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Beyond Beatty sequences: Complementary lattices

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Abstract. By taking square lattices as a two-dimensional analogue to Beatty sequences, we are motivated to define and explore the notion of complementary lattices. In particular, we present a continuous one-parameter family of complementary lattices. This main result then yields several novel examples of complementary sequences, along with a geometric proof of the fundamental property of Beatty sequences.

1 Introduction

Nearly a century ago, Sam Beatty [1] asked readers to show, given irrational numbers $\alpha, \beta > 1$ satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, that the sequences $\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \dots$ and $\lfloor \beta \rfloor, \lfloor 2\beta \rfloor, \lfloor 3\beta \rfloor, \dots$ are complementary, meaning that together these sequences contain every positive integer exactly once. (Recall that for $x \in \mathbb{R}$, the floor function $\lfloor x \rfloor$ is equal to the unique integer n for which $n \le x < n+1$.) Extensions to and generalizations of this fundamental property of Beatty sequences abound; the definitive bibliography on complementary sequences, for its time, is contained in [9]. A more recent work, referencing connections with Wythoff's game, is found in [4].

In this paper, we find that taking square lattices as two-dimensional analogues of Beatty sequences leads in fruitful directions. (Note that our approach is distinct from that taken in [5].) Section 2 anticipates our main result by highlighting three curiosities: two families of complementary sequences that we believe to be new, along with an unusual type of inequality. We next extend the notion of "complementary" to the setting of square lattices in the plane, enabling us to state and prove Theorem 3.1, modulo a geometric lemma that is postponed until Section 6. The gist of this main result is that, given an irrational number $\theta > 1$, the square lattice defined by the point $(\theta, 1)$ is complementary to its counterpart defined by $(1, \frac{1}{\theta})$, meaning that together these lattices cover each unit square precisely once. Section 4 demonstrates that the curiosities are all immediate corollaries of our main theorem, which is then employed once more in the next section to reveal Beatty's original result camouflaged within the diagram of complementary lattices.

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2 A Trio of Curiosities

We begin by presenting several pleasant, if not intriguing, observations. The first two, at least, appear to be loosely related, and one could believe that the style of argument for all three might have a similar algebraic flavor. What may not be as readily apparent is the fact that all three results (presented in a later section as Propositions 4.1, 4.2, and 4.3) follow more or less directly from a single underlying diagram involving a pair of complementary square lattices.

Curiosity 1 *For a positive irrational number $\theta > 1$, consider the sequences*

$$a_n = n + \lfloor \lfloor n\theta \rfloor \theta \rfloor, \quad b_n = n + \lfloor \lfloor n\tfrac{1}{\theta} + 1 \rfloor \tfrac{1}{\theta} \rfloor, \quad n \in \mathbb{N}.$$

These formulas are reminiscent of those given by the Lambek–Moser Theorem [7], which provides a universal method for generating pairs of complementary sequences, that is, disjoint sequences whose union is precisely the set of all natural numbers. Selecting $\theta = \frac{\pi}{2}$ to confirm, we find to our satisfaction that

$$\{a_n\} = \{2, 6, 9, 13, 15, 20, 22, 26, 30, 33, 37, 40, 44, 46, 51, \dots\},$$

$$\{b_n\} = \{1, 3, 4, 5, 7, 8, 10, 11, 12, 14, 16, 17, 18, 19, 21, 23, \dots\}$$

do indeed appear to be complementary sequences.[1] *The nested floor functions result in less regularity than in their more well-known relatives, the Beatty sequences [1], which can be written in the form $a_n = \lfloor n(1+\theta) \rfloor$ and $b_n = \lfloor n(1 + \frac{1}{\theta}) \rfloor$. The gaps between consecutive terms in the first sequence are either 2, 3, 4, or 5, whereas a Beatty sequence only ever features two gap sizes. In general, one can show that the first sequence can have up to four distinct gap sizes, with the maximum of four only attained when $\theta > 1.5$. We also note that the case $\theta = \frac{1}{2}(1 + \sqrt{5})$ plays a prominent role in the Lucas partitions discussed in [3].*

Curiosity 2 *As before, let $\theta > 1$ be a fixed positive irrational number, but this time consider positive integer solutions to*

$$\lfloor a + b\theta \rfloor = \lfloor a\theta - b \rfloor, \quad a, b \in \mathbb{N}.$$

Let S_1 be the set of all values of $\lfloor a + b\theta \rfloor$ (or equivalently, values of $\lfloor a\theta - b \rfloor$) that occur for solution pairs (a, b). Next, create set S_2 in the same manner, using common values arising from positive integer solutions to

$$\lfloor a\tfrac{1}{\theta} + b \rfloor = \lfloor a - b\tfrac{1}{\theta} \rfloor, \quad a, b \in \mathbb{N}.$$

