

A NOTE ON THE FUNDAMENTAL GROUP OF KODAIRA FIBRATIONS

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Abstract The fundamental group π of a Kodaira fibration is, by definition, the extension of a surface group Π_b by another surface group Π_g , i.e.

$$1 \rightarrow \Pi_g \rightarrow \pi \rightarrow \Pi_b \rightarrow 1.$$

Conversely, Catanese (2017) inquires about what conditions need to be satisfied by a group of that sort in order to be the fundamental group of a Kodaira fibration. In this short note we collect some restrictions on the image of the classifying map $m: \Pi_b \rightarrow \Gamma_g$ in terms of the coinvariant homology of Π_g . In particular, we observe that if π is the fundamental group of a Kodaira fibration with *relative irregularity* $g - s$, then $g \leq 1 + 6s$, and we show that this effectively constrains the possible choices for π , namely that there are group extensions as above that fail to satisfy this bound, hence it cannot be the fundamental group of a Kodaira fibration. A noteworthy consequence of this construction is that it provides examples of symplectic 4-manifolds that fail to admit a Kähler structure for reasons that eschew the usual obstructions.

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In this note we use the term *Kodaira fibration* to refer to an algebraic surface X endowed with the structure of a smooth fibre bundle with bundle map $f: X \rightarrow \Sigma_b$ over a Riemann surface Σ_b , with fibre a Riemann surface Σ_g so that the bundle map is not isotrivial as a fibration (in the sense of algebraic geometry, i.e. as a pencil of curves); in particular f is not a locally trivial *complex analytic* fibre bundle. It is well known that the base and fibre genera satisfy the conditions $b \geq 2$ and $g \geq 3$, see e.g. [3, § V.14]; moreover, by Arakelov's inequality, the signature of the intersection form of X is strictly positive, see e.g. [7, Corollary 42].

From the definition we deduce that X is aspherical and, denoting by $\pi := \pi_1(X)$ the fundamental group of X , we have a short exact sequence

$$1 \rightarrow \Pi_g \rightarrow \pi \rightarrow \Pi_b \rightarrow 1 \tag{1}$$

where Π_h is the fundamental group of a Riemann surface Σ_h . Namely, π is the extension of a surface group by a surface group. In [7, Question 16] the author inquires about what conditions need to be satisfied by a group of that sort in order to be the fundamental group of a Kodaira fibration, in terms of the *monodromy representation* $m: \Pi_b \rightarrow \Gamma_g$, where Γ_g is the genus- g mapping class group that classifies the bundle $f: X \rightarrow \Sigma_b$. This was not the very first time that such a question had been raised; previous instances include MathOverflow posts by I. Rivin and J. Bryan, implicitly [4] (prompted by Bryan's post), [11], and older ones, possibly starting with [12, 17]. We make reference to [7] as it is there that the question was framed in the form we discuss here. Further results on this question appeared recently in [1, 9].

As is well known, by the Parshin rigidity theorem, for any fixed pair (g, b) there can exist only finitely many extensions as in (1) that arise as the fundamental group of some Kodaira fibration, so *a priori* 'most' choices of m fail that condition. It is not clear, however, how to spell out explicit obstructions.

The question of [7] is asked under the obvious assumption that the image of m is infinite, so that the signature does not automatically vanish. To that, we can add the assumption that the first Betti number of π (and all its finite index subgroups) must be even, lest π fails to be Kähler. This further restricts, although it does not decide, possible choices for a group π as in (1); examples of this appear in [4, 12].

The purpose of this note is to show how subtler restrictions follow from Xiao's constraints [18] on the relative irregularity of holomorphic fibrations.

To start, assume that X is an aspherical surface with fundamental group π as in (1). As discussed by Kotschick and Hillman in [13, Proposition 1] and [10, Theorem 1], respectively, the existence of an epimorphism from π onto an hyperbolic surface group Π_b with a finitely generated kernel entails the existence of a holomorphic fibration $f: X \rightarrow \Sigma_b$ whose fibres are smooth and of multiplicity one. Namely, $f: X \rightarrow \Sigma_b$ is a smooth surface bundle over a surface. By the long exact sequence of the fibration, the fibre must have genus equal to g .

