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REPRESENTATIONS OF TRIANGULAR SUBALGEBRAS OF GROUPOID C^*-ALGEBRAS

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Abstract

We investigate the invariant subspace structure of subalgebras of groupoid C^*-algebras that are determined by automorphism groups implemented by cocycles on the groupoids. The invariant subspace structure is intimately tied to the asymptotic behavior of the cocycle.

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1. Introduction

In this paper we continue our investigations in the theory of representations of triangular subalgebras of groupoid C^*-algebras. This study follows the program founded by Arveson in [2, 3] to analyze representations of non-selfadjoint operator algebras. The idea is to try to associate with each (contractive) representation of an operator algebra a C^*-representation of its enveloping C^*-algebra in such a way that properties of the given representation can be inferred from the C^*-representation. Explicitly, given a contractive representation ρ of an operator algebra A on a Hilbert space H one would like to produce another Hilbert space K, such that $K \supseteq H$, and a C^*-representation π of $C^*(A)$, the enveloping C^*-algebra of A, on K, such that

$$\rho(a) = P_H \pi(a)|_H, \quad a \in A,$$

where P_H is the (orthogonal) projection onto H. Such a representation of A is, thus, a compression of a C^*-representation (called a *dilation* of ρ) to a subspace $H \subseteq K$. In

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general, a contractive representation need not have a dilation. However, those that do have been characterized by Arveson loc. cit. as the so-called completely contractive representations. It can be shown that when the representation ρ has a dilation, the Hilbert space H is a *semi-invariant* subspace for $\pi(A)$. This means that H may be written as $H = H_2 \ominus H_1$, where $H_1 \subseteq H_2$ and $\pi(A)H_i \subseteq H_i$. That is, H is the orthogonal difference of two nested invariant subspaces for $\pi(A)$. The converse also holds, that is, the compression of $\pi|A$ to a semi-invariant subspace defines a completely contractive representation of A. Hence the completely contractive representation theory of A can be studied by studying the C^*-representations of the C^*-envelope $C^*(A)$ of A, and their invariant (and semi-invariant) subspaces. Thus there appears to be an exact analogue of the program initiated by Sz.-Nagy for studying a contraction operator in terms of its (minimal) unitary dilation.

For subalgebras A of groupoid C^*-algebras of the kind we investigate, $C^*(A)$, is the groupoid C^*-algebra itself [17]. Moreover, our algebras A have the property that every contractive representation is completely contractive. Thus the structure of the contractive representations of A depends entirely on the invariant subspace structure of the algebras $\pi(A)$ obtained by letting π run through all the C^*-representations of $C^*(A)$. The representation theory of groupoid C^*-algebras has been developed by Renault in [26, 27]. Here we shall use Renault's analysis to describe the lattice of invariant subspaces for $\pi(A)$, where π is a C^*-representation of $C^*(A)$. We identify a subspace with the orthogonal projection onto it and we write the lattice of all such projections as $\operatorname{lat}\pi(A)$. Very roughly, our algebras are represented as block upper triangular matrix algebras and Renault's theory aids us in analyzing how a projection in $\operatorname{lat}\pi(A)$ decomposes.

In finite dimensions, the algebras we are studying are simply incidence algebras in the sense of [5]. We are motivated by the fact that for such algebras, the description of $\operatorname{lat}\pi(A)$ is very simple and classical. Suppose, indeed, that π is irreducible: then π is determined by a column, that is, the Hilbert space for π is the ℓ^2-space indexed by a column. If H_π is the subspace consisting of those functions supported on the partial order indexing A, then H_π is what algebraists call an indecomposable projective module over A. It is almost immediate, yet fundamental, that every indecomposable projective module over A is of this form. Further, the dilation result is tantamount to the assertion that every module admits a projective cover and this allows one to analyze, or 'resolve', an arbitrary module in terms of projective modules. Thus, in a sense, our goal is to see how far the structure of projective modules extends from the finite to the infinite dimensional setting.

In more detail, our setting is this. Let B be a nuclear C^*-algebra that has a diagonal subalgebra D in the sense of Kumjian [10]; that is, we assume D is a masa in B with certain properties that allow B to be represented (essentially) in terms of matrices indexed by an equivalence relation (= a principal groupoid) on the maximal ideal space of D. From the spectral theorem for bimodules [12] we may assert that any (norm closed) subalgebra A of B containing D must consist of all matrices supported on a transitive, reflexive subrelation of the equivalence relation indexing B. The algebras we consider here will also be assumed to satisfy: $A \cap A^* = D$ (that is, they will be *triangular*) and $A + A^*$ is dense in B. This means that the subrelation induces a total order on each equivalence class.