Observe that the second equation is obtained from the first by dividing the expressions inside the floor functions by θ. While one might predict that many of the same solution pairs will surface (which they do), the effect on sets S_1 and S_2 is dramatic, as the value $\theta = \sqrt{e}$ demonstrates:

[1] In the language of Lambek–Moser, this means $f^*(n) = \lfloor \lfloor n\frac{1}{\theta} + 1 \rfloor \frac{1}{\theta} \rfloor$ is the inverse function to $f(n) = \lfloor \lfloor n\theta \rfloor \theta \rfloor$, although it is not obvious from the formulas why this should be so.

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$$S_1 = \{1, 5, 11, 12, 15, 16, 22, 29, 33, 39, 40, 46, 50, 57, 63, 64, \ldots\},$$
$$S_2 = \{2, 3, 4, 6, 7, 8, 9, 10, 13, 14, 17, 18, 19, 20, 21, 23, 24, 25, \ldots\}.$$

Once again, the sets are perfectly complementary, but this time the terms within the sets appear to be even more erratic than before.

Curiosity 3 *Lastly, we turn our attention to a certain collection of four closed intervals. Let a, b, c, d, m, and n be positive integers with $m < c - 1$, $n > a$, and $n > d$ (so that all expressions below are positive), and define*

$$I_1 = \left[\frac{m+b}{a}, \frac{m+b+1}{a}\right], \qquad I_2 = \left[\frac{n-a}{b}, \frac{n-a+1}{b}\right],$$
$$I_3 = \left[\frac{d}{c-m}, \frac{d}{c-m-1}\right], \qquad I_4 = \left[\frac{c}{n-d+1}, \frac{c}{n-d}\right].$$

We claim that the intersection of such a collection of intervals will either be the empty set or, at most, a single point. For instance, taking $a = 6$, $b = 5$, $c = 11$, $d = 10$, $m = 5$, $n = 15$ yields the intervals

$$I_1 = \left[\tfrac{10}{6}, \tfrac{11}{6}\right], \qquad I_2 = \left[\tfrac{9}{5}, \tfrac{10}{5}\right], \qquad I_3 = \left[\tfrac{10}{6}, \tfrac{10}{5}\right], \qquad I_4 = \left[\tfrac{11}{6}, \tfrac{11}{5}\right],$$

whose intersection is the single point $\{\frac{11}{6}\}$. (Such examples require some care to construct; typically, the intersection of all four intervals is empty.) This property does not seem particularly noteworthy until it is revealed that the lack of overlap evaporates once we permit real numbers as opposed to integers. In fact, non-empty intersection appears to be the rule rather than the exception when we use small real numbers. Counterexamples persist even for larger real values; for instance, taking $a = 4.9$, $b = 4.2$, $c = 8.6$, $d = 7.3$, $m = 4.1$, and $n = 11.7$ results in an intersection of $I_1 \cap I_2 \cap I_3 \cap I_4 = [1.694, 1.857]$, rounded to four significant digits. The reader is invited to supply an algebraic proof of the above claim; the author found this task to be unexpectedly slippery.

3 Unit Square Coverage by Lattices

Recall that the integer lattice in $\mathbb{R}^2$ consists of all points both of whose coordinates are integers. More generally, given a point (x, y) in the Cartesian plane, we say that the square lattice based at (x, y) is the set of points

$$\mathcal{L}_{x,y} = \{(ax - by, ay + bx) \mid a, b \in \mathbb{Z}\}.$$

Geometrically, it is the image of the integer lattice under a rotation and dilation centered at the origin, mapping $(1, 0)$ to (x, y).

The integer lattice demarcates the *unit squares,* which for our purposes, will refer only to squares of side length 1 whose vertices have integer coordinates. Given a particular unit square and a set S of points in the plane, the coverage of that unit square by S is computed by adding 1 for each point of S in its interior, adding $\frac{1}{2}$ for each point along one of its edges, and adding $\frac{1}{4}$ for each point at a vertex. In this manner, each point of S contributes a total of 1 towards the coverage of the various unit squares, in a balanced fashion.