The first, trivial, observation is that the condition on the non-vanishing of the signature implies that the image of m cannot be contained in subgroups of Γ_g for which the signature of the corresponding surface bundle is zero; this entails the well-known facts that the image of m cannot be contained in the hyperelliptic subgroup, or in the Torelli subgroup, of the mapping class group.

We pause to give a different proof of the latter statement, which positions us in the spirit of this note.

Lemma 1.1 (Folklore). *Let X be an aspherical Kähler surface with fundamental group $\pi := \pi_1(X)$ as in (1). Then if the image of the monodromy representation $m: \Pi_b \rightarrow \Gamma_g$ is contained in the Torelli subgroup, it is trivial and $X = \Sigma_g \times \Sigma_b$.*

Proof. The Hochschild–Serre spectral sequence associated with the sequence in (1) gives, in low degrees, the long exact sequence

$$H_2(X; \mathbb{Z}) \xrightarrow{f_*} H_2(\Sigma_b; \mathbb{Z}) \rightarrow H_1(\Sigma_g; \mathbb{Z})_{\Pi_b} \rightarrow H_1(X; \mathbb{Z}) \xrightarrow{f_*} H_1(\Sigma_b; \mathbb{Z}) \rightarrow 0, \quad (2)$$

where $H_1(\Sigma_g; \mathbb{Z})_{\Pi_b}$ is the group of coinvariants under the action of Π_b on the homology of the fibre, i.e. the *homological* monodromy.

It is well known that $f_*: H_2(X; \mathbb{Z}) \rightarrow H_2(\Sigma_b; \mathbb{Z})$ is an epimorphism over rational coefficients. It follows that the *relative irregularity* $q_f = q(X) - q(\Sigma_b)$ is given by the half of the rank of the group of coinvariants (which must be even for X to have even first Betti number), i.e.

$$q_f = \frac{1}{2} \text{rk}_{\mathbb{Z}} H_1(\Sigma_g; \mathbb{Z})_{\Pi_b}.$$

As $m(\Pi_b)$ is contained in the Torelli group, the action of Π_b on $H_1(\Sigma_g; \mathbb{Z})$ is trivial, hence $q_f = g$. By a result of Beauville (see [6, Lemme]), this can occur if and only if X is birational to the product $\Sigma_g \times \Sigma_h$ with the fibration corresponding to the projection; as we assume that X is aspherical, this implies that X is actually isomorphic to $\Sigma_g \times \Sigma_h$. \square

As a Kodaira fibration is aspherical and non-isotrivial, it cannot be isomorphic to a product. The Lemma implies therefore that the image of m cannot be contained in the Torelli subgroup.

The take-home point of the previous proof is that we can use constraints on the relative irregularity of fibred surfaces (as we used Beauville’s) to control the coinvariants of the homological monodromy. This leads to the following.

Proposition 1.2. *Let X be an aspherical Kähler surface with fundamental group $\pi := \pi_1(X)$ as in (1). Then if the coinvariant homology $H_1(\Sigma_g; \mathbb{Z})_{\Pi_b}$ has rank $2g - 2s$, then $g \leq 1 + 6s$.*

Proof. Following *verbatim* the proof of Lemma 1.1, we have a holomorphic fibration $f: X \rightarrow \Sigma_b$ that is a smooth surface bundle over a surface, whose low-degree homology is described in (2). As the action of Π_b on $H_1(\Sigma_g; \mathbb{Z})$ is non-trivial, the fibration is necessarily non-trivial, hence by [18, Corollary 3] its relative irregularity satisfies the bound $q_f \leq \frac{5g+1}{6}$. By our assumption again we have $q_f = g - s$, whence the bound follows. \square

Corollary 1.3. *A Kodaira fibration of relative irregularity $q_f = g - 1$ has fibre of genus $g \leq 7$.*

