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A conjecture on the least stable mode for the energy stability of plane parallel flows

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In the energy stability theory, the critical Reynolds number is usually defined as the minimum of the first positive eigenvalue R_1 of an eigenvalue equation for all wavenumber pairs (α, β), where α and β are the streamwise and spanwise wavenumbers of the normal mode. We prove that $(\cos\theta \pm 1)R_1$ are decreasing functions of $\theta = \arctan(\beta/\alpha)$ for the parallel flows between no-slip or slip parallel plates with or without variations in temperature. Numerical results inspire us to conjecture that R_1 is also a decreasing function of θ for the parallel shear flows under the no-slip boundary condition and without variations in temperature. If the conjecture is correct, the least stable normal modes for the energy stability will be streamwise vortices for these base flows.

Key words: Bénard convection, nonlinear instability, Navier–Stokes equations

1. Introduction

The linear stability theory provides a necessary condition ($Re \leqslant Re_L$) for a base flow to be conditionally stable, and the energy stability theory (energy method) provides a sufficient condition ($Re \leqslant Re_E$) for a base flow to be globally stable, where Re_L and Re_E are the critical Reynolds numbers for the linear stability and energy stability, respectively. In the linear stability analysis of parallel flows, Squire's theorem implies that the least stable normal modes are two-dimensional waves with spanwise wavenumber $\beta = 0$ (Squire 1933; Knowles & Gebhart 1968). In the energy stability theory, however, there is no analogue to Squire's theorem. Actually, the least stable mode for the energy stability of the plane Poiseuille flow is a streamwise vortex with streamwise wavenumber $\alpha = 0$ (Busse 1969; Joseph & Carmi 1969).

Two proofs that the least stable mode for the energy stability of the plane Couette flow is a streamwise vortex were given by Joseph (1966) and Busse (1972). However, Joseph's proof was uncompleted because it failed to exclude the possibility that the least stable mode might be a two-dimensional wave (Busse 1972); Busse's proof required three particular eigenvalues to be calculated first. Consequently, there is no direct proof (without the need for calculating any particular eigenvalue) that the

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streamwise vortices are the least stable modes for the energy stability, even in the simplest case of the plane Couette flow.

The critical Reynolds number for the energy stability Re_E is usually calculated in two steps. The first (minimum) positive eigenvalues R_1 of an eigenvalue equation are calculated for all wavenumber pairs (α, β), where α and β are the streamwise and spanwise wavenumbers, and then the critical Reynolds number Re_E is the minimum R_1 in the wavenumber plane.

In this paper, we explore the variations of the positive eigenvalues in the wavenumber plane, and propose a conjecture on the first positive eigenvalue for a variety of parallel shear flows. This conjecture implies that the least stable modes for the energy stability are streamwise vortices for these base flows. We derive the eigenvalue equation for the energy stability in § 2, and prove two theorems on the monotonicity of the positive eigenvalues in the wavenumber plane in § 3. After the numerical results of the energy stability are introduced in § 4, we propose the conjecture in § 5. Section 6 concludes this work. We consider a special case in appendix A to show the gap in Joseph's proof that the least stable mode for the energy stability of the plane Couette flow is a streamwise vortex. Appendix B introduces the gradient descent algorithm used in search for the counter-examples to the conjecture.

2. Eigenvalue equation for the energy stability of plane parallel flows

When the fluctuation in the temperature of the fluid is small, the motion of the fluid between parallel plates is governed by the Boussinesq equation (Straughan 1992, p. 56),

$$\frac{\partial \boldsymbol{U}^*}{\partial t^*} + (\boldsymbol{U}^* \cdot \nabla^*)\boldsymbol{U}^* = -\frac{1}{\rho_r}\nabla^* P^* + \nu \nabla^{*2}\boldsymbol{U}^* + \frac{1}{\rho_r}F_x^* \boldsymbol{e}_x + [1 - \gamma(T^* - T_r^*)]\boldsymbol{g}, \tag{2.1}$$

$$\frac{\partial T^*}{\partial t^*} + (\boldsymbol{U}^* \cdot \nabla^*)T^* = \kappa \nabla^{*2} T^* + \frac{1}{\rho_r c_p} Q^*, \tag{2.2}$$

$$\nabla^* \cdot \boldsymbol{U}^* = 0, \tag{2.3}$$

$$\left(U^* \pm l^* \frac{\partial U^*}{\partial y^*}\right)\bigg|_{y^*=\pm h^*} - U^*_{w,\pm h^*} = V^*|_{y^*=\pm h^*} = \left(W^* \pm l^* \frac{\partial W^*}{\partial y^*}\right)\bigg|_{y^*=\pm h^*} = 0, \tag{2.4}$$

$$\left(T^* \pm l_T^* \frac{\partial T^*}{\partial y^*}\right)\bigg|_{y^*=\pm h^*} - T^*_{w,\pm h^*} = 0, \tag{2.5}$$

where x^*, y^* and z^* are the coordinates in the streamwise, wall-normal and spanwise directions, respectively; t^* is the time; $\boldsymbol{U}^* = (U^*, V^*, W^*)$, T^* and P^* are the velocity, temperature and pressure of the fluid, respectively; ρ_r is the reference density of the fluid at the reference temperature T_r^*; ν, γ, κ and c_p are the kinematic viscosity, coefficient of thermal expansion, thermal diffusivity and specific heat capacity at constant pressure of the fluid, respectively; $\boldsymbol{g} = -g\boldsymbol{e}_y$ is the gravitational acceleration; $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are the unit vectors in the streamwise and wall-normal directions. Both the streamwise component of the body force F_x^* and the volumetric heat source Q^* are functions of only the wall-normal coordinate y^*. The distance between the parallel plates is $2h^*$; $U^*_{w,\pm h^*}$ and $T^*_{w,\pm h^*}$ are the streamwise velocities and the temperatures of the walls at $y^* = \pm h^*$.

The region is assumed to be periodic in the streamwise and spanwise directions. The velocity of the fluid satisfies the slip boundary condition at the walls, and l^* is the slip length. The temperature of the fluid satisfies the convective boundary condition at the walls, and l_T^* is the ratio between the thermal conductivity of the fluid and the convective heat transfer coefficient of the flow. The no-slip boundary condition and the fixed temperature boundary condition correspond to $l^*=0$ and $l_T^*=0$, respectively.

Introducing non-dimensional quantities,

$$(x, y, z)=\frac{(x^*, y^*, z^*)}{h^*}, \quad \boldsymbol{U}=\frac{\boldsymbol{U}^*}{U_c^*}, \quad t=\frac{U_c^*}{h^*}t^*, \quad T=\frac{T^*-T_r^*}{\Delta T_c^*}, \tag{2.6a–d}$$

$$P=\frac{P^*+\rho_r g y^*}{\rho_r U_c^{*2}}, \quad F_x=\frac{h^*}{\rho_r U_c^{*2}}F_x^*, \quad Q=\frac{h^*}{\rho_r c_p U_c^* \Delta T_c^*}Q^*, \quad (l, l_T)=\frac{(l^*, l_T^*)}{h^*}, \tag{2.7a–d}$$

where U_c^* and ΔT_c^* are the characteristic velocity and temperature difference in the fluid, we have the non-dimensional governing equation,

$$\frac{\partial \boldsymbol{U}}{\partial t}+(\boldsymbol{U}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{U}=-\boldsymbol{\nabla}P+\frac{1}{Re}\nabla^2\boldsymbol{U}+F_x\boldsymbol{e}_x+\frac{Gr}{Re^2}T\boldsymbol{e}_y, \tag{2.8}$$

$$\frac{\partial T}{\partial t}+(\boldsymbol{U}\boldsymbol{\cdot}\boldsymbol{\nabla})T=\frac{1}{PrRe}\nabla^2 T+Q, \tag{2.9}$$

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{U}=0, \tag{2.10}$$

$$\left.\left(U\pm l\frac{\partial U}{\partial y}\right)\right|_{y=\pm1}-U_{w,\pm1}=V|_{y=\pm1}=\left.\left(W\pm l\frac{\partial W}{\partial y}\right)\right|_{y=\pm1}=0, \tag{2.11}$$

$$\left.\left(T\pm l_T\frac{\partial T}{\partial y}\right)\right|_{y=\pm1}-T_{w,\pm1}=0, \tag{2.12}$$

where the Reynolds number, the Prandtl number and the Grashof number are defined as

$$Re=\frac{U_c^*h^*}{\nu}, \quad Pr=\frac{\nu}{\kappa}, \quad Gr=\frac{g\gamma h^{*3}\Delta T_c^*}{\nu^2}. \tag{2.13a–c}$$

Here we assume $l\geqslant 0$ and $l_T\geqslant 0$. When $l_T>0$, the Nusselt number can be defined as l_T^{-1}.

The base flow $(\boldsymbol{U}_0, T_0, P_0)$ is a parallel flow (with $V_0=W_0=0$ and only depending on the wall-normal coordinate y), and satisfies

$$\frac{1}{Re}\frac{\mathrm{d}^2U_0}{\mathrm{d}y^2}+F_x=0, \tag{2.14}$$

$$-\frac{\mathrm{d}P_0}{\mathrm{d}y}+\frac{Gr}{Re^2}T_0=0, \tag{2.15}$$

$$\frac{1}{PrRe}\frac{\mathrm{d}^2T_0}{\mathrm{d}y^2}+Q=0, \tag{2.16}$$

$$\left.\left(U_0\pm l\frac{\mathrm{d}U_0}{\mathrm{d}y}\right)\right|_{y=\pm1}-U_{w,\pm1}=\left.\left(T_0\pm l_T\frac{\mathrm{d}T_0}{\mathrm{d}y}\right)\right|_{y=\pm1}-T_{w,\pm1}=0. \tag{2.17}$$

The governing equation of the disturbance $(\boldsymbol{u}, T', p)=(\boldsymbol{U}, T, P)-(\boldsymbol{U}_0, T_0, P_0)$ is

$$\frac{\partial \boldsymbol{u}}{\partial t}+U_0\frac{\partial \boldsymbol{u}}{\partial x}+v\frac{\mathrm{d}\boldsymbol{U}_0}{\mathrm{d}y}+(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{u}=-\boldsymbol{\nabla}p+\frac{1}{Re}\nabla^2\boldsymbol{u}+\frac{Gr}{Re^2}T'\boldsymbol{e}_y, \tag{2.18}$$

$$\frac{\partial T'}{\partial t} + U_0 \frac{\partial T'}{\partial x} + v \frac{\mathrm{d}T_0}{\mathrm{d}y} + (\boldsymbol{u} \boldsymbol{\cdot} \boldsymbol{\nabla}) T' = \frac{1}{PrRe} \nabla^2 T', \tag{2.19}$$

$$\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{u} = 0, \tag{2.20}$$

$$\left(u \pm l \frac{\partial u}{\partial y} \right) \bigg|_{y=\pm 1} = v|_{y=\pm 1} = \left(w \pm l \frac{\partial w}{\partial y} \right) \bigg|_{y=\pm 1} = \left(T' \pm l_T \frac{\partial T'}{\partial y} \right) \bigg|_{y=\pm 1} = 0. \tag{2.21}$$

