Notes

108.01 A use of Pythagorean triples in a problem in elementary geometry

Introduction

Mathematics contains many gems, for example, the golden section and the Pythagorean triple. I want to offer a sketch on these two gems. Namely, I give a geometric problem, the solution of which is associated with Pythagorean triples and the golden section.

Problem

The points A, B and C are given by $A(0, 0)$, $B(0, y)$, $C(x_1, 0)$ for $y > 0$, $x_1 > 0$. Let the angle ABC be α. We want to solve the following problem.

Find a point $D(x_2, 0)$ such that the segment AD will be seen at an angle 2α, with $x_2 > x_1$. Then $\tan\alpha = \dfrac{x_1}{y}$ and $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha} = \dfrac{x_2}{y}$. It follows that

$$x_2 = \frac{2y^2x_1}{y^2 - x_1^2}. \qquad (1)$$

Let us express the coordinate x_1 in terms of y and x_2. Using (1) we have

$$x_2x_1^2 + 2y^2x_1 - y^2x_2 = 0,$$

$$x_1 = \frac{-y^2 + y\sqrt{y^2 + x_2^2}}{x_2}, \text{ since } x_2 > 0.$$

Let us now consider triples of (x_1, x_2, y). For example, such a triple of numbers is

$$\left(\frac{1}{\phi}, \phi, \sqrt{\phi}\right).$$

Here, $\phi = \frac{1}{2}\left(\sqrt{5} + 1\right)$ is the golden section.

When will all the three numbers x_1, x_2 and y be integers?

The number $x_1 = \dfrac{-y^2 + y\sqrt{y^2 + x_2^2}}{x_2}$ must be an integer if both x_2 and y are integers. This means that the sum $y^2 + x_2^2$ must be the square of some integer z.

All natural number (Pythagorean number) solutions of the equation $y^2 + x_2^2 = z^2$ are obtained from the formulas [1]

$$y = 2mnl,$$
$$x_2 = \left(m^2 - n^2\right)l, \qquad (2)$$
$$z = \left(m^2 + n^2\right)l.$$

Here n, m, l are natural numbers with $n < m$.

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We can also apply another set of formulas [1]

$$\begin{aligned} y &= \left(m^2 - n^2\right) l, \\ x_2 &= 2mnl, \\ z &= \left(m^2 + n^2\right) l. \end{aligned} \tag{3}$$

In (2), the x_1 coordinate is expressed by

$$x_1 = \frac{-(2mnl)^2 + 2mnl\left(m^2 + n^2\right)l}{\left(m^2 - n^2\right)l} = \frac{2mn(m - n)l}{m + n}.$$

x_1 will be an integer if the equality $l = m + n$ is satisfied. Then for the triple of numbers (x_1, x_2, y) we get the formulas

$$\begin{aligned} x_1 &= 2mn(m - n), \\ x_2 &= \left(m^2 - n^2\right)(m + n), \\ y &= 2mn(m + n). \end{aligned} \tag{4}$$

In the case of set (3),

$$x_1 = \frac{-\left(m^2 - n^2\right)^2 l^2 + \left(m^2 - n^2\right)\left(m^2 + n^2\right)l^2}{2mnl} = \frac{\left(m^2 - n^2\right)nl}{m}$$

and x_1 will be integer for $l = m$. The integer triple (x_1, x_2, y) is then given by

$$\begin{aligned} x_1 &= n\left(m^2 - n^2\right), \\ x_2 &= 2m^2 n, \\ y &= \left(m^2 - n^2\right)m. \end{aligned} \tag{5}$$

We have found our two sets of integer triples in the form of (4) and (5).

Conclusion

Here we summarise our results. With the help of Pythagorean triples, we get two sets of integer triples (x_1, x_2, y), (4) and (5). Together with the point $A(0, 0)$, these three numbers define the coordinates $B(0, y)$, $C(x_1, 0)$ and $D(x_2, 0)$ of the right-angled triangles ABC and ABD such that if the side AC is seen at an angle α, then the side AD is seen at the angle 2α.

As shown above, the three numbers $\frac{1}{\phi}$, ϕ, $\sqrt{\phi}$ based on the golden section have a similar property.

Reference

1. Harold Davenport, *The higher arithmetic: an introduction to the theory of numbers*, Cambridge University Press (2008).

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ALEXANDER KRONBERG
141981 *Dubna Moscow obl., Russia*
e-mail: *kronberg.alexander@gmail.com*