It has been conjectured (see e.g. [2]) that Xiao’s bound on then relative irregularity could be improved to $q_f \leq \frac{g}{2} + 1$ (the *modified Xiao’s conjecture*; [2] makes some steps in that direction using the Clifford index of a fibration). If that were the case, we would get that the fibre genus is 4 at most (and similar improvements would occur to Proposition 1.2). As Kodaira fibrations have fibres of genus g at least 3, this would leave an even narrower room for the genera of possible fibres. In fact, with extra assumptions on the genus of the base, we can actually improve that result: as observed in [7], using standard inequalities for non-isotrivial smooth fibrations (see [6, 14]) it is not hard to prove that the holomorphic Euler characteristic of a Kodaira fibration satisfies the inequalities

$$3(b - 1)(g - 1) < 3\chi(X) < 4(b - 1)(g - 1), \tag{3}$$

so when $b = 2$ there are no Kodaira fibrations of genus 3 or 4 (regardless of the relative irregularity). We can then rephrase the above results.

Corollary 1.4. *If the modified Xiao’s conjecture holds true, a Kodaira fibration with fibre of genus g and base of genus $b = 2$ has relative irregularity $q_f \leq g - 2$.*

Remark. Without entering into much detail, we note that the inequalities in (3), or other standard inequalities for surfaces, do not seem to provide any further information for the case of $b = 2$ beyond those observed above; for example, when $g = 5$, we cannot exclude the existence of a Kodaira fibration with $\chi(X) = 5$ which may have $q_f = 4$ (although it is worth noting that such a surface would have $K_X^2 = 44$, hence slope $K_X^2/e(X) = 2.75$, higher than any known examples). In that sense, it seems necessary to invoke the modified Xiao's conjecture in the statement of Corollary 1.4. Similarly, if $b = 3$, for $g = 3$ we are aware of no constraints to the existence of a Kodaira fibration with $q_f = 2$ that would have again $\chi(X) = 5$, $K_X^2 = 44$. The information of (3) gets further diluted as b and g grow.

To complete the picture, we need to show that Proposition 1.2 is effective, namely that there do exist surface bundles over a surface that violate the constraints on the fibre genus contained therein. As previously observed, this entails that there exist actual groups π as in (1) with coinvariant homology of rank $2g - 2s$ that cannot be the fundamental groups of a Kodaira surface.

We will provide examples for some choice of the value s ; it is not too hard, with minor modifications of the constructions described below, to extend the class of such s : we leave this task to the interested reader. For good measure, the surface bundles we will describe have even Betti numbers and strictly positive signature, so there are no (to the best of our understanding) simpler obstructions to the existence of a Kodaira fibration.

Instead of reinventing the wheel, our construction will follow the template of [4] (to which we refer the reader), and exploit as a building block a remarkable surface bundle over a surface discovered in [8]. This manifold, which will be denoted as Z , has fibres of genus 3, base of genus 9, strictly positive signature $\sigma(Z) = 4$ and a section with self-intersection 0.

Proposition 1.5. *There exists an integer $s_0 \in [1, 3]$ such that for every $b \geq 9$ there exists a surface bundle $Z_{g,b}$ with fibre of genus g over a surface of genus b , of strictly positive signature, whose coinvariant homology $H_1(\Sigma_g; \mathbb{Z})_{\Pi_b}$ has rank $2g - 2s_0$ and $g > 1 + 6s_0$. Its fundamental group cannot occur as the fundamental group of a Kodaira fibration.*

Proof. We start by producing the example with $b = 9$. Consider the aforementioned surface bundle $\varphi: Z \rightarrow \Sigma_9$; as the fibres have genus 3, we must have $rk_{\mathbb{Z}} H_1(\Sigma_3; \mathbb{Z})_{\Pi_9} \in [0, 5]$. (It is probably not too difficult to actually determine the exact value—which must be strictly smaller than the maximal possible value $rk_{\mathbb{Z}} H_1(\Sigma_3; \mathbb{Z}) = 6$, otherwise the monodromy would be contained in the Torelli group—but our construction is independent of the outcome of such calculation; as this construction can be applied to surface bundles other than Z , we will keep the discussion general.)

At this point, depending on the parity of this rank, we proceed as follows.

- If the rank happens to be even, we will perform a *section sum* of Z and the product bundle $\Sigma_{h+1} \times \Sigma_9$ along sections of self-intersection 0, i.e. a normal connected sum in which the boundaries of fibred tubular neighbourhoods of the sections are identified with a base-preserving diffeomorphism, see [5]. (Here and in what follows, the normal connected sum operation is not determined uniquely by the data above, but also by

the isotopy class of the gluing map; all results we state, however, apply irrespective of those choices.)

