](x,\omega)} \Phi_{(x,\omega)}(x',\omega)/g(x,\omega)\, d\nu|_{\eta(x,\omega)}(x',\omega)\big)$ *is* $\overline{\mu}$*-integrable.*

By the key lemma, we see that

$$\frac{d\overline{\mu}_{(x,\omega)}}{d\nu|_{\eta(x,\omega)}}(x',\omega) = \frac{\Phi_{(x,\omega)}(x',\omega)}{\int_{\eta(x,\omega)} \Phi_{(x,\omega)}(x'',\omega)\, d\nu|_{\eta(x,\omega)}(x'',\omega)}$$
$$\bigg(= \frac{\Phi_{(x,\omega)}(x',\omega)}{\int_{X_i(x)} \Phi_{(x,\omega)}(x'',\omega)\, d\nu|_{X_i(x)}(x'')}\bigg).$$

Then the first assertion of Theorem 1 follows from the next lemma, which is an immediate consequence of Lemma 4.

LEMMA 7. $\overline{\mu}$*-a.e.,* $(x,\omega) \in \overline{X}$*, there exists a positive integer* $k(x,\omega) < \infty$ *satisfying for all* $(x',\omega) \in \eta(x,\omega)$*,*

$$H_{k(x,\omega)}^{-1} \leq \frac{d\overline{\mu}_{(x,\omega)}}{d\nu|_{\eta(x,\omega)}}(x',\omega) \leq H_{k(x,\omega)}.$$

Suppose that there exists a generalized indifferent periodic point x_0 with respect to ϕ. Then, there exists a fibre $\eta(x_0,\omega)$ on which

$$\inf_{(x',\omega)\in\eta(x_0,\omega)} \frac{d\overline{\mu}_{(x_0,\omega)}}{d\nu|_{\eta(x_0,\omega)}}(x',\omega) = 0,$$

so that the densities of conditional measures with respect to η fail to hold strict positivity, which shows the second assertion of Theorem 1.

At the end of this section, we shall consider piecewise C^0-invertible Markov systems of which natural extensions can be realized by piecewise diffeomorphisms. More specifically, suppose that there exists a piecewise C^0-invertible Markov system $(T_-, X_-, Q_- = \{Y_i\}_{i\in I})$ and a uniformly continuous semi-conjugacy map $\rho : \Sigma_- \to X_-$ such that $\rho\circ\sigma = T_-\circ\rho$. Then, we can obtain a realization of the invertible extension of $(T, X, Q; \mu)$ on a subspace of $X \times X_-$ as follows. We define $\overline{X} := \bigcup_{i\in I} X_i \times T_- Y_i \big(= \bigcup_{i\in I} TX_i \times Y_i\big)$, which is a subspace of the compact metric space $X \times X_-$, and define an invertible extension $\overline{T}$ of T on $\overline{X}$ by

$$\overline{T}(x,y) = (T|_{X_i}x, \psi_{-,i}\, y) \quad \text{if} \quad (x,y) \in X_i \times T_- Y_i,$$

where $\psi_{-,i} : T_- Y_i \to Y_i$ is the local inverse to $T_-|_{Y_i}$. $\overline{T}^{-1} : \overline{X} \to \overline{X}$ is defined by

$$\overline{T}^{-1}(x,y) = (\psi_i x, T_-|_{Y_i} y) \quad \text{if} \quad (x,y) \in TX_i \times Y_i.$$