If B were the $n \times n$ matrices, then an A satisfying our hypotheses would be (unitarily equivalent to) the full algebra of upper triangular matrices. More generally, this class of non-selfadjoint algebras contains the strongly maximal triangular AF algebras that have been studied extensively in recent years. (See [19, 21, 23, 22, 15, 8 and 24] to name a few pertinent references.)

In this paper we shall assume that not only are our algebras triangular in the sense just described, but also that they are analytic in the sense of [11] and [9] (see Section 2 for details and definitions). This hypothesis is satisfied for the upper triangular $n \times n$ matrices, but not for all strongly maximal triangular AF algebras. Although a bit restrictive, this hypothesis allows us to apply powerful techniques from ergodic theory and it puts into evidence phenomena that need further investigation in the future. Also, to simplify matters, we shall treat only C^*-representations π that are irreducible.

Given such an algebra A and an irreducible representation π of its enveloping C^*-algebra $C^*(A)$, we can associate with it a closed subset $\tilde{R}^{\mu}_{\infty}(c)$ of $\mathbb{R} \cup \{\infty\}$. Here μ is a measure associated with π and c is a cocycle describing A. (This subset is always a subgroup of $\mathbb{R}$ together, sometimes, with ∞ adjoined, and we shall therefore call such a subset a subgroup of $\mathbb{R} \cup \{\infty\}$.) We shall first show (Theorem 3.2) that $\operatorname{lat} \pi(A)$ is a nest (that is, it is totally ordered with respect to the usual order of projections) and, in fact, if it is not trivial, then every non-trivial projection in $\operatorname{lat} \pi(A)$ 'generates' the lattice. Then we examine the order type of this nest. It turns out that it depends on $\tilde{R}^{\mu}_{\infty}(c)$. If $\tilde{R}^{\mu}_{\infty}(c) = \{0\}$ then (Theorem 5.1) all the projections in $\operatorname{lat} \pi(A)$ are in $\pi(D)''$. In the finite dimensional setting, this is always the case. Note, too, that in general π can be disintegrated over the maximal ideal space X of D. Hence we can identify the representation space with $\int_X^{\oplus} K(u) d\mu(u)$ for some measure μ, that is ergodic because π is irreducible, and Hilbert space fibers $K(u)$, $u \in X$. Therefore, when $\tilde{R}^{\mu}_{\infty}(c) = \{0\}$, we have for μ-a.e. u, $\mathscr{L}(u) \equiv \{F(u) : F \in \operatorname{lat} \pi(A)\} = \{0, I\}$. Again, a little thought shows that in the finite dimensional situation, this recaptures the fact that the indecomposable projective modules over the algebra of upper triangular $n \times n$ matrices are the 'columns'.

If $\tilde{R}^{\mu}_{\infty}(c) = \{0, \infty\}$ we find (Theorem 4.1) that $\operatorname{lat} \pi(A) = \{0, I\}$; so that there is no non-trivial invariant subspace for $\pi(A)$. One can conclude from this that if ρ is a representation of A with an irreducible C^*-dilation π and if $\tilde{R}^{\mu}_{\infty}(c) = \{0, \infty\}$, then $\rho = \pi$, that is, ρ can be extended to a C^*-representation of $C^*(A)$. Using this we describe all (contractive) representations of a $\mathbb{Z}$-analytic subalgebra A, with irreducible C^*-dilation (Corollary 9.2).

The other two possibilities for $\tilde{R}^{\mu}_{\infty}(c)$ are $\mathbb{R} \cup \{\infty\}$ and $\lambda\mathbb{Z} \cup \{\infty\}$ for some $\lambda \in \mathbb{R}$. In both cases we show (Corollary 6.6 and Theorem 7.1) that either $\operatorname{lat}\pi(A) = \{0, I\}$ or the order type of $\mathscr{L}(u)$ (the lattice in the fiber over u as above) is the order type of $\tilde{R}^{\mu}_{\infty}(c)$.

In Section 8 we present an example of an irreducible representation π of the CAR algebra B and an analytic subalgebra A of B such that $\mathscr{L}(u)$ has the order type of $\mathbb{Z} \cup \{\pm\infty\}$.

Representations of triangular AF algebras were studied recently by Orr and Peters [19]. Most of the algebras they treat are analytic (such as the standard embedding or the refinement embedding algebras) and our analysis generalizes some of their results (see, for example, the remark following our Theorem 9.1).

Without further mention, all our Hilbert spaces will be complex and separable. All operators will be bounded and linear. All C^*-algebras will be separable and all locally compact spaces will be Hausdorff and second countable. The measures will be Radon measures and positive.

