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# The Bombieri–Vinogradov Theorem on Higher Rank Groups and its Applications

Yujiao Jiang and Guangshi Lü

*Abstract.* We study the analogue of the Bombieri–Vinogradov theorem for $SL_m(\mathbb{Z})$ Hecke–Maass form $F(z)$. In particular, for $SL_2(\mathbb{Z})$ holomorphic or Maass Hecke eigenforms, symmetric-square lifts of holomorphic Hecke eigenforms on $SL_2(\mathbb{Z})$, and $SL_3(\mathbb{Z})$ Maass Hecke eigenforms under the Ramanujan conjecture, the levels of distribution are all equal to $1/2$, which is as strong as the Bombieri–Vinogradov theorem. As an application, we study an automorphic version of Titchmarch's divisor problem; namely for $a \neq 0$,

$$\sum_{n \leq x} \Lambda(n)\rho(n)d(n-a) \ll x \log\log x,$$

where $\rho(n)$ are Fourier coefficients $\lambda_f(n)$ of a holomorphic Hecke eigenform $f$ for $SL_2(\mathbb{Z})$ or Fourier coefficients $A_F(n,1)$ of its symmetric-square lift $F$. Further, as a consequence, we get an asymptotic formula

$$\sum_{n \leq x} \Lambda(n)\lambda_f^2(n)d(n-a) = E_1(a)x\log x + O(x\log\log x),$$

where $E_1(a)$ is a constant depending on $a$. Moreover, we also consider the asymptotic orthogonality of the Möbius function against the arithmetic function $\rho(n)d(n-a)$.

## 1 Introduction and Main Results

The distributive properties of primes along arithmetic progressions have many applications in number theory that appeal to a number of number theorists. Let $\Lambda$ be the von Mangoldt function, and let $\varphi$ be the Euler function. Then the famous Siegel–Walfisz theorem asserts that for $q \leq \log^A x$, one has uniformly

$$\sum_{\substack{n \leq x \\ (a,q)=1 \\ n \equiv a \bmod q}} \Lambda(n) = x\varphi(q)^{-1} + O\Big(x \exp\big(-c\sqrt{\log x}\big)\Big).$$

Whenever the modulus $q$ gets much larger, this problem becomes more subtle and extremely difficult, and it is one of the most formidable obstacles in this area of study.

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For example, even under the Generalized Riemann Hypothesis, one only has

$$\sum_{\substack{n\le x\\ (a,q)=1\\ n\equiv a \bmod q}} \Lambda(n) = x\varphi(q)^{-1} + O\big(x^{\frac12}\log^2 x\big).$$

The miraculous Bombieri–Vinogradov theorem states that for any large constant $A>0$, there is a $B=B(A)>0$ depending on $A$ satisfying

$$\sum_{q\le x^{1/2}\log^{-B}x}\max_{(a,q)=1}\Big|\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\Lambda(n)-x\varphi(q)^{-1}\Big| \ll x\log^{-A}x, \tag{1.1}$$

which implies that the primes are equally distributed in arithmetic progressions over moduli $q\le x^{1/2}\log^{-B}x$ on average, and can be regarded as a fine substitute for the Generalized Riemann Hypothesis in many applications.

There are high-rank analogues of the classical Bombieri–Vinogradov theorem. Let $\tau(n)$ be the Ramanujan $\tau$-function. It was Perelli [35] who first proved that

$$\sum_{q\le x^{2/5}\log^{-B}x}\max_{(a,q)=1}\Big|\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\Lambda(n)\tau(n)\Big| \ll x^{13/2}\log^{-A}x.$$

It is transparent that Perelli's approach still works for any GL(2) holomorphic cusp form $f$, namely,

$$\sum_{q\le x^{2/5}\log^{-B}x}\max_{(a,q)=1}\Big|\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\Lambda(n)\lambda_f(n)\Big| \ll x\log^{-A}x,$$

where $\lambda_f(n)$ is the normalized $n$-th Fourier coefficient of $f$.

Our aim is to synthetically surpass and generalize previous results in this direction to Hecke–Maass forms for $SL_m(\mathbb{Z})$ with $m\ge 2$ (see Section 2). Set $A_F(n_1,\dots,n_{m-1})$ to be the Fourier coefficients of an even Hecke–Maass form $F$. Here we normalize its Fourier coefficient by assuming $A_F(1,\dots,1)=1$. We introduce the notion of *the level of distribution on Fourier coefficients at primes*. If for any $A>0$, we have

$$\sum_{q\le Q}\max_{(a,q)=1}\Big|\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\Lambda(n)A_F(n,1,\dots,1)\Big| \ll x\log^{-A}x$$

holds for $Q=x^{\vartheta_m-\varepsilon}$, we call $\vartheta_m$ the level of distribution on Fourier coefficients $A_F(n,1,\dots,1)$ at primes.

For GL(2) holomorphic cusp forms, Perelli's result implies that the level of distribution on the Fourier coefficients $\lambda_f(n)$ at primes equals $2/5$. Recently, we, together with Yan [19], showed that for an even Hecke–Maass form $F(z)$ for $SL_m(\mathbb{Z})$ with $m\le 3$, or the symmetric power lift of an even Hecke–Maass form for $SL_2(\mathbb{Z})$ if $m=4,5$, we have $\vartheta_m\le\frac{2(m+1)}{3(m^2+1)}$. Under the Generalized Riemann Hypothesis (GRH) for the twisted $L$-functions $L(s,F\otimes\chi)$ and the Generalized Ramanujan Conjecture (GRC)

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for $F$, Iwaniec and Kowalski [16] showed that

$$\sum_{\substack{n\le x\\(a,q)=1\\n\equiv a \bmod q}} \Lambda(n)A_F(n,1,\dots,1) \ll x^{\frac{1}{2}}\log x.$$

The aim of this paper is to investigate which cases the levels of distribution can match that for the Bombieri–Vinogradov theorem. Moveover, in order to enhance and generalize our previous results to Maass–Hecke forms on higher rank groups, we need to make two mild technical assumptions that hold for many cases.

(A) For any primitive Dirichlet character $\chi$, there exists no exceptional zero for the twisted $L$-function $L(s,F\otimes\chi)$.

(B) (Hypothesis H) For any fixed $\nu\ge 2$,

$$\sum_p \frac{|a_F(p^\nu)|^2(\log p)^2}{p^\nu} < \infty,$$

where the arithmetic function $a_F(n)$ is defined as in (2.7).

***Remark 1.1*** Assumption (A) on exceptional zeros is introduced in Section 2.2. It holds for $m = 2,3$ by the work of Hoffstein–Ramakrishnan [15] and Banks [1]. Actually, the Siegel-type theorem is sufficient for our goal, such as the symmetric third power $\mathrm{sym}^3 F$ or symmetric fourth power $\mathrm{sym}^4 F$ of a cuspidal representation $F$ of $\mathrm{GL}_2(\mathcal{A}_{\mathbb{Q}})$. Assumption (B) is the so-called Hypothesis H introduced by Rudnick and Sarnak [38], which is much weaker than the GRC mentioned in (2.4). For $m = 2,3$, Hypothesis H follows from the Rankin–Selberg theory [38]. The $\mathrm{GL}_4(\mathcal{A}_{\mathbb{Q}})$ case and the symmetric fourth power $\mathrm{sym}^4 F$ of a cuspidal representation $F$ of $\mathrm{GL}_2(\mathcal{A}_{\mathbb{Q}})$ were proved by Kim [21] based on his proof of the (weak) functoriality of the exterior square $\wedge^2 F$ from a cuspidal representation $F$ of $\mathrm{GL}_4(\mathcal{A}_{\mathbb{Q}})$.

Our main result is the following theorem.

***Theorem 1.2*** *Let $L(s,F)$ be the $L$-function associated with a Hecke–Maass form $F$ for $\mathrm{SL}_m(\mathbb{Z})$. Let $A_F(n,1,\dots,1)$ denote the $n$-th coefficient of the Dirichlet series for $L(s,F)$. Then under the hypotheses* (A) *and* (B)*, we have for $\varepsilon>0, A>0$,*

$$\sum_{q\le Q}\max_{(a,q)=1}\Big|\sum_{\substack{n\le x\\n\equiv a \bmod q}} \Lambda(n)A_F(n,1,\dots,1)\Big| \ll x\log^{-A} x,$$

*where*

$$Q=\begin{cases} x^{\frac{1}{2}}\log^{-B}x & \text{if } m=2,\\ x^{\frac{2-(m-1)\theta_m}{m+1+2\theta_m}}\log^{-B}x & \text{if } m\ge 3,\end{cases}$$

*with some $B>0$ depending on $A$ and $m$, and $\theta_m$ is as in* (2.5)*. Moreover, we could also get levels of distribution that do not depend on $\theta_m$, namely when $m>2$,*

$$Q = x^{\frac{2m^2-1}{(m^2+1)(2m+1)}-\varepsilon}. \tag{1.2}$$

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***Theorem 1.3*** *Let $\mu$ denote the Möbius function. With the same notation and hypotheses as in Theorem 1.2, we then have*

$$\sum_{q\leq Q}\max_{(a,q)=1}\Bigg|\sum_{\substack{n\leq x\\ n\equiv a \bmod q}}\mu(n)A_F(n,1,\dots,1)\Bigg| \ll x\log^{-A}x,$$

*where*

$$Q=\min\left\{x^{\frac{2-(m-1)\theta_m}{m+1+2\theta_m}},x^{\frac{(m+1)(1-2\theta_m)}{2(m+1+2\theta_m)}}\right\}\log^{-B}x$$

*with some $B>0$ depending on $A$ and $m$.*

***Remark 1.4*** As explained in Remark 1.1, Theorem 1.2 and Theorem 1.3 hold unconditionally for the cases $m=2,3$, or $\mathrm{sym}^3 F$, $\mathrm{sym}^4 F$. Here $F$ is a Hecke–Maass form for $SL_2(\mathbb{Z})$.

***Theorem 1.5*** *Let $f$ be a holomorphic Hecke eigenform of weight $k$ for $SL_2(\mathbb{Z})$, and let $\lambda_f(n)$ be the $n$-th Fourier coefficient. We have*

$$\sum_{q\leq x^{\frac{1}{2}}\log^{-B}x}\max_{(a,q)=1}\Bigg|\sum_{\substack{n\leq x\\ n\equiv a \bmod q}}\Lambda(n)\lambda_f(n)\Bigg| \ll x\log^{-A}x$$

*and*

$$\sum_{q\leq x^{\frac{1}{2}}\log^{-B}x}\max_{(a,q)=1}\Bigg|\sum_{\substack{n\leq x\\ n\equiv a \bmod q}}\mu(n)\lambda_f(n)\Bigg| \ll x\log^{-A}x,$$

*where $B>0$ is some constant depending on $A$.*

***Theorem 1.6*** *Let $F$ be the symmetric-square lift of a holomorphic Hecke eigenform of weight $k$ for $SL_2(\mathbb{Z})$. Then we have*

$$\sum_{q\leq x^{\frac{1}{2}}\log^{-B}x}\max_{(a,q)=1}\Bigg|\sum_{\substack{n\leq x\\ n\equiv a \bmod q}}\Lambda(n)A_F(n,1)\Bigg| \ll x\log^{-A}x$$

*and*

$$\sum_{q\leq x^{\frac{1}{2}}\log^{-B}x}\max_{(a,q)=1}\Bigg|\sum_{\substack{n\leq x\\ n\equiv a \bmod q}}\mu(n)A_F(n,1)\Bigg| \ll x\log^{-A}x,$$

*where $B>0$ is some constant depending on $A$.*

***Remark 1.7*** It is known from Theorems 1.2, 1.5, and 1.6 that the level of distribution $\vartheta_m$ is equal to $1/2$, when $F$ is a holomorphic cusp form or Maass cusp form for $SL_2(\mathbb{Z})$, the symmetric-square lift of a holomorphic Hecke eigenform for $SL_2(\mathbb{Z})$, or a Maass cusp form for $SL_3(\mathbb{Z})$ under Ramanujan conjecture. That is to say the corresponding Riemann Hypothesis holds on average.

***Corollary 1.8*** *Let $\lambda_f(n)$ be the n-th Fourier coefficient of a holomorphic cusp form $f$ for $\mathrm{SL}_2(\mathbb{Z})$. Then we have*

$$\sum_{q\le x^{\frac12}\log^{-B}x}\max_{(a,q)=1}\Big|\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\Lambda(n)\lambda_f^2(n)-\frac{x}{\varphi(q)}\Big|\ll x\log^{-A}x,$$

*where $B>0$ is some constant depending on $A$.*

**Proof** From the multiplicative property, we have

$$\lambda_f(p)^2=\lambda_f(p^2)+1.$$

Here, $\lambda_f(p^2)=A_F(p,1)$ with $F=\mathrm{sym}^2 f$. Combining the classical Bombieri–Vinogradov theorem (1.1) with Theorem 1.6, this corollary follows. ∎

Another interesting problem in analytic number theory is to understand the behaviour of arithmetical functions $\beta(n)$ at primes, which means to estimate the sum of type

$$\sum_{n\le x}\Lambda(n)\beta(n). \tag{1.3}$$

When $\beta(n)\equiv 1$, the estimation of (1.3) is closely related to the location of the zeros of Riemann $\zeta$-function. When $\beta(n)$ is multiplicative, the problem can be related to the analytic properties of a corresponding $L$-function in a similar way. (We refer the reader to [16, Chapter 5] for details.) If $\beta(n)$ has a shift in the argument, which destroys the multiplicativity, the approach above to estimate (1.3) is not available. Similarly, it is also very interesting to investigate the sum dual to (1.3),

$$\sum_{n\le x}\mu(n)\beta(n).$$

The "Möbius randomness principle" [16, p. 338] asserts that the sum above produces a considerable cancelation if the sequence $(\beta(n))$ is "reasonable". Sarnak [39] has recently posed a more precise conjecture in this direction, which says that

$$\sum_{n\le x}\mu(n)\beta(n)=o\Big(\sum_{n\le x}|\beta(n)|\Big),$$

whenever $\beta$ arises from a dynamical system of zero entropy. This expresses an orthogonality property between the Möbius function against a sequence with zero entropy. We refer the reader to [9, 26, 45] for recent developments on this theme. A classical example is Titchmarsh's divisor problem, which considers the average value of the divisor function $d(n)$ at shifted primes, in other words, the estimation of

$$\mathcal{F}(x)=\sum_{n\le x}\Lambda(n)d(n-a), \tag{1.4}$$

where $a>0$ is fixed. His original work in [42, 43] showed that $\mathcal{F}(x)$ admitted an asymptotic formula

$$\mathcal{F}(x)=E_1(a)x\log x+O(x\log\log x) \tag{1.5}$$

conditionally on the Generalized Riemann Hypothesis, where $E_1(a)$ is a constant depending on $a$. Later, this result was proved unconditionally by Linnik [25], and also considered in [14, 37]. The best known result at present is

$$\mathcal{F}(x) = E_1(a)x\log x + E_2(a)x + O\left(\frac{x}{\log^A x}\right)$$

for any $A > 0$, where the implied constant depends only on $a$ and $A$, due to Bombieri, Friedlander, and Iwaniec [4] and to Fouvry [8], independently.