Furthermore, for $(x,y) \in \overline{X}$, $\eta(x,y) = X_i(x) \times \{y\}$ and $\big(\bigvee_{k=0}^{\infty} \overline{T}^{-k}\overline{Q}\big)(x,y) = \{x\} \times T_- Y_i$ if $x \in X_i$. We can easily verify that

$$\eta(x,y) \subset \xi(x,y) \subset \{(x',y') \in \overline{X} \mid d_{X\times X_-}(\overline{T}^{-n}(x,y), \overline{T}^{-n}(x',y')) \to 0\ (n \to \infty)\}$$

and

$$\bigg(\bigvee_{k=0}^{\infty} \overline{T}^{-k}\overline{Q}\bigg)(x,y) \subset \{(x',y') \in \overline{X} \mid d_{X\times X_-}(\overline{T}^{n}(x,y), \overline{T}^{n}(x',y')) \to 0\ (n \to \infty)\}.$$

Indeed, we have the next lemma. We denote $\sigma(n) := \sup_{A\in\bigvee_{k=0}^{n-1} T^{-k}Q} \operatorname{diam} A$ and $\sigma_-(n) := \sup_{A\in\bigvee_{k=0}^{n-1} T_-^{-k}Q_-} \operatorname{diam} A$.

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LEMMA 8.

(1) *For $\overline{\mu}$-a.e. $(x', y) \in \xi(x, y)$*

$$d_{X\times X_-}(\overline{T}^{-n}(x, y), \overline{T}^{-n}(x', y)) \le d_X(\psi_{\omega_{-n},\dots\omega_{-1}}x, \psi_{\omega_{-n},\dots\omega_{-1}}x') \le \sigma(n),$$

where $\omega := (\omega_{-1}, \omega_{-2}, \dots, \omega_{-n} \dots) \in \Sigma_-$ *such that* $\rho(\omega) = y$.

(2) *For $\overline{\mu}$-a.e.* $(x, y') \in \big(\bigvee_{k=0}^{\infty} \overline{T}^{-k}\overline{Q}\big)(x, y)$,

$$d_{X\times X_-}(\overline{T}^{n}(x, y), \overline{T}^{n}(x, y')) \le d_{X_-}(\psi_{-,i_n,i_{n-1},\dots,i_1}y, \psi_{-,i_n,i_{n-1},\dots,i_1}y') \le \sigma_-(n),$$

where $x = \bigcap_{n=0}^{\infty} \mathrm{cl}(\mathrm{int}\, X_{i_{n+1}})$.

By the above lemma, we can say that each element $\xi(x, y)$ of ξ is an unstable leaf with expansion rate $\sigma(n)^{-1}$ and each element $\big(\bigvee_{k=0}^{\infty} \overline{T}^{-k}\overline{Q}\big)(x, y)$ of $\bigvee_{k=0}^{\infty} \overline{T}^{-k}\overline{Q}$ is a stable leaf with contraction rate $\sigma_-(n)$. As we will see in §3, for both the complex continued fraction and the inhomogeneous Diophantine algorithm, rates of decay of $\sigma(n)$ and $\sigma_-(n)$ are *polynomial*. It follows from these observations that the natural extension $\overline{\mu}$ of μ gives an analogy of u-Gibbs measures for partially hyperbolic systems introduced by Pesin and Sinai in [**14**].

THEOREM 2. *Under the assumptions in Theorem 1, the conditional measures of $\overline{\mu}$ with respect to the unstable leaves $\{\eta(x, y)\}_{(x,y)\in\overline{X}}$ are absolutely continuous with respect to the conformal measures ν on each unstable fibre of η. In particular, if ϕ admits a generalized indifferent periodic point, then the densities of conditional measures $\overline{\mu}$ along the unstable direction fail to hold strict positivity.*

In [**21**], a version of local product structure (the so-called *weak local product structure*) was established for the invertible extension of invariant ergodic weak Gibbs measures μ for ϕ of WBV with $P_{\mathrm{top}}(T, \phi) < \infty$. In particular, if (T, X, Q) is an FRS countable Markov shift and ϕ is a uniformly continuous potential with $P_{\mathrm{top}}(T, \phi) < \infty$ satisfying $\mathrm{Var}_1(\phi) < \infty$, then the natural extension of (T, μ) possesses *asymptotically almost local product structure* in the sense of Pesin, Barreira and Schmeling [**2**]. We recall the following inequalities obtained in Lemma 7:

$$H_{k(x,\omega)}^{-1}\, d\nu|_{X_i(x)}(x') \le d\overline{\mu}_{(x,\omega)}(x', \omega) \le H_{k(x,\omega)}\, d\nu|_{X_i(x)}(x').$$