2. Preliminaries

Throughout the paper B will always denote a nuclear C^*-algebra having a diagonal D in the sense of Kumjian [10]. A *normalizer* of D is simply an element $b \in B$ such that b^*db and bdb^* are in D whenever $d \in D$. Such a normalizer is called *free* if $b^2 = 0$. The set of normalizers and free normalizers of D will be denoted $N(D)$ and $N_f(D)$, respectively. We say that D is a *diagonal* in B if D is a masa in B containing an approximate identity for B, if B is not unital, such that there is a faithful expectation $\mathbf{P}$ from B onto D whose kernel is spanned by the free normalizers of D.

The example to keep in mind is $B = M_n(C)$ and $D = D_n$ the algebra of all diagonal matrices. $\mathbf{P}$ is the obvious map onto D and a normalizer is a matrix that can be written as a product dc where $d \in D_n$ and c is a permutation matrix.

In the general situation, the elements of B can be thought of as 'generalized matrices' whose coordinates are in some equivalence relation. More precisely, Kumjian's representation theorem [10] asserts that there is a $\mathbf{T}$-groupoid E over an r-discrete, locally compact, principal groupoid G, whose unit space $G^{(0)}$ may be identified with the maximal ideal space of D, such that B is isomorphic to $C^*_{red}(G, E)$. We shall now explain what all this means.

We assume the reader is familiar with the terminology, notation and basic facts from [26]. We will use them freely, except that we will use 's' for the 'domain mapping' on a groupoid that Renault denotes by 'd'. We fix, once and for all, a locally compact

r-discrete principal groupoid G. It is in fact an equivalence relation on $G^{(0)}$ but its topology is usually *different* from the relative topology inherited from $G^{(0)} \times G^{(0)}$. The equivalence classes of G are countable and we assume that the counting measures on them give rise to a Haar system, denoted by $\{\lambda^u : u \in G^{(0)}\}$.

A $\mathbf{T}$-groupoid E over G is a (locally compact groupoid) central extension of G by the trivial circle bundle $G^{(0)} \times \mathbf{T}$ (which has the obvious structure of a groupoid with unit space $G^{(0)}$). Thus we have an exact sequence of continuous groupoid homomorphisms

$$G^{(0)} \to G^{(0)} \times \mathbf{T} \xrightarrow{i} E \xrightarrow{j} G$$

where j is onto. Through i we find it convenient to view E as a principal $\mathbf{T}$-bundle with bundle projection j. For $t \in \mathbf{T}$ and $\gamma \in E$, we write t_γ for $i((r(\gamma), t))\gamma$. The centrality assumption amounts to assuming that for all $(\gamma_1, \gamma_2) \in E^{(2)}$ and all $t_1, t_2 \in \mathbf{T}$, we have $(t_1\gamma_1, t_2\gamma_2) \in E^{(2)}$ and $(t_1\gamma_1)(t_2\gamma_2) = t_1t_2(\gamma_1\gamma_2)$.

Let $C_c(G, E)$ denote the space of continuous complex valued, compactly supported functions f on E (that is, the closure of the support of the function is compact), such that $f(t\gamma) = tf(\gamma)$, $t \in \mathbf{T}$, $\gamma \in E$. These, of course, are just the continuous, compactly supported cross sections of the line bundle determined by E. Then with respect to pointwise addition, scalar multiplication, and inductive limit topology, $C_c(G, E)$ becomes a topological $*$-algebra under the operations:

$$(f * g)(\beta) = \int f(\alpha)g(\alpha^{-1}\beta)d\lambda^{r(\beta)}(\dot{\alpha})$$

and

$$f^*(\beta) = \overline{f(\beta^{-1})},$$

where $\dot{\alpha} = j(\alpha)$. (This notation for $j(\alpha)$ will be used frequently.) Note that the integrand is a $\mathbf{T}$-invariant function of α, so $f * g$ makes sense.

A *representation* of $C_c(G, E)$ is simply a $*$-homomorphism π from $C_c(G, E)$ into the algebra $B(H)$ of all bounded operators on a Hilbert space H that is continuous with respect to the inductive limit topology of $C_c(G, E)$ and the weak operator topology on $B(H)$. Renault [27] proves that the quantity

$$\sup\{\|\pi(f)\| : \pi \text{ is a representation}\}$$

is finite for each $f \in C_c(G, E)$ and defines a C^*-norm on $C_c(G, E)$. The completion of $C_c(G, E)$ with respect to this norm is denoted $C^*(G, E)$. The quotient of $C^*(G, E)$ by the common kernel of all the representations induced off the unit space is denoted $C^*_{red}(G, E)$. We will not need these here because when $C^*_{red}(G, E)$ is nuclear, which is one of our hypotheses, $C^*_{red}(G, E) = C^*(G, E)$ ([12]). Kumjian's theorem then,

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under the assumption that B is nuclear, asserts that we may represent B as $C^*(G, E)$ for a suitable $\mathbf{T}$-groupoid E over an r-discrete principal groupoid G on the maximal ideal space of D. Henceforth, we make no distinction between B and $C^*(G, E)$.