We focus on investigating the shifted convolution sum at primes associated with the divisor function $d(n)$ and some arithmetic functions $\rho(n)$, where $\rho(n)$ are Fourier coefficients $\lambda_f(n)$ of a normalized holomorphic cusp form $f$ with weight $k$ for $SL_2(\mathbb{Z})$ or Fourier coefficients $A_F(n,1)$ of its symmetric-square lift $F = \mathrm{sym}^2 f$. It means to estimate the sum

$$\mathcal{F}_{2,a}(x) = \sum_{n\le x} \Lambda(n)\rho(n)d(n-a), \tag{1.6}$$

where $a \ne 0$ is fixed. This can be viewed as an automorphic version of Titchmarsh's divisor problem. We shall also consider the dual sum to (1.6).

$$\mathcal{M}_{2,a}(x) = \sum_{n\le x} \mu(n)\rho(n)d(n-a).$$

First, we get directly from (1.4) that

$$\sum_{n\le x} \Lambda(n)|\rho(n)|d(n-a) \ll \mathcal{F}(x) \ll x\log x$$

due to Deligne's bound. In fact, it is known that the order of $d(n)$ is $\log n$ on average in $n$, so that

$$\sum_{n\le x} \Lambda(n)|\rho(n)|d(n-a) \quad \text{and} \quad \sum_{n\le x} \Lambda(n)|\rho(n)|\log(n-a)$$

have the same order in the average sense. What is more, the second sum satisfies

$$\begin{aligned}\sum_{n\le x} \Lambda(n)|\rho(n)|\log(n-a) &\gg (\log^2 x) \sum_{\frac{x}{2}<p\le x} |\rho(p)| \\ &\gg (\log^2 x) \sum_{\frac{x}{2}<p\le x} |\rho(p)|^2 \gg x\log x.\end{aligned}$$

The last step above is due to the prime number theorem for Rankin–Selberg $L$-functions. Therefore, the shifted convolution sum $\sum_{n\le x} \Lambda(n)|\rho(n)|d(n-a)$ has the order of $x\log x$ in the average sense.

We show that there exist some cancelations for sums $\mathcal{F}_{2,a}(x)$ and $\mathcal{M}_{2,a}(x)$.

***Theorem 1.9*** *Let $\mathcal{F}_{2,a}(x)$ and $\mathcal{M}_{2,a}(x)$ be defined as above. For any $a \ne 0$, we have*

$$\mathcal{F}_{2,a}(x) \ll x\log\log x, \quad \mathcal{M}_{2,a}(x) \ll x\log^{2\delta_j-1} x\log\log x,$$

*where*

$$\delta_j = \begin{cases} \frac{8}{3\pi}, & \text{if } j=1,\ \rho(n)=\lambda_f(n), \\ \frac{3\sqrt{3}}{2\pi}, & \text{if } j=2,\ \rho(n)=A_{\mathrm{sym}^2 f}(n,1), \end{cases}$$

*and the implied constant depends on $f$ only.*

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***Remark 1.10*** Recall the second author [28] showed that

$$\sum_{n\le x} |\rho(n)d(n-1)| \sim x\log^{\delta_j} x.$$

So the sequence composed of $\rho(n)d(n-1)$ fulfills the Sarnak conjecture due to $\delta_j < 1$. In fact, the power $2\delta_j - 1$ in the upper bound for $\mathcal{M}_{2,a}(x)$ can be improved to $\delta_j - 1$ when $a = 1$. The reason is that $G_2 \ll x\log^{\delta_j - 1} x \log\log x$ by inserting (1.7) directly.

**Proof** By the definition of divisor function $d(n)$,

$$d(n-a) = \sum_{q|(n-a)} 1,$$

we deduce that

$$\begin{aligned}
&\mathcal{F}_{2,a}(x)\\
&\quad = \sum_{n\le x}\Lambda(n)\rho(n)\Bigg(\sum_{\substack{q|(n-a)\\ q\le x^{\frac12}\log^{-B}x}} 1 + \sum_{\substack{q|(n-a)\\ x^{\frac12}\log^{-B}x<q\le x^{\frac12}\log^{B}x}} 1 + \sum_{\substack{q|(n-a)\\ q>x^{\frac12}\log^{B}x}} 1\Bigg)\\
&\quad = M_1 + M_2 + M_3.
\end{aligned}$$

Our first task is to estimate the sum $M_2$. The Brun–Titchmarsh theorem [25, Lemma 1.3.1] gives

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}} \Lambda(n) \ll \frac{x}{\varphi(q)}$$

for any $q \le x^{1-\varepsilon}$. A slight estimate of Titchmarsh [42, Equation (3.2)] states that

$$\sum_{q\le x}\frac{1}{\varphi(q)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)}\log x + O(1).$$

Further, these yield

$$\begin{aligned}
M_2 &= \sum_{x^{\frac12}\log^{-B}x<q\le x^{\frac12}\log^{B}x}\ \sum_{\substack{n\le x\\ n\equiv a \bmod q}} \Lambda(n)\rho(n)\\
&\ll \sum_{x^{\frac12}\log^{-B}x<q\le x^{\frac12}\log^{B}x}\ \sum_{\substack{n\le x\\ n\equiv a \bmod q}} \Lambda(n)\\
&\ll x \sum_{x^{\frac12}\log^{-B}x<q\le x^{\frac12}\log^{B}x} \frac{1}{\varphi(q)} \ll x\log\log x.
\end{aligned}$$

For the other two parts, after applying Theorem 1.5 or Theorem 1.6, we get

$$\begin{aligned}
M_1 &= \sum_{q\le x^{\frac12}\log^{-B}x}\ \sum_{\substack{n\le x\\ n\equiv a \bmod q}} \Lambda(n)\rho(n)\\
&\ll \sum_{q\le x^{\frac12}\log^{-B}x}\Bigg|\sum_{\substack{n\le x\\ n\equiv a \bmod q}} \Lambda(n)\rho(n)\Bigg| \ll x\log^{-A}x.
\end{aligned}$$

Note that the analogue of Bombieri–Vinogradov theorem used here still holds even if we remove the restrictive condition $(a,q) = 1$, since $\Lambda(n)$ is the closely related

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function defined over primes. Then $M_3$ can be estimated similarly to $M_1$ by passing from the divisor $q|(n-a)$ to its co-divisor $q' := (n-a)/q$.

The argument for estimating $\mathcal{M}_{2,a}(x)$ is analogous to that of $\mathcal{F}_{2,a}(x)$. To begin with,

$$\begin{aligned}\mathcal{M}_{2,a}(x) \\ &= \sum_{n\le x}\mu(n)\rho(n)\Bigg(\sum_{\substack{q|(n-a)\\ q\le x^{\frac12}\log^{-B}x}} 1 + \sum_{\substack{q|(n-a)\\ x^{\frac12}\log^{-B}x<q\le x^{\frac12}\log^{B}x}} 1 + \sum_{\substack{q|(n-a)\\ q> x^{\frac12}\log^{B}x}} 1\Bigg)\\ &= G_1+G_2+G_3.\end{aligned}$$

For each $G_i$, the situation is a little bit different from $M_i$. Shiu [Theorem 2][40] established upper bounds of the right order of magnitude for some multiplicative functions. For our special case of the multiplicative function $|\mu(n)\rho(n)|$, it gives

$$\sum_{\substack{n\le x\\ n\equiv l \bmod k}} |\mu(n)\rho(n)| \ll \frac{x}{\varphi(k)}\frac{1}{\log x}\exp\Bigg(\sum_{p\le x}\frac{|\rho(p)|}{p}\Bigg)$$

with $(l,k)=1$ and $k\le x^{1-\varepsilon}$. Owing to the recent proof of the Sato–Tate conjecture of Barnet–Lamb, Geraghty, Harris, and Taylor [2], it is easy to see that

$$\sum_{p\le x}|\lambda_f(p^j)| \sim \delta_j\frac{x}{\log x}$$

with $\delta_1=\frac{8}{3\pi}$, $\delta_2=\frac{3\sqrt{3}}{2\pi}$. Then, by partial summation, we get

$$\sum_{\substack{n\le x\\ n\equiv l \bmod k}} |\mu(n)\rho(n)| \ll \frac{x\log^{\delta_j-1}x}{\varphi(k)}. \tag{1.7}$$

Obviously,

$$G_2 \ll \sum_{x^{\frac12}\log^{-B}x<q\le x^{\frac12}\log^{B}x}\ \sum_{\substack{n\le x\\ n\equiv a \bmod q}} |\mu(n)\rho(n)|.$$

Assume that $(a,q)=d$, the change of variables $a\to ld, q\to kd$ and $n\to nd$ with $(l,k)=1$ yields

$$\begin{aligned}G_2 &\ll \sum_{d\le x^{\frac12}\log^{B}x}|\mu(d)\rho(d)| \sum_{\frac{x^{\frac12}\log^{-B}x}{d}<k\le\frac{x^{\frac12}\log^{B}x}{d}}\ \sum_{\substack{n\le \frac{x}{d}\\ n\equiv l \bmod k}}|\mu(n)\rho(n)|\\ &\ll x(\log^{2\delta_j-1}x)\log\log x.\end{aligned}$$

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To treat $G_1$ and $G_3$, we start to estimate

$$\begin{aligned}&\sum_{q\le x^{\frac12}\log^{-B}x}\Big|\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\mu(n)\rho(n)\Big|\\ &\ll \sum_{d\le x^{\frac12}\log^{-B}x}|\mu(d)\rho(d)|\sum_{k\le\frac{x^{\frac12}\log^{-B}x}{d}}\max_{(l,k)=1}\Big|\sum_{\substack{n\le\frac{x}{d}\\ n\equiv l \bmod k}}\mu(n)\rho(n)\Big|\\ &\ll x\log^{-A}x.\end{aligned}$$

Hence, $G_1+G_3\ll x\log^{-A}x$. This completes the proof of Theorem 1.9. ■

Combining (1.5) with Theorem 1.9, we obtain the following corollary.

***Corollary 1.11*** *Let $\lambda_f(n)$ be the $n$-th Fourier coefficient of a holomorphic cusp form $f$ for $\mathrm{SL}_2(\mathbb{Z})$. For $a\neq 0$, we have*

$$\sum_{n\le x}\Lambda(n)\lambda_f^2(n)d(n-a)=E_1(a)x\log x+O(x\log\log x),$$

*where $E_1(a)$ is a constant depending on $a$.*

The reader should take some caution with our use of the constant $\varepsilon$. Any statement including $\varepsilon$ is meant simply as the claim that the statement is true for any sufficiently small positive $\varepsilon$, which may vary from one line to the next. Moreover, the meaning of "the implied constants depend on $F$ only" is that the implied constants depend only on the Langlands parameters of $F$ rather than other information.

## 2 Preliminaries

In this section, we sum up some known results on $\mathrm{SL}_m(\mathbb{Z})$ Maass forms and their $L$-functions. As usual, denote by $F(z)$ an even Hecke–Maass form of type $\nu=(\nu_1,\nu_2,\dots,\nu_{m-1})$, which admits a Fourier expansion

$$(2.1)\qquad \begin{aligned}F(z)=&\sum_{\gamma\in U_{m-1}(\mathbb{Z})\backslash\mathrm{SL}_{m-1}(\mathbb{Z})}\sum_{n_1\ge1}\cdots\sum_{n_{m-2}\ge1}\sum_{n_{m-1}\ne0}\frac{A_F(n_1,\dots,n_{m-1})}{\prod_{k=1}^m|n_k|^{\frac{k(m-k)}{2}}}\\ &\times W_J\left(\begin{pmatrix}n_1\dots|n_{m-1}| & & & \\ & \ddots & & \\ & & n_1 & \\ & & & 1\end{pmatrix}\cdot\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}z,\nu,\psi_{1,\dots,1,\frac{n_{m-1}}{|n_{m-1}|}}\right),\end{aligned}$$

where $U_{m-1}(\mathbb{Z})$ is the group of $(m-1)\times(m-1)$ upper triangular matrices with 1s on the diagonal and integral entries above the diagonal, $A_F(n_1,\dots,n_{m-1})\in\mathbb{C}$, and $W_J$ is the Jacquet–Whittaker function. As for the symmetric-square lift $F$ of an $\mathrm{SL}_2(\mathbb{Z})$ holomorphic Hecke eigenform $f$, its coefficients are given by

$$A_F(n,1)=\sum_{ml^2=n}\lambda_f(m).$$

Here, $\lambda_f(n)$ is the $n$-th Fourier coefficient of $f$.

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We assume that $F$ is arithmetically normalized, *i.e.*, $A_F(1,\ldots,1) = 1$. It is well known that the Fourier coefficients $A_F(n_1,\ldots,n_{m-1})$ satisfy

$$(2.2)\qquad \begin{aligned} &A_F(n_1n_1',\ldots,n_{m-1}n_{m-1}') = A_F(n_1,\ldots,n_{m-1})A_F(n_1',\ldots,n_{m-1}'),\\ &\qquad\qquad\qquad\qquad \text{if } (n_1\cdots n_{m-1}, n_1'\cdots n_{m-1}') = 1,\\ &A_F(n,1,\ldots,1)A_F(n_1,\ldots,n_{m-1}) = \\ &\qquad \sum_{\substack{\prod_{\ell=1}^{m} c_\ell = n\\ c_1|n_1,\ldots,c_{m-1}|n_{m-1}}} A_F\Big(\frac{n_1c_m}{c_1},\frac{n_2c_1}{c_2},\ldots,\frac{n_{m-1}c_{m-2}}{c_{m-1}}\Big),\\ &A_F(n_{m-1},\ldots,n_1) = \overline{A_F(n_1,\ldots,n_{m-1})}. \end{aligned}$$

See, *e.g.*, Goldfeld [11, Theorem 9.3.11]. One of the open problems concerning Fourier coefficients is the Generalized Ramanujan Conjecture (GRC), which predicts that

$$(2.3)\qquad |A_F(n,1,\ldots,1)| \le d_m(n),$$

where $d_m(n)$ is the $m$-th dimensional divisor function. The up-to-date numerical bounds towards the GRC are established in [20] and [27]:

$$(2.4)\qquad |A_F(n,1,\ldots,1)| \le n^{\theta_m} d_m(n),$$

where

$$(2.5)\qquad \theta_2 = \frac{7}{64},\quad \theta_3 = \frac{5}{14},\quad \theta_4 = \frac{9}{22},\quad \theta_m = \frac{1}{2} - \frac{1}{m^2+1}\quad (m \ge 5).$$

Obviously, Deligne's impressive work [6] implies that the symmetric-square lift $F = \mathrm{sym}^2 f$ of a holomorphic Hecke eigenform $f$ satisfies the GRC.