If ϕ is of WBV with $\sup_{n\ge 1} C_n < \infty$ so that $k(x, \omega)$ is a uniform constant in (x, ω), then $\overline{\mu}$ possesses the LPS. Indeed, we see that there exists $1 \le H < \infty$ satisfying on each rectangle $X_i(x) \times \sigma[i]$ $\big(=\bigcup_{\omega\in\Sigma_-:(x,\omega)\in\overline{X}} \eta(x, \omega)\big)$,

$$H^{-1}\nu(A)\mu_-(B) \le \overline{\mu}(\pi_+^{-1}(A) \cap \pi_-^{-1}(B)) \le H\nu(A)\mu_-(B)$$

for any cylinders $A \subset X_i(x)$ and $B \subset \sigma[i]$. On the other hand, if ϕ admits an indifferent periodic point, then by (i) in Lemma 2 the WBV sequence $\{C_n\}_{n\ge 1}$ diverges as $n \to \infty$ so that uniformity of $k(x, \omega)$ in (x, ω) fails to hold. Hence, we cannot establish the LPS in the usual sense. On the other hand, it follows from [**21**, Theorem 3.2] that $\overline{\mu}$ possesses the weak local product structure.

3. *Examples*

In this section, we show higher dimensional intermittent systems which are piecewise C^1-invertible FRS Markov systems with non-hyperbolic periodic orbits and admit smooth representations of their dual systems.

Example A. (Inhomogeneous Diophantine approximations [**18, 20, 22–25**]) We define $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 1, -y \le x < -y + 1\}$ and $T : X \to X$ by

$$T(x, y) = \left(\frac{1}{x} - \left[\frac{1-y}{x}\right] + \left[-\frac{y}{x}\right], -\left[-\frac{y}{x}\right] - \frac{y}{x}\right),$$

where $[x] = \max\{n \in \mathbb{Z} \mid n \le x\} (x \in \mathbb{N})$ and $[x] = \max\{n \in \mathbb{Z} \mid n < x\}$ $(x \in \mathbb{Z}\backslash\mathbb{N})$. For this map, the potential $\phi = -\log|\det DT|$ admits indifferent periodic points $(1, 0)$ and $(-1, 1)$ with period 2, i.e., $|\det DT^2(1, 0)| = |\det DT^2(-1, 1)| = 1$. We introduce an index set $I = \{(a, b) \in \mathbb{Z}^2 \mid a > b > 0 \text{ or } a < b < 0\}$ and a partition $Q = \{X_{(a,b)} \mid (a, b) \in I\}$ of X, where $X_{(a,b)} = \{(x, y) \in X \mid a = [(1-y)/x] - [-y/x], b = -[-y/x]\}$. Then, (T, X, Q) is a countable FRS Markov system which satisfies $\sigma(n) = O(n^{-1})$. In [**18, 20, 23**], all assumptions in Theorems 1–2 were verified for $\phi = -\log|\det DT|$. B_1 is a union of cylinders of rank 1 away from the indifferent periodic points so that $D_n = \{x \in X \mid R(x) > n\}$ consists of cylinders of rank n containing these periodic points. We can see that $\nu(D_n) = 1/((n + 1)(2n + 1))$ and the weak Gibbs sequence $\{C_n\}_{n\ge1}$ for the T-invariant exact weak Gibbs measure μ for ϕ which is equivalent to the Lebesgue measure satisfies $C_n = O(n^3)$. We define a two-dimensional transformation T_- defined on $X_- := \{(x, y) \mid 0 \le y < 1, 0 \le y - x < 1\}$ by $T_-(x, y) = (1/x - c, y/x - d)$, where $c = -[-(1 - y)/x] + [y/x], d = [y/x]$. We define for each $(c, d) \in I$ $Y_{(c,d)} = \{(x, y) \in X_- \mid c = -[-(1 - y)/x] + [y/x], d = [y/x]\}$. Then $Q_- = \{Y_{(c,d)}\}_{(c,d)\in I}$ is a countable FRS Markov partition of X_- and $(T_-, X_-, Q_- = \{Y_{(c,d)}\}_{(c,d)\in I})$ gives a (piecewise) smooth representation of the dual system of (T, X, Q). T_- admits indifferent periodic points $(1, 1)$ and $(-1, 0)$ with period 2 and $\sigma_-(n) = O(n^{-1})$.