- If the rank happens to be odd, we will first perform a section sum of Z and a standard building block Q , where Q can be defined as the torus bundle over a surface of genus 9 obtained by fibre summing the Kodaira–Thurston manifold—a torus bundle over a torus—with the product bundle $T^2 \times \Sigma_8$. (Note that $b_1(Q) = b_1(\Sigma_8) + 1$.) Next, we will proceed with a further section sum with $\Sigma_h \times \Sigma_9$.

Either way, the resulting manifold will be a surface bundle $Z_{g,9}$ with fibre of genus $g := h + 4$ and even first Betti number; the monodromy representation of section sums is elucidated in [5, §2.2] and a little bookkeeping for the constructions above shows that in all cases $rk_{\mathbb{Z}}H_1(\Sigma_g; \mathbb{Z})_{\Pi_9} = 2g - 2s_0$ for some $s_0 \in [1, 3]$. Moreover, by Novikov additivity (and by the choice of the building block using in the stabilizations, all having vanishing signature) the bundle $Z_{g,9}$ has signature $\sigma(Z_{g,9}) = \sigma(Z) > 0$. We can then increase the genus of the base by fibre sum with $\Sigma_g \times \Sigma_c$ to get a surface bundle $Z_{g,b}$ with base a surface of any genus $b \geq 9$. Again, Novikov additivity and a straightforward Mayer–Vietoris argument show that the signature and the coinvariant homology keep the properties discussed above.

As $g \geq 4$ is a free parameter in the construction above, we can make it large as wanted, in particular larger than $1 + 6s_0$, in which case its fundamental group is not the fundamental group of a Kodaira fibration by Proposition 1.2. □

We can make a minor variation to the construction above to illustrate another phenomenon, which shows that at times we can gain insight into the (non-)existence of the structure of a Kodaira fibration by reiterated application of Proposition 1.2 to finite covers.

Proposition 1.6. *There exists a surface bundle $W_{g+1,b}$ with fibre of genus $g + 1$ over a surface of genus $b \geq 9$, which satisfies the constraints of Proposition 1.2 but admits an unramified cover $S_{g+1,5b-6}$ that fails them. Its fundamental group cannot occur as the fundamental group of a Kodaira fibration.*

Proof. The construction of our examples will build on the construction of Proposition 1.5. Consider the (say right-handed) trefoil knot $T(2, 3)$. As detailed, e.g. in [15, Chapter 10], the trefoil is a fibred knot of genus 1, whose exterior can thus be identified with the mapping torus of an automorphism $\phi : F \rightarrow F$ of a surface F on genus 1 with one boundary component ∂F , fixed pointwise by the monodromy ϕ . With a suitable choice of a basis for $H_1(F; \mathbb{Z}) = \mathbb{Z}^2$, the homological monodromy is given by

$$\phi_* = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4}$$

Next, perform a 0-surgery of S^3 along the trefoil knot; the resulting closed 3-manifold N has $b_1(N) = 1$ and admits a torus fibration over S^1 , whose fibre comes from capping off F with a disk. The 4-manifold $S^1 \times N$ is then a torus bundle over a torus, and the coinvariant homology of the fibre is trivial, as the homology monodromy representation is induced by that of the trefoil knot and coincides with that in (4). Moreover, the signature

of this bundle vanishes. Next, fibre sum with $T^2 \times \Sigma_{b-1}$ to get a torus bundle M with base Σ_b ; by the usual arguments, the coinvariant homology of the fibre and the signature vanish. The torus bundle M admits a section of self-intersection 0 (inherited from the dual of the trefoil knot in N).