We still need the theory of automorphic $L$-functions, among which the Godement–Jacquet $L$-function is initially defined in the absolutely convergent half-plane by

$$L(s,F) = \sum_{n=1}^{\infty} \frac{A_F(n,1,\ldots,1)}{n^s}.$$

Then it can be analytically continued to an entire function that satisfies the functional equation

$$\pi^{-\frac{ms}{2}} \prod_{i=1}^{m} \Gamma\left(\frac{s-\lambda_i(\nu)}{2}\right) L(s,F) = \pi^{-\frac{m(1-s)}{2}} \prod_{i=1}^{m} \Gamma\Big(\frac{1-s-\widetilde{\lambda}_i(\nu)}{2}\Big) L(1-s,\widetilde{F}),$$

where $\widetilde{F}$ is the dual form of $F$ and $\lambda_i(\nu)$ are the Langlands parameters. Suppose that $\chi$ is a primitive Dirichlet character modulo $q$. The twisted $L$-function

$$L(s,F\otimes\chi) = \sum_{n=1}^{\infty} \frac{A_F(n,1,\ldots,1)\chi(n)}{n^s}$$

admits an analytic continuation to an entire function satisfying

$$\begin{aligned} &\Big(\frac{q}{\pi}\Big)^{\frac{ms}{2}} \prod_{i=1}^{m} \Gamma\Big(\frac{s+\delta-\lambda_i(\nu)}{2}\Big) L(s,F\otimes\chi) = \\ &\qquad \Big(\frac{\tau(\chi)}{i^\delta\sqrt{q}}\Big)^m \Big(\frac{q}{\pi}\Big)^{\frac{m(1-s)}{2}} \prod_{i=1}^{m} \Gamma\Big(\frac{1-s+\delta-\widetilde{\lambda}_i(\nu)}{2}\Big) L(1-s,\widetilde{F}\otimes\overline{\chi}). \end{aligned}$$

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The $L$-function $L(s,F)$ can be written as an Euler product over primes,

$$L(s,F)=\prod_p\prod_{1\le j\le m}\Big(1-\frac{\alpha_F(p,j)}{p^s}\Big)^{-1},$$

where $\{\alpha_F(p,j)\}, 1\le j\le m$, are the local parameters at $p$, satisfying the relation

$$X^m+\sum_{\ell=1}^{m-1}(-1)^\ell A_F(\overbrace{1,\dots,1}^{\ell-1\text{ terms}},p,1,\dots,1)X^{m-\ell}+(-1)^m\in\mathbb{C}[X].$$

This guarantees that

$$A_F(\overbrace{1,\dots,1}^{\ell-1},p,1,\dots,1)=\sum_{1\le j_1<\dots<j_\ell\le m}\alpha_F(p,j_1)\dots\alpha_F(p,j_\ell)$$

for $1\le\ell\le m-1$.

Statement (2.3) is equivalent to $|\alpha_F(p,j)|=1$ for all local places $p$ and $j=1,\dots,m$. The numerical bound (2.4) is equivalent to

$$|\alpha_F(p,j)|\le p^{\theta_m}. \tag{2.6}$$

The logarithmic derivative of $L(s,F)$ is given by

$$-\frac{L'}{L}(s,F)=\sum_{n=1}^\infty\frac{\Lambda_F(n)}{n^s}=\sum_{n=1}^\infty\frac{\Lambda(n)a_F(n)}{n^s}. \tag{2.7}$$

It is not hard to derive that $a_F(n)$ are multiplicative and satisfy

$$a_F(p^k)=\sum_{j=1}^m\alpha_F(p,j)^k.$$

The reciprocal $L$-function $L(s,F)^{-1}$ is given as

$$L^{-1}(s,F)=\sum_{n=1}^\infty\frac{\mu_F(n)}{n^s}.$$

Here

$$\mu_F(n)=\begin{cases}0 & \text{if } p^{m+1}|n,\\ \prod_{p^\ell\|n}(-1)^\ell\sum_{1\le j_1<\dots<j_\ell\le m}\alpha_F(p,j_1)\cdots\alpha_F(p,j_\ell) & \text{otherwise.}\end{cases}$$

Clearly, $\mu_F(n)$ is a multiplicative function.

Our proof is based on the Vaughan identity for the Godement–Jacquet $L$-function $L(s,F)$. We introduce

$$M(s)=\sum_{n\le M}\mu_F(n)n^{-s},\quad N(s)=\sum_{n\le N}\Lambda_F(n)n^{-s}.$$

By comparing the following identities

$$\frac{L'}{L}(s,F)=\Big(\frac{L'}{L}(s,F)+N(s)\Big)\big(1-L(s,F)M(s)\big)-N(s)$$
$$+L'(s,F)M(s)+L(s,F)M(s)N(s),$$

and

$$L(s,F)^{-1} = \Big(\frac{1}{L(s,F)} - M(s)\Big)\Big(1 - L(s,F)M(s)\Big) + 2M(s) - L(s,F)M^2(s),$$

we have the following lemma.

***Lemma 2.1*** *Assume $M \ge 1$ and $N \ge 1$. Hence, for any integer $n > N$, one has*

$$\begin{aligned}\Lambda_F(n) = \sum_{\substack{b|n\\ b\le M}} \mu_F(b) A_F\Big(\frac{n}{b},1,\dots,1\Big)\log\Big(\frac{n}{b}\Big)\\ + \sum_{\substack{bc|n\\ b>M,c>N}}\sum \mu_F(b)\Lambda_F(c)A_F\Big(\frac{n}{bc},1,\dots,1\Big)\\ - \sum_{\substack{bc|n\\ b\le M,c\le N}}\sum \mu_F(b)\Lambda_F(c)A_F\Big(\frac{n}{bc},1,\dots,1\Big).\end{aligned}$$

*Analogously, for any integer $n > M$, one has*

$$\mu_F(n) = \sum_{\substack{bc|n\\ b>M,c>M}}\sum \mu_F(b)\mu_F(c)A_F\Big(\frac{n}{bc},1,\dots,1\Big) - \sum_{\substack{bc|n\\ b\le M,c\le M}}\sum \mu_F(b)\mu_F(c)A_F\Big(\frac{n}{bc},1,\dots,1\Big).$$

### 2.1 Fourier Coefficients of Automorphic Forms Over Arithmetic Progression

We intend to derive non-trivial bounds as sharp as possible for

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}} A_F(n,1,\dots,1),$$

where $(a,q) = 1$. If $f$ is an $SL_2(\mathbb{Z})$ holomorphic cusp form, Smith [41] first showed that

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}} \lambda_f(n) \ll x^{\frac{1}{3}+\varepsilon} \tag{2.8}$$

uniformly for $q \le x^{\frac{2}{3}}$. We have the following two propositions, which will be proved in Section 2.1.2.

***Proposition 2.2*** Let $(a,q) = 1$, and let $F(z)$ be an even Hecke–Maass form for $SL_m(\mathbb{Z})$ as in (2.1). Then we have

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}} A_F(n,1,\dots,1) \ll \begin{cases} x^{\frac{1}{3}(1+\theta_m)+\varepsilon} & \text{if } m=2, q \ll x^{\frac{2}{3}(1+\theta_m)},\\ q^{\frac{2\theta_m}{m+1}} x^{(1-\frac{2}{m+1})(1+\theta_m)} d^T(q)(\log^2 q)\log^2 x & \text{if } m\ge 3, q \ll x^{\frac{2(1+\theta_m)}{m+1+2\theta_m}},\end{cases} \tag{2.9}$$

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where $T$ is some positive constant depending only on $m$.

The result in Proposition 2.2 depends on the numerical value of the GRC. However, the current admissible values of $\theta_m$ in (2.5) only give trivial bounds when $m$ becomes a slightly larger. The reason is that we have used it to estimate the short-interval sum

$$\sum_{x\le n\le x+y} |A_F(n,1,\dots,1)|, \tag{2.10}$$

where $y \ll x$. In fact we can apply another method to estimate (2.10), which instead gives nontrivial estimates for all cases in (2.9).

***Proposition 2.3*** Let $(a,q) = 1$, and let $F(z)$ be an even Hecke–Maass form for $\mathrm{SL}_m(\mathbb{Z})$ as in (2.1). Then we have

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}} A_F(n,1,\dots,1) \ll \begin{cases} x^{\frac{107}{297}+\varepsilon} & \text{if } m=2, q \ll x^{\frac{214}{297}},\\ x^{\frac{m(m-1)}{m^2+1}+\varepsilon} q^{\frac{\theta_m}{m}} & \text{if } m\ge 3, q \ll x^{\frac{2m^2}{(m^2+1)(m+1)}}. \end{cases} \tag{2.11}$$

The Voronoi formula in Lemma 2.7 is similar to the case where $m = 3$ in Lemma 2.6. The Ramanujan conjecture for holomorphic cusp forms holds. Thus, by Proposition 2.2, we can establish the following proposition.

***Proposition 2.4*** Let $(a,q) = 1$, and let $F$ be the symmetric-square lift of a holomorphic Hecke eigenform of weight $k$ for $\mathrm{SL}_2(\mathbb{Z})$. Then we have

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}} A_F(n,1) \ll x^{\frac{1}{2}} d^T(q)(\log^2 q)\log^2 x, \tag{2.12}$$

where $T$ is some positive constant depending only on $m$.

**Proof** This is very similar to the proof of Proposition 2.2 in the special case $m = 3$ with $\theta_3$ replaced by zero. ∎

### 2.1.1 Preliminary Lemmas

Voronoi summation formulas for $\mathrm{GL}(m)$ are powerful tools to study the distribution of Fourier coefficients over arithmetic progression. In the case of full modular group $\mathrm{SL}_2(\mathbb{Z})$, Good [13] established such a formula for holomorphic cusp forms and Meurman [29] for Maass cusp forms. These can be generalized to arbitrary level and nebentypus with obvious minor modifications. For the case of $\mathrm{SL}_3(\mathbb{Z})$, Voronoi formulas for Fourier coefficients of automorphic forms on $\mathrm{GL}(3)$ twisted by additive characters were first proved by Miller and Schmid [31] using the technique of automorphic distributions. Later, a Voronoi formula was generalized to $\mathrm{GL}(m)$ with $m \ge 4$ by Goldfeld and Li [12] and Miller and Schmid [32].

***Lemma 2.5*** *If $F(z)$ is a Hecke–Maass form and nonnegative Laplacian eigenvalue $1/4+\mu^2$ on $\mathrm{SL}_2(\mathbb{Z})$, let $g$ be a compactly supported smooth function on $(0,\infty)$. Then*

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we have

$$\sum_{n=1}^{\infty} A_F(n) e\Big(\frac{hn}{c}\Big) g(n) = \sum_{\pm} \sum_{n=1}^{\infty} A_F(\mp n) e\Big(\pm \frac{\overline{h}n}{c}\Big) g^{\pm}(n),$$

where

$$g^{-}(y) = -\frac{\pi}{c \cosh \pi\mu} \int_0^{\infty} g(x) \{Y_{2i\mu} + Y_{-2i\mu}\}\Big(\frac{4\pi\sqrt{xy}}{c}\Big)\, dx,$$
$$g^{+}(y) = \frac{4 \cosh \pi\mu}{c} \int_0^{\infty} g(x) K_{2i\mu}\Big(\frac{4\pi\sqrt{xy}}{c}\Big)\, dx.$$

*Here* $(h, c) = 1$, $\overline{h}$ *denotes the multiplicative inverse of* $h \bmod c$, *and* $Y_{\pm 2i\mu}$, $K_{2i\mu}$ *are all Bessel functions.*

**Lemma 2.6** *Fix* $m \geq 3$. *Let* $F$ *be an even Maass form for* $\mathrm{SL}_m(\mathbb{Z})$ *and let* $g$ *be a compactly supported smooth function on* $(0, \infty)$. *Then we have*

$$\begin{aligned}(2.13)\qquad & \sum_{n\neq 1}^{\infty} A_F(1, \dots, 1, n) e_q(an) g(|n|) \\ &= q \sum_{d_1|q} \sum_{d_2|\frac{q}{d_1}} \cdots \sum_{d_{m-2}|\frac{q}{d_1\cdots d_{m-3}}} \sum_{n\neq 0} \frac{A_F(n, d_{m-2}, \dots, d_2, d_1)}{d_1 \cdots d_{m-2}|n|} \\ &\quad \times KL_{m-2}((-1)^m \overline{a}, n; \mathbf{d}, q) G\Big(\frac{|n| \prod_{i=1}^{m-2} d_i^{m-i}}{q^m}\Big),\end{aligned}$$

*where* $(a, q) = 1$, $\overline{a}$ *denotes the multiplicative inverse of* $a \bmod q$, *and* $KL_{m-2}(a, n; \mathbf{d}, q)$ *is the Kloosterman sum*

$$\begin{aligned} & KL_{m-2}(a, n; \mathbf{d}, q) \\ &= \sideset{}{^*}\sum_{t_1 \left(\bmod \frac{q}{d_1}\right)} e\Big(\frac{at_1}{\frac{q}{d_1}}\Big) \sideset{}{^*}\sum_{t_2 \left(\bmod \frac{q}{d_1 d_2}\right)} e\Big(\frac{\overline{t}_1 t_2}{\frac{q}{d_1 d_2}}\Big) \cdots \\ &\quad \times \sideset{}{^*}\sum_{t_{m-2} \left(\bmod \frac{q}{d_1\cdots d_{m-2}}\right)} e\Big(\frac{\overline{t}_{m-3} t_{m-2}}{\frac{q}{d_1\cdots d_{m-2}}}\Big) e\Big(\frac{n\overline{t}_{m-2}}{\frac{q}{d_1\cdots d_{m-2}}}\Big)\end{aligned}$$

*with* $\mathbf{d} = (d_1, \dots, d_{m-2})$. *Here* $G$ *is an integral transform of* $g$ *given by*

$$(2.14)\qquad G(x) = \frac{1}{2\pi i} \int_{\Re s = -\sigma} \widetilde{g}(s) x^s \frac{\widetilde{\gamma}(1-s)}{\gamma(s)}\, ds,$$

*where*

$$\widetilde{g}(s) = \int_0^{\infty} g(x) x^s \frac{dx}{x}$$

*is the Mellin transform of* $g$ *and*

$$\gamma(s) = \pi^{-ms/2} \prod_{i=1}^{m} \Gamma\Big(\frac{s - \lambda_i(\nu)}{2}\Big),$$
$$\widetilde{\gamma}(s) = \pi^{-ms/2} \prod_{i=1}^{m} \Gamma\Big(\frac{s - \widetilde{\lambda}_i(\nu)}{2}\Big).$$

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It is known that the symmetric-square lift of a holomorphic modular form is associated with $\mathrm{GL}_3(\mathbb{Z})$ automorphic distribution, and so we will use the GL(3) Voronoi summation formula proved by Miller and Schmid [31, Theorem 1.18]. The version here is given in [10, Theorem 4.2]).