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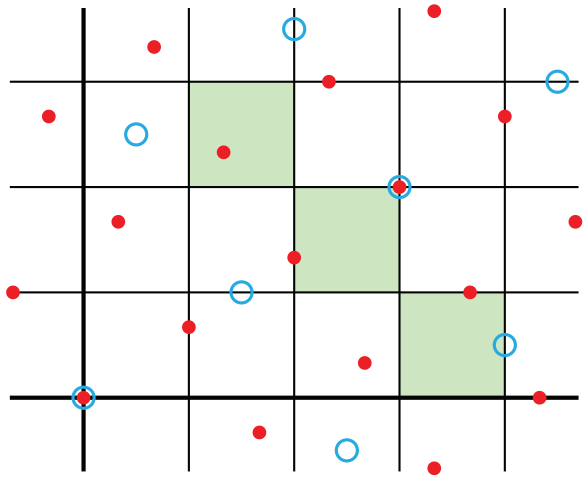

Figure 1: The pair of lattices $\mathcal{L}_{1,2/3}$ and $\mathcal{L}_{3/2,1}$ cover the unit squares.

Figure 1 illustrates two square lattices: the first is based at $(1, \frac{2}{3})$, denote d $\mathcal{L}_{1,2/3}$ and marked by solid dots; the other is $\mathcal{L}_{3/2,1}$, based at $(\frac{3}{2}, 1)$ and marked by small circles. The three shaded squares each have a total coverage of 1; from top-left to bottom-right this is because of one interior point, one edge point plus two vertex points, and two edge points, respectively. In fact, every unit square in the plane has a coverage of 1 by this pair of lattices—when this occurs, we say that the square lattices are *complementary*. Our main result states that there is a continuous family of complementary lattices of this sort.

Theorem 3.1 *Let $\theta > 1$ be a positive irrational number. Then the lattices $\mathcal{L}_{\theta,1}$ and $\mathcal{L}_{1,1/\theta}$ are complementary.*

This theorem provides a natural generalization of Beatty sequences to a planar setting, where we interpret complementary Beatty sequences as one-dimensional lattices (the integer multiples of α and β), in which every unit interval $[n, n+1]$ for $n \in \mathbb{N}$ has a coverage of 1. As we shall see in Theorem 5.1, the Beatty sequences for $\alpha = 1 + \theta$ and $\beta = 1 + \frac{1}{\theta}$ reside in a neat manner within the diagram containing our pair of square lattices, and spotting them in this context motivates a pleasing geometric proof of their complementary nature.

Proof Due to four-fold rotational symmetry, we need only show that unit squares in the first quadrant have total coverage of 1 by our square lattices. Because the line $x = \theta y$, displayed as a dashed line in Figure 2, plays an important role in the proof by containing the centers of the various rotations we will need, we refer to it as the *pivotal line*. The integer multiples of the lattice base points, namely, $(m\theta, m)$ and $(m, \frac{m}{\theta})$ for $m \in \mathbb{Z}$, all lie on the pivotal line. These points are indicated by small circles and solid dots, respectively, and are situated where the pivotal line crosses horizontal and

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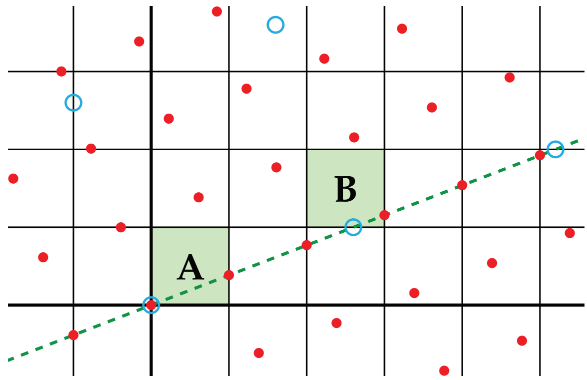

Figure 2: The pivotal line passing through unit squares for $\theta \approx 2.6$.

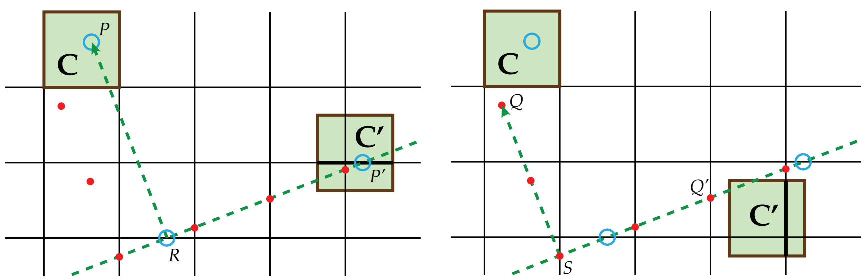

Figure 3: Rotations of a unit square about lattice points on the pivotal line.

vertical grid lines. Because θ is irrational, none of them coincides with an integer lattice point, nor with one another, except at the origin.

