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Scaling laws and warning signs for bifurcations of SPDEs

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Critical transitions (or tipping points) are drastic sudden changes observed in many dynamical systems. Large classes of critical transitions are associated with systems, which drift slowly towards a bifurcation point. In the context of stochastic ordinary differential equations, there are results on growth of variance and autocorrelation before a transition, which can be used as possible warning signs in applications. A similar theory has recently been developed in the simplest setting for stochastic partial differential equations (SPDEs) for self-adjoint operators in the drift term. This setting leads to real discrete spectrum and growth of the covariance operator via a certain scaling law. In this paper, we develop this theory substantially further. We cover the cases of complex eigenvalues, degenerate eigenvalues as well as continuous spectrum. This provides a fairly comprehensive theory for most practical applications of warning signs for SPDE bifurcations.

Key words: Critical transition, tipping point, warning sign, scaling law, stochastic partial differential equation

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1 Introduction

In many areas of science,we frequently observe events that appear rather abruptly. Some examples are epileptic seizures [32, 33] and asthma attacks [41] in medicine, market collapses in economics [19, 31], epidemic outbreaks [34, 28], engineering system failures [8] and population/habitat changes in ecology [6, 5]. Although these *critical transitions* seem – *a priori* – unrelated, there are many unifying features. The events happen rather *fast* after a long period of *slow* change, there are special *thresholds* or *tipping points* to be crossed, and *stochastic fluctuations* are always present. Using stochastic fluctuations to estimate the presence of, and the distance to, a tipping point has been a successful strategy already proposed by Wiesenfeld in 1985 [42] and tested in the context of chemical experiments [22]. One exploits that the main deterministic driving forces near bifurcation are weakened (also known as *critical slowing down* or *intermittency* [16]) and measures the relatively amplified noisy fluctuations. If we would not have fluctuations/perturbations, then it would be impossible to detect critical slowing down (e.g. consider the case of a subcritical pitchfork bifurcation). The strategy to exploit slowing down in combination with noise has been (re-)discovered also in many application areas, recently, mainly in ecology [5, 38] and climate science [30, 38]. Yet, to actually obtain predictive power of warning signs is often highly non-trivial from a practical [12] as well as statistical [43, 4] viewpoint.

Therefore, a detailed mathematical theory must be developed to better understand the assumptions, limitations and opportunities of warning signs for critical transitions. For systems modelled by stochastic ordinary differential equations (SODEs), a detailed theory can be found in [23]; see also [2] for relevant background. However, if we discard all spatial components, we may miss important aspects of the theory, which could also be very important in practical applications [11, 13]. This leads one to consider stochastic partial differential equations (SPDEs), where warning signs have only been investigated so far for propagation failure of travelling waves numerically [24] and with a combination of analytical/numerical methods for stationary patterns in [15]. The work [15] is our main starting point. It focuses on system of the form

$$\begin{aligned} \mathrm{d}u &= Lu + f(u,p)\ \mathrm{d}t + \sigma B\ \mathrm{d}W, \\ \mathrm{d}p &= \varepsilon g(u,p)\ \mathrm{d}t, \end{aligned} \qquad 0<\sigma,\varepsilon \ll 1, \tag{1.1}$$

where $(x,t) \in \mathcal{I} \times [0,\infty)$, $\mathcal{I}$ is an interval, L is a spatial differential operator, $u = u(x,t)$, $p = p(x,t)$, the nonlinearities f and g are sufficiently smooth maps, $W = W(x,t)$ is a space-time generalized Wiener process, B is a given linear operator, $0 < \sigma \ll 1$ controls the noise level and $0 < \varepsilon \ll 1$ is the time-scale separation between the fast u variable and the slow p variable; see also Section 2 for the technical setting. Suppose $f(0,p) = 0$, so that $u_* \equiv 0$ is a homogeneous steady state for any p for the deterministic ($\sigma = 0$) partial differential equation (PDE). The local stability of $u_* \equiv 0$ is determined by studying the operator

$$A = A(p) := L + D_u f(0,p),$$

where D_u is the Fréchet derivative and we have to pick a function space to obtain a well-defined spectral problem. The basic idea to induce a critical transition in the fast–slow SPDE (1.1) is that the slow dynamics

$$\partial_t p = \varepsilon g(0,p)$$

changes, so that for some p, say $p < 0$, we obtain that spec$(A(p))$ is contained in $\{z : \mathrm{Re}(z) < 0\}$ while for some other p, say $p > 0$, the spectrum contains parts in $\{z : \mathrm{Re}(z) > 0\}$. In particular, this means the fast PDE dynamics

$$\partial_t u = Lu + f(u,p)$$

undergoes a bifurcation at $p = 0$ as p is varying [20]. Since it is very difficult to control the interplay between σ, ε and the location of spec$(A(p))$ [25], the first natural approximation is to consider the fast subsystem singular limit $\varepsilon = 0$ and just view p as a parameter [26]. In [15], this situation is considered for the linearised problem

$$\mathrm{d}U = A(p)U\ \mathrm{d}t + \sigma B\ \mathrm{d}W, \qquad U = U(x,t), \tag{1.2}$$

see also Section 2. Several further key assumptions are made in [15]:

(GK1): spec$(A(p))$ contains eigenvalues with multiplicity one;
(GK2): $A(p)$ is self-adjoint;
(GK3): the noise term is independent of p;
(GK4): the spatial domain is $\mathcal{I} = [0, L_*]$ for some $L_* > 0$.

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Under these assumptions, one can show [15] that the covariance operator Cov(u) diverges, when projected onto certain Fourier modes as $p \to 0^-$. One can also determine an explicit asymptotic scaling power law in p. Furthermore, the scaling can be numerically computed in examples, such as the cubic–quintic Allen–Cahn equation as shown in [27, Figures 3, 4 and 8]. The results just summarised above are a natural generalisation to SPDEs for the well-known fast–slow SODE setting [1, 23].