***Lemma 2.7*** *Let F be the symmetric-square lift of a holomorphic Hecke eigenform of weight k for* $\mathrm{SL}_2(\mathbb{Z})$. *Let g be a compactly supported smooth function on* $(0,\infty)$. *Then we have*

$$\begin{aligned}\sum_{n=1}^{\infty} A_F(1,n)e_q(an)g(n) &= q\sum_{d_1|q}\sum_{n\geq 1}\frac{A_F(n,d_1)}{d_1 n}S(\overline{a},n;q/d_1)G_+\Big(\frac{nd_1^2}{q^2}\Big)\\ &\quad + q\sum_{d_1|q}\sum_{n\geq 1}\frac{A_F(n,d_1)}{d_1 n}S(\overline{a},-n;q/d_1)G_-\Big(\frac{nd_1^2}{q^2}\Big),\end{aligned}\tag{2.15}$$

*where for* $\eta\in\{0,1\}$,

$$\begin{aligned}G_\eta(x) &= \frac{1}{2\pi i}\int_{\Re s=-\sigma}\widetilde{g}(s)(\pi^3 x)^s\frac{\Gamma\big(\frac{1-s+k-\eta}{2}\big)\Gamma\big(\frac{1-s+k-1+\eta}{2}\big)\Gamma\big(\frac{1-s+1-\eta}{2}\big)}{\Gamma\big(\frac{s+k-\eta}{2}\big)\Gamma\big(\frac{s+k-1+\eta}{2}\big)\Gamma\big(\frac{s+1-\eta}{2}\big)}\,ds,\\ G_+(x) &= \frac{1}{2\pi^{3/2}}\big(G_0(x)-iG_1(x)\big),\\ G_-(x) &= \frac{1}{2\pi^{3/2}}\big(G_0(x)+iG_1(x)\big).\end{aligned}$$

We also need a lemma of Kıral and Zhou [24], which shows that the average of the hyper-Kloosterman sum on the right hand side of (2.13) against a Dirichlet character becomes a product of $(m-2)$ Gauss sums. The following lemma is only a special case of [24, Lemma 3.4].

***Lemma 2.8*** *Let* $\chi$ *be a Dirichlet character modulo c that is induced from the primitive character* $\chi^*$ *modulo* $c^*$. *Let* $\mathbf{d}=(d_1,\dots,d_{m-2})$ *be a tuple of positive integers, and assume that the divisibility condition* $d_1\cdots d_{m-3}d_{m-2}|c$ *is met. To simplify notation, we set*

$$\xi_i=\frac{c}{d_1\cdots d_i}.\tag{2.16}$$

*Consider the summation*

$$S:=\sum_{\substack{a \bmod c\\(a,c)=1}}\chi(a)KL_{m-2}(a,n;\mathbf{d},c).$$

*The quantity S is zero unless the divisibility condition*

$$d_{m-2}c^*\ \Big|\ \frac{c}{d_1\cdots d_{m-3}},$$

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*is satisfied. Under such divisibility condition, S can be written as a product of Gauss sums*

$$S = g(\chi^*, c, d_1) g(\chi^*, \xi_1, d_2) \cdots g(\chi^*, \xi_{m-3}, d_{m-2}) g(\chi^*, \xi_{m-2}, n),$$

*where the Gauss sum of $\chi$ is defined by*

$$g(\chi^*, c, m) = \sum_{\substack{u \mod c \\ (u,c)=1}} \chi(u) e\Big(\frac{mu}{c}\Big),$$

*and the standard Gauss sum for $\chi^*$ is given as $\tau(\chi^*) = g(\chi^*, c^*, 1)$.*

We formulate the asymptotic formula for the integral transform $G(x)$ in the following lemma, which is a transcription of [36, Theorem 1.1].

***Lemma 2.9*** *Let F be a Hecke-Maass form for $\mathrm{SL}_m(\mathbb{Z})$. Let $m \geq 3$ be an integer. Let $G(x)$ be defined as in (2.14) with $g(y) = \phi\big(\frac{y}{X}\big)$, where $\phi(x)$ is a fixed smooth function of compact support on $[a, b]$ with $b > a > 0$. Then for any $x > 0$, $xX \gg 1$, and $K > m/2$, we have*

$$\begin{aligned} G(x) = x \sum_{k=0}^{K} c_k \int_0^\infty &(xy)^{1/(2m)-1/2-k/m} g(y) \\ &\times \Big\{ i^{k+(m-1)/2} e\big(m(xy)^{1/m}\big) + (-i)^{k+(m-1)/2} e\big(-m(xy)^{1/m}\big) \Big\}\, dy \\ &+ O\big((xX)^{-K/m+1/2+\varepsilon}\big), \end{aligned}$$

*where $c_k$, $k = 0, \ldots, K$, are constants depending on $m$ and $\{\lambda_j(\nu)\}_{j=1,\ldots,m}$ with $c_0 = -1/\sqrt{m}$, and the implied constant depends at most on F, g, K, a, b, and ε.*

### 2.1.2 Proof of Propositions 2.2 and 2.3

Throughout this section, we choose a smooth compactly supported function $g$ with $g(x) = 1$ for $x \in [X, 2X]$, $g(x) = 0$ for $x \geq 2X + X/J$ and $x \leq X - X/J$, and $g^{(j)}(x) \ll (X/J)^{-j}$ for $1 \leq J < X$ and all integers $j \geq 0$. Here $J$ is a parameter that will be chosen for optimizing the estimates later. Then we have

$$\tag{2.17} \begin{aligned} \sum_{\substack{n \sim X \\ n \equiv a \bmod q}} A_F(1, \ldots, 1, n) = &\sum_{n \equiv a \bmod q} A_F(1, 1, \ldots, n) g(n) \\ &+ O\Bigg(\Bigg(\sum_{\substack{X - X/J < n \leq X \\ n \equiv a \bmod q}} + \sum_{\substack{2X \leq n \leq 2X + X/J \\ n \equiv a \bmod q}}\Bigg) |A_F(1, 1, \ldots, n)|\Bigg), \end{aligned}$$

where the condition $n \sim X$ denotes $X < n \leq 2X$. Inserting (2.4), the error terms on the right-hand side of (2.17) can be controlled by

$$\tag{2.18} O\Big(\frac{X^{1+\theta_m+\varepsilon}}{Jq}\Big).$$

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We detect the congruence condition with additive characters in the smoothed sum and obtain that

$$\sum_{n\equiv a \bmod q} A_F(1,\dots,1,n)g(n) = \frac{1}{q}\sum_{c|q}\sum_{h \bmod c}^{*} e\Big(-\frac{ah}{c}\Big)\sum_{n} A_F(1,1,\dots,n)e\Big(\frac{nh}{c}\Big)g(n).$$

We first consider the cases where $m \geq 3$. The Voronoi summation formula in Lemma 2.6 yields

$$\begin{aligned}\sum_{n\equiv a \bmod q} & A_F(1,\dots,1,n)g(n) \\ &= \frac{1}{q}\sum_{c|q} c \sum_{d_1|c}\sum_{d_2|\frac{c}{d_1}}\cdots\sum_{d_{m-2}|\frac{c}{d_1\cdots d_{m-3}}}\sum_{n\neq 0}\frac{A_F(n,d_{m-2},\dots,d_2,d_1)}{d_1\cdots d_{m-2}|n|} \\ &\quad\times T_{m-1}(-a,n;\mathbf{d},c)G\Big(\frac{|n|\prod_{i=1}^{m-2} d_i^{m-i}}{c^m}\Big),\end{aligned}$$

where

$$T_{m-1}(-a,n;\mathbf{d},c) = \sum_{h \bmod c}^{*} e\Big(-\frac{ah}{c}\Big) KL_{m-2}((-1)^m\overline{h},n;\mathbf{d},c).$$

For our requirement, we will give an upper bound for the hyper-Kloosterman sum on the right-hand side of (2.13) after averaging against an additive character, and expect that $T_{m-1}(-a,n;\mathbf{d},c)$ cancels to order square root of the number of terms. In fact, by using variable substitution,

$$T_{m-1}(-a,n;\mathbf{d},c) = KL_{m-1}\big((-1)^{m-1}a,n;\mathbf{d_0},c\big),$$

where $\mathbf{d_0} = (1,d_1,\dots,d_{m-2})$.

***Lemma 2.10*** *Let $m \geq 3$, and let $T_{m-1}(-a,n;\mathbf{d},c)$ be defined as above. Then we have*

$$T_{m-1}(-a,n;\mathbf{d},c) \ll (\xi_{m-2},n)\xi_{m-2}^{\frac{m-1}{2}}(\log\xi_{m-2})\, d^T(\xi_{m-2}),$$

*where the definition of the symbols $\xi_{m-2}$ is in (2.16), and $T$ is some positive constant depending only on $m$.*

**Proof** The classical hyper-Kloosterman sum is defined as

$$S(a_1,\dots,a_m;q) = \sum_{\mathbf{x} \bmod q}^{*} e_q(a_1x_1+\cdots+a_mx_m),$$

where $\prod x_i \equiv 1 \pmod q$. We will use Weinstein's version of Deligne's result [46], which states

$$S(a_1,\dots,a_m;q) \leq 2^{\frac{m+1}{2}} m^{\nu(q)} q^{\frac{m-1}{2}}(a_1,a_m,q)^{\frac{1}{2}}\cdots(a_{m-1},a_m,q)^{\frac{1}{2}}. \tag{2.19}$$

Here $\nu(q)$ denotes the number of different prime factors of $q$, and $m^{\nu(q)} \ll d_m(q)$ for any fixed $m$. The additive character $e(-\frac{ah}{c})$ can be expressed in terms of the multiplicative ones by Gauss sums (see [16, Equation (3.11)]),

$$e\Big(-\frac{ah}{c}\Big) = \frac{1}{\varphi(c)} \sum_{\chi \mod c} \overline{\chi}(-ah)\tau(\chi).$$

Note that the condition $(ah, c) = 1$ holds here. Hence, we derive that

$$T_{m-1}(-a, n; \mathbf{d}, c) = \frac{1}{\varphi(c)} \sum_{\chi \mod c} \overline{\chi}(-a)\tau(\chi) \sideset{}{^*}\sum_{h \mod c} \chi(\overline{h}) KL_{m-2}((-1)^m \overline{h}, n; \mathbf{d}, c). \tag{2.20}$$

By Lemma 2.8, the innermost sum on the right-hand side of (2.20) is zero unless the divisibility condition $c^*|\xi_{m-2}$ holds. And in this case, after changing the variable $(-1)^m\overline{h} \to h$, the summation over $h$ mod $c$ is equal to

$$\overline{\chi}((-1)^m) g(\chi^*, c, d_1) g(\chi^*, \xi_1, d_2) \cdots g(\chi^*, \xi_{m-3}, d_{m-2}) g(\chi^*, \xi_{m-2}, n).$$

Combining with (see [34, Theorem 9.12])

$$g(\chi^*, c, a) = \tau(\chi^*) \frac{\varphi(c)}{\varphi(\frac{c}{(c,a)})} \mu\Big(\frac{c}{c^*(c,a)}\Big) \chi^*\Big(\frac{c}{c^*(c,a)}\Big) \overline{\chi^*}\Big(\frac{a}{(c,a)}\Big),$$

we obtain

$$\begin{aligned} T_{m-1}(-a, n; \mathbf{d}, c) &= \sum_{c^*|\xi_{m-2}} \frac{1}{\varphi(\frac{\xi_{m-2}}{(\xi_{m-2},n)})} \mu\Big(\frac{c}{c^*}\Big)\mu\Big(\frac{\xi_1}{c^*}\Big) \cdots \mu\Big(\frac{\xi_{m-2}}{c^*}\Big)\mu\Big(\frac{\xi_{m-2}}{(\xi_{m-2},n)}\Big) \\ &\quad \times \sum_{\chi^* \mod c^*} \chi^*\Big(\frac{c\xi_1\cdots\xi_{m-2}}{c^{*m}(\xi_{m-2},n)} \cdot \overline{\frac{(-1)^{m+1}an}{(\xi_{m-2},n)}}\Big) \tau(\chi^*)^m. \end{aligned} \tag{2.21}$$

Put the variable

$$b = \frac{c\xi_1\cdots\xi_{m-2}}{c^{*m}(\xi_{m-2},n)} \cdot \overline{\frac{(-1)^{m+1}an}{(\xi_{m-2},n)}}.$$

Now our task is to estimate

$$\sum_{\chi^* \mod c^*} \chi^*(b)\tau(\chi^*)^m. \tag{2.22}$$

Clearly, this summation over all primitive characters $\chi^*$ mod $c^*$ is multiplicative in $c^*$. More precisely, if $c^* = c_1^* c_2^*$, $(c_1^*, c_2^*) = 1$, $\chi_1^*, \chi_2^*$ are primitive characters modulus $c_1^*$ and $c_2^*$, respectively. We first have $\chi^* = \chi_1^*\chi_2^*$; further,

$$\sum_{\chi^* \mod c^*} \chi^*(b)\tau(\chi^*)^m = \sum_{\chi_1^* \mod c_1^*} \chi_1^*(bc_2^*)\tau(\chi_1^*)^m \sum_{\chi_2^* \mod c_2^*} \chi_1^*(bc_1^*)\tau(\chi_2^*)^m.$$

This property allows us to reduce the problem of evaluation (2.22) for any $c^*$ to that for prime power moduli $c^* = p^\alpha, \alpha \geq 1$. By the definition of Gauss sum $\tau(\chi^*)$ and changing the order of summations, we have

$$\sum_{\chi^* \bmod p^\alpha} \chi^*(b)\tau(\chi^*)^m = \sideset{}{^*}\sum_{a_1,\dots,a_m \bmod p^\alpha} e\Big(\frac{a_1+\cdots+a_m}{p^\alpha}\Big) \sum_{\chi^* \bmod p^\alpha} \chi^*(ba_1\cdots a_m).$$

Using the orthogonality of characters, we infer a more general result ([16, Equation (3.8)])

$$\sideset{}{^*}\sum_{\chi \bmod r} \chi(m) = \sum_{d|(m-1,r)} \varphi(d)\mu\Big(\frac{r}{d}\Big), \tag{2.23}$$

if $(r,m)=1$; and zero otherwise. By the relation above, we have

$$\begin{aligned}\sum_{\chi^* \bmod p^\alpha} \chi^*(b)\tau(\chi^*)^m &= \sum_{d|p^\alpha} \varphi(d)\mu\Big(\frac{p^\alpha}{d}\Big) \sideset{}{^*}\sum_{\substack{a_1,\dots,a_m \bmod p^\alpha \\ a_1\cdots a_m\equiv \overline{b} \bmod d}} e\Big(\frac{a_1+\cdots+a_m}{p^\alpha}\Big)\\ &= \varphi(p^\alpha) \sideset{}{^*}\sum_{\substack{a_1,\dots,a_m \bmod p^\alpha \\ a_1\cdots a_m\equiv \overline{b} \bmod p^\alpha}} e\Big(\frac{a_1+\cdots+a_m}{p^\alpha}\Big)\\ &\quad - \varphi(p^{\alpha-1}) \sideset{}{^*}\sum_{\substack{a_1,\dots,a_m \bmod p^\alpha \\ a_1\cdots a_m\equiv \overline{b} \bmod p^{\alpha-1}}} e\Big(\frac{a_1+\cdots+a_m}{p^\alpha}\Big).\end{aligned}$$