There are three types of unit squares in the first quadrant to consider. The first is the single unit square labeled **A** in Figure 2. It has coverage 1, since the points in our lattices land on a vertex (the origin) twice, and along an edge once. Next we consider the remaining unit squares crossed by the pivotal line, such as the square labeled **B**. This also has coverage 1, due to the two lattice points situated where the pivotal line intersects the boundary. Observe that the solid dots are on vertical sides, so the nearest other solid dots, which are 1 unit above or below, are too far away to contribute to coverage. Similarly, the nearest other small circles are $\theta > 1$ units away vertically, so also cannot land within this unit square.

It remains to analyze the unit squares disjoint from the pivotal line, such as the one labeled **C** in Figure 3. Since lattice points off the pivotal line have irrational coordinates, they must land in the interior of (as opposed to on the edge of) some unit square. We wish to show that **C** contains precisely one point in its interior from among the points in the two square lattices combined.

We adopt a geometric approach. Recall that each square lattice is obtained by rotating and dilating the integer lattice so that the image of the x-axis is the pivotal

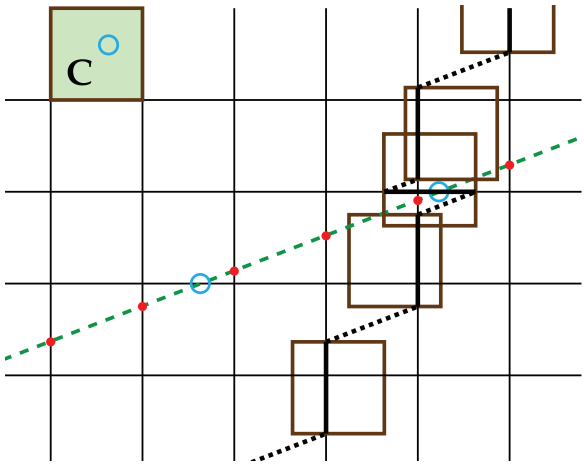

Figure 4: The pivotal line intersects the set of enclosed segments exactly once.

line. It follows that given a point P in the lattice marked by small circles, the foot of the perpendicular from P to the pivotal line is itself a lattice point R, as depicted in Figure 3. Hence, P is located within the unit square **C** if and only if the image **C**′ of that unit square under a 90° clockwise rotation about R encloses one of the small circles along the pivotal line, such as P'. Since these occur where the pivotal line cross horizontal grid lines, it is equivalent to check whether the horizontal segment enclosed by **C**′, highlighted in bold in Figure 3, intersects the pivotal line.

In the same manner, a point Q in the other lattice is exterior to unit square **C** if and only if the image square **C**′ under a 90° rotation about S fails to enclose lattice point Q' where the pivotal line meets the corresponding vertical grid line. This is apparent on the right in Figure 3; the vertical segment enclosed by **C**′ (highlighted again in bold) is situated entirely below the pivotal line. Since **C**′ does not enclose a solid dot on the pivotal line, neither does **C** contain point Q.

These observations prompt the following strategy. Suppose we are given a unit square **C** in the first quadrant that does not intersect the pivotal line. For each lattice point R on the pivotal line of the form $(m\theta, m)$ with $m \in \mathbb{N}$, find the image of **C** under a 90° clockwise rotation about R and note the horizontal segment enclosed by the image, marked in boldface in Figure 4. Do the same for each lattice point S of the form $(m, \frac{m}{\theta})$, except instead note the vertical segment enclosed by the image. According to the above discussion, the total number of lattice points of either type within **C** is equal to the number of times the pivotal line intersects this collection of horizontal and vertical segments.

As Lemma 6.1 will show, the pivotal line intersects this collection of segments exactly once, but the reason is already visually apparent in Figure 4. The dotted segments connecting the endpoints of the boldface segments turn out to be parallel to the pivotal line, so the boldface segments effectively present a wall with no gaps or overlap in the direction of the pivotal line. Moreover, the pivotal line cannot coincide

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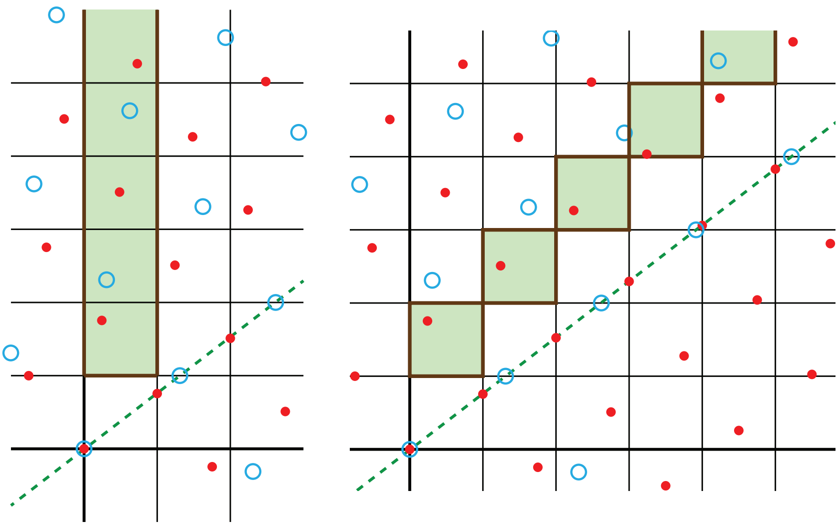