Example B. (Complex continued fraction [**17, 18, 21, 23, 24**]) Let $X = \{z = x_1\alpha + x_2\overline{\alpha} \mid -1/2 \le x_1, z_2 \le 1/2\}$, where $\alpha = 1 + i$ and define $T : X \to X$ by $T(z) = 1/z - [1/z]_1$. Here $[z]_1$ denotes $[x_1 + 1/2]\alpha + [x_2 + 1/2]\overline{\alpha}$, where z is written in the form $z = x_1\alpha + x_2\overline{\alpha}$, $[x] = \max\{n \in \mathbb{Z} \mid n \le x\} (x \in \mathbb{N})$ and $[x] = \max\{n \in \mathbb{Z} \mid n < x\}$ $(x \in \mathbb{Z} - \mathbb{N})$. For this transformation, the potential $\phi = -\log|T'|$ has an indifferent periodic orbit $\{1, -1\}$ of period 2 and two indifferent fixed points at i and $-i$. We define for each $n\alpha + m\overline{\alpha} \in I := \{m\alpha + n\overline{\alpha} \mid (m, n) \in \mathbb{Z}^2 - (0, 0)\}$, $X_{n\alpha+m\overline{\alpha}} = \{z \in X \mid [1/z]_1 = n\alpha + m\overline{\alpha}\}$. Then, we have a countable FRS Markov partition $Q = \{X_a\}_{a\in I}$ of X which satisfies $\sigma(n) = O(n^{-1})$. All assumptions in Theorems 1–2 were verified for $\phi = -\log|T'|$ in [**17, 18, 21, 23, 24**]. B_1 is a union of cylinders of rank 1 away from the indifferent periodic points so that $D_n = \{x \in X|R(x) > n\}$ consists of cylinders of rank n touching these periodic points. We can see that $\nu(D_n) = O(n^{-2})$ and the weak Gibbs sequence $\{C_n\}_{n\ge1}$ for the T-invariant exact weak Gibbs measure μ which is equivalent to the Lebesgue measure satisfies $C_n = O(n^4)$. Define $X_- := \{w \in \mathbb{C} \mid |w| \le 1\}$ and $T_- : X_- \to X_-$ by $T_-w = 1/w - [1/w]_{(-)}$, where $[w]_{(-)} = a$ if $w \in a + U_k$ for some

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$a \in J_k$ and each U_k, J_k are defined by

$$U_0 = X_-, \quad U_1 := \{w \in X_- \mid |w + \alpha| \geq 1\}, \quad U_2 = -i \times U_1, \quad U_3 = -i \times U_2,$$
$$U_4 = -i \times U_3, \quad U_5 = U_1 \cap U_2, \quad U_6 = -i \times U_5, \quad U_7 = -i \times U_6, \quad U_8 = -i \times U_7$$

and

$$J_1 = \{n\alpha \mid n > 0\}, \quad J_2 = -i \times J_1, \quad J_3 = -i \times J_2, \quad J_4 = -i \times J_3,$$
$$J_5 = \{n\alpha + m\overline{\alpha} \mid n, m > 0\}, \quad J_6 = -i \times J_5, \quad J_7 = -i \times J_6, \quad J_8 = -i \times J_7.$$