Define the energy of the disturbance as

$$E(Pr, \lambda) = \int_{\Omega} \frac{u^2 + v^2 + w^2 + \lambda Pr T'^2}{2} \, \mathrm{d}V, \tag{2.22}$$

where $\lambda > 0$ is a coupling parameter (Joseph 1966), and $\mathrm{d}V = \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$. The region is $\Omega = (0, L_x) \times (-1, 1) \times (0, L_z)$, where L_x and L_z are the wavelengths of the disturbance in the streamwise and spanwise directions, respectively. From (2.18)–(2.21), we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{\Omega} \left(-uv \frac{\mathrm{d}U_0}{\mathrm{d}y} - \lambda Pr v T' \frac{\mathrm{d}\tilde{T}_0}{\mathrm{d}y} \right) \mathrm{d}V - \frac{1}{Re} (\|\boldsymbol{\nabla} \boldsymbol{u}\|^2 + \lambda \|\boldsymbol{\nabla} T'\|^2), \tag{2.23}$$

where

$$\tilde{T}_0 = T_0 - \frac{Gr}{\lambda Pr Re^2} y, \tag{2.24}$$

$$\begin{aligned} \|\boldsymbol{\nabla} \boldsymbol{u}\|^2 = {} & \int_{\Omega} \left(\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right) \mathrm{d}V + l \int_0^{L_z} \int_0^{L_x} \left[\left(\frac{\partial u}{\partial y} \bigg|_{y=1} \right)^2 + \left(\frac{\partial u}{\partial y} \bigg|_{y=-1} \right)^2 \right] \mathrm{d}x \, \mathrm{d}z \\ & + l \int_0^{L_z} \int_0^{L_x} \left[\left(\frac{\partial w}{\partial y} \bigg|_{y=1} \right)^2 + \left(\frac{\partial w}{\partial y} \bigg|_{y=-1} \right)^2 \right] \mathrm{d}x \, \mathrm{d}z, \end{aligned} \tag{2.25}$$

$$\|\boldsymbol{\nabla} T'\|^2 = \int_{\Omega} \left(\frac{\partial T'}{\partial x_j} \frac{\partial T'}{\partial x_j} \right) \mathrm{d}V + l_T \int_0^{L_z} \int_0^{L_x} \left[\left(\frac{\partial T'}{\partial y} \bigg|_{y=1} \right)^2 + \left(\frac{\partial T'}{\partial y} \bigg|_{y=-1} \right)^2 \right] \mathrm{d}x \, \mathrm{d}z, \tag{2.26}$$

and the Einstein summation convention is used.

In the energy stability analysis, the critical Reynolds number $Re_E(Pr, \lambda)$ is defined by

$$\frac{1}{Re_E(Pr, \lambda)} = \max \frac{\displaystyle\int_{\Omega} \left(-uv \frac{\mathrm{d}U_0}{\mathrm{d}y} - \lambda Pr v T' \frac{\mathrm{d}\tilde{T}_0}{\mathrm{d}y} \right) \mathrm{d}V}{\|\boldsymbol{\nabla} \boldsymbol{u}\|^2 + \lambda \|\boldsymbol{\nabla} T'\|^2}, \tag{2.27}$$

where the maximum is searched among all divergence-free disturbances satisfying the boundary condition (2.21). Therefore, we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leqslant \left(\frac{1}{Re_E(Pr, \lambda)} - \frac{1}{Re} \right) (\|\boldsymbol{\nabla} \boldsymbol{u}\|^2 + \lambda \|\boldsymbol{\nabla} T'\|^2) \leqslant 0, \tag{2.28}$$

when $Re \leqslant Re_E(Pr, \lambda)$. We also have $E \to 0$ when $Re < Re_E(Pr, \lambda)$, because $\|\boldsymbol{u}\|$ and $\|T'\|$ are controlled by $\|\boldsymbol{\nabla} \boldsymbol{u}\|$ and $\|\boldsymbol{\nabla} T'\|$ in the bounded domain according to the Poincaré inequality.

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The Euler–Lagrange equation corresponding to (2.27) is

$$\nabla^2 u - \frac{R}{2}\frac{\mathrm{d}U_0}{\mathrm{d}y}v - \frac{\partial q}{\partial x} = 0, \tag{2.29}$$

$$\nabla^2 v - \frac{R}{2}\frac{\mathrm{d}U_0}{\mathrm{d}y}u - \frac{\lambda Pr R}{2}\frac{\mathrm{d}\tilde{T}_0}{\mathrm{d}y}T' - \frac{\partial q}{\partial y} = 0, \tag{2.30}$$

$$\nabla^2 w - \frac{\partial q}{\partial z} = 0, \tag{2.31}$$

$$\nabla^2 T' - \frac{PrR}{2}\frac{\mathrm{d}\tilde{T}_0}{\mathrm{d}y}v = 0, \tag{2.32}$$

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u} = 0, \tag{2.33}$$

$$\left.\left(u \pm l\frac{\partial u}{\partial y}\right)\right|_{y=\pm 1} = v|_{y=\pm 1} = \left.\left(w \pm l\frac{\partial w}{\partial y}\right)\right|_{y=\pm 1} = \left.\left(T' \pm l_T\frac{\partial T'}{\partial y}\right)\right|_{y=\pm 1} = 0, \tag{2.34}$$

where R and $q(x, y, z)$ are the Lagrange multipliers.

From (2.25), (2.26) and (2.29)–(2.34), we have

$$\begin{aligned} \|\boldsymbol{\nabla}\boldsymbol{u}\|^2 + \lambda\|\boldsymbol{\nabla}T'\|^2 &= -\int_{\Omega}(u\nabla^2 u + v\nabla^2 v + w\nabla^2 w + \lambda T'\nabla^2 T')\,\mathrm{d}V \\ &= R\int_{\Omega}\left(-uv\frac{\mathrm{d}U_0}{\mathrm{d}y} - \lambda Pr v T'\frac{\mathrm{d}\tilde{T}_0}{\mathrm{d}y}\right)\mathrm{d}V. \end{aligned} \tag{2.35}$$

Comparing (2.35) with (2.27), we notice that the critical Reynolds number $Re_E(Pr, \lambda)$ is just the minimum positive eigenvalue R.

Introducing the normal mode $(u, v, w, T', q) = (\hat{u}, \hat{v}, \hat{w}, \hat{T}, \hat{q})\exp[\mathrm{i}(\alpha x + \beta z)] + \mathrm{c.c.}$, where α and β are the streamwise and spanwise wavenumbers, and c.c. denotes the complex conjugate, we have

$$(\mathrm{D}^2 - k^2)\hat{u} - \frac{R}{2}(\mathrm{D}U_0)\hat{v} - \mathrm{i}k(\cos\theta)\hat{q} = 0, \tag{2.36}$$

$$(\mathrm{D}^2 - k^2)\hat{v} - \frac{R}{2}(\mathrm{D}U_0)\hat{u} - \frac{\lambda PrR}{2}(\mathrm{D}\tilde{T}_0)\hat{T} - \mathrm{D}\hat{q} = 0, \tag{2.37}$$

$$(\mathrm{D}^2 - k^2)\hat{w} - \mathrm{i}k(\sin\theta)\hat{q} = 0, \tag{2.38}$$

$$(\mathrm{D}^2 - k^2)\hat{T} - \frac{PrR}{2}(\mathrm{D}\tilde{T}_0)\hat{v} = 0, \tag{2.39}$$

$$\mathrm{i}k(\cos\theta)\hat{u} + \mathrm{D}\hat{v} + \mathrm{i}k(\sin\theta)\hat{w} = 0, \tag{2.40}$$

$$(\hat{u} \pm l\mathrm{D}\hat{u})(\pm 1) = \hat{v}(\pm 1) = (\hat{w} \pm l\mathrm{D}\hat{w})(\pm 1) = (\hat{T} \pm l_T\mathrm{D}\hat{T})(\pm 1) = 0, \tag{2.41}$$

where $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}y$, $k = (\alpha^2 + \beta^2)^{1/2}$ and $\theta = \arctan(\beta/\alpha)$. If we denote the positive eigenvalues of (2.36)–(2.41) as $R_1(k, \theta, Pr, \lambda) \leqslant R_2(k, \theta, Pr, \lambda) \leqslant \cdots$, then the critical Reynolds number for the energy stability is

$$Re_E(Pr, \lambda) = \min_{k,\theta} R_1(k, \theta, Pr, \lambda). \tag{2.42}$$

Note that the nth positive eigenvalue R_n and the corresponding eigenvector may be only piecewise continuously differentiable where $R_n = R_{n+1}$ or $R_n = R_{n-1}$ (figure 1).

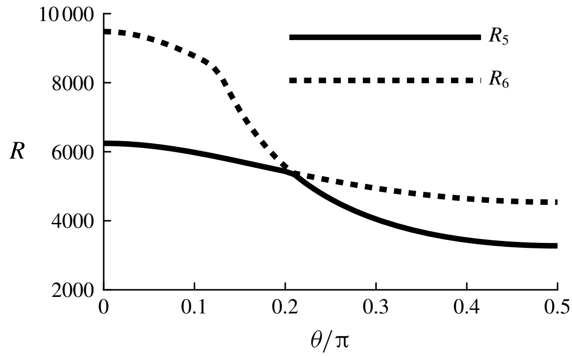

FIGURE 1. The fifth and sixth positive eigenvalues for the energy stability of the parallel shear flow with $DU_0 = \sin(8\pi y)$, $Pr\, D\tilde{T}_0 = 0$, $l = 0$ and $k = 1$.

Due to the symmetry of (2.36)–(2.41), only the cases with $k > 0$ and $0 \leqslant \theta \leqslant \pi/2$ need to be considered.

The critical Reynolds number for the energy stability Re_E depends on the definition of the energy of the disturbance (2.22), and is therefore a function of the Prandtl number Pr and the coupling parameter λ. For given Prandtl number, an optimal critical Reynolds number for the energy stability can be further defined as the maximum of $Re_E(Pr, \lambda)$ for all $\lambda > 0$ (Joseph 1966).

3. The monotonicity of the positive eigenvalues R_n ($n \geqslant 1$) in the wavenumber plane

3.1. *The expressions for $\partial R_n/\partial k$ and $\partial R_n/\partial \theta$*

To study the monotonicity of R_n in the (k, θ) plane, we first derive the expressions of $\partial R_n/\partial k$ and $\partial R_n/\partial \theta$.