The bundle E is trivial over $G^{(0)}$, which is closed and open in G. Therefore, we may view $C_0(G^{(0)}) \subseteq C^*(G, E)$ in the obvious way. This containment identifies D with $C_0(G^{(0)})$. In this situation, a normalizer b has the property that $j(\mathrm{supp}(b))$ is an open G-set in G. That is, $j(\mathrm{supp}(b))$ is an open subset τ of G such that r and s, when restricted to τ, are one-to-one. Thus we may view τ as a partially defined homeomorphism on $G^{(0)}$ mapping the open set $r(\tau)$ onto the open set $s(\tau)$. (Strictly speaking, τ is the graph of this homeomorphism, but we will not distinguish between the two.) Hence, we have in this general situation an analogy with what happens for $M_n(\mathbb{C})$: Every normalizer can be written as a diagonal element times a permutation 'element'. The essential difference, really, is that the permutation 'element' is not really in $C^*(G, E)$, but only in some sort of Borel completion of $C^*(G, E)$. It would be in $C^*(G, E)$ if τ were compact as well as open and if E were the trivial extension. In this case, $C^*(G, E) \cong C^*(G)$ and the permutation 'element' would be χ_τ, the characteristic function of τ.

We turn now to the representations of the objects we have been discussing, $C^*(G, E)$, G and E. The key philosophical point that needs to be made is that while groups act on sets, groupoids act on fibered sets. So, while groups are often represented by unitary operators on Hilbert space, groupoids have unitary representations on Hilbert bundles. There is nothing fundamentally mysterious here, although the technicalities can be somewhat daunting. What we will be describing are infinite dimensional analogues of the elementary fact that finite dimensional incidence algebras have (block) matrix representations.

We begin with some measure theory. Given a measure μ on $G^{(0)}$, we obtain a measure ν on G simply by integrating: $\nu(F) = \int \lambda^u(F)d\mu(u)$. The measure μ is called *quasi-invariant* if ν and ν^{-1} are mutually absolutely continuous, where ν^{-1} is simply the image of ν under the map $\gamma \to \gamma^{-1}$. If μ is quasi-invariant and if Δ is defined to be the Radon-Nikodym derivative, $d\nu^{-1}/d\nu$, then there is a conull Borel set $F \subseteq G^{(0)}$ such that the restriction of Δ to $G|_F$ is a homomorphism from $G|_F$ into the multiplicative positive reals. (The groupoid $G|_F$ is called an *inessential contraction* or *reduction* of G.) It is customary to call Δ the *modular function* of μ. This fundamental fact has a long history. The form in which we are stating it may be attributed to Ramsay [25].

The groupoid E has the same unit space $G^{(0)}$ as G, and a Haar system of its own, denoted $\{\sigma^u\}_{u\in G^{(0)}}$, given by the formula $\sigma^u(F) = \int_G \int_{\mathbf{T}} 1_F(t\cdot\gamma)\,dt\,d\lambda^u(\gamma)$, where dt is Haar measure on $\mathbf{T}$. It is easy to see that a measure μ on $G^{(0)}$ is quasi-invariant relative to E if and only if it is so relative to G. Moreover, the modular functions on G and E, Δ_G and Δ_E respectively, are related by the formula $\Delta_E(\gamma) = \Delta_G(\dot{\gamma})$ for all

$\gamma \in E|_F$, where F is the μ-conull set in $G^{(0)}$ used to produce Δ_G.

We adopt the notation of Ramsay in [25] and write $G^{(0)} * \mathscr{H}$ for a Hilbert bundle over $G^{(0)}$. This means that we are given a family of Hilbert spaces $\{\mathscr{H}(u)\}_{u \in G^{(0)}}$ and $G^{(0)} * \mathscr{H}$ is defined to be $\{(u, \xi)\colon\ \xi \in \mathscr{H}(u)\}$, that is, $G^{(0)} * \mathscr{H}$ is the union of the $\mathscr{H}(u)$ made disjoint by identifying $\mathscr{H}(u)$ with $\{u\} \times \mathscr{H}(u)$. The natural projection from $G^{(0)} * \mathscr{H}$ to $G^{(0)}$ is denoted π. It is assumed that there is an analytic Borel structure on $G^{(0)} * \mathscr{H}$ defined by π and a countable number of sections (see [25] for details). The space of square integrable sections (with respect to some measure μ) is a Hilbert space denoted by $L^2(\mu, \mathscr{H})$ or $\int^{\oplus} \mathscr{H}(u) d\mu(u)$. The set $\mathrm{End}(G^{(0)} * \mathscr{H}) := \{(u, S, v) \mid S : \mathscr{H}(v) \to \mathscr{H}(u)$ is linear and bounded$\}$ has a natural Borel structure defined using the sections defining $G^{(0)} * \mathscr{H}$. It contains the subgroupoid $\mathrm{Iso}(G^{(0)} * H)$ consisting of those (u, S, v) such that S is a Hilbert space isomorphism. It is not hard to see that $\mathrm{Iso}(G^{(0)} * \mathscr{H})$ is a Borel subset of $\mathrm{End}(G^{(0)} * \mathscr{H})$.