Take the surface bundle $Z_{g,b}$, which retains a section of self-intersection 0, and perform a section sum with M . The resulting manifold $W_{g+1,b}$ is a surface bundle with fibre and surface genus as in the indices, and $\sigma(W_{g+1,b}) > 0$. The homology monodromy representation in $SL(2g+2, \mathbb{Z})$ factors in block diagonal form, with a 2×2 block as in (4) that comes from the homological monodromy of the trefoil knot. The coinvariant homology of the fibre satisfies $rk_{\mathbb{Z}} H_1(\Sigma_{g+1}; \mathbb{Z})_{\Pi_b} = 2g - 2s_0$, as the monodromy of the 2×2 block coming from M increases by 1 in the genus of the fibre but does not contribute to the coinvariant homology. Choose then g such that $1 + 6s_0 < g + 1 \leq 7 + 6s_0$; the latter inequality ensures that $W_{g+1,b}$ does not violate the inequality of Proposition 1.2, while the rationale for the first inequality will be apparent momentarily. Recall now that the homological monodromy of the trefoil knot is periodic of period 6, as can be verified from (4) by computing explicitly $(\phi_*)^6 = I$. Then it is not too hard to see the existence of a 6-fold (unramified) cyclic cover of $W_{g+1,b}$, which we will denote $S_{g+1,6b-5}$, induced by a 6-fold cyclic cover $\Sigma_{6b-5} \rightarrow \Sigma_b$ of the base determined by a normal subgroup $\Pi_{6b-5} \leq \Pi_b$. The covering map on $W_{g+1,b}$ is trivial on the fibres, hence the resulting cover $S_{g+1,6b-5}$ is again a surface bundle of fibre Σ_{g+1} over the base Σ_{6b-5} , as recorded in the notation. Its coinvariant homology satisfies $rk_{\mathbb{Z}} H_1(\Sigma_{g+1}; \mathbb{Z})_{\Pi_{6b-5}} \geq 2g + 2 - 2s_0$: there is in fact a gain, in rank, of at least two due to the periodicity of the 2×2 block in the homological monodromy representation in $SL(2g+2; \mathbb{Z})$. (Further gain would depend on the monodromy of Z , as well as the choice of the gluing data in the section sums above.) Application of Proposition 1.2 entails that this manifold cannot carry the structure of Kodaira fibration, as having $rk_{\mathbb{Z}} H_1(\Sigma_{g+1}; \mathbb{Z})_{\Pi_{6b-5}} \geq 2g + 2 - 2s_0$ would require having at least $g + 1 \leq 1 + 6s_0$, which violates the above choice of $g + 1 > 1 + 6s_0$. (If $rk_{\mathbb{Z}} H_1(\Sigma_{g+1}; \mathbb{Z})_{\Pi_{6b-5}}$ happens to be odd, this would follow as well for simple parity reasons). As the finite cover of a Kodaira fibration must be a Kodaira fibration, $W_{g+1,b}$ does not carry the structure of Kodaira fibration either. In summary, reiterated use of Proposition 1.2 for finite covers may unveil further information. \square

Remarks

(1) The surface bundles $Z_{g,b}$ and $W_{g+1,b}$ identified in the proofs of Propositions 1.5 and 1.6 admit a symplectic structure by [16]. They fail, however, to admit a Kähler structure. We recap the argument: if they did admit a Kähler structure, as their fundamental group surjects onto Π_b with finitely generated kernel, they would admit a holomorphic fibration over Σ_b . By the aforementioned results of Kotschick and Hillman, the fibres would be smooth with multiplicity one; as the signature is non-zero, these would be Kodaira fibrations, in violation of Proposition 1.2. To our knowledge, this type of obstruction to the existence of a Kähler structure on a symplectic manifold has not been spelled out in this form before.

(2) We conclude this note observing the rather annoying fact that none of the discussion above has any bearing on the *a priori* more basic question of whether a group as in (1) can occur as the fundamental group of some projective (or even Kähler) variety (compare Question (5) of [11]). In fact, we have no means to exclude the possibility that any of the

groups we dismissed above could be the fundamental group of some projective variety, and hence, by standard arguments, of some algebraic surface Y . As the fundamental group of Y would surject onto Π_b , such a surface would admit a fibration with base Σ_b . By the above considerations, reversing the aforementioned result of [10, 13], we can only infer that the fibration would be necessarily non-smooth, Y would not be aspherical, and its Euler characteristic would have to exceed $(2g - 2)(2b - 2)$.

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