In this paper, we manage to drop and/or generalise all the assumptions (GK1)–(GK4). We are going to allow for degenerate Jordan blocks lifting (GK1). We also consider complex eigenvalues and parameter-dependent noise thereby removing (GK2) and (GK3). Furthermore, we are going to consider essential spectrum frequently arising for differential operators on unbounded domains. In this context, we consider rather general classes of linear operators A. These results are a major generalisation in contrast to classical differential operators on bounded domains as in (GK4). The last generalisation may look slightly unnatural at first sight, but it is crucial as modulation/amplitude equations [21] for SPDEs [3] are posed on unbounded domains. Modulation equations can be viewed as normal forms for local pattern formation [18].

Our results in this paper show that we can essentially always expect diverging covariance for *generic* noise terms, either in the form

$$\langle \mathrm{Cov}(u_{k^*}), u_{k^*} \rangle = \mathcal{O}(h(p)) \text{ as } p \to 0^-, \qquad \lim_{p \to 0^-} h(p) = +\infty \tag{1.3}$$

in the Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ for some function u_{k^*} and explicitly computable $h(p)$, or more generally for essential spectrum in the form

$$\lim_{p \to 0^-} \|\mathrm{Cov}(u)\| = +\infty, \tag{1.4}$$

where $\| \cdot \|$ is a norm on linear operators. If the parametric dependence is chosen, so that the noise degenerates, we show that other behaviours are possible. For precise technical statements, we refer to Sections 3 and 4. In summary, this completes the theory of warning signs for SPDEs bifurcating from a homogeneous steady state in the vast majority of cases of practical relevance.

The paper is structured as follows: in Section 2, we briefly present the mathematical background required for our study. In Section 3, we consider the case of discrete spectrum for $A = A(p)$. Here, we manage to lift the assumptions (GK1)–(GK3) and prove a result of the form (1.3). Then, we obtain a result of the form (1.4) for essential spectrum in Section 4. The proof shows when we can characterise the precise scaling laws also for essential spectrum as stated in (1.3). We conclude with a summary and an outlook in Section 5.

2 Background and framework

Consider an evolution equation on a Hilbert space H of the form

$$\partial_t U = A(p)U, \qquad U = U(t) \in H, p \in \mathbb{R}, \tag{2.1}$$

for a linear operator $A = A(p) : \mathcal{D}(A) \subset H \to H$. Assume that A is the infinitesimal generator of a strongly continuous semigroup e^{tA} [35]. Suppose the steady state $U_* = 0$ is stable for (2.1), for $p < 0$, i.e. the spectrum of $A(p)$ is contained in the left half of the complex plane for $p < 0$. Suppose at $p = 0$, the spectrum crosses the imaginary axis $\mathrm{i}\mathbb{R}$, so that we have an instability that we interpret as the linearised problem for the drift part of (1.2).

Next, we briefly introduce the framework for SPDEs we need from [10]. Consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and a non-negative self-adjoint trace class operator Q on a Hilbert space H. By the spectral theorem, Q has a countable orthonormal basis $\{q_k\}_{k=1}^{\infty}$ of eigenfunctions, with corresponding eigenvalues $\rho_k \geq 0$ such that $Qq_k = \rho_k q_k$. A stochastic process W is a Q-Wiener process on H if

$$W_t = \sum_{j=1}^{\infty} \sqrt{\rho_j}\, q_j\, \beta_t^j, \qquad \text{almost surely (a.s.),}$$

where $\beta^j = \beta_t^j$ are independent and identically distributed $\mathcal{F}_t$-adapted Brownian motions and the series converges in $L^2(\Omega, H)$. The identity matrix I is not a trace class operator. Nevertheless, one can (uniquely) construct a Wiener process with covariance matrix that is not trace class by showing that the series

$$W_t = \sum_{j=1}^{\infty} Q^{1/2}\, q_j\, \beta_t^j \tag{2.2}$$

converges in a larger Hilbert space H_1 (in particular, H_1 has to be such that the embedding $J : Q^{1/2}H \to H_1$ is a Hilbert–Schmidt operator, see [10, Proposition 4.7]). The processes defined by convergence of the series (2.2) are called *generalised Wiener processes*. A *cylindrical Wiener process* (or space-time white noise) is the generalised Wiener process with covariance matrix I. One can then define integration with respect to Q-Wiener processes and generalised Wiener processes. A general linear additive-noise SPDE can be written in the following form:

$$\mathrm{d}U = AU\,\mathrm{d}t + \sigma B\,\mathrm{d}W_t \qquad U(0) = U_0, \tag{2.3}$$

where we assume $B \in L_0^2$ with L_0^2 denoting the space of Hilbert–Schmidt operators [10] and that U_0 is an $\mathcal{F}_0$-measurable random variable. An H-valued predictable process $\{U(t)\}_{t\in[0,T]}$ is called a *mild solution* of (2.3) if a.s. we have

$$U(t) = \mathrm{e}^{tA}U_0 + \sigma \int_0^t \mathrm{e}^{(t-s)A}B\,\mathrm{d}W_s. \tag{2.4}$$

Under the assumptions above, a unique mild solution is guaranteed to exist [10, Theorem 5.4]. Since we assume that $\mathrm{e}^{tA}U_0$ decays exponentially for $p < 0$ and we always take the limit $p \to 0^-$, we directly start on the deterministic steady state from now on and assume

$$U(0) = U_0 \equiv 0.$$

We have the following expression for the covariance operator of the second term in (2.4) as given in [10, Theorem 5.2]

$$V(t) := \mathrm{Cov}\left(\sigma \int_0^t \mathrm{e}^{(t-s)A}B\,\mathrm{d}W_s\right) = \sigma^2 \int_0^t \mathrm{e}^{\tau A}BQB^*\mathrm{e}^{\tau A^*}\,\mathrm{d}\tau, \tag{2.5}$$

where B^* denotes the adjoint of B. The asymptotic limit $V_\infty := \lim_{t\to\infty} V(t) = \lim_{t\to\infty} \mathrm{Cov}(U(t))$ satisfies the Lyapunov equation

$$\langle AV_\infty g, h\rangle + \langle V_\infty A^* g, h\rangle = -\sigma^2\langle BQB^* g, h\rangle \tag{2.6}$$

for all h, g such that the expression is well defined (see [9, Lemma 2.45]). Hence, we must study the different behaviours of $V(t)$, respectively V_∞, as $p \to 0^-$ to understand the scaling of the covariance to leading-order as we approach the transition at $p = 0$ [15].