DEFINITION 2.1. A (*unitary*) *representation* of (G, E) is a triple $(\mu, G^{(0)} * \mathscr{H}, U)$ where μ is a quasi-invariant measure on $G^{(0)}$, $G^{(0)} * \mathscr{H}$ is a Hilbert bundle, and U is a *Borel* homomorphism from $E|_F$, for some inessential reduction of E, to $\mathrm{Iso}(G^{(0)} * \mathscr{H}|_F)$, where $G^{(0)} * \mathscr{H}|_F$ is the restriction of $G^{(0)} * \mathscr{H}$ to F, such that $U(t\gamma) = \bar{t}U(\gamma)$, $t \in \mathbf{T}$, $\gamma \in E$.

A bit more explicitly, $U(\gamma)$ really is a triple $(r(\gamma), U^-(\gamma), s(\gamma))$ where $U^-(\gamma)$ is a Hilbert space isomorphism from $\mathscr{H}(s(\gamma))$ onto $\mathscr{H}(r(\gamma))$. However, we will blur the distinction between $U(\gamma)$ and $U^-(\gamma)$. It is important to note that for representations $(\mu, G^{(0)} * \mathscr{H}, U)$ of (G, E), the support of $G^{(0)} * \mathscr{H}$ $(= \{u \in G^{(0)} \mid \mathscr{H}(u) \neq \{0\}\})$ is invariant for $G|_F$.

Two representations $(\mu_i, G^{(0)} * \mathscr{H}_i, U_i)$, of (G, E), $i = 1, 2$, are *equivalent* in case there is a reduction $E|_F$ of E, which is inessential for both μ_1 and μ_2, and a bundle isomorphism

$$V : G^{(0)} * \mathscr{H}_1|_F \to G^{(0)} * \mathscr{H}_2|_F,$$

given by a Borel field $\{V(u)\}_{u \in F}$ of Hilbert space isomorphisms, such that on F, μ_1 is equivalent to μ_2 and such that

$$V(r(\gamma))U_1(\gamma) = U_2(\gamma)V(s(\gamma))$$

for ν-almost all $\gamma \in E$, where ν is either the integral of $\{\sigma^u\}_{u \in G^{(0)}}$ with respect to μ_1 or with respect to μ_2.

A representation $(\mu, G^{(0)} * \mathscr{H}, U)$ of (G, E) can be integrated to give a representation π of $C_c(G, E)$ defined by the formula

$$(\pi(f)\xi)(u) = \int f(\gamma)\Delta^{1/2}(\gamma)U(\gamma)\xi(s(\gamma))\, d\lambda^u(\gamma),$$

$f \in C_c(G, E)$, $\xi \in \int^{\oplus} \mathscr{H}(u) d\mu(u)$. Here, Δ is the modular function of μ on E. Note that the integrand is $\mathbf{T}$-invariant and so the integral makes sense. Note, too, that if $a \in N(C_0(G^{(0)}))$ with $j(\operatorname{supp}(a)) = \tau$, then

$$(\pi(a)\xi)(u) = \Delta^{1/2}(\gamma)a(\gamma)U(\gamma)\xi(\tau(u))$$

for any $\gamma \in E$ such that $j(\gamma) = (u, \tau(u))$. Henceforth, we shall view G explicitly as a subset of $G^{(0)} \times G^{(0)}$ and we shall write elements of G both as ordered pairs and as lower case Greek letters, whichever is convenient. Renault's disintegration theorem [27] asserts, conversely, that every representation of $C_c(G, E)$ is the integrated form of some (necessarily unique) representation of (G, E). Passing to completions, then, the same bijective correspondence exists between C^*-representations of $C^*(G, E)$ and representations of (G, E). Furthermore, equivalent unitary representations of (G, E) integrate to unitarily equivalent C^*-representations of $C^*(G, E)$ and conversely, a unitary equivalence between C^*-representations of $C^*(G, E)$ is implemented by an equivalence between their disintegrated forms.

DEFINITION 2.2. A subalgebra $A \subseteq B = C^*(G, E)$ is called *triangular* if it is norm closed and $A \cap A^* = D$. It is said to be a *strongly maximal triangular* algebra if A is triangular and $A + A^*$ is dense in B.