By (2.19), we have

$$\sideset{}{^*}\sum_{\substack{a_1,\dots,a_m \bmod p^\alpha \\ a_1\cdots a_m\equiv \overline{b} \bmod p^\alpha}} e\Big(\frac{a_1+\cdots+a_m}{p^\alpha}\Big) \le 2^{\frac{m+1}{2}} m p^{\frac{m+1}{2}\alpha}.$$

Moreover, we only need to deal with

$$\sideset{}{^*}\sum_{\substack{a_1,\dots,a_m \bmod p^\alpha \\ a_1\cdots a_m\equiv \overline{b} \bmod p^{\alpha-1}}} e\Big(\frac{a_1+\cdots+a_m}{p^\alpha}\Big).$$

When $\alpha=1$, it is clear that

$$\left(\sideset{}{^*}\sum_{\substack{a_1,\dots,a_m \bmod p^\alpha \\ a_1\cdots a_m\equiv \overline{b} \bmod p^{\alpha-1}}} e\Big(\frac{a_1+\cdots+a_m}{p^\alpha}\Big)\right) = 1.$$

When $\alpha>1$, we first put $a_i = s_i p^{\alpha-1} + t_i$, $i=1,2,\dots,m$. Then the condition that $a_i$ runs through a complete set of residues prime to $p^\alpha$ is equivalent to

$$0 \le s_i < p, \quad 0 \le t_i < p^{\alpha-1}, \quad (t_i,p)=1.$$

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Thus, we have

$$\sum_{\substack{a_1,\dots,a_m \mod p^\alpha \\ a_1\cdots a_m\equiv \overline{b} \mod p^{\alpha-1}}} e\left(\frac{a_1+\cdots+a_m}{p^\alpha}\right) = \sideset{}{^*}\sum_{\substack{t_1,\dots,t_m \mod p^{\alpha-1} \\ t_1\cdots t_m\equiv \overline{b} \mod p^{\alpha-1}}} e\left(\frac{t_1+\cdots+t_m}{p^\alpha}\right) \times \sum_{0\le s_i<p} e\left(\frac{s_1+\cdots+s_m}{p}\right) = 0.$$

To sum up, we get

$$\sum_{\chi^* \mod c^*} \chi^*(b)\tau(\chi^*)^m \ll d^T(c)c^{*\frac{m+1}{2}}, \tag{2.24}$$

where $T$ is some positive constant depending only on $m$. Finally, this lemma follows from inserting (2.24) into (2.21) and trivial estimate $\varphi(n) \gg n/\log n$. ∎

Our next task is to evaluate the exponential integral $G(x)$ in Voronoi summation formula (2.15).

***Lemma 2.11*** *Let $G(x)$ be defined as in (2.14) and $g(x)$ as in the beginning of this section. Then we have*

$$G(x) \ll \begin{cases} J^{-A} & \text{if } x > J^{m+\varepsilon}X^{-1}, \\ J^{\frac{m-1}{2}} & \text{if } J^m X^{-1} < x \le J^{m+\varepsilon}X^{-1}, \\ (xX)^{\frac{1}{2}-\frac{1}{2m}} & \text{if } X^{-1} \ll x \le J^m X^{-1}, \\ (xX)^{\frac{1}{2}}J^{\varepsilon} & \text{if } x \ll X^{-1}. \end{cases}$$

**Proof** After integrating by parts $j$ times, the Mellin transform of $g(x)$ satisfies

$$\widetilde{g}(\sigma+i\tau) = \frac{1}{s\cdots(s+j-1)}\int_0^\infty g^{(j)}(x)x^{s+j-1}dx \ll \frac{X^\sigma}{J}\left(\frac{J}{(1+|\tau|)}\right)^j$$

for any $j \ge 1$. Moreover, for $\sigma > -1$, Stirling's approximation gives

$$\frac{\widetilde{\gamma}(1-s)}{\gamma(s)} \ll_\pi (1+|\tau|)^{m(\frac{1}{2}-\sigma)},$$

so that

$$G(x) \ll_{\pi,\sigma,j} (xX)^{-\sigma}J^{j-1}\int_{-\infty}^{\infty}\frac{1}{(1+|\tau|)^{j-m\sigma-\frac{m}{2}}}d\tau.$$

Choose a proper $\sigma$ such that $j = m\sigma + \frac{m}{2} + 1 + \varepsilon$ is an integer. Then we have

$$G(x) \ll_\sigma J^{\frac{m}{2}+\varepsilon}\left(\frac{J^m}{xX}\right)^\sigma.$$

If $x > J^{m+\varepsilon}X^{-1}$, then $G(x)$ can be made to be $O(J^{-A})$ for $A$ arbitrarily large by taking $\sigma$ large enough.

For the second and third assertions, we learn from Lemma 2.9 that

$$G(x) \ll_{\pi,\varepsilon} x \sum_{k=0}^{K} \Big| \int_0^\infty (xy)^{\frac{1}{2m}-\frac{1}{2}-\frac{k}{m}} g(y) \Big( e\big(m(xy)^{\frac{1}{m}}\big) + e\big(-m(xy)^{\frac{1}{m}}\big) \Big) dy \Big| + (xX)^{-\frac{K}{m}+\frac{1}{2}+\varepsilon}$$

for $x \gg X^{-1}$, $K > m/2$. Then we get by partial integration that

$$\begin{aligned} G(x) &\ll \sum_{k=0}^{K} (xX)^{\frac{1}{2}+\frac{1}{2m}-\frac{k+j}{m}} J^{j-1} + (xX)^{-\frac{K}{m}+\frac{1}{2}+\varepsilon} \\ &\ll (xX)^{\frac{1}{2}+\frac{1}{2m}-\frac{j}{m}} J^{j-1} \end{aligned}$$

for any $j \geq 1$. If $J^m X^{-1} < x \leq J^{m+\varepsilon} X^{-1}$, we take $j = \frac{m+1}{2}$ when $m$ is odd, and otherwise $j = \frac{m}{2} + 1$. However, $G(x)$ is always less than $J^{\frac{m-1}{2}}$ in this case. If $X^{-1} \ll x \leq J^m X^{-1}$, we only require $j = 1$.

For smaller $x \ll X^{-1}$, we take $\sigma = -\frac{1}{2}$ (using the result of Jacquet–Shalika). This gives $G(x) \ll (xX)^{\frac{1}{2}} J^{\varepsilon}$. This gives a complete proof of Lemma 2.11. ■

Now we continue the argument. By Lemma 2.10, we have

$$\begin{aligned} &\sum_{n \equiv a \bmod q} A_F(1,\dots,1,n) g(n) \\ &\ll \frac{d^T(q)\log q}{q} \sum_{c|q} c^{\frac{m+1}{2}} \sum_{d_1|c} \sum_{d_2|\frac{c}{d_1}} \cdots \sum_{d_{m-2}|\frac{c}{d_1\cdots d_{m-3}}} \\ &\quad \times \sum_{n\neq 0} \frac{(\xi_{m-2},n)|A_F(n,d_{m-2},\dots,d_2,d_1)|}{\prod_{i=1}^{m-2} d_i^{\frac{m+1}{2}} |n|} \Big| G\Big( \frac{|n|\prod_{i=1}^{m-2} d_i^{m-i}}{c^m} \Big) \Big|. \end{aligned}$$

It follows by Lemma 2.11 that the contribution from the terms with

$$|n| \prod_{i=1}^{m-2} d_i^{m-i}/c^m > J^{m+\varepsilon} X^{-1}$$

is negligibly small. For the smaller values of $|n|\prod_{i=1}^{m-2} d_i^{m-i}/c^m$, we use the other estimates in Lemma 2.11. Thus, we decompose the sum into three parts $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ with

$$\begin{aligned} \frac{(cJ)^m}{X\prod_{i=1}^{m-2} d_i^{m-i}} \leq |n| \leq \frac{(cJ)^{m+\varepsilon}}{X\prod_{i=1}^{m-2} d_i^{m-i}}, \\ \frac{c^m}{X\prod_{i=1}^{m-2} d_i^{m-i}} \leq |n| \leq \frac{(cJ)^{m}}{X\prod_{i=1}^{m-2} d_i^{m-i}} \end{aligned}$$

and

$$|n| < \frac{c^m}{X\prod_{i=1}^{m-2} d_i^{m-i}},$$

respectively.

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First we will estimate $\mathcal{R}_1$. We have

$$\mathcal{R}_1 \ll J^{\frac{m-1}{2}} \frac{d^T(q)\log q}{q} \sum_{c|q} c^{\frac{m+1}{2}} \sum_{d_1|c} \sum_{d_2|\frac{c}{d_1}} \cdots \sum_{d_{m-2}|\frac{c}{d_1\cdots d_{m-3}}} \frac{1}{\prod_{i=1}^{m-2} d_i^{\frac{m+1}{2}}} \times \sum_{\frac{(cJ)^m}{X\prod_{i=1}^{m-2} d_i^{m-i}} \le |n| \le \frac{(cJ)^{m+\varepsilon}}{X\prod_{i=1}^{m-2} d_i^{m-i}}} \frac{(\xi_{m-2}, n)|A_F(n, d_{m-2}, \dots, d_2, d_1)|}{|n|}. \tag{2.25}$$

By the multiplicative relation (2.2) of Fourier coefficients and (2.5), one has

$$|A_F(n, d_{m-2}, \dots, d_2, d_1)| \le \Big( \prod_{i=1}^{m-2} d_i^{m-i} |n| \Big)^{\theta_m} d^T(n d_{m-2} \cdots d_2 d_1).$$

And from (2.29), it is easy to see that

$$\sum_{|n|\le x} |A_F(n, 1, \dots, 1)| \le x. \tag{2.26}$$

Hence, we have

$$\begin{aligned}
&\sum_{|n|\le x} (\xi_{m-2}, n)|A_F(n, d_{m-2}, \dots, d_2, d_1)| \\
&\quad\le \sum_{l|\xi_{m-2}} l \sum_{|n|\le x/l} |A_F(ln, d_{m-2}, \dots, d_2, d_1)| \\
&\quad\le \sum_{l|\xi_{m-2}} l \sum_{r|(d_1 d_2\cdots d_{m-2} l)^\infty} \sum_{\substack{|n|\le x/(|r|l),\\ (n, r d_1 d_2 \cdots d_{m-2})=1}} |A_F(rln, d_{m-2}, \dots, d_2, d_1)| \\
&\quad\le \sum_{l|\xi_{m-2}} l \sum_{r|(d_1 d_2\cdots d_{m-2} l)^\infty} |A_F(rl, d_{m-2}, \dots, d_2, d_1)| \\
&\quad\quad \times \sum_{|n|\le x/(|r|l)} |A_F(n, 1, \dots, 1)| \\
&\quad\ll \prod_{i=1}^{m-2} d_i^{\frac{m-i}{2}+\varepsilon} \xi_{m-2}^{\theta_m} d^T(\xi_{m-2}) x.
\end{aligned}$$

For convenience, we have enlarged the power of $d_i$, namely $\frac{m-i}{2}$ instead of $(m-i)\theta_m$. This cannot make any difference to our final result. By partial summation for (2.25), we get

$$\mathcal{R}_1 \ll J^{\frac{m-1}{2}} q^{\frac{m-1}{2}+\theta_m} d^T(q)(\log^2 q)\log J.$$

Next we estimate $\mathcal{R}_2$ and $\mathcal{R}_3$ similarly and obtain

$$\mathcal{R}_2 \ll J^{\frac{m-1}{2}} q^{\frac{m-1}{2}+\theta_m} d^T(q)\log q, \quad \mathcal{R}_3 \ll J^{\varepsilon} q^{\frac{m-1}{2}+\theta_m+\varepsilon}.$$

Joining the estimates of $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, we have

$$\sum_{n\equiv a \bmod q} A_F(1, \dots, 1, n) g(n) \ll J^{\frac{m-1}{2}} q^{\frac{m-1}{2}+\theta_m} d^T(q)(\log^2 q)\log J.$$

For $q \le X^{\frac{2(1+\theta_m)}{m+1+2\theta_m}}$, we can choose $J = X^{\frac{2}{m+1}(1+\theta_m)}/q^{1+\frac{2\theta_m}{m+1}}$, which proves that

$$\sum_{\substack{n\sim X\\ n\equiv a \bmod q}} A_F(1, \dots, 1, n) \ll q^{\frac{2\theta_m}{m+1}} X^{(1-\frac{2}{m+1})(1+\theta_m)} d^T(q)(\log^2 q)\log X.$$

On summing up all the dyadic intervals, it follows that

$$\sum_{\substack{n\le x\\ n\equiv a \bmod q}} A_F(1,\dots,1,n) \ll q^{\frac{2\theta_m}{m+1}} x^{(1-\frac{2}{m+1})(1+\theta_m)} d^T(q)(\log^2 q)\log^2 x.$$

Since $F(z)$ is an eigenfunction of all the Hecke operators,

$$A_F(n,1,\dots,1) = \overline{A_F(1,\dots,1,n)}.$$

So the proof of Proposition 2.2 for the cases $m \ge 3$ is completed.