Denote (2.36)–(2.40) as $\boldsymbol{L}\hat{\boldsymbol{u}} = \boldsymbol{0}$, where

$$\boldsymbol{L} = \begin{bmatrix} D^2 - k^2 & -\frac{1}{2}R(DU_0) & 0 & 0 & -ik\cos\theta \\ -\frac{1}{2}R(DU_0) & D^2 - k^2 & 0 & -\frac{1}{2}\lambda Pr R(D\tilde{T}_0) & -D \\ 0 & 0 & D^2 - k^2 & 0 & -ik\sin\theta \\ 0 & -\frac{1}{2}Pr R(D\tilde{T}_0) & 0 & D^2 - k^2 & 0 \\ ik\cos\theta & D & ik\sin\theta & 0 & 0 \end{bmatrix}, \quad \hat{\boldsymbol{u}} = \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{T} \\ \hat{q} \end{bmatrix}. \tag{3.1a,b}$$

For any $\lambda > 0$, assume the geometric multiplicity of the nth eigenvalue $R_n(k, \theta, Pr)$ to be 1 in an open set $S \subset (0, +\infty) \times [0, \pi/2] \times [0, +\infty)$. If the nth positive eigenvalue and the corresponding eigenvector of (2.36)–(2.41) are $(R_n, \hat{\boldsymbol{u}}_n)$ for parameters $(k, \theta, Pr) \in S$, and are $(R_n + dR_n, \hat{\boldsymbol{u}}_n + d\hat{\boldsymbol{u}}_n)$ for parameters $(k + dk, \theta + d\theta, Pr + dPr) \in S$, we can neglect the higher-order terms and obtain

$$\boldsymbol{L}(d\hat{\boldsymbol{u}}_n) + (dk)\frac{\partial \boldsymbol{L}}{\partial k}\hat{\boldsymbol{u}}_n + (d\theta)\frac{\partial \boldsymbol{L}}{\partial \theta}\hat{\boldsymbol{u}}_n + (dPr)\frac{\partial \boldsymbol{L}}{\partial Pr}\hat{\boldsymbol{u}}_n + (dR_n)\frac{\partial \boldsymbol{L}}{\partial R_n}\hat{\boldsymbol{u}}_n = \boldsymbol{0}, \tag{3.2}$$

$$[d\hat{u}_n \pm lD(d\hat{u}_n)](\pm 1) = (d\hat{v}_n)(\pm 1) = [d\hat{w}_n \pm lD(d\hat{w}_n)](\pm 1) = 0, \tag{3.3}$$

$$[d\hat{T}_n \pm l_T D(d\hat{T}_n)](\pm 1) = 0. \tag{3.4}$$

Define the inner product between two vectors $\hat{\boldsymbol{u}}'$ and $\hat{\boldsymbol{u}}''$ as

$$\langle \hat{\boldsymbol{u}}', \hat{\boldsymbol{u}}'' \rangle = \int_{-1}^{1} (\bar{\hat{u}}'\hat{u}'' + \bar{\hat{v}}'\hat{v}'' + \bar{\hat{w}}'\hat{w}'' + \lambda \bar{\hat{T}}'\hat{T}'' + \bar{\hat{q}}'\hat{q}'')\,\mathrm{d}y, \tag{3.5}$$

where the overlines denote the complex conjugates. Because $\boldsymbol{L}$ is a self-adjoint operator with respect to the inner product (3.5) and the boundary conditions (2.41), (3.3) and (3.4), we have

$$\langle \hat{\boldsymbol{u}}_n, \boldsymbol{L}(\mathrm{d}\hat{\boldsymbol{u}}_n) \rangle = \langle \boldsymbol{L}\hat{\boldsymbol{u}}_n, \mathrm{d}\hat{\boldsymbol{u}}_n \rangle = \langle \boldsymbol{0}, \mathrm{d}\hat{\boldsymbol{u}}_n \rangle = 0, \tag{3.6}$$

and then it follows from (3.2) that

$$\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}}{\partial k}\hat{\boldsymbol{u}}_n \right\rangle \mathrm{d}k + \left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}}{\partial \theta}\hat{\boldsymbol{u}}_n \right\rangle \mathrm{d}\theta + \left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}}{\partial Pr}\hat{\boldsymbol{u}}_n \right\rangle \mathrm{d}Pr + \left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}}{\partial R_n}\hat{\boldsymbol{u}}_n \right\rangle \mathrm{d}R_n = 0. \tag{3.7}$$

Therefore, we have

$$\frac{\partial R_n}{\partial k} = -\frac{\left\langle \hat{\boldsymbol{u}}_n, \dfrac{\partial \boldsymbol{L}}{\partial k}\hat{\boldsymbol{u}}_n \right\rangle}{\left\langle \hat{\boldsymbol{u}}_n, \dfrac{\partial \boldsymbol{L}}{\partial R_n}\hat{\boldsymbol{u}}_n \right\rangle}, \tag{3.8}$$

$$\frac{\partial R_n}{\partial \theta} = -\frac{\left\langle \hat{\boldsymbol{u}}_n, \dfrac{\partial \boldsymbol{L}}{\partial \theta}\hat{\boldsymbol{u}}_n \right\rangle}{\left\langle \hat{\boldsymbol{u}}_n, \dfrac{\partial \boldsymbol{L}}{\partial R_n}\hat{\boldsymbol{u}}_n \right\rangle}. \tag{3.9}$$

The inner products in (3.7)–(3.9) are

$$\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}}{\partial k}\hat{\boldsymbol{u}}_n \right\rangle = -2k \int_{-1}^{1} (|\hat{u}_n|^2 + |\hat{v}_n|^2 + |\hat{w}_n|^2 + \lambda |\hat{T}_n|^2)\,\mathrm{d}y - \frac{1}{k}\int_{-1}^{1} (\bar{\hat{q}}_n \mathrm{D}\hat{v}_n + \hat{q}_n \mathrm{D}\bar{\hat{v}}_n)\,\mathrm{d}y, \tag{3.10}$$

$$\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}}{\partial \theta}\hat{\boldsymbol{u}}_n \right\rangle = -\int_{-1}^{1} (\bar{\hat{q}}_n \hat{\eta}_n + \hat{q}_n \bar{\hat{\eta}}_n)\,\mathrm{d}y, \tag{3.11}$$

$$\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}}{\partial Pr}\hat{\boldsymbol{u}}_n \right\rangle = -\frac{\lambda R_n}{2}\int_{-1}^{1} (\mathrm{D}\tilde{T}_0)(\bar{\hat{v}}_n \hat{T}_n + \hat{v}_n \bar{\hat{T}}_n)\,\mathrm{d}y, \tag{3.12}$$

$$\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}}{\partial R_n}\hat{\boldsymbol{u}}_n \right\rangle = -\frac{1}{2}\int_{-1}^{1} (\mathrm{D}U_0)(\bar{\hat{u}}_n \hat{v}_n + \hat{u}_n \bar{\hat{v}}_n)\,\mathrm{d}y - \frac{\lambda Pr}{2}\int_{-1}^{1} (\mathrm{D}\tilde{T}_0)(\bar{\hat{v}}_n \hat{T}_n + \hat{v}_n \bar{\hat{T}}_n)\,\mathrm{d}y, \tag{3.13}$$

where $\hat{\eta}_n = \mathrm{i}k\hat{u}_n \sin\theta - \mathrm{i}k\hat{w}_n \cos\theta$ is the wall-normal component of the disturbance vorticity.

Using (2.36)–(2.41) in (3.10)–(3.13), we have

$$\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}}{\partial k}\hat{\boldsymbol{u}}_n \right\rangle = -\frac{2}{k}\int_{-1}^{1} [2(|\mathrm{D}\hat{v}_n|^2 + k^2|\hat{v}_n|^2) + |\hat{\eta}_n|^2 + \lambda k^2 |\hat{T}_n|^2]\,\mathrm{d}y + \frac{1}{k^3}(I_v + I_\eta + \lambda k^2 I_T), \tag{3.14}$$

$$\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}}{\partial \theta}\hat{\boldsymbol{u}}_n \right\rangle = \frac{R_n}{2}\int_{-1}^{1}(\mathrm{D}U_0)(\bar{\hat{v}}_n\hat{w}_n + \hat{v}_n\bar{\hat{w}}_n)\,\mathrm{d}y, \tag{3.15}$$

$$\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}}{\partial Pr}\hat{\boldsymbol{u}}_n \right\rangle = \frac{2\lambda}{Pr}I_T, \tag{3.16}$$

$$\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}}{\partial R_n}\hat{\boldsymbol{u}}_n \right\rangle = \frac{1}{k^2R_n}(I_v + I_\eta + \lambda k^2 I_T), \tag{3.17}$$

where

$$I_v = \int_{-1}^{1}|(\mathrm{D}^2-k^2)\hat{v}_n|^2\,\mathrm{d}y + l[|(\mathrm{D}^2\hat{v}_n)(1)|^2 + |(\mathrm{D}^2\hat{v}_n)(-1)|^2], \tag{3.18}$$

$$I_\eta = \int_{-1}^{1}(|\mathrm{D}\hat{\eta}_n|^2 + k^2|\hat{\eta}_n|^2)\,\mathrm{d}y + l[|(\mathrm{D}\hat{\eta}_n)(1)|^2 + |(\mathrm{D}\hat{\eta}_n)(-1)|^2], \tag{3.19}$$

$$I_T = \int_{-1}^{1}(|\mathrm{D}\hat{T}_n|^2 + k^2|\hat{T}_n|^2)\,\mathrm{d}y + l_T[|(\mathrm{D}\hat{T}_n)(1)|^2 + |(\mathrm{D}\hat{T}_n)(-1)|^2]. \tag{3.20}$$

3.2. *The dependence of the positive eigenvalues R_n ($n \geqslant 1$) on k*

THEOREM 1. *For any continuously differentiable real functions $\mathrm{D}U_0$ and $\mathrm{D}\tilde{T}_0$, and any $\lambda > 0$, $Pr \geqslant 0$, $l \geqslant 0$, $l_T \geqslant 0$ and $\theta \in [0, \pi/2]$, kR_n is an increasing function of k when $k > 0$, and R_n is a decreasing function of k when $0 < k < k_0(l, l_T)$, where R_n is the nth positive eigenvalue of (2.36)–(2.41) ($n \geqslant 1$), and $k_0(l, l_T) = \min\{k_0'(l), \sigma(\max\{l, l_T\})\}$. Here $k_0'(l)$ is the solution of the equation*

$$(\sqrt{3}k_0')\tan(\sqrt{3}k_0') + k_0'\tanh(k_0') + 4k_0'^2 l = 0 \tag{3.21}$$

in the interval $(\pi/\sqrt{12}, \pi/\sqrt{3})$; $\sigma(l)$ is the solution of the equation $\sigma^{-1}\cot\sigma = l$ in the interval $(0, \pi/2]$. Specifically, when $l = l_T = 0$ and $0 < k < k_0(0, 0) = k_0'(0) \approx 1.534$, R_n is a decreasing function of k.

The following lemmas, which can be proved easily with the variational method, will be used in the proof of theorem 1.

LEMMA 1. *For any $l \geqslant 0$ and any complex-valued function $\hat{f}$, if $(\hat{f} \pm l\mathrm{D}\hat{f})(\pm 1) = 0$, then*

$$\int_{-1}^{1}|\mathrm{D}\hat{f}|^2\,\mathrm{d}y + l[|(\mathrm{D}\hat{f})(1)|^2 + |(\mathrm{D}\hat{f})(-1)|^2] \geqslant [\sigma(l)]^2\int_{-1}^{1}|\hat{f}|^2\,\mathrm{d}y, \tag{3.22}$$

where $\sigma(l)$ is the solution of the equation $\sigma^{-1}\cot\sigma = l$ in the interval $(0, \pi/2]$. The equality in (3.22) holds when $\hat{f} = C\cos(\sigma y)$, where C is any complex constant.