3 Discrete spectrum

We start by considering the problem of discrete spectrum, motivated by many classical differential operators A on bounded domains. Our goal is to generalise the following result already obtained in [15]:

Theorem 3.1 *Consider* (2.3) *with*

$$A = p\,\mathrm{Id} + \mathcal{A}$$

where $\mathcal{A}$ has a discrete real spectrum with eigenvalues $\lambda_k \leq 0$, eigenfunctions u_k and that there exists a unique k^* such that $\lambda_{k^*} = 0$. Also assume the genericity condition $\langle BQB^* u_{k^*}, u_{k^*}\rangle \neq 0$ to be satisfied. Then, the covariance operator V(t) satisfies

$$\left\langle \lim_{t\to\infty} V(t)u_k, u_j \right\rangle = -\sigma^2 \frac{\langle BQB^* u_k, u_j\rangle}{2p + \lambda_k + \lambda_j} \quad \forall j, k \in \mathbb{N} \tag{3.1}$$

and in particular

$$\left\langle \lim_{t\to\infty} V(t)u_{k^*}, u_{k^*} \right\rangle = \mathcal{O}\left(\frac{1}{p}\right) \quad \text{as } p \to 0^-. \tag{3.2}$$

Proof See [15], Proposition 3.1. □

The assumptions on the operator $\mathcal{A}$ guarantee that the spectrum of $A = (p\,\mathrm{Id} + \mathcal{A})$ is strictly contained in $(-\infty, 0)$ for $p < 0$. For $p = 0$, the spectrum spec(A) contains the point 0, which corresponds to the eigenfunction u_{k^*}. In the language of dynamical systems, in this case, the steady state $u_* \equiv 0$ is non-hyperbolic and a centre manifold $W^c_{\mathrm{loc}}(0)$ appears. Being linear, the centre manifold is explicitly given by the linear subspace $W^c_{\mathrm{loc}}(0) = \mathrm{span}\{u_{k^*}\}$. The asymptotic result in Theorem 3.1 can then be restated as saying that the component of the covariance operator along the centre manifold diverges as the critical transition is approached. Hence, this is a very natural first analog to the results for SODEs in [23].

3.1 Imaginary eigenvalues

As a first step, we relax the real discrete spectrum assumption on the operator A to obtain

Theorem 3.2 *Consider the SPDE* (2.3)*, i.e.*

$$\mathrm{d}U = AU\,\mathrm{d}t + \sigma B\,\mathrm{d}W.$$

Suppose $A = A(p)$ has a discrete spectrum with eigenvalues $\lambda_k(p)$ with $\mathrm{Re}(\lambda_k(p)) < 0$ for all k and $p < 0$, and eigenfunctions u_k. If k^* is such that λ_{k^*} is a purely imaginary eigenvalue for

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$p^* = 0$ and the genericity condition $\langle BQB^* u_{k^*}, u_{k^*}\rangle \neq 0$ is satisfied, the covariance operator V(t) satisfies

$$\left\langle \lim_{t\to\infty} V(t)u_k, u_j \right\rangle = -\sigma^2 \frac{\langle BQB^* u_k, u_j\rangle}{\lambda_k + \bar{\lambda}_j} \quad \forall j,k \in \mathbb{N} \tag{3.3}$$

where $\bar{\lambda}_j$ is the complex conjugate of λ_j. In particular, we find

$$\left\langle \lim_{t\to\infty} V(t)u_{k^*}, u_{k^*} \right\rangle = \mathcal{O}\left(\frac{1}{\mathrm{Re}(\lambda_{k^*})}\right) \quad \text{as } p \to 0^-. \tag{3.4}$$

Proof The proof is a calculation using the Lyapunov equation

$$\langle AV_\infty g, h\rangle + \langle V_\infty A^* g, h\rangle = -\sigma^2 \langle BQB^* g, h\rangle$$

which holds, in particular, for the eigenfunctions u_k. Therefore, we obtain

$$\begin{aligned} &\langle AV_\infty u_k, u_j\rangle + \langle V_\infty A^* u_k, u_j\rangle = -\sigma^2 \langle BQB^* u_k, u_j\rangle, \\ \Rightarrow \quad &\lambda_j \langle V_\infty u_k, u_j\rangle + \bar{\lambda}_k \langle V_\infty u_k, u_j\rangle = -\sigma^2 \langle BQB^* u_k, u_j\rangle, \\ &\Rightarrow \quad (\lambda_j + \bar{\lambda}_k)\langle V_\infty u_k, u_j\rangle = -\sigma^2 \langle BQB^* u_k, u_j\rangle. \end{aligned}$$

This proves the first claim (3.3). Setting $k = j$ one has

$$\langle V_\infty u_j, u_j\rangle = -\sigma^2 \frac{\langle BQB^* u_j, u_j\rangle}{2\mathrm{Re}(\lambda_j)},$$

therefore, for $j = k^*$, the second claim (3.4) also follows. □

3.2 Jordan blocks

In the previous sections, we have shown divergence of the variance along the component corresponding to eigenfunctions of the operator A corresponding to the eigenvalue crossing the imaginary axis. In case A has Jordan blocks, one might ask whether such behaviour is also observed, when projecting along the generalised eigenfunctions $\{u^l_{k^*}\}_{l=1,\dots,m_{k^*}}$. Here, m_{k^*} denotes the dimension of the Jordan block corresponding to λ_{k^*}. For arbitrary k, setting $u^0_k := 0$, we have the formula

$$Au^l_k = u^{l-1}_k + \lambda_k u^l_k.$$

We find that the variance diverges also along generalised eigenfunctions, with the rate of divergence depending on the order l of the corresponding generalised eigenfunction.