If $m = 2$, using the Voronoi formula in Lemma 2.5, we have

$$\sum_{n\equiv a \bmod q} A_F(n)g(n) = \frac{1}{q}\sum_{c|q}\sum_{\pm}\sum_{n=1}^{\infty} A_F(\mp n)S(-a,\pm n;c)g^{\pm}(n). \tag{2.27}$$

Observe that the greatest common divisor $(a,c)$ is 1. We will make use of the usual Weil bound for Kloosterman sum $S(-a,\pm n;c)$,

$$S(-a,\pm n;c) \ll c^{\frac{1}{2}+\varepsilon}.$$

We estimate $g^{\pm}(n)$ by successive applications of integration by parts and the relations

$$\frac{d}{dz}\big(z^s K_s(z)\big) = -z^s K_{s-1}(z), \quad \frac{d}{dz}\big(z^s Y_s(z)\big) = z^s Y_{s-1}(z);$$
$$K_s(z) \ll_s z^{-1/2}, \quad Y_s(z) \ll_s z^{-1/2}, \qquad z > 0.$$

We get

$$g^{\pm}(n) \ll \Big(\frac{cJ}{\sqrt{nX}}\Big)^{j+\frac{1}{2}} \frac{X}{cJ^{\frac{3}{2}}}. \tag{2.28}$$

It suggests that we can neglect the contribution to (2.27) for these term $n > \frac{(cJ)^{2+\varepsilon}}{X}$. The choice $j = 1$ in (2.28) shows that the remaining terms (for which $n$ are at most $O(\frac{(cJ)^{2+\varepsilon}}{X})$) contribute

$$\frac{1}{q}\sum_{c|q} c^{\frac{1}{2}+\varepsilon} \sum_{|n|\le \frac{(cJ)^{2+\varepsilon}}{X}} |A_F(n)g^{\pm}(n)| \ll \frac{1}{q}\sum_{c|q} c^{1+\varepsilon}X^{\frac{1}{4}} \sum_{|n|\le\frac{(cJ)^{2+\varepsilon}}{X}} |A_F(n)|n^{-\frac{3}{4}}$$
$$\ll (qJ)^{\frac{1}{2}+\varepsilon}.$$

Taking $J = X^{\frac{2}{3}(1+\theta_m)}/q$ yields

$$\sum_{\substack{n\sim X\\ n\equiv a \bmod q}} A_F(n) \ll X^{\frac{1}{3}(1+\theta_m)}$$

for $q \le X^{\frac{2}{3}(1+\theta_m)}$. Proposition 2.2 for this case follows by summing all the dyadic intervals.

Now we turn to proving Proposition 2.3. Let $L(s, F\times\widetilde{F})$ denote the Rankin–Selberg $L$-function of $F$. It is defined by

$$L(s,F\times\widetilde{F}) = \zeta(ms)\sum_{n_1=1}^{\infty}\cdots\sum_{n_{m-1}=1}^{\infty}\frac{|A_F(n_1,\dots,n_{m-1})|^2}{(n_1^{m-1}n_2^{m-2}n_{m-1})^s} := \sum_{n=1}^{\infty}\frac{A_{F\times\widetilde{F}}(n)}{n^s},$$

where

$$A_{F\times\widetilde{F}}(n)=\sum_{n=n_0^m n_1^{m-1} n_2^{m-2} n_{m-1}} |A_F(n_1,\dots,n_{m-1})|^2.$$

Then we derive by the refinement of Landau's Lemma [3, Theorem 3.2] that

$$\sum_{n\le x} A_{F\times\widetilde{F}}(n)=c_F x+O\big(x^{\frac{m^2-1}{m^2+1}}\big), \tag{2.29}$$

for some constant $c_F>0$. Therefore, by Cauchy's inequality we get

$$\begin{aligned}\sum_{\substack{x\le n\le x+y\\ n\equiv a \bmod q}} |A_F(n,1,\dots,1)| &\ll \Big(1+\frac{y}{q}\Big)^{\frac12}\Big(\sum_{x\le n\le x+y}|A_F(n,1,\dots,1)|^2\Big)^{\frac12}\\ &\ll \Big(1+\frac{y}{q}\Big)^{\frac12}\big(x^{\frac{m^2-1}{2(m^2+1)}}+y^{\frac12}\big).\end{aligned} \tag{2.30}$$

When $m=2$, inserting Lemma 3.1 and inequality (2.6) in [17], we obtain

$$\sum_{x<n<x+y}|A_F(n)|^8 \ll x^{\varepsilon}\sum_{x<n<x+y}\sum_{d|n}|A_F(d^4)|^2 \ll x^{\varepsilon}y+x^{\frac{25}{27}+\varepsilon}.$$

Further, we get

$$\begin{aligned}\sum_{\substack{x\le n\le x+y\\ n\equiv a \bmod q}} |A_F(n)| &\ll \Big(1+\frac{y}{q}\Big)^{\frac78}\Big(\sum_{x\le n\le x+y}|A_F(n)|^8\Big)^{\frac18}\\ &\ll \Big(1+\frac{y}{q}\Big)^{\frac78}\big(x^{\frac{25}{216}+\varepsilon}+x^{\varepsilon}y^{\frac18}\big).\end{aligned} \tag{2.31}$$

Instead of (2.18), we insert the estimates (2.30) and (2.31) into (2.17), respectively; then the corresponding result follows.

## 2.2 Prime Number Theorem for the Twisted $L$-function $L(s,F\otimes\chi)$

Let $\chi$ be a primitive character modulo $q$, and let $F(z)$ be a Hecke–Maass form for $\mathrm{SL}_m(\mathbb{Z})$. Actually, $\chi$ corresponds to a Hecke character of the idele class group $\mathbb{A}^\times/\mathbb{Q}^\times$ trivial on $R_+^\times$. We know from [5, p. 305] that $F\otimes\chi$ is a cuspidal automorphic form for $\mathrm{GL}(m)$ with central character $\chi^m$. We apply [16, Theorem 5.13] to the twisted $L$-function $L(s,F\otimes\chi)$ and get

$$\sum_{n\le x}\Lambda_{F\otimes\chi}(n)=-\frac{x^\beta}{\beta}+O\big(q^{\frac m2}x\exp(-c\sqrt{\log x})\big), \tag{2.32}$$

where $\beta$ is the exceptional real zero, $c$ is a positive absolute constant and the implied constant is absolute. Note that

$$\Lambda_{F\otimes\chi}(n)=\Lambda(n)a_F(n)\chi(n)=\Lambda_F(n)\chi(n).$$

If $m=1$, Siegel's famous theorem shows that $L(s,\chi)$ has no zeros in the interval

$$\Big[1-\frac{c(\varepsilon)}{q^\varepsilon},1\Big]. \tag{2.33}$$

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This estimate implies the Siegel–Walfisz theorem, which states that for any fixed $A > 0, q \le \log^A x$,

$$\sum_{n\le x} \Lambda(n)\chi(n) \ll x\exp(-c\sqrt{\log x}).$$

The existence of the exceptional zero, even the analog of Siegel's theorem (2.33) does not seem to be known for $L$-function of $F\otimes\chi$ for all $m \ge 1$, but some cases are known. It has been proved that the exceptional zero does not exist by Hoffstein and Ramakrishnan [15] for $m = 2$ and by Banks [1] for $m = 3$. Moreover, Siegel-type theorem is available by the works of Kim and Shahidi [22, 23], Kim [20], and Molteni [33, Theorem 2.32] when $m = 4, 5$ and $F$ is the symmetric power lift of a Hecke-Maass form for $SL_2(\mathbb{Z})$, which means that $L(s, F\otimes\chi)$ has no zeros in the interval

$$\Big[1 - \frac{c(F,\varepsilon)}{q^{\varepsilon}}, 1\Big]. \tag{2.34}$$

Therefore, together with (2.32), we will obtain the following proposition.

***Proposition 2.12*** Let $\chi$ mod $q$ be any primitive character, and let $F(z)$ be a Hecke–Maass form for $SL_m(\mathbb{Z})$. Then we have

$$\sum_{n\le x} \Lambda_F(n)\chi(n) \ll q^{\frac{m}{2}} x\exp(-c\sqrt{\log x}) \tag{2.35}$$

unconditionally for $2 \le m \le 3$ and under the assumption that there exists no exceptional zero for $m \ge 4$.

Moreover, if $F(z)$ is the symmetric power lift of a Hecke-Maass form for $SL_2(\mathbb{Z})$ with $m = 4, 5$, we have for any fixed $A > 0, q \le \log^A x$,

$$\sum_{n\le x} \Lambda_F(n)\chi(n) \ll x\exp(-c\sqrt{\log x}) \tag{2.36}$$

where the implied constant depends on $F, A$.

**Proof** Clearly, (2.35) directly follows from (2.32). For the assertion (2.36), taking $\varepsilon = \frac{1}{2A}$ in (2.34), we have

$$\beta \le 1 - c(F,A)q^{-\frac{1}{2A}} \le 1 - c(F,A)(\log x)^{-\frac{1}{2}}, \tag{2.37}$$

due to $q \le \log^A x$, where $c(F, A)$ is some constant depending only on $F$ and $A$. Thus, (2.36) follows from inserting (2.37) into (2.32). ∎

## 2.3 Estimates of Some Arithmetic Functions

In this section, we will use the Rankin–Selberg theory to evaluate sums of arithmetic functions.

***Lemma 2.13*** *Assume that Hypothesis H holds. Let $F(z)$ be a Hecke-Maass form for $\mathrm{SL}_m(\mathbb{Z})$. Then we have*

$$\sum_{n\le x} d(n)|\mu_F(n)|^2 \ll x\log x,$$
$$\sum_{n\le x} d(n)|A_F(n,1,\dots,1)|^2 \ll x\log x,$$
$$\sum_{n\le x} d(n)|a_F(n)|^2 \ll x\log x,$$
$$\sum_{n\le x} |\mu_F(n)|^2 \ll x,$$

*where the implied constants depend on F only.*

**Proof** To the arithmetic function $d(n)|\mu_F(n)|^2$, we attach the Dirichlet series

$$D(\mu_F,s)=\sum_{n=1}^{\infty} d(n)|\mu_F(n)|^2 n^{-s}.$$

We decompose the attached Dirichlet series $D(\mu_F,s)$ into some functions whose properties are well known. From the definition of $\mu_F(n)$, we know that it is multiplicative. Then Dirichlet series $D(\mu_F,s)$ has Euler product

$$D(\mu_F,s)=\prod_p\Big(1+\frac{2|\mu_F(p)|^2}{p^s}+\frac{3|\mu_F(p^2)|^2}{p^{2s}}+\dots+\frac{(m+1)|\mu_F(p^m)|^2}{p^{ms}}\Big).$$

Here, $\mu_F(p)=-\sum_{j=1}^m \alpha_F(p,j)$. We can treat this infinite series as a rational function in $p^{-s}$. In particular, the coefficient of $p^{-s}$ is

$$2\sum_{j=1}^m\sum_{i=1}^m \alpha_F(p,j)\overline{\alpha_F(p,i)}.$$

Recall that

$$L(s,F\times F)=\prod_p\prod_{1\le j\le m}\prod_{1\le i\le m}\bigg(1-\frac{\alpha_F(p,j)\overline{\alpha_F(p,i)}}{p^s}\bigg)^{-1}.$$

Then we see that $D(\mu_F,s)$ has the same coefficients of $p^{-s}$ as $L^2(s,F\times F)$. Write

$$D(\mu_F,s)=L^2(s,F\times F)U(s).$$

Then a straightforward calculation shows that $U(s)=\prod_p U_p(s)$, where

$$U_p(s)=1+O\bigg(\sum_{\nu\ge2}\frac{|a_F(p^\nu)|^2}{p^{\nu\sigma}}\bigg).$$

Put $\eta_m:=\frac{1}{2}(1-2\theta_m-\varepsilon)>0$, where $\theta_m$ is given by (2.5) and $\varepsilon>0$ is sufficiently small. In view of (2.7), we see that

$$|a_F(p^\nu)|\le mp^{\theta_m\nu}$$

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for all primes $p$ and integers $\nu \geq 1$. From this we deduce that, for any $\sigma \geq 1-\varepsilon$,

$$\sum_{\nu \geq [1/(2\eta_m)]+2} \sum_p \frac{|a_F(p^\nu)|^2}{p^{\nu\sigma}} \ll \sum_p \sum_{\nu \geq [1/(2\eta_m)]+2} \frac{1}{p^{\nu(1-2\theta_m-\varepsilon)}}$$
$$\ll \sum_p \sum_{\nu \geq [1/(2\eta_m)]+2} \frac{1}{p^{2\eta_m \nu}} \ll \sum_p \frac{1}{p^{1+2\eta_m}} \ll 1.$$

Further, we derive that

$$\log|U(s)| \ll \sum_p \log|U_p(s)| \ll \sum_p \sum_{\nu \geq 2} \frac{|a_F(p^\nu)|^2}{p^{\nu\sigma}}$$
$$\ll \sum_{2 \leq \nu \leq [1/(2\eta_m)]+2} \sum_p \frac{|a_F(p^\nu)|^2}{p^{\nu\sigma}} + 1 \ll 1$$

providing $\sigma = 1$ under Hypothesis H. Thus, $U(s)$ converges absolutely in $\operatorname{Re} s \geq 1$. By a standard use of the Wiener–Ikehara Theorem, we have

$$\sum_{n \leq x} d(n)|\mu_F(n)|^2 \ll x \log x.$$

Because we can prove other results in a similar way to the first assertion, we choose to omit them completely. ∎

## 3 Proof of the Theorems for the von Mangoldt Function

It follows from the definition of $\Lambda_F(n)$, (2.4) and (2.6) that

$$\sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \Lambda(n) A_F(n,1,\ldots,1) = \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \Lambda_F(n) + O\big(q^{-1} x^{\frac{1}{2}+\theta_m+\varepsilon}\big). \tag{3.1}$$

Set

$$W_F(x,\chi) = \sum_{n \leq x} \Lambda_F(n)\chi(n).$$

By using the orthogonality relation of Dirichlet characters and splitting the sum dyadically, we get

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \Lambda_F(n) &= \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \sum_{n \leq x} \Lambda_F(n)\chi(n) \\ &\ll \frac{1}{\varphi(q)} \sum_{r \mid q} \Big| \sum_{\chi \bmod r}^{*} \overline{\chi}(a) W_F(x, \chi\chi^0) \Big|, \end{aligned} \tag{3.2}$$

where $\chi^0 \bmod q$ is the principal character. For $\chi \bmod r$ and $\chi^0 \bmod q$ in the last line,

$$W_F(x,\chi\chi^0) - W_F(x,\chi) \ll \sum_{\substack{n \leq x \\ (n,q)>1}} |\Lambda_F(n)| \ll x^{\theta_m} (\log q) \log x \ll x^{\theta_m+\varepsilon}.$$

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Therefore, we can replace $W_F(x,\chi\chi^0)$ by $W_F(x,\chi)$ up to an error $x^{\theta_m+\varepsilon}$. Hence, we obtain from (3.1) and (3.2) that

$$\begin{aligned}\sum_{q\le Q}\max_{(a,q)=1}\Big|\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\Lambda(n)A_F(n,1,\ldots,1)\Big| &= \sum_{q\le Q}\frac{1}{\varphi(q)}\sum_{r|q}\Big|\sideset{}{^*}\sum_{\chi \bmod r}\overline{\chi}(a)W_F(x,\chi)\Big| + x^{\frac12+\theta_m+\varepsilon}\\ &\ll \big(\log^2 Q\big)\sum_{r\le Q}\frac{1}{r}\Big|\sideset{}{^*}\sum_{\chi \bmod r}\overline{\chi}(a)W_F(x,\chi)\Big| + x^{\frac12+\theta_m+\varepsilon}.\end{aligned}$$

The term $x^{\frac12+\theta_m+\varepsilon}$ is acceptable. By dyadic arguments, it suffices to estimate

$$\sum_{r\sim R}\Big|\sideset{}{^*}\sum_{\chi \bmod r}\overline{\chi}(a)W_F(x,\chi)\Big| \tag{3.3}$$

with $R\le Q$.