Figure 5: Subsets of unit squares producing complementary sequences.

with any dotted segment, for this would imply that some lattice point was situated along an edge of **C**. Therefore, the pivotal line must cross the collection of segments in a single location, which completes the proof. ■

The theorem actually holds for every nonzero irrational number θ, because the pairs of square lattices in these cases either match those for $\theta > 1$ (when $-1 < \theta < 0$) or can be obtained from them via a reflection over the line $y = x$ (when $\theta < -1$ or $0 < \theta < 1$). Therefore, we only needed to treat the case $\theta > 1$.

The sequence of points in the first quadrant where the pivotal line intersects horizontal and vertical grid lines will continue to play a significant role later in the proofs of Theorem 5.1 and Lemma 6.1. Consequently, the complementary lattices presented here are closely tied to cutting sequences and Sturmian words, introduced in [8] and surveyed more recently in [2].

4 The Curiosities Explained

By selecting strategic subsets of unit squares, one can obtain a variety of results involving complementary sequences. For example, considering the column of unit squares shaded in Figure 5 leads to Curiosity 1 presented earlier.

Theorem 4.1 *Let $\theta > 1$ be irrational and define*

$$a_n = n + \big\lfloor \lfloor n\theta \rfloor \theta \big\rfloor, \quad b_n = n + \big\lfloor \lfloor n\tfrac{1}{\theta} + 1 \rfloor \tfrac{1}{\theta} \big\rfloor, \quad n \in \mathbb{N}.$$

Then the sequences $\{a_n\}$ and $\{b_n\}$ are complementary.

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Proof We identify all points in the lattices $\mathcal{L}_{\theta,1}$ and $\mathcal{L}_{1,1/\theta}$ falling within the shaded region in Figure 5, defined by $0 \le x \le 1$ and $y \ge 1$. Recall that a point P in the first lattice has the form $(m\theta - n, m + n\theta)$. Clearly, we must have $m, n \ge 1$ in order for P to be within our region. The x-coordinate then gives $0 \le m\theta - n \le 1$, or equivalently, $n = \lfloor m\theta \rfloor$, since m and n are positive. The particular unit square containing P will rest on the horizontal grid line

$$y = \lfloor m + n\theta \rfloor = m + \lfloor n\theta \rfloor = m + \lfloor \lfloor m\theta \rfloor \theta \rfloor, \quad m \in \mathbb{N}.$$

The same reasoning shows that point Q in the second lattice at $(m - n\frac{1}{\theta}, m\frac{1}{\theta} + n)$ falls within our region whenever $0 \le m - n\frac{1}{\theta} \le 1$, or $m - 1 \le n\frac{1}{\theta} \le m$, yielding $m = \lfloor n\frac{1}{\theta} \rfloor + 1$. This time the unit square containing Q is at height

$$y = \left\lfloor m\tfrac{1}{\theta} + n \right\rfloor = n + \left\lfloor m\tfrac{1}{\theta} \right\rfloor = n + \left\lfloor \lfloor n\tfrac{1}{\theta} + 1 \rfloor \tfrac{1}{\theta} \right\rfloor, \quad n \in \mathbb{N}.$$

According to Theorem 3.1, we know that each unit square in the shaded region contains exactly one lattice point. It follows that the two sequences, which are the y-values indexing these unit squares, must be complementary. ■

One can create variations on this theme by translating the shaded region or allowing it to extend downwards as well as upwards. However, anomalies arise due to the existence of lattice points on the edges of unit squares where the lines $x = \theta y$ and $y = -\theta x$ cross our region. For instance, when $\theta = \sqrt[3]{6}$, the sequences

$$a_n = n + \left\lfloor (\lfloor n\theta \rfloor - 7)\theta \right\rfloor, \quad b_n = n + \left\lfloor (\lfloor n\tfrac{1}{\theta} \rfloor + 8)\tfrac{1}{\theta} \right\rfloor, \quad n \in \mathbb{Z}$$

nearly partition the integers, except 4 and -13 each appear in both sequences, while 3 and -15 do not appear in either of them.