The assumption that B is nuclear implies that the spectral theorem for bimodules is valid [12]. Thus, if A is a norm closed subalgebra of B that contains D, then there is an open subset $P \subseteq G$ such that $G^{(0)} \subseteq P$, $P \circ P \subseteq P$ (where $P \circ P = \{\alpha\beta : (\alpha, \beta) \in G^{(2)} \cap P \times P\}$), and such that A is the closure in $C^*(G, E)$ of $\{f \in C_c(G, E) : \operatorname{supp}(f) \subseteq j^{-1}(P)\}$ [12, Theorem 4.1]. We call this closure $A(P)$. The algebra $A = A(P)$ is triangular if and only if $P \cap P^{-1} = G^{(0)}$ (where $P^{-1} = \{\alpha^{-1} : \alpha \in P\}$) and it is strongly maximal triangular if and only if, in addition, we have $P \cup P^{-1} = G$. In this case P totally orders every equivalence class. When P is given, we write $u \leq v$ if $(u, v) \in P$.

A class of strongly maximal triangular algebras arises as follows. Let $c : G \to \mathbb{R}$ be a continuous homomorphism, that is, $c(\alpha\beta) = c(\alpha) + c(\beta)$ whenever $(\alpha, \beta) \in G^{(2)}$. Such a c is called a *cocycle* on G and we write $Z^1(G, \mathbb{R})$ for the set of all cocycles. If $c^{-1}(\{0\}) = G^{(0)}$ the cocycle is called *faithful*. For a faithful cocycle write $P(c) = \{\alpha \in G \mid c(\alpha) \geq 0\}$. Then $P = P(c)$ is open and satisfies $P \circ P \subseteq P$, $P \cap P^{-1} = G^{(0)}$ and $P \cup P^{-1} = G$. Hence $A = A(P(c))$ is a strongly maximal triangular algebra. Such an algebra is called an *analytic subalgebra* of B.

The reason for the terminology is the following. With any cocycle c we can associate a group of automorphisms $\alpha = \{\alpha_t\}_{t \in \mathbb{R}}$ of B by defining

$$\alpha_t(f)(\gamma) = e^{itc(\gamma)} f(\gamma), \qquad f \in C^*(G, E).$$

Then the elements of $A = A(P(c))$ are precisely the elements of B with non-negative Arveson spectrum. In other words, for $a \in B$, a lies in $A(P(c))$ if and only if, for every bounded linear functional f on B, the function $t \to f(\alpha_t(a))$ can be extended to a bounded analytic function in the open upper half plane.

We shall now cite some terminology and basic facts from the spectral theory of automorphism groups. (For more details see [1, 11 and 18]). Given an automorphism group $\alpha = \{\alpha_t : t \in \mathbb{R}\}$ that is strongly continuous (that is, $t \to f(\alpha_t(a))$ is continuous for every $a \in B$ and f in the dual of B) and given $a \in B$, we define $\mathrm{sp}_\alpha(a) = \{s \in \mathbb{R} : \hat{f}(s) = 0$ whenever $f \in L^1(\mathbb{R})$ and $\alpha_f(a) = 0\}$. Here $\alpha_f(a) = \int f(t)\alpha_t(a)\,dt$ and $\hat{f}(s) = \int e^{its} f(t)\,dt$. For every subset $\Sigma \subseteq \mathbb{R}$ the spectral subspace is $B^\alpha(\Sigma) = \overline{\{a \in B : \mathrm{sp}_\alpha(a) \subseteq \Sigma\}}$ (where the closure is in the norm topology). If Σ is closed in $\mathbb{R}$, $B^\alpha(\Sigma) = \{a \in B : \mathrm{sp}_\alpha(a) \subseteq \Sigma\}$. In general, if Σ_1 and Σ_2 are subsets of $\mathbb{R}$, we have

$$B^\alpha(\Sigma_1)B^\alpha(\Sigma_2) \subseteq B^\alpha(\Sigma_1 + \Sigma_2).$$

When $B = C^*(G, E)$, α is defined by a cocycle c, and $a \in B$, then $\mathrm{sp}_\alpha(a)$ may be alternately described using the following equation:

$$\begin{aligned}\mathrm{sp}_\alpha(a) &= \{s \in \mathbb{R} : \hat{f}(s) = 0 \text{ when } f \in L^1(\mathbb{R}) \text{ and } \forall \gamma \in E,\ \int e^{itc(\dot{\gamma})} a(\gamma) f(t)dt = 0\} \\ &= \{s \in \mathbb{R} : \hat{f}(s) = 0 \text{ when } f \in L^1(\mathbb{R}) \text{ and } \forall \gamma \in E,\ a(\gamma)\hat{f}(c(\dot{\gamma})) = 0\} \\ &= \overline{c(j(\mathrm{supp}(a)))} \quad (\text{here } \mathrm{supp}(a) = \{\gamma \in E : a(\gamma) \neq 0\}).\end{aligned}$$