Theorem 3.3 *Consider (2.3) and suppose $A = A(p)$ has a discrete spectrum with eigenvalues λ_k with $\mathrm{Re}(\lambda_k) < 0$ for all k and $p < 0$. Further assume that k^* is such that λ_{k^*} is a purely imaginary eigenvalue for $p^* = 0$ with generalised eigenfunctions*

$$\{u^l_{k^*}\}_{l=1,\dots,m_{k^*}}.$$

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If the genericity condition $\langle BQB^* u^1_{k^*}, u^1_{k^*}\rangle \neq 0$ *is satisfied, the covariance operator* $V(t)$ *satisfies*

$$\left\langle \lim_{t\to\infty} V(t)u^l_{k^*}, u^m_{k^*}\right\rangle = \mathcal{O}\left(\frac{1}{\mathrm{Re}(\lambda_{k^*})^{l+m-1}}\right) \quad \text{as } p \to 0^- \tag{3.5}$$

for each $l, m \geq 1$.

Proof We aim to prove it by induction on $l+m$. First of all, suppose $l+m=2$, then the only non-trivial case is $l=m=1$ and the claim has already been proven. Therefore, we have the first step for induction. We then assume the claim holds for all l, m s.t. $l+m\leq n$, and we want to prove it for all l, m s.t. $l+m=n+1$. Fix such l and m. The Lyapunov equation implies

$$2\mathrm{Re}(\lambda_k)\langle V_\infty u^l_k, u^m_k\rangle + \langle V_\infty u^{l-1}_k, u^m_k\rangle + \langle V_\infty u^l_k, u^{m-1}_k\rangle = -\sigma^2\langle BQB^* u^l_k, u^m_k\rangle.$$

Using the induction assumption $l+m\leq n$ for the last two terms on the right-hand side, we get

$$2\mathrm{Re}(\lambda_k)\langle V_\infty u^l_k, u^m_k\rangle = \mathcal{O}\left(\frac{1}{\mathrm{Re}(\lambda_k)^{l+m-2}}\right) - \sigma^2\langle BQB^* u^l_k, u^m_k\rangle.$$

Therefore, we may conclude that

$$\langle V_\infty u^l_k, u^m_k\rangle = \mathcal{O}\left(\frac{1}{\mathrm{Re}(\lambda_k)^{l+m-1}}\right) - \sigma^2\frac{\langle BQB^* u^l_k, u^m_k\rangle}{2\mathrm{Re}(\lambda_k)} = \mathcal{O}\left(\frac{1}{\mathrm{Re}(\lambda_k)^{l+m-1}}\right),$$

which proves the claim. □

3.3 Noise and operator dependent on a parameter

Another interesting case to study is when both A and σ depend on the parameter p. We expect that, if the noise near the bifurcation is too small, the variance does not diverge anymore. Indeed, for constant noise, we observe that the system exhibits slow recovery when the critical transition is approached. If the noise decreases too fast, this could balance the critical slowing down and prevent the divergence of the variance. We show in the next result that it is enough to guarantee $\sigma^2 \gg \lambda_{k^*}$ to avoid such a problem.

Theorem 3.4 *Consider the SPDE*

$$\mathrm{d}U = AU\,\mathrm{d}t + \sigma(p)B\,\mathrm{d}W. \tag{3.6}$$

Assume that $A = A(p)$ *has a discrete spectrum with eigenvalues* λ_k *with* $\mathrm{Re}(\lambda_k) < 0$ *for all* k *and* $p<0$, *and eigenfunctions* u_k. *Assume* k^* *is such that* λ_{k^*} *is a purely imaginary eigenvalue for* $p^* = 0$ *and set*

$$\Xi := \lim_{p\to 0^-}\frac{\sigma^2(p)}{2\lambda_{k^*}(p)}.$$

The following holds:

$$\lim_{p\to 0^-}\langle V_\infty u_{k^*}, u_{k^*}\rangle = \langle BQB^* u_{k^*}, u_{k^*}\rangle\,\Xi. \tag{3.7}$$

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In particular, if we also assume the genericity condition $\langle BQB^* u_{k^*}, u_{k^*}\rangle \neq 0$ *we have*

$$\langle V_\infty u_{k^*}, u_{k^*}\rangle = \mathcal{O}\left(\frac{\sigma^2}{\lambda_{k^*}}\right) \quad \textit{as} \quad p \to 0^-. \tag{3.8}$$

Proof We can replicate the same computations as in the preceding sections to obtain

$$\langle AV_\infty u_k, u_j\rangle + \langle V_\infty A^* u_k, u_j\rangle = -\sigma^2(p)\langle BQB^* u_k, u_j\rangle$$
$$\Rightarrow \quad \langle V_\infty u_k, u_k\rangle = -\sigma^2(p)\frac{\langle BQB^* u_k, u_k\rangle}{2\mathrm{Re}(\lambda_k)}.$$

Again as before, taking the limit for $p \to 0^-$ gives the required result. □

The last result significantly generalises an SODE result for a particular model equation obtained in [23, Section 7.5]. It shows that one must ensure that the noise source does not interact and/or depend in a degenerate way on the distance to the critical transition to be able to obtain a warning sign.

4 Continuous spectrum

We have shown that extended versions of Theorem 3.1 still hold for general discrete spectra, including both complex eigenvalues and Jordan blocks. We also found an asymptotic lower bound on the noise σ, which guarantees the result to hold in presence of non-constant external perturbations. To further generalise the results in [15], we want to also consider differential operators on unbounded domains. This naturally leads to the possibility that A has a continuous spectrum. For example, this is the case for the one-dimensional Laplacian that is a fundamental operator in modelling diffusion and will be the starting point of our discussion.

We remark here the main difference with the previous case: in the discrete setting we identified the eigenfunction corresponding to the eigenvalue crossing the imaginary axis and showed that the component of the covariance along that direction tends to infinity as the critical transition is approached. However, if essential spectrum crosses the imaginary axis, there exists no eigenfunction. For this reason, we start considering the norm of the variance and we show that it diverges to infinity (which is of course a weaker result). Later, we prove a stronger result using Weyl's theorem on approximating eigenfunctions.