LEMMA 2. *For any $l \geqslant 0$ and any complex-valued function $\hat{g}$, if $\hat{g}(\pm 1) = (\mathrm{D}\hat{g} \pm l\mathrm{D}^2\hat{g})(\pm 1) = 0$, then*

$$\int_{-1}^{1}(|\mathrm{D}^2\hat{g}|^2 + k^2|\mathrm{D}\hat{g}|^2)\,\mathrm{d}y + l[|(\mathrm{D}^2\hat{g})(1)|^2 + |(\mathrm{D}^2\hat{g})(-1)|^2]$$
$$\geqslant [\sigma'(k, l)]^2\int_{-1}^{1}(|\mathrm{D}\hat{g}|^2 + k^2|\hat{g}|^2)\,\mathrm{d}y, \tag{3.23}$$

where $\sigma'(k, l)$ *is the solution of the equation* $\sigma' \tan\sigma' + k\tanh k + (\sigma'^2 + k^2)l = 0$ *in the interval* $(\pi/2, \pi)$. *The equality in* (3.23) *holds when* $\hat{g} = C'[\cos(\sigma')\cosh(ky) - \cosh(k)\cos(\sigma' y)]$, *where* C' *is any complex constant.*

To prove theorem 1, it is sufficient to prove for any given $\lambda > 0$, $l \geqslant 0$ and $l_T \geqslant 0$,

$$\frac{\partial}{\partial k}(kR_n) \geqslant 0, \quad (k > 0), \tag{3.24}$$

$$\frac{\partial R_n}{\partial k} \leqslant 0, \quad (0 < k < k_0(l, l_T)), \tag{3.25}$$

in each subset $S \subset (0, +\infty) \times [0, \pi/2] \times [0, +\infty)$ where the geometric multiplicity of the nth eigenvalue $R_n(k, \theta, Pr)$ is 1.

Substituting (3.14) and (3.17) into (3.8), we have $\partial R_n/\partial k \geqslant -R_n/k$, and then (3.24) holds.

Using lemma 1 for $\hat{\eta}_n$ and $\hat{T}_n$, and using lemma 2 for $\hat{v}_n$, we have

$$I_\eta \geqslant [k^2 + (\sigma(l))^2] \int_{-1}^{1} |\hat{\eta}_n|^2 \,\mathrm{d}y, \tag{3.26}$$

$$I_T \geqslant [k^2 + (\sigma(l_T))^2] \int_{-1}^{1} |\hat{T}_n|^2 \,\mathrm{d}y, \tag{3.27}$$

$$I_v \geqslant [k^2 + (\sigma'(k, l))^2] \int_{-1}^{1} (|\mathrm{D}\hat{v}_n|^2 + k^2|\hat{v}_n|^2) \,\mathrm{d}y. \tag{3.28}$$

Substituting (3.26)–(3.28) into (3.14), we have

$$\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}}{\partial k}\hat{\boldsymbol{u}}_n \right\rangle \geqslant \frac{(\sigma'(k, l))^2 - 3k^2}{k^3} \int_{-1}^{1} (|\mathrm{D}\hat{v}_n|^2 + k^2|\hat{v}_n|^2) \,\mathrm{d}y + \frac{(\sigma(l))^2 - k^2}{k^3} \int_{-1}^{1} |\hat{\eta}_n|^2 \,\mathrm{d}y + \frac{\lambda[(\sigma(l_T))^2 - k^2]}{k} \int_{-1}^{1} |\hat{T}_n|^2 \,\mathrm{d}y. \tag{3.29}$$

When $0 < k \leqslant k_0'(l)$, we have $[\sigma'(k, l)]^2 - 3k^2 \geqslant [\sigma'(k_0'(l), l)]^2 - 3[k_0'(l)]^2 = 0$, because $\sigma'(k, l)$ is a decreasing function of k for any given $l \geqslant 0$. Noticing that $\sigma(l)$ is a decreasing function of l when $l \geqslant 0$, we have $[\sigma(l)]^2 - k^2 \geqslant 0$ and $[\sigma(l_T)]^2 - k^2 \geqslant 0$ when $0 < k \leqslant \sigma(\max\{l, l_T\})$. Therefore, we have $\langle \hat{\boldsymbol{u}}_n, (\partial \boldsymbol{L}/\partial k)\hat{\boldsymbol{u}}_n \rangle \geqslant 0$ when $0 < k < k_0(l, l_T) = \min\{k_0'(l), \sigma(\max\{l, l_T\})\}$, and then (3.25) holds because of (3.8) and (3.17).

Specifically, when $l = l_T = 0$, we have $k_0(0, 0) = k_0'(0)$ because $k_0'(0) \approx 1.534 < \pi/2 = \sigma(0)$. Thus we have proved theorem 1.

A direct corollary of theorem 1 is that $R_n = O(k^{-1})$ when $k \to 0$. Another corollary is that the least stable mode in the energy method must have $k \geqslant k_0(l, l_T)$. The contours of k_0 in the (l, l_T) plane are plotted in figure 2.

3.3. *The dependence of the positive eigenvalues* R_n $(n \geqslant 1)$ *on* θ

THEOREM 2. *For any continuously differentiable real functions* $\mathrm{D}U_0$ *and* $\mathrm{D}\tilde{T}_0$, *and any* $\lambda > 0$, $Pr \geqslant 0$, $l \geqslant 0$, $l_T \geqslant 0$ *and* $k > 0$, $(\cos\theta \pm 1)R_n$ *are decreasing functions of* θ *when* $0 \leqslant \theta \leqslant \pi/2$, *where* R_n *is the nth positive eigenvalue of* (2.36)–(2.41) $(n \geqslant 1)$.

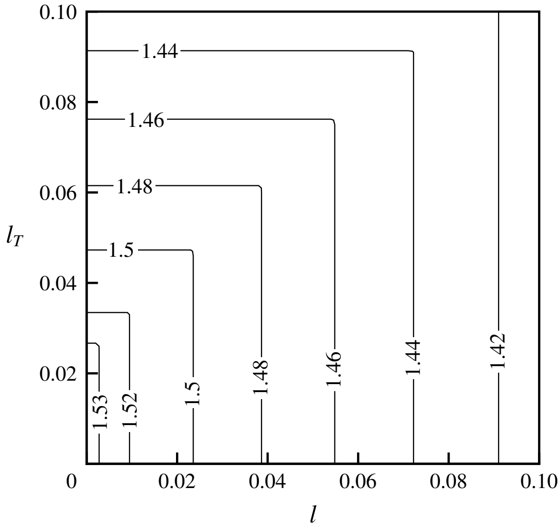

FIGURE 2. The contours of k_0 in the (l, l_T) plane.

To prove theorem 2, it is sufficient to prove for any given $\lambda > 0$, $l \geqslant 0$ and $l_T \geqslant 0$,

$$\frac{\partial}{\partial\theta}[(\cos\theta \pm 1)R_n] \leqslant 0 \tag{3.30}$$

in each subset $S \subset (0, +\infty) \times [0, \pi/2] \times [0, +\infty)$ where the geometric multiplicity of the nth eigenvalue $R_n(k, \theta, Pr)$ is 1.

Substituting (3.13) and (3.15) into (3.9), we have

$$\frac{1}{R_n}\frac{\partial R_n}{\partial\theta} = \frac{\displaystyle\int_{-1}^{1} (\mathrm{D}U_0)(\bar{\hat{v}}_n\hat{w}_n + \hat{v}_n\bar{\hat{w}}_n)\,\mathrm{d}y}{\displaystyle\int_{-1}^{1} (\mathrm{D}U_0)(\bar{\hat{u}}_n\hat{v}_n + \hat{u}_n\bar{\hat{v}}_n)\,\mathrm{d}y + \lambda Pr\int_{-1}^{1} (\mathrm{D}\tilde{T}_0)(\bar{\hat{v}}_n\hat{T}_n + \hat{v}_n\bar{\hat{T}}_n)\,\mathrm{d}y}. \tag{3.31}$$

(i) When $\theta = 0$, we have $\hat{w}_n = 0$ from (2.38) and (2.41), and then $\partial R_n/\partial\theta = 0$ follows from (3.31). As a result,

$$\frac{\partial}{\partial\theta}[(\cos\theta \pm 1)R_n] = 0. \tag{3.32}$$

(ii) When $\theta = \pi/2$, if $(\hat{u}_n, \hat{v}_n, \hat{w}_n, \hat{T}_n, \hat{q}_n)$ is an eigenvector corresponding to the eigenvalue R_n, then $(\bar{\hat{u}}_n, \bar{\hat{v}}_n, -\bar{\hat{w}}_n, \bar{\hat{T}}_n, \bar{\hat{q}}_n)$ is also an eigenvector corresponding to the same eigenvalue. In each subset S where the geometric multiplicity of the eigenvalue R_n is 1, there exists a complex constant C such that $(\bar{\hat{u}}_n, \bar{\hat{v}}_n, -\bar{\hat{w}}_n, \bar{\hat{T}}_n, \bar{\hat{q}}_n) = C(\hat{u}_n, \hat{v}_n, \hat{w}_n, \hat{T}_n, \hat{q}_n)$, and then $\bar{\hat{v}}_n\hat{w}_n + \hat{v}_n\bar{\hat{w}}_n = 0$. Therefore, we have $\partial R_n/\partial\theta = 0$ from (3.31), and then

$$\frac{\partial}{\partial\theta}[(\cos\theta \pm 1)R_n] = -R_n \leqslant 0. \tag{3.33}$$

(iii) When $0 < \theta < \pi/2$, using (2.36)–(2.41) in (3.15), we have

$$\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial\boldsymbol{L}}{\partial\theta}\hat{\boldsymbol{u}}_n \right\rangle = -\frac{(\sin^2\theta)I_v - (1+\cos^2\theta)I_\eta - \lambda k^2(\sin^2\theta)I_T}{k^2\sin\theta\cos\theta}. \tag{3.34}$$

Define

$$I(\mu) = \int_{-1}^{1} |\hat{h}_\mu|^2 \,\mathrm{d}y + l[|\hat{h}_\mu(1)|^2 + |\hat{h}_\mu(-1)|^2], \tag{3.35}$$

for $\mu \in \mathbb{R}$, where

$$\hat{h}_\mu = (\mathrm{D}^2 - k^2)\hat{v}_n + \mu \mathrm{D}\hat{\eta}_n. \tag{3.36}$$

Using (2.36)–(2.41) in (3.35), we have

$$I(\mu) = I_v - \mu(\tan\theta)(I_v - I_\eta - \lambda k^2 I_T) + \mu^2 \left(I_\eta - k^2 \int_{-1}^{1} |\hat{\eta}_n|^2 \,\mathrm{d}y \right). \tag{3.37}$$

Substituting (3.17) and (3.34) into (3.9), we have

$$\frac{1}{R_n}\frac{\partial R_n}{\partial \theta} = \frac{(\sin^2\theta)I_v - (1+\cos^2\theta)I_\eta - \lambda k^2(\sin^2\theta)I_T}{(\sin\theta\cos\theta)(I_v + I_\eta + \lambda k^2 I_T)}, \tag{3.38}$$

and then

$$\begin{aligned} &\frac{1}{R_n}\frac{\partial}{\partial\theta}[(\cos\theta \pm 1)R_n] \\ &= \pm\frac{(\sin^2\theta)I_v - (1\pm\cos\theta)^2 I_\eta - \lambda k^2(\sin^2\theta)(1\pm 2\cos\theta)I_T}{(\sin\theta\cos\theta)(I_v + I_\eta + \lambda k^2 I_T)} \\ &= -\frac{(\sin^2\theta) I\left(\dfrac{\cos\theta\pm 1}{\sin\theta}\right) + \lambda k^2(\sin^2\theta)I_T + k^2(1\pm\cos\theta)^2 \displaystyle\int_{-1}^{1}|\hat{\eta}_n|^2\,\mathrm{d}y}{(\sin\theta)(I_v + I_\eta + \lambda k^2 I_T)} \\ &\leqslant 0, \end{aligned} \tag{3.39}$$

where we have used (3.37) for $\mu = (\cos\theta \pm 1)/\sin\theta$. Noticing that $R_n > 0$, we have proved theorem 2.