Hence the spectral subspace $B^\alpha(\Sigma)$ is $\overline{\{a \in B : \overline{c(j(\mathrm{supp}(a)))} \subseteq \Sigma\}}$, and if Σ is closed, then $B^\alpha(\Sigma) = \{a \in B : j(\mathrm{supp}(a)) \subseteq c^{-1}(\Sigma)\}$. In particular,

$$A(P(c)) = \{a \in B : j(\mathrm{supp}(a)) \subseteq c^{-1}([0, \infty))\} = B^\alpha([0, \infty)).$$

We can also define the spectrum of the automorphism group α by

$$\mathrm{sp}(\alpha) = \overline{\cup\{\mathrm{sp}_\alpha(a) : a \in B\}}.$$

This is a closed subset of $\mathbb{R}$ but in general it has no algebraic properties that are useful to us. One can modify this set to get the *Connes spectrum* $\Gamma(\alpha)$ of α, which is a closed subgroup of $\mathbb{R}$ (see [20, 8.8.2] for the definition). For an automorphism group α given by a cocycle c it turns out ([26, p. 112]) that $\Gamma(\alpha) = R_\infty(c)$, where $R_\infty(c) = \cap\{\overline{c(G \cap (U \times U))} : U \subseteq G^{(0)}$ is open$\}$. In [29] it was shown that, when G is minimal, $R_\infty(c)$ (more precisely, $\tilde{R}_\infty(c)$, which is what one gets if the closure of $c(G \cap (U \times U))$ above is taken in $\mathbb{R} \cup \{\infty\}$) is an isometric isomorphism invariant for the algebra $A(P(c))$. Here we shall need a 'measure-theoretic' version of $R_\infty(c)$. Recall that a Borel measure μ on $G^{(0)}$ is *ergodic* if, whenever we have a Borel subset $F \subseteq G^{(0)}$ with $\mu(r(s^{-1}(F))\Delta F) = 0$ (that is, F is *almost invariant*) then F is either null or conull.

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DEFINITION 2.3. (see [28, Definition 3.1]). Let μ be a given ergodic measure on $G^{(0)}$. Let $c : G \to \mathbb{R}$ be a Borel homomorphism (that is, a Borel cocycle).

(1) $R^{\mu}_{\infty}(c)$ is the set of all $t \in \mathbb{R}$ with the property that for every $\epsilon > 0$ and every Borel set $B \subseteq G^{(0)}$ with $\mu(B) > 0$ we have $\nu((B \times B) \cap c^{-1}(t-\epsilon, t+\epsilon)) > 0$.
(2) We say that $\infty \in \tilde{R}^{\mu}_{\infty}(c)$ if for every $M > 0$ and every Borel set $B \subseteq G^{(0)}$ with $\mu(B) > 0$ we have $\nu((B \times B) \cap c^{-1}[M, \infty)) > 0$.
(3) We write $\tilde{R}^{\mu}_{\infty}(c)$ for $R^{\mu}_{\infty}(c)$ if $\infty \notin \tilde{R}^{\mu}_{\infty}(c)$ and $\tilde{R}^{\mu}_{\infty}(c) = R^{\mu}_{\infty}(c) \cup \{\infty\}$ otherwise.

Some basic properties are given in the following proposition. The proofs can be found in [28].

PROPOSITION 2.4. *Let μ be an ergodic Borel measure on $G^{(0)}$ and let $c : G \to \mathbb{R}$ be a Borel cocycle. Then*

(1) *$R^{\mu}_{\infty}(c)$ is a closed subgroup of R; that is, it is either $\{0\}$ or R or $\lambda\mathbb{Z}$ for some $\lambda \in \mathbb{R}$.*
(2) *$\tilde{R}^{\mu}_{\infty}(c)$ is either $\{0\}$, $\{0, \infty\}$, $\mathbb{R} \cup \{\infty\}$ or $\lambda\mathbb{Z} \cup \{\infty\}$ for some $\lambda \in \mathbb{R}$.*
(3) *If $a < b < \infty$ and $[a, b] \cap R^{\mu}_{\infty}(c) = \emptyset$, then every Borel subset of $G^{(0)}$ with positive measure has a subset F, with positive measure, satisfying $(F \times F) \cap c^{-1}([a, b]) = \emptyset$.*
(4) *c is a Borel coboundary (that is, there is a Borel function $g : G^{(0)} \to \mathbb{R}$ such that $c(x, y) = g(y) - g(x)$ for ν - a.e. $(x, y) \in G$) if and only if $\tilde{R}^{\mu}_{\infty}(c) = \{0\}$.*
(5) *If $R^{\mu}_{\infty}(c) = \lambda Z$, then there is a coboundary a such that $c(u, v) + a(u, v) \in \lambda Z$ for all (u, v) in G.*