4.1 The one-dimensional Laplace operator

Consider the operator ∂_{xx} on the Sobolev space $H^2(\mathbb{R})$. We want to study as a simple starting point the following modified stochastic heat equation:

$$\mathrm{d}U = (p\,\mathrm{Id} + \partial_{xx})U\,\mathrm{d}t + \mathrm{d}W, \qquad U(0) = U_0, \tag{4.1}$$

where we set $\sigma = 1$, $B = \mathrm{Id}$ for simplicity of the exposition. The Laplacian is a self-adjoint operator on $H^2(\mathbb{R})$. We recall that, by the spectral theorem [36], self-adjoint operators are unitarily equivalent to multiplication operators. In particular, the Fourier transform

$$\mathcal{F}(h)(k) := \hat{h}(k) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}kx} h(x)\,\mathrm{d}x$$

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unitarily maps ∂_{xx} into the multiplication operator of multiplication by $|k|^2$. This can be used to prove the divergence of the norm of V_∞ as follows.

Theorem 4.1 *Consider the SPDE (4.1). Then*

$$\lim_{p\to 0^-} \|V_\infty\|_{B(L^2(\mathbb{R}))} = +\infty, \tag{4.2}$$

where $\|\cdot\|_{B(L^2(\mathbb{R}))}$ *is the norm on linear operators induced by the* $L^2(\mathbb{R})$ *norm.*

Proof First of all, we compute the covariance using (2.5):

$$V(t) = \int_0^t S(r)BQB^*S^*(r)\,\mathrm{d}r = \int_0^t S(r)S^*(r)\,\mathrm{d}r = \int_0^t \mathrm{e}^{2r(p+\partial_{xx})}\,\mathrm{d}r.$$

This implies that in the limit $t\to\infty$ it holds

$$V_\infty = \int_0^\infty \mathrm{e}^{2r(p+\partial_{xx})}\,\mathrm{d}r.$$

Then, we take the norm and obtain

$$\begin{aligned}
\|V_\infty\|_{B(L^2(\mathbb{R}))} &= \left\|\int_0^\infty \mathrm{e}^{2r(p+\partial_{xx})}\,\mathrm{d}r\right\|_{B(L^2(\mathbb{R}))} = \sup_{h\in L^2(\mathbb{R}),\|h\|=1} \left\|\int_0^\infty \mathrm{e}^{2r(p+\partial_{xx})}h\,\mathrm{d}r\right\|_{L^2(\mathbb{R})}\\
&= \sup_{h\in H^2(\mathbb{R}),\|h\|=1} \left\|\int_0^\infty \mathcal{F}^{-1}\mathrm{e}^{2r(p-|\cdot|^2)}\mathcal{F}(h)\,\mathrm{d}r\right\|_{L^2(\mathbb{R})}\\
&= \sup_{h\in H^2(\mathbb{R}),\|h\|=1} \left\|\frac{1}{\sqrt{2\pi}}\int_0^\infty\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{2r(p-|k|^2)}\hat{h}(k)\,\mathrm{d}k\,\mathrm{d}r\right\|_{L^2_x(\mathbb{R})}\\
&\overset{\text{Fubini}}{=} \sup_{h\in H^2(\mathbb{R}),\|h\|=1} \left\|\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx}\hat{h}(k)\int_0^\infty \mathrm{e}^{2r(p-|k|^2)}\,\mathrm{d}r\,\mathrm{d}k\right\|_{L^2_x(\mathbb{R})}\\
&\overset{p<0}{=} \sup_{h\in H^2(\mathbb{R}),\|h\|=1} \left\|\frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx}\frac{\widehat{h}(k)}{p-|k|^2}\,\mathrm{d}k\right\|_{L^2_x(\mathbb{R})}\\
&= \sup_{h\in H^2(\mathbb{R}),\|h\|=1} \left\|\frac{1}{2}\mathcal{F}^{-1}\left[\frac{\widehat{h}(\cdot)}{p-|\cdot|^2}\right]\right\|_{L^2(\mathbb{R})}\\
&= \sup_{h\in H^2(\mathbb{R}),\|h\|=1} \left\|\frac{1}{2}\frac{\widehat{h}(\cdot)}{p-|\cdot|^2}\right\|_{L^2(\mathbb{R})} \overset{*}{\geq} \left\|\frac{1}{2}\frac{\mathrm{e}^{-|\cdot|^2/2}}{p-|\cdot|^2}\right\|_{L^2(\mathbb{R})}.
\end{aligned}$$

In * we set $h = \mathrm{e}^{-x^2/2}$, while Fubini's Theorem can be applied since

$$|\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{2r(p-|k|^2)}\hat{h}(k)| \leq \mathrm{e}^{2rp}(1+|k|^2)\hat{h}(k) \in L^2(\mathbb{R}_r^+\times\mathbb{R}_k).$$

Applying the limes inferior on both sides of the inequality and using Fatou's Lemma yields

$$\begin{aligned}
\liminf_{p\to 0^-}\|V_\infty\|^2_{B(L^2(\mathbb{R}))} &\geq \liminf_{p\to 0^-}\left\|\frac{1}{2}\frac{\mathrm{e}^{-|\cdot|^2/2}}{p-|\cdot|^2}\right\|^2_{L^2(\mathbb{R})} = \liminf_{p\to 0^-}\int_{\mathbb{R}}\frac{1}{4}\frac{\mathrm{e}^{-x^2}}{(p-x^2)^2}\,\mathrm{d}x\\
&\geq \int_{\mathbb{R}}\liminf_{p\to 0^-}\frac{1}{4}\frac{\mathrm{e}^{-x^2}}{(p-x^2)^2}\,\mathrm{d}x = \int_{\mathbb{R}}\frac{\mathrm{e}^{-x^2}}{x^4}\,\mathrm{d}x = +\infty,
\end{aligned}$$

which concludes the proof. □

We stress that Theorem 4.1 uses only some particular properties of the Laplacian. It relies on three main ingredients: the existence of the diagonalising map $\mathcal{F}$, its interchangeability with the integration via Fubini's Theorem and the divergence of the last integral. For all self-adjoint operators, the existence of a diagonalising map is guaranteed by the spectral theorem. Once such map is known, it might be relatively easy to check that also the other requirements for the proof are satisfied. Nevertheless, since there exists no explicit formula for the diagonalising map, whether this holds or not has to be studied case by case.