For each $a \in I$, we define $Y_a := \{w \in X_- \mid [1/w]_{(-)} = a\}$ and $Q_- := \{Y_a \mid a \in I\}$. Then (T_-, X_-, Q_-) is a piecewise C^1-invertible FRS Markov system with $\mathcal{U} = \{U_i \mid i = 1, 2, , \ldots, 8\}$ and gives a (piecewise) smooth representation of the dual system of (T, X, Q). $\sigma_-(n) = O(n^{-1})$ and T_- admits an indifferent periodic orbit $\{1, -1\}$ of period 2 and two indifferent fixed points at i and $-i$.

4. *Proofs*

Proof of Proposition 1. Equi-Hölder continuity of a family $\{\phi^* \circ \psi_{\underline{i}}\}_{\underline{i} \in I^*}$ and uniformly expanding property for T^* allow us to have a Hölder continuous function h^* satisfying $\mathcal{L}_{\phi^* - RP_{\mathrm{top}}(T,\phi)} h^* = h^*$, which is away from zero and infinity. Since ν satisfies $\mathcal{L}^*{}_{\phi^* - RP_{\mathrm{top}}(T,\phi)} \nu = \nu$, $\mu^* = h^* \nu$ gives a T^*-invariant finite measure. If $\nu\big(\bigcap_{n \geq 0} D_n\big) = 0$, then a T-invariant σ-finite exact measure μ equivalent to ν is obtained by the following Schweiger's formula:

$$\frac{d\mu}{d\nu}(x) = \sum_{n=0}^{\infty} \sum_{X_{d(n)} \subset D_n} \exp\left[\sum_{i=0}^{n-1} \phi T^i(\psi_{d(n)} x) - n P_{\mathrm{top}}(T, \phi)\right] 1_{T^n X_{d(n)}}(x) h^*(\psi_{d(n)} x). \quad (1)$$

In particular, if $R \in L^1(\nu)$ then $\mu(X) < \infty$. □

Proof of Lemma 1. Let $\{C_n\}_{n \geq 1}$ be the WBV sequence for ϕ. Then we see that $\sup_{x, y \in X_{i_1 \ldots i_n}} (\mathcal{L}_\phi^n 1(x)/\mathcal{L}_\phi^n 1(y)) \leq C_n$. It follows from this fact that $\hat{\phi}_n$ satisfies the WBV property. Next, we note that $d(\nu_n T)|_{X_i}/d\nu|_{X_i} = \exp[-\hat{\phi}_n]$ (for all $i \in I$). Then, the desired result follows from [**19, 23**]. □

Proof of Lemma 3. Since $\overline{\mu}(F_n) = \overline{\mu}(\overline{T}^{-n} F_n) = \overline{\mu}(\pi_+^{-1}\{x \in X \mid R(x) > n\})$, we have the first assertion. The second assertion follows from $\int_{X^*} R \, d\nu = \sum_{n=1}^{\infty} \nu(\{x \in X^* \mid R(x) = n\}) < \infty$ and $\mu \sim \nu$. □

Proof of Lemma 4. We recall that for all $(x', \omega) \in \eta(x, \omega)(= X_i(x) \times \{\omega\})$,

$$\Phi_{(x,\omega)}(x', \omega) = \limsup_{n \to \infty} \exp\left[\sum_{h=1}^{n} (-\phi)(\psi_{\omega_{-h} \ldots \omega_{-1}} x) - \sum_{h=1}^{n} (-\phi)(\psi_{\omega_{-h} \ldots \omega_{-1}} x')\right]$$