According to theorem 2, we have

$$\frac{R_n\left(k, \dfrac{\pi}{2}, Pr, \lambda\right)}{1+\cos\theta} \leqslant R_n(k, \theta, Pr, \lambda) \leqslant \min\left\{ \frac{R_n\left(k, \dfrac{\pi}{2}, Pr, \lambda\right)}{1-\cos\theta}, \frac{2R_n(k, 0, Pr, \lambda)}{1+\cos\theta} \right\}, \tag{3.40}$$

for any $k > 0$, $0 < \theta \leqslant \pi/2$, $Pr \geqslant 0$, $\lambda > 0$, $l \geqslant 0$, $l_T \geqslant 0$ and $n \geqslant 1$. From (2.42) and (3.40), we have

$$\frac{1}{2}\min_{k>0} R_1\left(k, \frac{\pi}{2}, Pr, \lambda\right) \leqslant Re_E(Pr, \lambda) \leqslant \min_{k>0} R_1\left(k, \frac{\pi}{2}, Pr, \lambda\right). \tag{3.41}$$

Calculating the critical Reynolds number for the energy stability Re_E according to (2.42) requires solving the eigenvalue equation (2.36)–(2.41) for all wavenumber pairs (α, β) or (k, θ). However, the critical Reynolds number can be estimated by only solving the eigenvalue equation for the streamwise vortices with $\alpha = k\cos\theta = 0$ according to (3.41).

As a direct result of (3.41), the minimum R_1 for all two-dimensional waves is also bounded from below by a half of the minimum R_1 for all streamwise vortices, i.e.

$$\min_{k>0} R_1(k, 0, Pr, \lambda) \geqslant \frac{1}{2}\min_{k>0} R_1\left(k, \frac{\pi}{2}, Pr, \lambda\right), \tag{3.42}$$

for any $Pr \geqslant 0$, $\lambda > 0$, $l \geqslant 0$ and $l_T \geqslant 0$. Kaiser & Schmitt (2001) have proved an inequality stronger than (3.42),

$$\min_{k>0} R_1(k, 0) \geqslant \frac{16}{27} \min_{k>0} R_1\left(k, \frac{\pi}{2}\right), \tag{3.43}$$

for $Pr\, \mathrm{D}\tilde{T}_0 = 0$ and $l = l_T = 0$.

4. Numerical results

In order to calculate the first positive eigenvalue R_1 of the eigenvalue equation (2.36)–(2.41) for various base flows, we first discretise the equation using 129 Chebyshev–Gauss–Lobatto collocation points, and then solve it with the QZ function in MATLAB (Dongarra, Straughan & Walker 1996). The first positive eigenvalue R_1 is found to decrease with increasing θ for any given k in the energy stability analysis of the plane Couette flow, the plane Poiseuille flow, the shear flow with a cubic velocity profile and the inclined buoyancy layer when $\lambda = 1$ (figure 3). In figure 3, we plot the contours of R_1 in the (k, θ) plane, instead of plotting them in the usual wavenumber plane (α, β) as in the previous works (Reddy & Henningson 1993; Xiong & Tao 2017). Although Sagalakov & Shtern (1971) also plotted their figure 2 in the (k, θ) plane, the contour $R_1 = 35$ in their figure is slightly different from that in our figure 3(*c*), implying that R_1 increases with increasing θ at $(k, \theta) \approx (1.9, \pi/2)$. Noticing that their contour $R_1 = 35$ and the boundary $\theta = \pi/2$ do not intersect at a right angle at $(k, \theta) \approx (1.9, \pi/2)$, we believe our result is more accurate. Actually, we have shown $\partial R_1/\partial\theta = 0$ at the boundaries $\theta = 0$ and $\theta = \pi/2$ in the proof of theorem 2.

Under the no-slip boundary condition and the slip boundary condition with the slip length $l = 0.1$, the least stable modes in the energy stability analysis of the plane Couette flow have $(k, \theta) = (1.558, \pi/2)$ (figure 3*a*) and $(k, \theta) = (1.426, \pi/2)$ (figure 3*c*), respectively. These wavenumbers k are close to the corresponding lower bounds $k_0(0, 0) = 1.534$ and $k_0(0.1, 0) = 1.411$ given in theorem 1 (figure 2).

When there is temperature variation in the base flow ($Pr\, \mathrm{D}\tilde{T}_0 \neq 0$), the first positive eigenvalue R_1 may increase with θ, e.g. when $\mathrm{D}U_0 = \cos(\pi y) + \cos(3\pi y)$, $Pr\, \mathrm{D}\tilde{T}_0 = 10 \sin(\pi y)$, $\lambda = 1$ and $l = l_T = 0$ (figure 4*a*). Figure 4(*b*) shows that theorem 2 still holds for this base flow.

For shear flows under the no-slip boundary condition ($l = 0$) and without variations in temperature ($Pr\, \mathrm{D}\tilde{T}_0 = 0$), we define

$$J(k, \theta, \boldsymbol{a}) = -\frac{1}{R_1}\frac{\mathrm{d}R_1}{\mathrm{d}\theta} = -\frac{\displaystyle\int_{-1}^{1} (\mathrm{D}U_0)(\bar{\hat{v}}_1 \hat{w}_1 + \hat{v}_1 \bar{\hat{w}}_1)\, \mathrm{d}y}{\displaystyle\int_{-1}^{1} (\mathrm{D}U_0)(\bar{\hat{u}}_1 \hat{v}_1 + \hat{u}_1 \bar{\hat{v}}_1)\, \mathrm{d}y}, \tag{4.1}$$

where we have used (3.31). Here $\mathrm{D}U_0 = \sum_{m\geqslant 0} a_m T_m(y)$, T_m is the Chebyshev polynomial of degree m, and $\boldsymbol{a} = (a_0, a_1, \ldots)$ is the coefficient. We also define

$$J_{min}(k, \theta) = \min_{\boldsymbol{a}} J(k, \theta, \boldsymbol{a}). \tag{4.2}$$

Note that R_1 and $\mathrm{D}U_0$ only appear in the form of $R_1 \mathrm{D}U_0$ in (2.36)–(2.41), provided that $Pr\, \mathrm{D}\tilde{T}_0 = 0$. The eigenvector $\hat{\boldsymbol{u}}_1$ therefore does not change when $\mathrm{D}U_0$ is multiplied

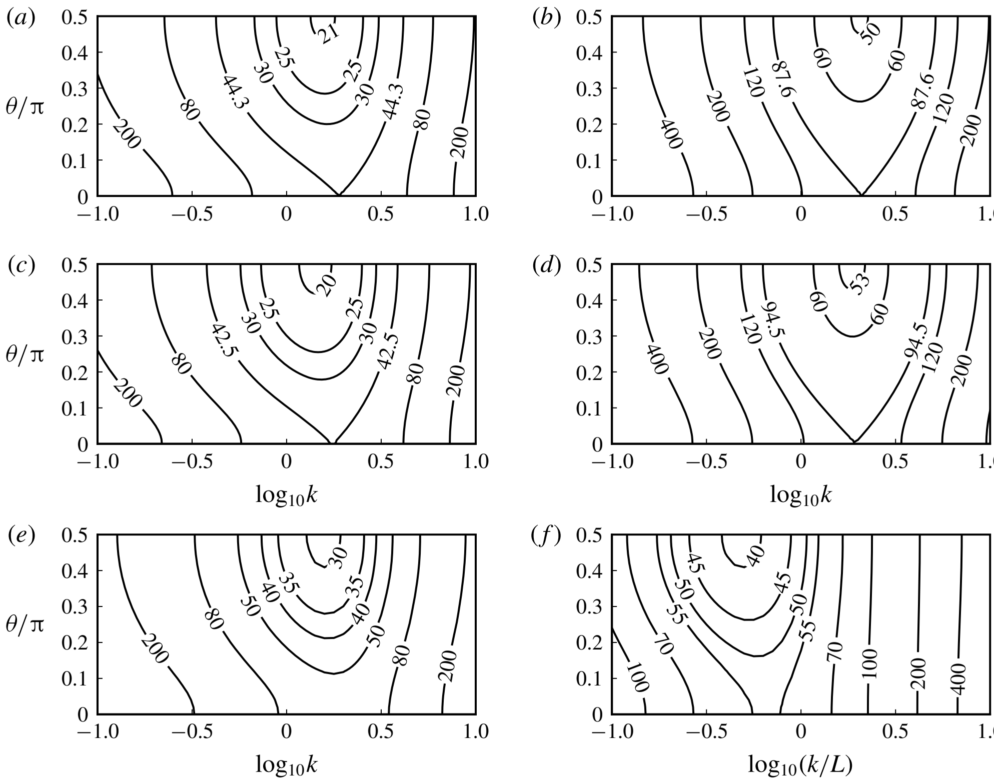

FIGURE 3. The contours of the first positive eigenvalue R_1 for the energy stability of various base flows. (*a*) The plane Couette flow with the no-slip boundary condition ($DU_0 = 1$, $Pr\,D\tilde{T}_0 = 0$ and $l = 0$) (Reddy & Henningson 1993). (*b*) The plane Poiseuille flow with the no-slip boundary condition ($DU_0 = -2y$, $Pr\,D\tilde{T}_0 = 0$ and $l = 0$) (Reddy & Henningson 1993). (*c*) The plane Couette flow with the slip boundary condition ($DU_0 = (1 + l)^{-1}$, $Pr\,D\tilde{T}_0 = 0$ and $l = 0.1$). (*d*) The plane Poiseuille flow with the slip boundary condition ($DU_0 = -2(1 + 3l)^{-1}y$, $Pr\,D\tilde{T}_0 = 0$ and $l = 0.1$). (*e*) The shear flow with a cubic velocity profile ($DU_0 = 1 - 3y^2$, $Pr\,D\tilde{T}_0 = 0$ and $l = 0$) (Sagalakov & Shtern 1971). (*f*) The inclined buoyancy layer ($DU_0 = -0.5L^2 e^{-y'}(\sin y' - \cos y')$, $Pr = 0.72$, $D\tilde{T}_0 = -0.5L^2 e^{-y'}(\sin y' + \cos y')$, $\lambda = 1$ and $l = l_T = 0$, where $y' = L(y + 1)$ and $L = 20$) (Xiong & Tao 2017).

by a non-zero real constant, and then $J(k, \theta, C\boldsymbol{a}) = J(k, \theta, \boldsymbol{a})$ for any real constant $C \neq 0$.