4.2 Multiplication operators

As we have remarked, a possible generalisation of the preceding result can be obtained considering general self-adjoint operators. This involves applying the spectral theorem and diagonalising the operator. Being the diagonalising map not known, it is hard to give a formal statement that includes all the self-adjoint operators. Instead, we will assume the operator to be already diagonalised: namely, we consider multiplication operators. Such operators can be characterised as follows.

Theorem 4.2 (Structure of multiplication operators, [14, 35]) *Let $\mathcal{X}$ be a metric space and μ a positive measure on the Borel sigma-algebra of $\mathcal{X}$ such that $\mu(\Lambda) < \infty$ for any bounded Borel set $\Lambda \subset \mathcal{X}$. For a (possibly unbounded) measurable function $f : \mathcal{X} \to \mathbb{R}$, the linear operator T_f in $L^2(\mathcal{X}, \mu)$ defined by*

$$(T_f u)(x) := f(x)u(x), \quad D(T_f) = \{u \in L^2(\mathcal{X}, \mu) | fu \in L^2(\mathcal{X}, \mu)\}$$

is self-adjoint. Its spectrum coincides with the essential range of f and its point/discrete spectrum is given by

$$\operatorname{spec}_p(T_f) = \{\mu(f^{-1}(\lambda)) > 0\}.$$

Consider now the following stochastic evolution equation on $H = L^2(\mathcal{X}, \mu)$

$$\mathrm{d}U = (p\,\mathrm{Id} + T_f)U\,\mathrm{d}t + \mathrm{d}W, \qquad U = U(t). \tag{4.3}$$

Set $M := \sup\{x : x \in \operatorname{essran}(f)\} = \operatorname{esssup}(f)$. For the operator $p\,\mathrm{Id} + T_f$, the spectrum is given by the set $p + \operatorname{essran}(f)$. It is contained in the left half of the complex plane as long as $p + M < 0$. Therefore, the associated dynamical system undergoes a bifurcation at $p^* = -M$. We compute the norm of the variance as in the previous section assuming $p < p^* = -M$:

$$\begin{aligned}
\|V_\infty\|_{B(H)}^2 &= \left\| \int_0^\infty \mathrm{e}^{2r(p+T_f)}\,\mathrm{d}r \right\|_{B(H)}^2 \\
&= \sup_{h\in D(T_f), \|h\|=1} \left\| \int_0^\infty \mathrm{e}^{2r(p+f)} h\,\mathrm{d}r \right\|_H^2 \\
&\overset{p<-M}{=} \sup_{h\in D(T_f), \|h\|=1} \left\| \frac{h}{2(p+f)} \right\|_H^2 \\
&= \sup_{h\in D(T_f), \|h\|=1} \int_{\mathcal{X}} \left| \frac{h(x)}{2(p+f(x))} \right|^2 \mathrm{d}\mu(x).
\end{aligned}$$

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Now, we look at the spectrum of f crossing the imaginary axis (i.e. $p \to -M^-$). The divergence of the last expression depends, of course, on the function f and on the L^2 space we consider. Suppose we set μ equal to the Lebesgue measure and $\mathcal{X} \subset \mathbb{R}$, which is the most straightforward generalisation of the Laplacian case. To simplify the treatment, we also assume that f attains its essential supremum at only one point $x^* \in \mathcal{X}$ and that it is continuous in a neighbourhood of that point. We assume continuity around x^* in order for the limit $\lim_{x\to 0^-} f(x)$ to be well-defined and independent of the sequence converging to 0^-. Moreover, without loss of generality we set $x^* = 0$ and $p^* = 0$. By definition of M and continuity of f, we have $\lim_{x\to 0} f(x) = 0$. Assume now $\mu(\mathcal{X}) < \infty$. Let $\theta(x)$ be a smooth function such that $\lim_{x\to 0} \frac{f(x)}{\theta(x)} = 1$ and that θ is bounded from above and below outside any neighbourhood of x^* (intuitively θ represents the order of f at $x^* = 0$). Then, the function h defined by $h(x) = x^{-1/2}\theta(x)$ is in $L^2 = L^2(\mathcal{X}, \mu)$ and θ can be chosen so that h has unit norm. We obtain

$$\|V_\infty\|^2_{B(L^2)} = \sup_{h\in D(T_f), \|h\|=1} \int_{\mathcal{X}} \left| \frac{h(x)}{2(p+f(x))} \right|^2 \mathrm{d}x$$

$$\geq \int_X \left| \frac{\theta(x)}{2(p+f(x))x^{1/2}} \right|^2 \mathrm{d}x$$

and take the limit inferior as before

$$\liminf_{p\to 0^-} \|V_\infty\|^2_{B(H)} \geq \liminf_{p\to 0^-} \int_{\mathcal{X}} \left| \frac{\theta(x)}{2(p+f(x))x^{1/2}} \right|^2 \mathrm{d}x$$

$$\overset{\text{Fatou}}{\geq} \int_X \frac{\theta^2(x)}{f^2(x)} \frac{1}{4x} \, \mathrm{d}x = +\infty.$$

Remark 4.3 *The assumption $\mu(X) < +\infty$ can be relaxed by multiplying h by a function g such that g/f is square integrable in a neighbourhood of infinity. Requiring that the essential supremum is attained at a unique point x^* can also be easily avoided by studying each of the points separately. In any case, for each of them the analysis is similar.*

We have shown the following:

Theorem 4.4 *Consider a map $f : \mathbb{R} \to \mathbb{C}$ and the stochastic evolution equation (4.3) over the domain $D(T_f) \subset L^2(\mathbb{R}, \mu)$ for some sigma-finite measure μ. Assume f to be continuous in a neighbourhood of the points at which it attains its essential supremum. Then,*

$$\lim_{p\to -esssup(f)^-} \|V_\infty\|_{B(L^2)} = +\infty. \tag{4.4}$$

4.3 The general case

We have seen how the assumption of discrete spectrum allows for identifying the direction along which the covariance operator diverges. We have argued that such an approach cannot be used in the general setting because it involves considering eigenfunctions for the operator A. Indeed, if the operator has continuous spectrum, eigenfunctions do not exist. Nevertheless, one can find 'approximate' eigenfunctions for elements at the boundary of the spectrum.