If $R\le \log^C x$ with an arbitrary $C>0$, then by Proposition 2.12,

$$W_F(x,\chi)\ll x\exp(-c\sqrt{\log x})$$

for some $c>0$, and hence (3.3) is true in this case.

If $\log^C x<R\le Q$, applying Vaughan's identity of $\Lambda_F(n)$ in Lemma 2.1 with $X=Y<x$ for Godement–Jacquet $L$-function $L(s,F)$, we have

$$\sum_{n\le x}\Lambda_F(n)\chi(n)=S_1+S_2-S_3+S_4,$$

where

$$\begin{aligned}S_1&=\sum_{n\le X}\Lambda_F(n)\chi(n),\\ S_2&=\sum_{b\le X}\mu_F(b)\chi(b)\sum_{c\le x/b}A_F(c,1,\ldots,1)\log c\,\chi(c),\\ S_3&=\sum_{b\le X}\sum_{c\le X}\sum_{d\le x/bc}\mu_F(b)\Lambda_F(c)A_F(d,1,\ldots,1)\chi(bcd)\\ &=\sum_{m\le X^2}\Bigg(\sum_{\substack{bc=m\\ b\le X,c\le X}}\mu_F(b)\Lambda_F(c)\Bigg)\chi(m)\sum_{d\le x/m}A_F(d,1,\ldots,1)\chi(d),\\ S_4&=\sum_{b>X}\sum_{c>X}\sum_{d\le x/bc}\mu_F(b)\Lambda_F(c)A_F(d,1,\ldots,1)\chi(bcd)\\ &=\sum_{X<m<x/X}\Bigg(\sum_{\substack{bd=m\\ b>X}}\mu_F(b)A_F(d,1,\ldots,1)\Bigg)\chi(m)\sum_{X<c\le x/m}\Lambda_F(c)\chi(c).\end{aligned}$$

Put

$$b_F(m)=\sum_{\substack{bc=m\\ b\le X,c\le X}}\mu_F(b)\Lambda_F(c),\qquad c_F(m)=\sum_{\substack{bd=m\\ b>X}}\mu_F(b)A_F(d,1,\ldots,1).$$

By Cauchy's inequality and Lemma 2.13, we can estimate the square moments of $\Lambda_F(m), b_F(m)$ and $c_F(m)$

$$\sum_{m\le x}\left|\Lambda_F(m)\right|^2 \ll \left(\log^2 x\right)\sum_{n\le x} d(n)|a_F(n)|^2 \ll x\log^3 x, \tag{3.4}$$

$$\begin{aligned}\sum_{m\le x}\left|b_F(m)\right|^2 &\ll \sum_{m\le x}\sum_{\substack{bc=m\\ b\le X, c\le X}} |\mu_F(b)|^2|\Lambda_F(c)|^2 d(m)\\ &\ll \left(\log^2 x\right)\sum_{b\le x} d(b)|\mu_F(b)|^2\sum_{c\le x/b} d(c)|a_F(c)|^2\\ &\ll x\log^5 x,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\sum_{m\le x}\left|c_F(m)\right|^2 &\ll \sum_{m\le x}\sum_{\substack{bd=m\\ b>X}} |\mu_F(b)|^2|A_F(d,1,\dots,1)|^2 d(m)\\ &\ll \sum_{b\le x} d(b)|\mu_F(b)|^2\sum_{c\le x/b} d(d)|A_F(d,1,\dots,1)|^2\\ &\ll x\log^3 x.\end{aligned} \tag{3.6}$$

First, to estimate the contribution from $S_1$, we use Cauchy's inequality and get, from (3.4),

$$\begin{aligned}\sum_{r\sim R}\left|\sum_{\chi \bmod r}^{*} \overline{\chi}(a)S_1\right| &\ll R^2\sum_{n\le X}|\Lambda_F(n)| \ll R^2X^{\frac{1}{2}}\left(\sum_{n\le X}|\Lambda_F(n)|^2\right)^{\frac{1}{2}}\\ &\ll R^2X\log^2 X.\end{aligned} \tag{3.7}$$

For the corresponding sum of $S_2$, we split $S_2$ in the following way:

$$\begin{aligned}S_2 &= \left(\sum_{b\le H}+\sum_{H<b\le X}\right)\mu_F(b)\chi(b)\sum_{c\le x/b} A_F(c,1,\dots,1)\,(\log c)\,\chi(c)\\ &=: S_2' + S_2'',\end{aligned}$$

where $H < X$. We treat $S_2'$ and derive that

$$\begin{aligned}\sum_{r\sim R}\left|\sum_{\chi \bmod r}^{*}\overline{\chi}(a)S_2'\right| &=\\ &\sum_{r\sim R}\left|\sum_{b\le H}\mu_F(b)\sum_{c\le x/b} A_F(c,1,\dots,1)\,(\log c)\sum_{\chi \bmod r}^{*}\chi(\overline{a}bc)\right|.\end{aligned} \tag{3.8}$$

Since $(r, abc) = 1$, we know from (2.23) that the summation over all primitive characters $\chi$ is

$$\sum_{d|(\overline{a}bc-1,r)} \varphi(d)\mu\left(\frac{r}{d}\right).$$

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Inserting this into (3.8), we have

$$\sum_{r\sim R}\Big|\sideset{}{^*}\sum_{\chi \bmod r}\overline{\chi}(a)S_2'\Big| = \sum_{r\sim R, d|r}\varphi(d)\Big|\sum_{\substack{b\le H\\(b,r)=1}}\mu_F(b)\sum_{\substack{c\le x/b\\ c\equiv a\overline{b} \bmod d}} A_F(c,1,\dots,1)\log c\Big|.$$

We can remove the smooth factor $\log n$ in the innermost sum by partial summation. Then applying the classical estimate

$$\sum_{n\le x} d^T(n) \ll x\log^{2^T-1}x \tag{3.9}$$

and Proposition 2.2 (the bound for $m=2$ is weaker than the one that the value $m=2$ inserts into the corresponding result $m\ge 3$; we will only use the bound for $m\ge 3$ for convenience), we deduce the following estimate:

$$\begin{aligned}\sum_{r\sim R}\Big|\sideset{}{^*}\sum_{\chi \bmod r}\overline{\chi}(a)S_2'\Big| &\ll x^{\frac{m-1}{m+1}(1+\theta_m)}(\log^2 x)\sum_{r\sim R, d|r} d^T(d)d^{1+\frac{2\theta_m}{m+1}}(\log^2 d)\sum_{\substack{b\le H\\(b,r)=1}}\frac{|\mu_F(b)|}{b^{\frac{m-1}{m+1}(1+\theta_m)}}\\ &\ll x^{\frac{m-1}{m+1}(1+\theta_m)}(\log^2 x)R^{2+\frac{2\theta_m}{m+1}}(\log^T R)\big(1+H^{\frac{2-(m-1)\theta_m}{m+1}}\big).\end{aligned}$$

Before estimating $S_2''$, we recall a lemma of Vaughan [44].

***Lemma 3.1*** *Let $a_m, (m=1,2,\dots,M)$ and $b_n, (n=1,2,\dots,N)$ be complex numbers. Then*

$$\sum_{q\le Q}\sideset{}{^*}\sum_{\chi \bmod q}\Big|\sum_{\substack{m=1\\ mn\le X}}^{M}\sum_{n=1}^{N} a_m b_n\chi(mn)\Big| \ll \Big((M+Q^2)(N+Q^2)\sum_m |a_m|^2\sum_n |b_n|^2\Big)^{\frac12} \times (\log Q)\log(MNx).$$

In order to estimate $S_2''$, splitting the range of summation over $b$ into intervals of the form $b\sim M$ with $H<M\le X$ and applying Lemma 3.1, we get

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}|S_2''| \ll \log^c x\big(x^{\frac12}+RM^{\frac12}+Rx^{\frac12}M^{-\frac12}+R^2\big) \times\Big(\sum_{b\sim M}|\mu_F(b)|^2\Big)^{\frac12}\Big(\sum_{c\le x/M}|A_F(c,1,\dots,1)|^2\Big)^{\frac12},$$

for some positive constant $c$. Inserting the estimate

$$\sum_{n\le x}|\mu_F(n)|^2 \ll \sum_{n\le x} d(n)|\mu_F(n)|^2 \ll x\log x, \tag{3.10}$$

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and (2.26), we have

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}|S_2''| \ll (\log^c x)\,x\Big(x + R(Mx)^{\frac12} + RxM^{-\frac12} + R^2x^{\frac12}\Big) \tag{3.11}$$
$$\ll (\log^c x)\,x\Big(x + R(Xx)^{\frac12} + RxH^{-\frac12} + R^2x^{\frac12}\Big).$$

Combining (3.8) and (3.11), we obtain

$$\sum_{r\sim R}\Big|\sideset{}{^*}\sum_{\chi \bmod r}\overline{\chi}(a)S_2\Big| \ll (\log^c x)\Big(x^{\frac{m-1}{m+1}(1+\theta_m)}R^{2+\frac{2\theta_m}{m+1}}\Big(1+H^{\frac{2-(m-1)\theta_m}{m+1}}\Big)\Big). \tag{3.12}$$

The treatment of $S_3$ is similar to that of $S_2$. If we use the estimate (3.5) instead of (3.10), then

$$\sum_{r\sim R}\Big|\sideset{}{^*}\sum_{\chi \bmod r}\overline{\chi}(a)S_3\Big| \ll (\log^c x)\Big(x^{\frac{m-1}{m+1}(1+\theta_m)}R^{2+\frac{2\theta_m}{m+1}}\Big(1+H^{\frac{2-(m-1)\theta_m}{m+1}}\Big) \tag{3.13}$$
$$+x + RXx^{\frac12} + RxH^{-\frac12} + R^2x^{\frac12}\Big).$$

The sum $S_4$ is of the same form as $S_2''$, except the coefficients $c_F(n)$ instead of $\mu_F(n)$ and $\Lambda_F(n)$ instead of $A_F(n,1,\dots,1)\log n$. Thus, it follows from the estimates (3.4) and (3.6) that

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}|S_4| \ll (\log^c x)\Big(x + RxX^{-\frac12} + R^2x^{\frac12}\Big). \tag{3.14}$$

Now we choose the parameters. We see that each of the estimates (3.7), (3.12), (3.13), and (3.14) is $\ll Qx\log^{-A}x$, provided

$$X = x^{\frac13}, \quad H = \log^{2A}x,$$
$$\log^C x < R \le Q = \min\Big\{x^{\frac12}\log^{-B}x, x^{\frac{2-(m-1)\theta_m}{m+1+2\theta_m}}\log^{-B}x\Big\}.$$

Due to Kim and Sarnak, $\theta_2 = \frac{7}{64}$, so that $Q = x^{\frac12}\log^{-B}x$ if $m = 2$. If $m \ge 3$,

$$Q = x^{\frac{2-(m-1)\theta_m}{m+1+2\theta_m}}\log^{-B}x.$$

The proof of (1.2) in Theorem 1.2, Theorem 1.5, or Theorem 1.6 follows the same approach as above, if we insert the bounds (2.11), (2.8), or (2.12) instead of Proposition 2.2.

## 4 Proof of the Theorems Related to Möbius Function

To prove our theorems, we need some lemmas. By the definition of $\mu_F(n)$, it is clear that $\mu_F(n)$ is equal to $\mu(n)A_F(n,1,\dots,1)$ at primes. For general positive integers, they can transfer to each other by the following Dirichlet convolution.

***Lemma 4.1*** *Let $g(n)$ and $h(n)$ be arithmetic functions defined over square-full integers with $|g(n)|, |h(n)| \le n^{\theta_m}d^T(n)$. Then we have*

$$\mu_F(n) = \sum_{d|n}\mu(d)A_F(d,1,\dots,1)g\Big(\frac{n}{d}\Big)$$

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*and*

$$\mu(n)A_F(n,1,\dots,1)=\sum_{d\mid n}\mu_F(d)h\Big(\frac{n}{d}\Big),$$

*where $T$ is some constant depending only on $m$.*

**Proof** We will achieve these two Dirichlet convolutions by comparing the Euler products of the corresponding Dirichlet series

$$\frac{1}{L(s,F)}\quad\text{and}\quad D(s)=\sum_{n=1}^{\infty}\mu(n)A_F(n,1,\dots,1)n^{-s}.$$

In fact, they admit Euler products

$$\frac{1}{L(s,F)}=\prod_p\Big(1-\frac{A_F(p,1,\dots,1)}{p^s}+\cdots+(-1)^{m-1}\frac{A_F(1,\dots,1,p)}{p^{(m-1)s}}+(-1)^m\frac{1}{p^{ms}}\Big)$$

and

$$D(s)=\prod_p\Big(1-\frac{A_F(p,1,\dots,1)}{p^s}\Big).$$

Then we deduce that

$$\begin{aligned}\frac{1}{L(s,F)}&=D(s)\prod_p\Big(1+\frac{A_F(1,p,1,\dots,1)}{p^{2s}}+\cdots\Big)\\&=D(s)G(s),\end{aligned}\tag{4.1}$$

where the omitted terms in the last brackets denote the higher power terms of $p^{-ks}$ with $k\ge 3$, and the Dirichlet series

$$G(s):=\prod_p\Big(1+\frac{g(p^2)}{p^{2s}}+\frac{g(p^3)}{p^{3s}}+\cdots\Big)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}.$$

Due to

$$g(p^k)=\sum_{\substack{k=k_1+k_2\\0\le k_1\le m,k_2\ge 0}}(-1)^{k_1}A_F(\underbrace{1,\dots,1,p}_{\text{position }k_1},\dots,1)A_F^{k_2}(p,1,\dots,1)$$

for $k\ge 2$ and the current estimates for Fourier coefficients, we have

$$|g(p^k)|\le p^{k\theta_m}d^T(p^k)$$

for some constant $T$ depending only on $m$. Thus, the arithmetic function $g(n)$ is defined over square-full positive integers and satisfies $|g(n)|\le n^{\theta_m}d^T(n)$. Moreover, (4.1) in the half-plane $\Re s>1$ is equivalent to

$$\mu_F(n)=\sum_{d\mid n}\mu(d)A_F(d,1,\dots,1)g\Big(\frac{n}{d}\Big).$$

The second statement can be shown by the same method as employed in the first one and so is omitted. ■

***Lemma 4.2*** *Let $g(n)$ and $h(n)$ be as in Lemma 4.1. Then we have*

$$\sum_{n\le x}|g(n)| \ll x^{\frac{1}{2}+\theta_m}\log^T x, \quad \sum_{n\le x}|h(n)| \ll x^{\frac{1}{2}+\theta_m}\log^T x,$$

*where $T$ is some constant depending only on $m$.*

**Proof** Let $l(n)$ denote the characteristic function for the set of all square-full positive integers. It is well known from Lemma 4.1 that $|g(n)|$ and $|h(n)|$ are all less than

$$l(n)n^{\theta_m}d^T(n).$$

Hence, it reduces to estimating the sum

$$\sum_{n\le x} l(n)n^{\theta_m}d^T(n).$$

By using elementary methods, Erdös and Szekeres [7] proved that

$$\sum_{n\le x} l(n) = \frac{\zeta(3/2)}{\zeta(3)}x^{\frac{1}{2}} + O(x^{\frac{1}{3}}).$$

Taking (3.9) into consideration, we obtain by the Cauchy–Schwarz inequality that

$$\begin{aligned}\sum_{n\le x} l(n)n^{\theta_m}d^T(n) &\le x^{\frac{1}{4}+\theta_m}\Big(\sum_{n\le x} l(n)\Big)^{\frac{1}{2}}\Big(\sum_{n\le x} n^{-\frac{1}{2}}d^{2T}(n)\Big)^{\frac{1}{2}}\\ &\le x^{\frac{1}{2}+\theta_m}\log^{4^T} x.\end{aligned}$$

This completes the proof of Lemma 4.2. ∎

***Proposition 4.3*** Let $\chi$ mod $q$ be any primitive character with $q \le \log^A x$ for any fixed $A > 0$, and let $F(z)$ is a Hecke–Maass form for $\mathrm{SL}_m(\mathbb{Z})$. Then we have for any $k \ge 1$,

$$\sum_{\substack{n\le x\\(k,n)=1}} \mu_F(n)\chi(n) \ll d(k)x\exp\big(-c(\log x)^{\frac{1}{3}}\big) \tag{4.2}$$

unconditionally for $2 \le m \le 3$ and under the assumption that there exists no exceptional zero for $m \ge 4$, where the implied constant depends on $F, A$. When $F(z)$ is the symmetric power lift of a Hecke–Maass form for $\mathrm{SL}_2(\mathbb{Z})$ with $m = 4, 5$, (4.2) also holds.