We use two methods to approximate J_{min} at (k_j, θ_s), where $k_j = 10^{-1+(j-1)/5}$ $(1 \leqslant j \leqslant 11)$ and $\theta_s = (s/20)\pi$ $(1 \leqslant s \leqslant 9)$. In the following calculations, 65 Chebyshev–Gauss–Lobatto collocation points are used. In the first calculation, for given wavenumber pair (k_j, θ_s), J_{min} is approximated by the minimum of $J(k_j, \theta_s, \boldsymbol{a})$ among all $\boldsymbol{a}$ such that $|a_m| \leqslant 5$ $(0 \leqslant m \leqslant 5)$ are integers and $a_m = 0$ for $m \geqslant 6$ (figure 5*a*). In the second calculation, J_{min} is approximated with a gradient descent method among all $\boldsymbol{a}$ such that a_m $(0 \leqslant m \leqslant 15)$ are real numbers and $a_m = 0$ for $m \geqslant 16$. The minimum found with the gradient descent method is generally a local minimum, and depends on the initial value of the coefficient $\boldsymbol{a}$. The result shown in figure 5(*b*) is the minimum of J calculated from 100 sets of initial $\boldsymbol{a}$ that are

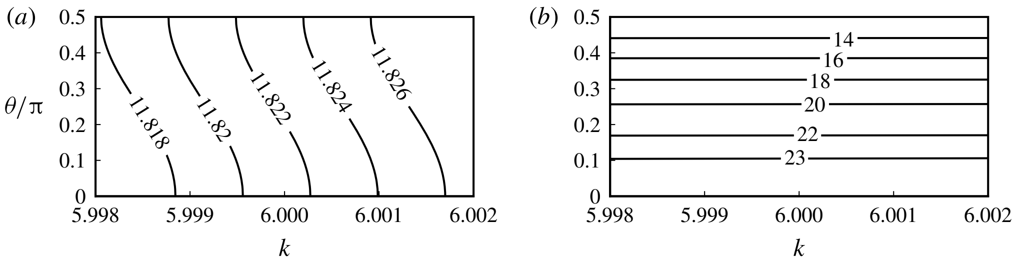

FIGURE 4. The contours of R_1 (*a*) and $(1+\cos\theta)R_1$ (*b*) for the energy stability of the mixed convection with $DU_0=\cos(\pi y)+\cos(3\pi y)$, $Pr\, D\tilde{T}_0=10\sin(\pi y)$, $\lambda=1$ and $l=l_T=0$.

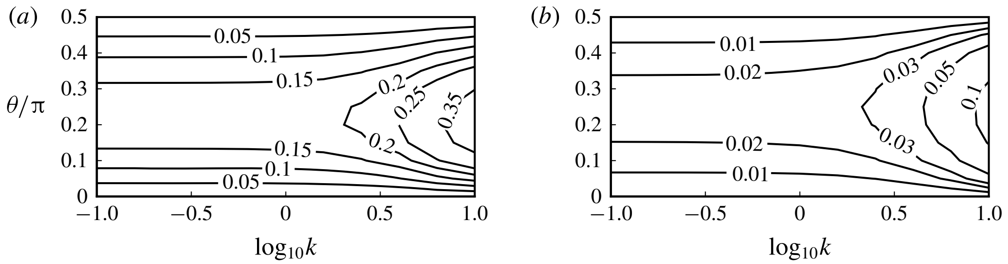

FIGURE 5. The contours of J_{min} in the wavenumber plane approximated with two methods: (*a*) J_{min} is approximated by the minimum of J for all $DU_0=\sum_{0\leqslant m\leqslant 5} a_m T_m(y)$, where T_m is the Chebyshev polynomial of degree m, and $|a_m|\leqslant 5$ $(0\leqslant m\leqslant 5)$ are integers; (*b*) J_{min} is approximated by the local minimum of J among all $DU_0=\sum_{0\leqslant m\leqslant 15} a_m T_m(y)$ with a gradient descent method.

randomly chosen. The detail of the gradient descent algorithm will be introduced in appendix B. In both calculations, we did not find $J<0$ for any base flow at any (k_j,θ_s) $(1\leqslant j\leqslant 11$ and $1\leqslant s\leqslant 9)$, which means R_1 is probably a decreasing function of θ for the base flows that we have examined. Note that $J=0$ when $\theta=0$ or $\theta=\pi/2$, according to the proof of theorem 2.

5. Conjecture

In the parallel shear flows without variations in temperature, we have $DT_0=0$ and $Gr=0$, and then $D\tilde{T}_0=0$ due to (2.24). Under this circumstance, Pr, λ and l_T have no influence on the eigenvalue equation (2.36)–(2.41). According to the numerical results in § 4, we propose the following conjecture:

CONJECTURE 1. *For any continuously differentiable real function* DU_0, *if* $D\tilde{T}_0=0$ *and* $l=0$, *then the first positive eigenvalue of* (2.36)–(2.41) $R_1(k,\theta)$ *is a decreasing function of* θ *when* $0\leqslant\theta\leqslant\pi/2$ *for any* $k>0$.

If conjecture 1 is correct, the least stable mode for the energy stability of any parallel shear flow between no-slip walls without variations in temperature will be a streamwise vortex, because

$$Re_E=\min_{k>0} R_1\left(k,\frac{\pi}{2}\right) \tag{5.1}$$

follows from (2.42), and $\theta=\pi/2$ implies the streamwise wavenumber $\alpha=k\cos\theta=0$.

Conjecture 1 implies that

$$\frac{\partial R_1}{\partial \theta} \leqslant 0 \tag{5.2}$$

in each subset $S \subset (0, +\infty) \times [0, \pi/2]$ where the geometric multiplicity of the first eigenvalue $R_1(k, \theta)$ is 1.

According to (3.9), (3.11) and (3.17), conjecture 1 is equivalent to the following conjecture:

CONJECTURE 2 (A conjecture equivalent to conjecture 1). For any continuously differentiable real function DU_0, if $D\tilde{T}_0 = 0$, $l = 0$, $k > 0$ and $0 \leqslant \theta \leqslant \pi/2$, then the eigenvector corresponding to the first positive eigenvalue of (2.36)–(2.41) satisfies

$$-\int_{-1}^{1} (\bar{\hat{q}}_1 \hat{\eta}_1 + \hat{q}_1 \bar{\hat{\eta}}_1)\, dy \geqslant 0. \tag{5.3}$$

Now the monotonicity of the eigenvalue in conjecture 1 has been transformed to a more tractable inequality of the eigenvector in conjecture 2, where more mathematical tools can be used in the proof, such as the lemmas used in the proof of theorem 1.

Using (3.15) instead of (3.11), we have another condition which is equivalent to (5.3),

$$\frac{R_1}{2} \int_{-1}^{1} (DU_0)(\bar{\hat{v}}_1 \hat{w}_1 + \hat{v}_1 \bar{\hat{w}}_1)\, dy \geqslant 0. \tag{5.4}$$

When $D\tilde{T}_0 = 0$, we have $\hat{T}_1 = 0$ from (2.39) and (2.41), and then $I_T = 0$. The condition (5.3) is therefore also equivalent to

$$(1 + \cos^2 \theta) I_\eta - (\sin^2 \theta) I_v \geqslant 0 \tag{5.5}$$

after using (3.34) instead of (3.11).

6. Conclusion

In this paper, we explore the monotonicity of the positive eigenvalues of the eigenvalue equation for the energy stability of plane parallel flows in the (k, θ) plane, where $k = (\alpha^2 + \beta^2)^{1/2}$ and $\theta = \arctan(\beta/\alpha)$, and α and β are the streamwise and spanwise wavenumbers of the normal mode. We prove that the nth positive eigenvalue R_n decreases with increasing k when $0 < k < k_0(l, l_T)$, and kR_n increases with k when $k > 0$. We also prove that $(\cos\theta \pm 1)R_n$ are decreasing functions of θ when $0 \leqslant \theta \leqslant \pi/2$. The above results apply to all parallel base flows between no-slip or slip parallel plates with or without variations in temperature, including the plane Couette flow, the plane Poiseuille flow and the inclined buoyancy layer. When there is temperature variation in the base flow, the difference between T_0 and $\tilde{T}_0$ should be noted, i.e. (2.24). If T_0 instead of $\tilde{T}_0$ is given, the above results hold for any given $Gr/(\lambda Pr Re^2)$.

Our theorems are also illustrated with the computations of the energy stability of various basic flows when the coupling parameter $\lambda = 1$. The first positive eigenvalue R_1 is found to decrease with increasing θ in the computations for many parallel shear flows under the no-slip boundary condition and without variations in temperature. Therefore, we conjecture that R_1 is a decreasing function of θ for all parallel shear flows between no-slip walls without variations in temperature. We also propose an equivalent conjecture to relate the monotonicity of the eigenvalue to an inequality of the eigenvector (5.3), and derive two equivalent expressions of (5.3), which are (5.4) and (5.5).

If our conjecture is correct, then the least stable mode for the energy stability of any parallel shear flow under the no-slip boundary condition and without variations in temperature must be a streamwise vortex. The importance of this conjecture to the energy stability theory is similar to the importance of Squire's theorem to the linear stability theory. Actually, Squire's theorem implies that the neutral stable Reynolds number for the linear stability always increases with θ for given $k>0$; our conjecture anticipates the opposite for the energy stability.

In the special case with $\mathrm{D}\tilde{T}_0=\mathrm{D}U_0$, $l=l_T\geqslant 0$ and $\lambda=1$, we prove that $(1+Pr^2)^{1/2}R_n(k,\theta,Pr)$ only depends on k and $(1+Pr^2)^{-1/2}\cos\theta$ in appendix A, and explain why the proof by Joseph (1966) is not completed to prove that the least stable mode for the energy stability of the plane Couette flow is a streamwise vortex.

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Appendix A. A special case with $\mathrm{D}\tilde{T}_0=\mathrm{D}U_0$, $l=l_T\geqslant 0$ and $\lambda=1$

THEOREM 3. *For any continuously differentiable real function* $\mathrm{D}U_0$ *and any* $l=l_T\geqslant 0$, *if* $\mathrm{D}\tilde{T}_0=\mathrm{D}U_0$ *and* $\lambda=1$, *then there exist functions* $\{f_n\}_{n\geqslant 1}$ *of* k *and* $(1+Pr^2)^{-1/2}\cos\theta$ *such that*

$$R_n(k,\theta,Pr)=(1+Pr^2)^{-1/2}f_n(k,(1+Pr^2)^{-1/2}\cos\theta), \tag{A 1}$$

where R_n *is the nth positive eigenvalue of* (2.36)–(2.41) $(n\geqslant 1)$.