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Theorem 4.5 (Weyl's criterion) *Consider a closed linear operator $\mathcal{A}$ on a Hilbert space H. If $\lambda \in \partial\mathrm{spec}(\mathcal{A})$, there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ in H such that $\|u_k\| = 1$ and*

$$\lim_{k\to\infty} \|\mathcal{A}u_k - \lambda u_k\| = 0.$$

For completeness, and since it might not be well-known, we provide a proof of Weyl's criterion. We denote by $\mathrm{res}(\mathcal{A}) := \mathbb{C} \setminus \mathrm{spec}(\mathcal{A})$ the resolvent set of $\mathcal{A}$. We need the following auxiliary result.

Lemma 4.6 *If $\mathcal{A}$ is a closed operator and $z \in \mathrm{res}(\mathcal{A})$, then $\|(A - z)^{-1}\| \geq \mathrm{dist}(z, \mathrm{spec}(\mathcal{A}))^{-1}$.*

Proof Fix $z \in \mathrm{res}(\mathcal{A})$ and $\lambda \in \mathrm{spec}(\mathcal{A})$. Then $|z - \lambda| \geq \|(\mathcal{A} - z)^{-1}\|^{-1}$ (indeed, by Proposition 2.9 in [39], if λ satisfies $|z - \lambda| < \|(\mathcal{A} - z)^{-1}\|^{-1}$ then $\lambda \in \mathrm{res}(\mathcal{A})$). This also implies

$$\mathrm{dist}(z, \mathrm{spec}(\mathcal{A})) = \inf_{\lambda\in\mathrm{spec}(\mathcal{A})} |z - \lambda| \geq \frac{1}{\|(\mathcal{A} - z)^{-1}\|}$$

so that rearranging gives the claimed result. □

Proof of Theorem 4.5 Since $\lambda \in \partial\mathrm{spec}(\mathcal{A})$, we can find a sequence $\{\lambda_k\} \subset \mathrm{res}(\mathcal{A})$ such that $\lambda_k \to \lambda$. For each λ_k, we can apply Lemma 4.6 and for each k

$$\|(A - \lambda_k)^{-1}\| \geq \mathrm{dist}(\lambda_k, \mathrm{spec}(\mathcal{A}))^{-1}.$$

We can also find $v_k \in D((A - \lambda_k)^{-1})$ such that $\|v_k\| = 1$ and

$$\|(A - \lambda_k)^{-1} v_k\| \geq \frac{1}{2}\mathrm{dist}(\lambda_k, \mathrm{spec}(\mathcal{A}))^{-1}.$$

Then, the sequence

$$u_k := \frac{(A - \lambda_k)^{-1} v_k}{\|(A - \lambda_k)^{-1} v_k\|}$$

is normalised and satisfies the claim. Indeed, one computes

$$\begin{aligned}\|\mathcal{A}u_k - \lambda u_k\| &= \|\mathcal{A}u_k - \lambda_k u_k + \lambda_k u_k - \lambda u_k\| \leq \|\mathcal{A}u_k - \lambda_k u_k\| + |\lambda_k - \lambda|\|u_k\| \\ &= \frac{\|v_k\|}{\|(A - \lambda_k)^{-1} v_k\|} + |\lambda_k - \lambda|\|u_k\| = \frac{1}{\|(A - \lambda_k)^{-1} v_k\|} + |\lambda_k - \lambda| \\ &\leq 2\mathrm{dist}(\lambda_k, \mathrm{spec}(\mathcal{A})) + |\lambda_k - \lambda| \to 0.\end{aligned}$$

Therefore, Weyl's criterion follows. □

Since our bifurcation point necessarily involves spectrum on the boundary, we can exploit this theorem to prove a result similar to the one obtained in the discrete setting.

Theorem 4.7 *Consider the stochastic evolution equation*

$$\mathrm{d}U = (p\,\mathrm{Id} + \mathcal{A})U\,\mathrm{d}t + \sigma B\,\mathrm{d}W \tag{4.5}$$

and assume $\mathrm{spec}(\mathcal{A}) = \{\lambda_*\} \cup \mathrm{spec}_-(\mathcal{A})$*, with* $\mathrm{Re}(\lambda_*) = 0$ *and* $\mathrm{spec}_-(\mathcal{A}) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$*. Also assume that* $BQB^* \geq c > 0$*; in particular this holds for* $B = Q = Id$*. Then, there exists a sequence* $\{u_k\}_{k\in\mathbb{N}} \subset H$ *such that for each* k

$$\lim_{p\to 0^-} \langle V_\infty u_k, u_k\rangle = +\infty. \tag{4.6}$$

Proof Since $\lambda_* \in \partial\mathrm{spec}(\mathcal{A})$, by Weyl's criterion there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ s.t.

$$\lim_{k\to\infty} \|\mathcal{A}u_k - \lambda_* u_k\| = 0, \quad \|u_k\| = 1.$$

Our aim is to find a subsequence of $\{u_k\}$ that satisfies the claim. Define $e_k := \mathcal{A}u_k - \lambda_* u_k$, $\bar{e}_k := \mathcal{A}^* u_k - \bar{\lambda}_* u_k$. Note that we have

$$\lim_{k\to\infty} \|e_k\| = 0, \quad \lim_{k\to\infty} \|\bar{e}_k\| = 0.$$

As usual, the Lyapunov equation gives

$$\langle (p+\mathcal{A})V_\infty u_k, u_k\rangle + \langle V_\infty (p+\mathcal{A})^* u_k, u_k\rangle = -\sigma^2 \langle BQB^* u_k, u_k\rangle.$$