and note that for all $n > m$,

$$\exp\left[\sum_{h=0}^{n-1}\phi T^h(\psi_{\omega_{-n}\ldots\omega_{-m}\omega_{m-1}\ldots\omega_{-1}}x') - \sum_{h=0}^{n-1}\phi T^h(\psi_{\omega_{-n}\ldots\omega_{-m}\omega_{m-1}\ldots\omega_{-1}}x)\right]$$

$$= \exp\left[\sum_{h=0}^{n-m}\phi T^h(\psi_{\omega_{-n}\ldots\omega_{-m}}\circ\psi_{\omega_{m-1}\ldots\omega_{-1}}x') - \sum_{h=0}^{n-m}\phi T^h(\psi_{\omega_{-n}\ldots\omega_{-m}}\circ\psi_{\omega_{m-1}\ldots\omega_{-1}}x)\right]$$

$$\times \exp\left[\sum_{h=0}^{m-1}\phi T^h(\psi_{\omega_{m-1}\ldots\omega_{-1}}x') - \sum_{h=0}^{m-1}\phi T^h(\psi_{\omega_{m-1}\ldots\omega_{-1}}x)\right].$$

Then equi-Hölder continuity of the family

$$\left\{\sum_{k=0}^{n-1}\phi^* T^{*k}\psi_{\underline{i}_1\ldots\underline{i}_n}\;\middle|\;\forall(\underline{i}_1\ldots\underline{i}_n)\in I^{*n}, \forall n\geq 1\right\}$$

allows us to obtain a uniform constant $1 \leq C < \infty$ such that for all $(x,\omega), (x',\omega) \in A_m$,

$$C^{-1}\exp\left[\sum_{h=1}^{m}\{\phi(\psi_{\omega_{-h}\ldots\omega_{-1}}x') - \phi(\psi_{\omega_{-h}\ldots\omega_{-1}}x)\}\right] \leq \Phi_{(x,\omega)}(x',\omega)$$

$$\leq C\exp\left[\sum_{h=1}^{m}\{\phi(\psi_{\omega_{-h}\ldots\omega_{-1}}x') - \phi(\psi_{\omega_{-h}\ldots\omega_{-1}}x)\}\right].$$

By the WBV property for ϕ we can establish

$$C^{-1}C_m^{-1} \leq \Phi_{(x,\omega)}(x',\omega) \leq CC_m,$$

where $\{C_m\}_{m\geq 1}$ is the WBV sequence for ϕ. The equality

$$\frac{\Phi_{(x,\omega)}(x',\omega)}{\int_{\eta(x,\omega)}\Phi_{(x,\omega)}(x'',\omega)\,d\nu|_{\eta(x,\omega)}(x'',\omega)} = \frac{\Phi_{(x,\omega)}(x',\omega)}{\int_{X_i(x)}\Phi_{(x,\omega)}(x'',\omega)\,d\nu|_{X_i(x)}(x'')}$$

allows us to put $H_n = (CC_n)^2$. Thus, we complete the proof. □

Proof of Lemma 5. We can easily verify that for all $(x',\omega) \in [\overline{T}^{-n}\eta](x,\omega)$,

$$\Phi_{(x,\omega)}(x',\omega) = \Phi_{\overline{T}^n(x,\omega)}\overline{T}^n(x',\omega)\exp\left[\sum_{h=0}^{n-1}(-\phi)T^h(x') - \sum_{h=0}^{n-1}(-\phi)T^h(x)\right].$$

Then, it follows from conformality of ν that the desired equality holds. □

Proof of Lemma 6. It follows from Lemmas 4–5 that for all $n \geq 1$,

$$\int_{\overline{X}}\log\left[\int_{[\overline{T}^{-n}\eta](x,\omega)}\frac{\Phi_{(x,\omega)}(x',\omega)}{g(x,\omega)}\,d\nu|_{\eta(x,\omega)}(x',\omega)\right]d\overline{\mu}(x,\omega)$$

is bounded from below by

$$\sum_{m=1}^{\infty}\int_{A_m}\log\left[H_m^{-1}\nu|_{\eta(x,\omega)}([\overline{T}^{-n}\eta](x,\omega))\right]d\overline{\mu}(x,\omega)$$

$$= \sum_{m=1}^{\infty}\log(H_m^{-1})\overline{\mu}(A_m) + \int_{\overline{X}}\log\left[\nu\left(\bigvee_{i=0}^{n-1}T^{-i}Q\right)\right]\circ\pi_+(x,\omega)\,d\overline{\mu}(x,\omega).$$