To prove theorem 3, we rewrite the eigenvalue equation (2.36)–(2.41) by introducing $\phi=\arctan(Pr)$ and $R'_n(k,\theta,\phi)=(1+Pr^2)^{1/2}R_n(k,\theta,Pr)$.

When $\mathrm{D}\tilde{T}_0=\mathrm{D}U_0$, $l=l_T\geqslant 0$ and $\lambda=1$, we have

$$\boldsymbol{L}'(k,\theta,\phi,R'_n)\hat{\boldsymbol{u}}_n=\boldsymbol{0}, \tag{A 2}$$

$$(\hat{u}_n\pm l\mathrm{D}\hat{u}_n)(\pm 1)=\hat{v}_n(\pm 1)=(\hat{w}_n\pm l\mathrm{D}\hat{w}_n)(\pm 1)=(\hat{T}_n\pm l\mathrm{D}\hat{T}_n)(\pm 1)=0, \tag{A 3}$$

where

$$\boldsymbol{L}'=\begin{bmatrix} \mathrm{D}^2-k^2 & -\dfrac{\cos\phi}{2}R'_n(\mathrm{D}U_0) & 0 & 0 & -\mathrm{i}k\cos\theta \\ -\dfrac{\cos\phi}{2}R'_n(\mathrm{D}U_0) & \mathrm{D}^2-k^2 & 0 & -\dfrac{\sin\phi}{2}R'_n(\mathrm{D}U_0) & -\mathrm{D} \\ 0 & 0 & \mathrm{D}^2-k^2 & 0 & -\mathrm{i}k\sin\theta \\ 0 & -\dfrac{\sin\phi}{2}R'_n(\mathrm{D}U_0) & 0 & \mathrm{D}^2-k^2 & 0 \\ \mathrm{i}k\cos\theta & \mathrm{D} & \mathrm{i}k\sin\theta & 0 & 0 \end{bmatrix}. \tag{A 4}$$

After a similar discussion as in § 2, we have

$$\left\langle \hat{\boldsymbol{u}}_n,\frac{\partial\boldsymbol{L}'}{\partial k}\hat{\boldsymbol{u}}_n\right\rangle \mathrm{d}k+\left\langle \hat{\boldsymbol{u}}_n,\frac{\partial\boldsymbol{L}'}{\partial\theta}\hat{\boldsymbol{u}}_n\right\rangle \mathrm{d}\theta+\left\langle \hat{\boldsymbol{u}}_n,\frac{\partial\boldsymbol{L}'}{\partial\phi}\hat{\boldsymbol{u}}_n\right\rangle \mathrm{d}\phi+\left\langle \hat{\boldsymbol{u}}_n,\frac{\partial\boldsymbol{L}'}{\partial R'_n}\hat{\boldsymbol{u}}_n\right\rangle \mathrm{d}R'_n=0, \tag{A 5}$$

and then

$$\frac{\partial R_n'}{\partial \theta} = -\frac{\left\langle \hat{\boldsymbol{u}}_n, \dfrac{\partial \boldsymbol{L}'}{\partial \theta}\hat{\boldsymbol{u}}_n \right\rangle}{\left\langle \hat{\boldsymbol{u}}_n, \dfrac{\partial \boldsymbol{L}'}{\partial R_n'}\hat{\boldsymbol{u}}_n \right\rangle}, \tag{A 6}$$

$$\frac{\partial R_n'}{\partial \phi} = -\frac{\left\langle \hat{\boldsymbol{u}}_n, \dfrac{\partial \boldsymbol{L}'}{\partial \phi}\hat{\boldsymbol{u}}_n \right\rangle}{\left\langle \hat{\boldsymbol{u}}_n, \dfrac{\partial \boldsymbol{L}'}{\partial R_n'}\hat{\boldsymbol{u}}_n \right\rangle}, \tag{A 7}$$

where

$$\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}'}{\partial \theta}\hat{\boldsymbol{u}}_n \right\rangle = -\int_{-1}^{1} (\bar{\hat{q}}_n\hat{\eta}_n + \hat{q}_n\bar{\hat{\eta}}_n)\,\mathrm{d}y = \frac{R_n' \cos\phi}{2}\int_{-1}^{1} (\mathrm{D}U_0)(\bar{\hat{v}}_n\hat{w}_n + \hat{v}_n\bar{\hat{w}}_n)\,\mathrm{d}y, \tag{A 8}$$

$$\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}'}{\partial \phi}\hat{\boldsymbol{u}}_n \right\rangle = \frac{R_n'}{2}\int_{-1}^{1} (\mathrm{D}U_0)[(\sin\phi)(\bar{\hat{u}}_n\hat{v}_n + \hat{u}_n\bar{\hat{v}}_n) - (\cos\phi)(\bar{\hat{v}}_n\hat{T}_n + \hat{v}_n\bar{\hat{T}}_n)]\,\mathrm{d}y. \tag{A 9}$$

From (A 2)–(A 3), we have

$$(\mathrm{D}^2 - k^2)\hat{g}_n = 0, \tag{A 10}$$

$$(\hat{g}_n \pm l\mathrm{D}\hat{g}_n)(\pm 1) = 0, \tag{A 11}$$

where $\hat{g}_n = (\sin\theta \sin\phi)\hat{u}_n - (\cos\theta \sin\phi)\hat{w}_n - (\sin\theta\cos\phi)\hat{T}_n$. Then we have $\hat{g}_n = 0$, which leads to

$$(\cos\theta \sin\phi)\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}'}{\partial \theta}\hat{\boldsymbol{u}}_n \right\rangle = (\sin\theta\cos\phi)\left\langle \hat{\boldsymbol{u}}_n, \frac{\partial \boldsymbol{L}'}{\partial \phi}\hat{\boldsymbol{u}}_n \right\rangle, \tag{A 12}$$

because of (A 8) and (A 9), and then

$$(\cos\theta\sin\phi)\frac{\partial R_n'}{\partial\theta} = (\sin\theta\cos\phi)\frac{\partial R_n'}{\partial\phi}, \tag{A 13}$$

where we have used (A 6), (A 7) and (A 12).

For given $k > 0$, the tangent direction of the contour of R_n' at any point (θ_0, ϕ_0) in the (θ, ϕ) plane is therefore parallel to $(\cos\theta_0 \sin\phi_0, -\sin\theta_0\cos\phi_0)$, which is just the tangent direction of the contour of $\cos\theta\cos\phi$ at the same point. Consequently, $R_n'(k, \theta, \phi)$ only depends on k and $\cos\theta\cos\phi$, which means that $(1 + Pr^2)^{1/2}R_n(k, \theta, Pr)$ only depends on k and $(1 + Pr^2)^{-1/2}\cos\theta$. Theorem 3 is therefore proved.

According to the proof of theorem 3, we have

$$R_1'\left(k, \frac{\pi}{2}, \phi\right) = R_1'\left(k, \frac{\pi}{2}, \frac{\pi}{2}\right) = R_1'\left(k, \theta, \frac{\pi}{2}\right), \tag{A 14}$$

for any $0 \leqslant \theta \leqslant \pi/2$ and $0 \leqslant \phi \leqslant \pi/2$. Note that there is no singularity in (A 2)–(A 3) when $\phi = \pi/2$ $(Pr \to +\infty)$.

Theorem 3 is an extension of the work by Joseph (1966), who studied the case with $\mathrm{D}U_0 = \mathrm{D}\tilde{T}_0 = 1$ and $l = l_T = 0$. He implicitly used an expression similar to (A 13)

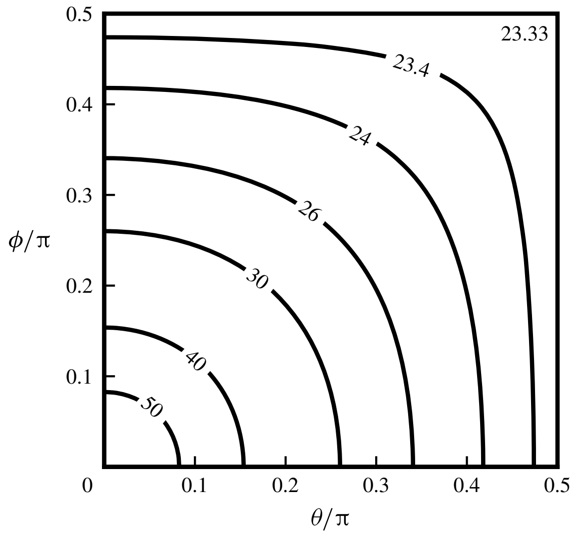

FIGURE 6. The contours of R_1' for $DU_0 = D\tilde{T}_0 = 1$, $k = 1$, $\lambda = 1$ and $l = l_T = 0$. The boundaries $\theta = \pi/2$ and $\phi = \pi/2$ constitute the contour $R_1' \approx 23.33$.

to declare that R_1' is independent of ϕ along the line $\theta = \theta_0(k, \phi)$, where $\theta_0(k, \phi)$ satisfies $R_1'(k, \theta_0, \phi) = \min_\theta R_1'(k, \theta, \phi)$. $\min_\theta R_1'(k, \theta, \phi)$ is therefore independent of ϕ, and then $\min_\theta R_1'(k, \theta, 0) = \min_\theta R_1'(k, \theta, \pi/2) = R_1'(k, \pi/2, 0)$, where (A 14) has been used. He concluded from $\theta_0(k, 0) = \pi/2$ that the minimum R_1' is achieved by a streamwise vortex in the case of the plane Couette flow ($\phi = 0$).

Although the conclusion of Joseph (1966) has been verified by previous computation (Reddy & Henningson 1993) and our computation (figure 6), there is a gap in his proof: equation (A 13) cannot guarantee $\partial R_1'/\partial\phi = 0$ if $\partial R_1'/\partial\theta = 0$ at $\theta = 0$ (Busse 1972). Actually, theorem 2 and theorem 3 cannot rule out the possibility that R_1' may be $(1 + \cos\theta\cos\phi)^{-1}$ or $1 - \cos\theta\cos\phi + \cos^2\theta\cos^2\phi$, and $\min_\theta R_1'(k, \theta, 0)$ may be achieved at $\theta = 0$ or $0 < \theta < \pi/2$.