Then, we can compute

$$\begin{aligned}\langle (p+\mathcal{A})V_\infty u_k, u_k\rangle + \langle V_\infty (p+\mathcal{A})^* u_k, u_k\rangle &= 2p\langle V_\infty u_k, u_k\rangle + \lambda_*\langle V_\infty u_k, u_k\rangle \\ &\quad + \langle V_\infty u_k, e_k\rangle + \bar{\lambda}_*\langle V_\infty u_k, u_k\rangle + \langle \bar{e}_k, V_\infty u_k\rangle \\ &= 2p\langle V_\infty u_k, u_k\rangle + \langle V_\infty u_k, e_k\rangle + \langle \bar{e}_k, V_\infty u_k\rangle.\end{aligned}$$

Therefore, we conclude that

$$\langle V_\infty u_k, u_k\rangle = \frac{-\langle V_\infty u_k, e_k\rangle - \langle \bar{e}_k, V_\infty u_k\rangle - \sigma^2\langle BQB^* u_k, u_k\rangle}{2p}.$$

We will now consider two cases: first, assume $\|V_\infty u_k\| < C$ for some $C > 0$ (uniformly in k). Then, by eventually discarding some of the pairs $(e_k, \bar{e}_k)$, we can also assume the bounds

$$\|e_k\| < \frac{1}{k}, \quad \|\bar{e}_k\| < \frac{1}{k}$$

Together, this gives

$$|\langle V_\infty u_k, e_k\rangle| \leq \|V_\infty u_k\|\|e_k\| \leq \frac{C}{k}.$$

And therefore:

$$\begin{aligned}\langle V_\infty u_k, u_k\rangle &\leq \frac{|\langle V_\infty u_k, e_k\rangle| + |\langle \bar{e}_k, V_\infty u_k\rangle| - \sigma^2\langle BQB^* u_k, u_k\rangle}{2p} \\ &\leq \frac{2C/k - \sigma^2\langle BQB^* u_k, u_k\rangle}{2p} \leq \frac{2C/k - \sigma^2 c}{2p}.\end{aligned}$$

The sequence $\{u_k\}_{k>2C/\sigma^2 c}$ satisfies the claim of the theorem. In this case, the stronger result

$$\langle V_\infty u_k, u_k\rangle = \mathcal{O}\left(\frac{1}{p}\right), \qquad \text{as } p \to 0^-$$

holds. Suppose now $\|V_\infty u_k\| < C$ does not hold for any $C > 0$, then, there exists a subsequence $\{u_{k_j}\} \subset \{u_k\}$ such that $\|V_\infty u_{k_j}\| \geq j$. But this implies that also $\langle V_\infty u_{k_j}, u_{k_j}\rangle = \|\sqrt{V_\infty} u_{k_j}\|^2 \to \infty$ as $j \to \infty$, which implies the claim. □

We conclude with some remarks. Note that, in the previous sections, we were not only able to prove that some component of the variance diverges, but also to compute its rate of divergence. In the last proof, we had to exclude the possibility that $\|V_\infty u_k\|$ has a divergent subsequence. If such a subsequence exists, the variance still diverges, but its rate of divergence is in general unknown. Therefore, we cannot conclude as before. Nevertheless, we observe that if we replace the genericity condition $\langle BQB^* u_{k^*}, u_{k^*}\rangle \neq 0$ with the assumption

$$\langle V_\infty u_k, e_k\rangle + \langle \bar{e}_k, V_\infty u_k\rangle + \sigma^2 \langle BQB^* u_k, u_k\rangle \neq 0 \qquad \text{for infinite values of } k,$$

we can indeed conclude that there exists a sequence such that

$$\langle V_\infty u_k, u_k\rangle = \mathcal{O}\left(\frac{1}{p}\right), \qquad \text{as } p \to 0^-.$$

Furthermore, Theorem 4.7 is actually stronger than the results obtained previously. Indeed, it shows that, under some assumptions, there exists a *whole sequence* of 'approximate eigenfunctions' such that the components of the covariance operator along this sequence diverge. Of course, such a sequence might be constant, as it is the classical case for discrete spectrum, which occurs for many differential operators on bounded domains.

As another remark, suppose we only look for one vector $u \in H$ such that

$$\langle V_\infty u, u\rangle = \mathcal{O}\left(\frac{1}{p}\right), \qquad \text{as } p \to 0^-.$$

Then, to obtain the claim on the asymptotic limit of the covariance operator, we only have to require that there exists a k such that

$$\langle V_\infty u_k, e_k\rangle + \langle \bar{e}_k, V_\infty u_k\rangle + \sigma^2 \langle BQB^* u_k, u_k\rangle \neq 0,$$

which is a much weaker condition.

5 Conclusion and outlook

In this work, we have given a relatively comprehensive view of stochastic evolution equation statistics upon approaching bifurcation points of the underlying deterministic system. In particular, we extended the work [15] for SPDEs with self-adjoint linear operators A on bounded domains in several directions. This included complex eigenvalues and non-trivial Jordan blocks for A, degenerate parameter-dependent noise term as well as a major generalisation to continuous spectrum frequently encountered for spatially extended systems.

The next natural steps are to apply the results we obtained here to various classes of model problems and data sets to check the role played by statistics and data availability. Although the field of critical slowing down for spatial systems has a long history in the context of physics [17], it is definitely worthwhile to build a general mathematical theory in the context of SPDEs. In fact, we mention that there is already a strongly surging interest in the area of warning signs for spatial systems recently, just see [7, 29, 37, 40] for a few examples. Hence, our study here in combination with [15] provides the detailed theoretical underpinnings, why and how spatial systems modelled by SPDEs display slowing down effects when measured through stochastic effects.

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Conflict of interest

None.

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