**Proof** Our strategy is to establish the relation between $\mu_F(n)$ and $\Lambda_F(n)$ so that Proposition 4.3 can be derived from Proposition 2.12. Similar to (3.1), we have

$$\begin{aligned}\sum_{p\le x} A(p,1,\dots,1)\chi(p)\log p &= \sum_{n\le x}\Lambda_F(n)\chi(n) + O(x^{\frac{1}{2}+\theta_m+\varepsilon})\\ &\ll x\exp\big(-c\sqrt{\log x}\big).\end{aligned} \tag{4.3}$$

Then we derive by partial summation that

$$\sum_{p\le x} A(p,1,\dots,1)\chi(p) \ll x\exp\Big(-c\sqrt{\log x}\Big). \tag{4.4}$$

Now we further adopt the technique of Iwaniec–Kowalski [16, p. 124], which was employed in [18] to estimate the uniform bound for the sum associated with Möbius function, additive characters and Fourier coefficients of GL(3) automorphic forms. First we write

$$\sum_{\substack{n\le x\\(k,n)=1}} \mu(n)A_F(n,1,\dots,1)\chi(n) = 1 - \sum_{\substack{r\le x\\(k,r)=1}} \mu(r)A_F(r,1,\dots,1)\chi(r) \times \sum_{\substack{p_r<p\le x/r\\(k,n)=1}} A_F(p,1,\dots,1)\chi(p), \tag{4.5}$$

since $(p,r)=1$, where $p_r$ denotes the largest prime divisor of $m$. Let $\omega(r)>0$ be the number of prime divisors of $r$. Then the summation conditions imply $r^{1/\omega(r)} \le p_r < x/r$. So $r \le x^{\frac{\omega(r)}{\omega(r)+1}}$. By using (4.4), the inner sum over $p$ is

$$\begin{aligned}\sum_{\substack{p_r<p\le x/r\\(k,p)=1}} A_F(p,1,\dots,1)\chi(p) &= \Big|\sum_{p_r<p\le x/r} A_F(p,1,\dots,1)\chi(p)\Big| + d(k)\Big(\frac{x}{r}\Big)^{\theta_m}\\ &\ll d(k)\frac{x}{r}\exp\Big(-c\sqrt{\log\frac{x}{r}}\Big)\\ &\ll d(k)\frac{x}{m}\exp\Big(-c\sqrt{\frac{\log x}{\omega(m)+1}}\Big)\\ &\ll d(k)\frac{x}{r}\exp\Big(-c^{\frac12}(\log x)^{\frac{k}{2(k+1)}}\Big)\exp\Big(\omega(r)^{\frac{k}{2}}\Big).\end{aligned} \tag{4.6}$$

In the last step, we use the geometric inequality,

$$B^k + \frac{A}{B} \gg A^{\frac{k}{k+1}}$$

for $A, B > 0$ and any fixed integer $k > 0$. On inserting (4.6) ($k = 2$) into (4.5), we deduce from Cauchy's inequality that

$$\begin{aligned}&\sum_{\substack{n\le x\\(k,n)=1}} \mu(n)A_F(n,1,\dots,1)\chi(n)\\ &\qquad\ll d(k)x\exp\Big(-c^{\frac12}(\log x)^{\frac13}\Big)\sum_{r\le x}\frac{|A_F(r,1,\dots,1)|d^2(r)}{r}\\ &\qquad\ll d(k)x\exp\Big(-c^{\frac12}(\log x)^{\frac13}\Big).\end{aligned} \tag{4.7}$$

Here we used (3.9) and the inequalities

$$\exp\big(\omega(m)\big) \le 2^{2\omega(m)} \le d^2(m),$$

$$\sum_{m\le x}\big|A_F(r,1,\dots,1)\big|^2 \le x.$$

Finally, we turn to estimate the sum

$$\sum_{\substack{n\le x\\(k,n)=1}} \mu_F(n)\chi(n).$$

By Lemma 4.1, we have

$$\sum_{\substack{n\le x\\(k,n)=1}}\mu_F(n)\chi(n)=\sum_{\substack{l\le x\\(k,l)=1}}g(l)\chi(l)\sum_{\substack{d\le x/l\\(k,d)=1}}\mu(d)A_F(d,1,\dots,1)\chi(d).$$

Breaking the summations into dyadic intervals, it suffices to estimate the sums of the form

$$S(L,D)=\sum_{\substack{l\sim L\\(k,l)=1}}g(l)\chi(l)\sum_{\substack{d\sim D\\(k,d)=1}}\mu(d)A_F(d,1,\dots,1)\chi(d)$$

with $LD\le x$. By Lemma 4.2 and (4.7), we have

$$\begin{aligned}\sum_{\substack{n\le x\\(k,n)=1}}\mu_F(n)\chi(n)&\ll\log^3 x\max_{LD\le x}S(L,D)\\&\ll d(k)x\exp\big(-c(\log x)^{\frac13}\big).\end{aligned}$$

■

Now we are ready to show the theorems for $\mu(n)$. Similar to the proof of theorems concerning $\Lambda(n)$, we have

$$\sum_{\substack{n\le x\\n\equiv a\bmod q}}\mu_F(n)\ll\frac{1}{\varphi(q)}\sum_{r|q}\Big|\sum_{\chi\bmod r}^{*}\overline{\chi}(a)\sum_{\substack{n\le x\\(q,n)=1}}\mu_F(n)\chi(n)\Big|.$$

Then we get

$$\sum_{q\le Q}\max_{(a,q)=1}\Big|\sum_{\substack{n\le x\\n\equiv a\bmod q}}\mu_F(n)\Big|\ll\log Q\sum_{kr\le Q}\max_{(a,kr)=1}\frac{1}{kr}\Big|\sum_{\chi\bmod r}^{*}\overline{\chi}(a)\sum_{\substack{n\le x\\(k,n)=1}}\mu_F(n)\chi(n)\Big|.$$

It is convenient to work with the sum

$$S_k(R,x):=\sum_{r\sim R}\max_{(a,r)=1}\Big|\sum_{\chi\bmod r}^{*}\overline{\chi}(a)\sum_{\substack{n\le x\\(k,n)=1}}\mu_F(n)\chi(n)\Big|$$

with $KR\le Q$. Clearly,

$$\sum_{q\le Q}\max_{(a,q)=1}\Big|\sum_{\substack{n\le x\\n\equiv a\bmod q}}\mu_F(n)\Big|\ll\log^3 Q\max_{KR\le Q}\frac{1}{KR}\sum_{k\sim K}S_k(R,x).\tag{4.8}$$

We evaluate $S_k(R,x)$ by the approach that has been used to estimate (3.3).

If $R\le\log^C x$ with an arbitrary $C>0$, then by Proposition 4.3,

$$\sum_{\substack{n\le x\\(k,n)=1}}\mu_F(n)\chi(n)\ll d(k)x\exp\big(-c(\log x)^{\frac13}\big)$$

for some $c > 0$, and hence

$$S_k(R,x) \ll R^2 d(k) x \exp\left(-c(\log x)^{\frac{1}{3}}\right). \tag{4.9}$$

If $\log^C x < R \le Q$, applying Vaughan's identity of $\mu_F(n)$ in Lemma 2.1 with $X < x$, we have

$$\sum_{\substack{n\le x\\(k,n)=1}} \mu_F(n)\chi(n) = M_1 - M_2 + M_3,$$

where

$$\begin{aligned}
M_1 &= \sum_{\substack{n\le X\\(k,n)=1}} \mu_F(n)\chi(n),\\
M_2 &= \sum_{\substack{b\le X\\(k,b)=1}} \sum_{\substack{c\le X\\(k,c)=1}} \sum_{\substack{d\le x/bc\\(k,d)=1}} \mu_F(b)\mu_F(c)A_F(d,1,\ldots,1)\chi(bcd)\\
&= \sum_{\substack{m\le X^2\\(k,m)=1}} \Bigg(\sum_{\substack{bc=m\\ b\le X, c\le X}} \mu_F(b)\mu_F(c)\Bigg)\chi(m) \sum_{\substack{d\le x/m\\(k,d)=1}} A_F(d,1,\ldots,1)\chi(d),\\
M_3 &= \sum_{\substack{b> X\\(k,b)=1}} \sum_{\substack{c> X\\(k,c)=1}} \sum_{\substack{d\le x/bc\\(k,d)=1}} \mu_F(b)\mu_F(c)A_F(d,1,\ldots,1)\chi(bcd)\\
&= \sum_{\substack{X<m<x/X\\(k,m)=1}} \Bigg(\sum_{\substack{bd=m\\ b>X}} \mu_F(b)A_F(d,1,\ldots,1)\Bigg)\chi(m) \sum_{\substack{X<c\le x/m\\(k,c)=1}} \mu_F(c)\chi(c).
\end{aligned}$$

Thus, we proceed as with the estimates of the contributions of $S_i$ with $i = 1,3,4$ as before. We can easily find out

$$\begin{aligned}\sum_{i=1}^{3}\sum_{r\sim R}\max_{(a,r)=1}\Bigg|\sideset{}{^*}\sum_{\chi \bmod r} \overline{\chi}(a)M_i\Bigg| \ll\\
(\log^c x)\left(x^{\frac{m-1}{m+1}(1+\theta_m)}R^{2+\frac{2\theta_m}{m+1}} + x + Rx^{\frac{5}{6}} + R^2x^{\frac{1}{2}}\right) + Rx\log^{-A} x,\end{aligned} \tag{4.10}$$

which implies that

$$S_k(R,x) \ll (\log^c x)\left(x^{\frac{m-1}{m+1}(1+\theta_m)}R^{2+\frac{2\theta_m}{m+1}} + x + Rx^{\frac{5}{6}} + R^2x^{\frac{1}{2}}\right) + Rx\log^{-A} x. \tag{4.11}$$

By combining (4.9) with (4.11), we have

$$\begin{aligned}
&\max_{KR\le Q}\frac{1}{KR}\sum_{k\sim K} S_k(R,x)\\
&\quad\ll \max_{\substack{KR\le Q\\ R\le \log^C x}}\frac{1}{KR}\sum_{k\sim K} S_k(R,x) + \max_{\substack{KR\le Q\\ \log^C x<R\le Q}}\frac{1}{KR}\sum_{k\sim K} S_k(R,x)\\
&\quad\ll (\log^c x)\left(Q^{1+\frac{2\theta_m}{m+1}}x^{\frac{m-1}{m+1}(1+\theta_m)} + Qx^{\frac{1}{2}}\right) + x\log^{-A} x.
\end{aligned}$$

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It follows from (4.8) that

$$\sum_{q\le Q}\max_{(a,q)=1}\Bigg|\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\mu_F(n)\Bigg| \ll (\log^c x)\left(Q^{1+\frac{2\theta_m}{m+1}}x^{\frac{m-1}{m+1}(1+\theta_m)}+Qx^{\frac12}\right)+x\log^{-A}x.$$

Notice that

$$\mu(n)A_F(n,1,\dots,1)=\sum_{d|n}\mu_F(d)h\Big(\frac{n}{d}\Big)$$

as in Lemma 4.1, and $h(d)$ satisfies some properties as in Lemma 4.2. Hence, we obtain

$$\begin{aligned}\sum_{q\le Q}\max_{(a,q)=1}&\Bigg|\sum_{\substack{n\le x\\ n\equiv a \bmod q}}\mu(n)A_F(n,1,\dots,1)\Bigg|\\ &\ll\sum_{d\le x}|h(d)|\sum_{q\le Q}\max_{(da,q)=1}\Bigg|\sum_{\substack{n\le x/d\\ n\equiv a\overline{d} \bmod q}}\mu_F(n)\Bigg|\\ &\ll\log x\max_{D\le x}\sum_{d\sim D}|h(d)|\sum_{q\le Q}\max_{(da,q)=1}\Bigg|\sum_{\substack{n\le x/d\\ n\equiv a\overline{d} \bmod q}}\mu_F(n)\Bigg|\\ &\ll(\log^c x)\,Q^{1+\frac{2\theta_m}{m+1}}x^{\frac{m-1}{m+1}(1+\theta_m)}\left(1+x^{\frac{3-m+4\theta_m}{2(m+1)}}\right)+x\log^{-A}x.\end{aligned}$$

Finally, on taking

$$Q=\min\left\{x^{\frac{2-(m-1)\theta_m}{m+1+2\theta_m}},x^{\frac{(m+1)(1-2\theta_m)}{2(m+1+2\theta_m)}}\right\}\log^{-B}x,$$

we complete the proof of the theorems. Note that $\theta_m=0$ when $A_F(n,1,\dots,1)$ are Fourier coefficients of $\mathrm{SL}_2(\mathbb{Z})$ holomorphic cusp forms or its symmetric lifts.

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*School of Mathematics and Statistics, Shandong University, Weihai, Weihai, Shandong 264209, China*
*e-mail:* yujiaoj@hotmail.com

*School of Mathematics, Shandong University, Jinan, Shandong 250100, China*
*e-mail:* gslv@sdu.edu.cn