Appendix B. The gradient descent algorithm for J_{min}

When $Pr\, D\tilde{T}_0 = 0$, $l = 0$ and $DU_0 = \sum_{0\leqslant m\leqslant M} a_m T_m(y)$, where T_m is the Chebyshev polynomial of degree m, and $\boldsymbol{a} = (a_0, a_1, \ldots, a_M)$ is the coefficient, we have

$$\boldsymbol{L}''(k, \theta, \boldsymbol{a}, R_1)\hat{\boldsymbol{u}}_1 = \boldsymbol{0}, \tag{B 1}$$

$$\hat{u}_1(\pm 1) = \hat{v}_1(\pm 1) = \hat{w}_1(\pm 1) = 0, \tag{B 2}$$

where

$$\boldsymbol{L}'' = \begin{bmatrix} D^2 - k^2 & -\frac{1}{2}R_1(DU_0) & 0 & -ik\cos\theta \\ -\frac{1}{2}R_1(DU_0) & D^2 - k^2 & 0 & -D \\ 0 & 0 & D^2 - k^2 & -ik\sin\theta \\ ik\cos\theta & D & ik\sin\theta & 0 \end{bmatrix}, \quad \hat{\boldsymbol{u}}_1 = \begin{bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \hat{w}_1 \\ \hat{q}_1 \end{bmatrix}, \tag{B 3a,b}$$

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and then

$$\boldsymbol{L}''(\mathrm{d}\hat{\boldsymbol{u}}_1) + (\mathrm{d}R_1)\frac{\partial \boldsymbol{L}''}{\partial R_1}\hat{\boldsymbol{u}}_1 + \sum_{0\leqslant m\leqslant M}(\mathrm{d}a_m)\frac{\partial \boldsymbol{L}''}{\partial a_m}\hat{\boldsymbol{u}}_1 = \boldsymbol{0}, \tag{B 4}$$

$$(\mathrm{d}\hat{u}_1)(\pm 1) = (\mathrm{d}\hat{v}_1)(\pm 1) = (\mathrm{d}\hat{w}_1)(\pm 1) = 0, \tag{B 5}$$

at given (k, θ).

Define the inner product between two vectors $\hat{\boldsymbol{u}}'$ and $\hat{\boldsymbol{u}}''$ as

$$\langle \hat{\boldsymbol{u}}', \hat{\boldsymbol{u}}''\rangle = \int_{-1}^{1} (\bar{\hat{u}}'\hat{u}'' + \bar{\hat{v}}'\hat{v}'' + \bar{\hat{w}}'\hat{w}'' + \bar{\hat{q}}'\hat{q}'')\,\mathrm{d}y. \tag{B 6}$$

It is straightforward to examine that $\boldsymbol{L}''$ is a self-adjoint operator with respect to this inner product and the boundary conditions, and then we have

$$\langle \hat{\boldsymbol{u}}_1, \boldsymbol{L}''(\mathrm{d}\hat{\boldsymbol{u}}_1)\rangle = \langle \boldsymbol{L}''\hat{\boldsymbol{u}}_1, \mathrm{d}\hat{\boldsymbol{u}}_1\rangle = \langle \boldsymbol{0}, \mathrm{d}\hat{\boldsymbol{u}}_1\rangle = 0, \tag{B 7}$$

and then it follows from (B 4) that

$$\left\langle \hat{\boldsymbol{u}}_1, \frac{\partial \boldsymbol{L}''}{\partial R_1}\hat{\boldsymbol{u}}_1\right\rangle (\mathrm{d}R_1) + \sum_{0\leqslant m\leqslant M}\left\langle \hat{\boldsymbol{u}}_1, \frac{\partial \boldsymbol{L}''}{\partial a_m}\hat{\boldsymbol{u}}_1\right\rangle (\mathrm{d}a_m) = 0. \tag{B 8}$$

The inner products in (B 8) are

$$\left\langle \hat{\boldsymbol{u}}_1, \frac{\partial \boldsymbol{L}''}{\partial R_1}\hat{\boldsymbol{u}}_1\right\rangle = -\frac{1}{2}\int_{-1}^{1}(\mathrm{D}U_0)(\bar{\hat{u}}_1\hat{v}_1 + \hat{u}_1\bar{\hat{v}}_1)\,\mathrm{d}y = \frac{1}{k^2R_1}(I_{v1} + I_{\eta 1}) \geqslant 0, \tag{B 9}$$

$$\left\langle \hat{\boldsymbol{u}}_1, \frac{\partial \boldsymbol{L}''}{\partial a_m}\hat{\boldsymbol{u}}_1\right\rangle = -\frac{R_1}{2}\int_{-1}^{1} T_m(\bar{\hat{u}}_1\hat{v}_1 + \hat{u}_1\bar{\hat{v}}_1)\,\mathrm{d}y, \tag{B 10}$$

where

$$I_{v1} = \int_{-1}^{1} |(\mathrm{D}^2 - k^2)\hat{v}_1|^2\,\mathrm{d}y, \tag{B 11}$$

$$I_{\eta 1} = \int_{-1}^{1} (|\mathrm{D}\hat{\eta}_1|^2 + k^2|\hat{\eta}_1|^2)\,\mathrm{d}y, \tag{B 12}$$

and $\hat{\eta}_1 = \mathrm{i}k\hat{u}_1\sin\theta - \mathrm{i}k\hat{w}_1\cos\theta$.

Without loss of generality, we assume

$$-\tfrac{1}{2}\int_{-1}^{1}(\mathrm{D}U_0)(\bar{\hat{u}}_1\hat{v}_1 + \hat{u}_1\bar{\hat{v}}_1)\,\mathrm{d}y = 1. \tag{B 13}$$

For $0 \leqslant m \leqslant M$, from (B 8), (B 9) and (B 13), we have

$$\frac{\partial R_1}{\partial a_m} = -\left\langle \hat{\boldsymbol{u}}_1, \frac{\partial \boldsymbol{L}''}{\partial a_m}\hat{\boldsymbol{u}}_1\right\rangle. \tag{B 14}$$

From (B 4), (B 5) and (B 14), we have

$$\boldsymbol{L}''\frac{\partial \hat{\boldsymbol{u}}_1}{\partial a_m} = \left\langle \hat{\boldsymbol{u}}_1, \frac{\partial \boldsymbol{L}''}{\partial a_m}\hat{\boldsymbol{u}}_1 \right\rangle \frac{\partial \boldsymbol{L}''}{\partial R_1}\hat{\boldsymbol{u}}_1 - \frac{\partial \boldsymbol{L}''}{\partial a_m}\hat{\boldsymbol{u}}_1, \tag{B 15}$$

$$\frac{\partial \hat{u}_1}{\partial a_m}(\pm 1) = \frac{\partial \hat{v}_1}{\partial a_m}(\pm 1) = \frac{\partial \hat{w}_1}{\partial a_m}(\pm 1) = 0. \tag{B 16}$$

Equation (B 13) also requires

$$\int_{-1}^{1}\left[(\mathrm{D}U_0)\left(\frac{\partial \bar{\hat{u}}_1}{\partial a_m}\hat{v}_1 + \bar{\hat{u}}_1\frac{\partial \hat{v}_1}{\partial a_m} + \frac{\partial \hat{u}_1}{\partial a_m}\bar{\hat{v}}_1 + \hat{u}_1\frac{\partial \bar{\hat{v}}_1}{\partial a_m}\right) + T_m(\bar{\hat{u}}_1\hat{v}_1 + \hat{u}_1\bar{\hat{v}}_1)\right]\mathrm{d}y = 0. \tag{B 17}$$

Substituting (B 13) into (4.1), we have

$$J(k, \theta, \boldsymbol{a}) = \tfrac{1}{2}\int_{-1}^{1}(\mathrm{D}U_0)(\bar{\hat{v}}_1\hat{w}_1 + \hat{v}_1\bar{\hat{w}}_1)\,\mathrm{d}y, \tag{B 18}$$

and then

$$\frac{\partial J}{\partial a_m} = \frac{1}{2}\int_{-1}^{1}\left[(\mathrm{D}U_0)\left(\frac{\partial \bar{\hat{v}}_1}{\partial a_m}\hat{w}_1 + \bar{\hat{v}}_1\frac{\partial \hat{w}_1}{\partial a_m} + \frac{\partial \hat{v}_1}{\partial a_m}\bar{\hat{w}}_1 + \hat{v}_1\frac{\partial \bar{\hat{w}}_1}{\partial a_m}\right) + T_m(\bar{\hat{v}}_1\hat{w}_1 + \hat{v}_1\bar{\hat{w}}_1)\right]\mathrm{d}y. \tag{B 19}$$

Noticing that $J(k, \theta, C\boldsymbol{a}) = J(k, \theta, \boldsymbol{a})$ for any real constant $C \neq 0$, instead of searching for the minimum of J, we search the minimum of

$$J'(k, \theta, \boldsymbol{a}) = J(k, \theta, \boldsymbol{a}) + \left(\sum_{0\leqslant m\leqslant M} a_m^2 - 1\right)^2 \tag{B 20}$$

to avoid the potential blow-up of $\|\boldsymbol{a}\|$. Therefore, we have

$$\frac{\partial J'}{\partial a_m} = \frac{\partial J}{\partial a_m} + 4a_m\left(\sum_{0\leqslant m'\leqslant M} a_{m'}^2 - 1\right). \tag{B 21}$$

The minimum of J is calculated with the AdaMax algorithm (Kingma & Ba 2017), which is a gradient-based optimisation algorithm. The detailed algorithm is as follows:

(i) For given wavenumber pair (k, θ), randomly choose initial $\boldsymbol{a}$ such that $\|\boldsymbol{a}\| = 1$. Set $t = 0$, $\boldsymbol{s} = (s_0, s_1, \ldots, s_M) = \boldsymbol{0}$ and $\boldsymbol{r} = (r_0, r_1, \ldots, r_M) = \boldsymbol{0}$.

(ii) Solve the eigenvalue equation (B 1)–(B 2) for the first positive eigenvalue R_1 and the corresponding eigenvector $\hat{\boldsymbol{u}}_1$. Equation (B 13) is satisfied by scaling $\hat{\boldsymbol{u}}_1$.

(iii) Solve (B 15)–(B 16) for a particular solution $\partial\hat{\boldsymbol{u}}_1/\partial a_m$ $(0 \leqslant m \leqslant M)$. Noticing that $(\partial\hat{\boldsymbol{u}}_1/\partial a_m) + C\hat{\boldsymbol{u}}_1$ is also a solution of (B 15)–(B 16) for any complex constant C, we require

$$\int_{-1}^{1}\left[(\mathrm{D}U_0)\left(\bar{\hat{u}}_1\frac{\partial \hat{v}_1}{\partial a_m} + \bar{\hat{v}}_1\frac{\partial \hat{u}_1}{\partial a_m}\right) + T_m(\bar{\hat{u}}_1\hat{v}_1)\right]\mathrm{d}y = 0, \tag{B 22}$$

and then (B 17) holds naturally.

(iv) For $0 \leqslant m \leqslant M$, calculate $\partial J'/\partial a_m$ from (B 19) and (B 21).

(v) Set $t+1$ to be the new t. For $0 \leqslant m \leqslant M$, set $0.9s_m + 0.1\partial J'/\partial a_m$ as the new s_m; set $\max\{0.999r_m, |\partial J'/\partial a_m|\}$ as the new r_m; set $a_m - 0.002s_m/((1-0.9^t)r_m)$ as the new a_m. Repeat (ii)–(v) until $t = 10\,000$ or until J cannot be decreased in the last 1000 steps. The parameters here are recommended by Kingma & Ba (2017).

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