By considering a subset of unit squares arranged along a diagonal, we arrive at a different type of result.

Theorem 4.2 *Given $\theta > 1$ irrational, define set S_1 to consist of all positive integers n for which there exist $a, b \in \mathbb{N}$ with $\lfloor a + b\theta \rfloor = \lfloor a\theta - b \rfloor = n$. Similarly, define S_2 to contain those positive integers n for which there exist $a, b \in \mathbb{N}$ satisfying $\left\lfloor a\frac{1}{\theta} + b \right\rfloor = \left\lfloor a - b\frac{1}{\theta} \right\rfloor = n$. Then S_1 and S_2 are complementary sequences.*

Proof Given a positive integer n, consider how the unit square whose lower left vertex is at (n, n) is covered by the lattices $\mathcal{L}_{\theta,1}$ and $\mathcal{L}_{1,1/\theta}$. If a single point from the first lattice lands in its interior, then by construction and Theorem 3.1, n will appear in S_1 but not S_2, and vice-versa. The only other possibility is that the pivotal line passes through this unit square, intersecting the lower and right edges, as pictured on the right in Figure 5. Due to the definition of the floor function, only the lattice point on the lower edge will satisfy the equations, causing this value of n to appear in set S_1 only. ■

As before, variations are possible. For instance, it is not hard to show that if S_1 contains all $n \in \mathbb{Z}$ for which there exist $a, b \in \mathbb{Z}$ satisfying

$$\lfloor a + b\theta \rfloor = \lfloor a\theta - b \rfloor + 7 = n,$$

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and similarly S_2 consists of all $n \in \mathbb{Z}$ for which there exist $a, b \in \mathbb{Z}$ with

$$\lfloor a\tfrac{1}{\theta} + b \rfloor = \lfloor a - b\tfrac{1}{\theta} \rfloor + 7 = n,$$

then, once again, S_1 and S_2 exactly partition $\mathbb{Z}$.

Theorem 4.3 *Let a, b, c, d, m, n be positive integers with $m < c - 1$, $n > a$, $n > d$, and define the closed intervals*

$$I_1 = \Big[\frac{m+b}{a}, \frac{m+b+1}{a}\Big], \qquad I_2 = \Big[\frac{n-a}{b}, \frac{n-a+1}{b}\Big],$$
$$I_3 = \Big[\frac{d}{c-m}, \frac{d}{c-m-1}\Big], \qquad I_4 = \Big[\frac{c}{n-d+1}, \frac{c}{n-d}\Big].$$

Then their intersection will either be the empty set or consist of a single point.

Proof Suppose to the contrary that their intersection were an interval of positive length, and choose an irrational number θ within that interval. Then $\theta \in I_1$ is equivalent to $m < a\theta - b < m + 1$, while $\theta \in I_2$ translates to $n < a + b\theta < n + 1$. In other words, $\theta \in I_1 \cap I_2$ means that a point of $\mathcal{L}_{\theta,1}$ is in the interior of the unit square whose lower left vertex is (m, n). In the same manner, $\theta \in I_3 \cap I_4$ implies that a point of $\mathcal{L}_{1,1/\theta}$ sits inside the same unit square, which produces a contradiction, by Theorem 3.1. ■

An instance in which the intersection of the four intervals is a single point corresponds to a unit square in the first quadrant above the pivotal line that nonetheless has coverage of 1 by complementary lattices due to two edge points, one from each lattice. This situation never occurs for irrational values of θ, but does happen in a periodic fashion throughout the plane when θ is rational.

5 Beatty Sequences and Further Questions

The author was first entranced by Beatty sequences as a student upon reading Honsberger's presentation in [6]. Given that square lattices serve as a suitable two-dimensional analogue of Beatty sequences, it seems fitting that a pair of Beatty sequences appears neatly tucked away within the diagram; not on the pivotal line, but along a secondary line making an angle of 45° with the pivotal line. This manifestation provides a less common, self-contained (in that it does not rely on Theorem 3.1) approach to understanding the Beatty sequence property.