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Since both μ and ν are weak Gibbs measures for ϕ, we see that for all $n \geq 1$ and for all $X_{i_1 \dots i_n} \in \bigvee_{j=0}^{n-1} T^{-j} Q$,

$$C_n^{-2} \leq \frac{\mu(X_{i_1 \dots i_n})}{\nu(X_{i_1 \dots i_n})} \leq C_n^2. \tag{2}$$

Then, by $H_\nu(Q) < \infty$, we have that for all $n \geq 1$,

$$\begin{aligned} \int_X \log\left[\nu\left(\bigvee_{i=0}^{n-1} T^{-i} Q\right)\right] \circ \pi_+(x,\omega)\, d\overline{\mu}(x,\omega) &= \int_X \log\left[\nu\left(\bigvee_{i=0}^{n-1} T^{-i} Q\right)\right](x)\, d\mu(x) \\ &= \sum_{X_{i_1 \dots i_n} \in \bigvee_{i=0}^{n-1} T^{-i} Q} \mu(X_{i_1 \dots i_n}) \log \nu(X_{i_1 \dots i_n}) \\ &\geq C_n^2 \sum_{X_{i_1 \dots i_n} \in \bigvee_{i=0}^{n-1} T^{-i} Q} \nu(X_{i_1 \dots i_n}) \log \nu(X_{i_1 \dots i_n}) > -\infty. \end{aligned}$$

We recall that $H_m = (CC_m)^2$ and $A_m \subset F_{m-1}$. Then, we see immediately that $\sum_{m=1}^{\infty} \log(H_m^{-1})\overline{\mu}(A_m) > -\infty$. □

Proof of Key Lemma. We have for each $n \geq 1$ that

$$nh_{\overline{\mu}}(\overline{T}) = H_{\overline{\mu}}(\overline{T}^{-n}\eta|\eta) = \int_X (-\log \overline{\mu}_{(x,\omega)}([\overline{T}^{-n}\eta](x,\omega)))\, d\overline{\mu}(x,\omega).$$

On the other hand, it follows from Lemma 5 and [**21**, Proposition 4.1] that

$$\begin{aligned} nh_\mu(T) = n\int_X (P_{\text{top}}(T,\phi) - \phi)\, d\mu &= \int_X \left(\sum_{j=0}^{n-1} P_{\text{top}}(T,\phi) - \phi\right) T^j(x)\, d\mu(x) \\ &= \int_X -\log\left[\frac{\left(\int_{[\overline{T}^{-n}\eta](x,\omega)} \Phi_{(x,\omega)}(x',\omega)\, d\nu|_{\eta(x,\omega)}(x',\omega)\right)}{g(x,\omega)}\right] d\overline{\mu}(x,\omega). \end{aligned}$$

Indeed, the inequalities (2) allow one to see that $H_\nu(Q) < \infty, \phi \in L^1(\nu)$ imply $H_\mu(Q) < \infty, \phi \in L^1(\mu)$ respectively, so that $h_\mu(T) = \int_X (P_{\text{top}}(T,\phi) - \phi)\, d\mu < \infty$ and $\int_X (g \circ \overline{T}^n(x,\omega)/g(x,\omega))\, d\overline{\mu} = 0$. Since $h_\mu(T) = h_{\overline{\mu}}(\overline{T})$ and η is generating, the desired result follows from the concavity of log. □

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