Theorem 5.1 (Beatty Sequences) *Given an irrational number $\theta > 1$, plot the positive integer multiples of $1 + \theta$ and $1 + \frac{1}{\theta}$. Then each unit interval of the form $(k, k + 1)$ for $k \in \mathbb{N}$ contains precisely one of these multiples.*

Proof As shown in Figure 6, mark the points where the line $x = \theta y$ intersects the horizontal and vertical grid lines in the first quadrant—these are the lattice points along the pivotal line. Next, rotate the origin by 90° clockwise about each such lattice point to obtain other lattice points along a secondary line, which is angled at 45° with respect to the pivotal line. Rotating the origin by 90° clockwise about a point (x, y)

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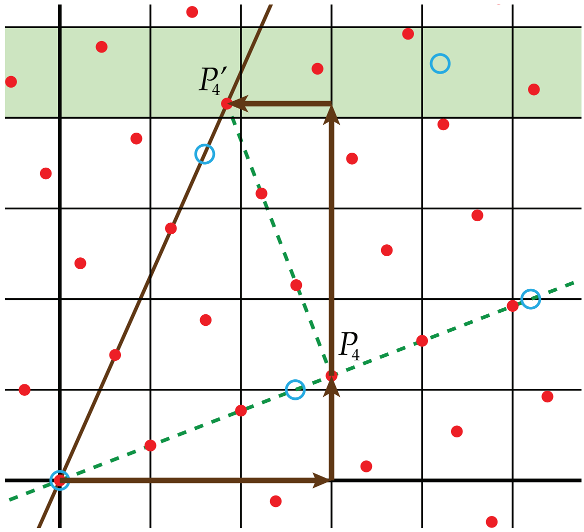

Figure 6: A Beatty sequence appearing within a diagram showing complementary lattices.

gives the point $(x - y, x + y)$, as illustrated by the arrows in Figure 6, which form the sides of a pair of congruent right triangles. Hence, the y-coordinates of these lattice points along the secondary line are $n + n\theta$ and $n + n\frac{1}{\theta}$, for $n \in \mathbb{N}$, precisely the positive integer multiples of $1 + \theta$ and $1 + \frac{1}{\theta}$. We wish to show that each horizontal strip $k < y < k + 1$ for $k \in \mathbb{N}$ contains one such lattice point.

This is nearly self-evident from the diagram. The k-th point $P_k(x_k, y_k)$ along the pivotal line marks the k-th instance where this line meets a horizontal or vertical grid line. Therefore, to reach it by moving along gridlines (the taxicab distance), we must travel k "blocks," plus a fraction of a block. In other words, $k < x_k + y_k < k + 1$, illustrated by arrows for P_4 in Figure 6. But the y-coordinate of the rotated point is equal to $x_k + y_k$, as noted previously. Hence, the k-th lattice point P_k' (and only this point) along the secondary line lies in the strip $k < y < k + 1$, as desired. ■

One can readily extend the notion of "complementary" to any number of lattices by requiring, as before, that the total coverage of each unit square by the set of lattices be equal to 1. However, beyond several isolated instances, further examples of complementary lattices are not readily forthcoming. There is the example of

$$\mathcal{L}_{11/10,\,3/10}, \quad \mathcal{L}_{3,2}, \quad \mathcal{L}_{3,2}, \quad \mathcal{L}_{3,2},$$

which can be manually verified, since $\mathcal{L}_{3,2}$ is a subset of $\mathcal{L}_{11/10,\,3/10}$, so we need only examine 13 unit squares and then invoke periodicity. More generally, empirical evidence suggests that for $n \in \mathbb{N}$, taking $a = (n^2 + 2)/(n^2 + 1)$ and $b = n/(n^2 + 1)$ results in $\mathcal{L}_{a,b}$ and three copies of $\mathcal{L}_{n,2}$ being complementary. (We permit ourselves to include the same lattice multiple times to form a collection of complementary lattices, which is thus technically a multiset.)

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Since every square lattice contains the origin, clearly we can have at most four lattices in a complementary set. This comment prompts several questions.

Question 1 *Uspensky [10] has shown that if k sequences, each of the form* $\{\lfloor n\alpha_i \rfloor\}$ *for* $n \geq 1$*, are complementary, then* $k \leq 2$*. Does a similar result hold for complementary lattices? More precisely, disregarding the four unit squares having one vertex at the origin, if k lattices provide total coverage of 1 to all remaining unit squares in the plane, must it be the case that* $k \leq 4$*?*

Question 2 *Do there exist other continuous families of complementary lattices, such as the pair featured in Theorem 3.1, or other discrete families, such as the set just noted above?*

Question 3 *What sorts of results can be found if we permit the lattices to be translated away from the origin? In particular, does there exist a continuous family of five or more complementary lattices of this sort?*

To illustrate the latter question, observe that $\mathcal{L}_{15/13,\,3/13}$, three copies of $\mathcal{L}_{3,3}$, and a fifth lattice $\mathcal{L}_{3,0}$ translated by $(1.5, 1.5)$ exactly cover all unit squares.

6 The Geometric Lemma

Recall that in the proof of Theorem 3.1, given an irrational $\theta > 1$, we mark where the pivotal line $x = \theta y$ meets horizontal and vertical gridlines in the first quadrant, which occurs at the points $(m\theta, m)$ and $(m, \frac{m}{\theta})$ for $m \in \mathbb{N}$. Label these points as $P_1, P_2, P_3, \dots$, ordered by distance from the origin. These are the points of $\mathcal{L}_{\theta,1}$ and $\mathcal{L}_{1,1/\theta}$ in the first quadrant that lie along the pivotal line. In our proof that these lattices are complementary, a key step involved knowing that certain segments were parallel to the pivotal line. We now present a synthetic proof of this fact.

To set the stage, let **C** be a unit square in the first quadrant. For each $k \geq 1$, rotate **C** clockwise by 90° about P_k to obtain the image square $\mathbf{C}_k$. If P_k is situated on a vertical gridline, let A_kB_k be the vertical segment enclosed by $\mathbf{C}_k$ lying along a vertical gridline, with A_k below B_k. Otherwise, when P_k lies along a horizontal gridline, let A_kB_k be the horizontal enclosed segment, with A_k to the right of B_k. (Note that because P_k is not an integer lattice point, there is no ambiguity since the enclosed segment cannot coincide with an edge of $\mathbf{C}_k$.) A portion of the set-up is illustrated in Figure 7.

Lemma 6.1 *With all geometric objects as defined above, segment* B_kA_{k+1} *is parallel to the pivotal line.*

Proof To orient the reader, these segments are indicated with short dashes on the left in Figure 7. Observe that points P_k and P_{k+1} cannot both be on horizontal gridlines, since the slope of the pivotal line is $\frac{1}{\theta} < 1$, so there are three cases to consider, depicted by segments B_3A_4, B_4A_5, and B_5A_6. The arguments are fairly similar, so we

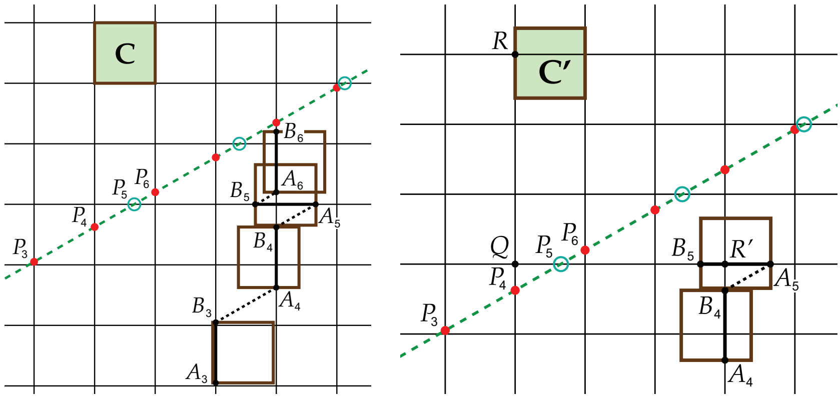

Figure 7: Rotations of a given unit square defining segments A_kB_k.

will only present the proof for B_4A_5 in detail, which represents the case that P_k is on a vertical gridline and P_{k+1} is on a horizontal gridline.

Let Q be the intersection of the gridlines through P_4 and P_5. Rotating **C** about P_4 is equivalent to translating **C** up by QP_4 to **C**′, rotating by 90° clockwise about Q, then translating down again, illustrated on the right in Figure 7. The advantage to this sequence of steps is that we "pick up the enclosed segment" in the process. In other words, the upper edge of **C** within the vertically translated square **C**′ ultimately maps to segment A_4B_4. This is because, by construction, the image of **C** extends a horizontal distance of QP_4 beyond a vertical gridline.

In the same manner, rotating **C** about P_5 is equivalent to translating **C** to the left by distance QP_5, rotating by 90° clockwise about Q, then translating to the right again. This time the left edge of **C**, within the horizontally translated square, eventually maps to A_5B_5. The crucial observation to make is that point R, the upper left vertex of unit square **C**, maps to R' under both rotations, then is translated to B_4 in one case and to A_5 in the other. Therefore, segment B_4A_5 is parallel (and in fact congruent) to P_4P_5, as desired. ■

We note that the geometric lemma may also be verified via coordinate geometry, starting with the observation that a rotation by 90° clockwise about $P_k(x_k, y_k)$ maps a point (u, v) to the point $(v - y_k + x_k, -u + x_k + y_k)$. One would then proceed to argue that the slope of segments P_3P_4 and B_3A_4 are equal, for instance. The reader can fill in the remainder of this satisfying demonstration.

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