3. The lattice of invariant projections of $\pi(A(P))$

We now assume that $B = C^*(G, E)$ as above, $A = A(P)$ is an analytic triangular subalgebra of B associated with a faithful cocycle c on G; that is, $P = P(c) = \{(x, y) \in G : c(x, y) \geq 0\}$. Also we fix an irreducible representation π of B. As we saw above, there is a (unitary) representation $(\mu, G^{(0)} * K, U)$ of (G, E) such that π is its integrated form; that is,

$$(\pi(f)\xi)(u) = \int f(\gamma)\Delta^{1/2}(\gamma)U(\gamma)\xi(s(\gamma))\, d\lambda^u(\dot{\gamma}), \qquad f \in C_c(G, E)$$

and the space on which π acts can be identified with $\int^{\oplus} K(u)d\mu(u)$. We write $K = \int^{\oplus} K(u)d\mu(u)$. For $a \in C_0(G^{(0)})$ and $\xi \in \int^{\oplus} K(u)d\mu(u)$, $(\pi(a)\xi)(u) = a(u)\xi(u)$. Hence every projection Q in $B(K)$ whose range is invariant for $\pi(A)$ is decomposable (since it commutes with $\pi(A) \cap \pi(A)^* = \pi(C_0(G^{(0)}))$); that is, $Q = \int^{\oplus} Q(u)d\mu(u)$. As was noted above, for every $a \in N(C_0(G^{(0)}))$ with $j(\mathrm{supp}(a)) = \tau$, a G-set, we have $(\pi(a)\xi)(u) = \Delta^{1/2}(\gamma)a(\gamma)U(\gamma)\xi(\tau(u))$ for any $\gamma \in E$ such that $\dot{\gamma} = (u, \tau(u))$. Thus $\pi(a)Q(K) \subseteq Q(K)$ if and only if $U(\gamma)Q(\tau(u))U(\gamma)^* \leq Q(u)$ for ν-a.e. $\dot{\gamma} = (u, \tau(u))$.

LEMMA 3.1. *With the assumptions above we have the following:*

(1) *μ is ergodic.*

(2) *Every projection $Q \in B(K)$ whose range is $\pi(A)$-invariant can be written as $Q = \int^{\oplus} Q(u)d\mu$, where $U(\gamma)Q(v)U(\gamma)^* \leq Q(u)$ for σ-almost every $\gamma \in E$ with $j(\gamma) = (u, v) \in P$.*

PROOF. (1) Suppose $F \subseteq G^{(0)}$ is a Borel subset satisfying $\mu(F \Delta s(r^{-1}(F))) = 0$. Write $Q = \int^{\oplus} Q(u)d\mu(u) \in B(K)$ where $Q(u) = \chi_F(u)I_{K(u)}$. Then for σ-a.e. $\gamma \in E$ and $j(\gamma) = (u, v)$ we have $U(\gamma)Q(v)U(\gamma)^* \leq Q(u)$ because this fails only if $v \in F$ and $u \notin F$. Therefore, for every $a \in N(C_0(G^{(0)}))$ that is supported on a G-set, we have $\pi(a)Q(K) \subseteq Q(K)$; hence $Q(K) \in \pi(B)' = \mathbb{C}I$. Thus F is either null or conull.

(2) This follows from the discussion preceding the lemma, since the supports of all $a \in N(C_0(G^{(0)})) \cap A$ cover $j^{-1}(P)$ (see [12]).

Since μ is ergodic it must be either concentrated on a single orbit $[u]$ in $G^{(0)}$ ($[u] = \{v : (u, v) \in G\}$) or $\mu([u]) = 0$ for all $u \in G^{(0)}$. In the latter case μ is said to be *properly ergodic*.

We recall that $\operatorname{lat}\pi(A)$ is the lattice of all projections $Q \in B(K)$ whose range is left invariant by every $\pi(f)$, $f \in A$.

We shall use the notation $[M]$, where $M \subseteq K$ is a subset of the Hilbert space K, to denote the (closed) linear space spanned by M. Also, when an automorphism group α of our C^*-algebra B is fixed, we write $B(\Sigma)$ for the spectral subspace determined by α and the set Σ in place of $B^{\alpha}(\Sigma)$.

The following is the main result of this section.

THEOREM 3.2. *Let π be an irreducible representation of $B = C^*(G; E)$ and let $(\mu, G^{(0)} * K, U)$ be the associated representation of E. Then*

(1) $\operatorname{lat}\pi(A(P))$ *is a nest (that is, it is totally ordered).*

(2) *If μ is properly ergodic, then* $\operatorname{lat}\pi(A(P))$ *is either $\{0, I\}$ or a continuous nest. Furthermore,* $\operatorname{lat}\pi(A(P)) = \{Q_t : t \in \mathbb{R}\} \cup \{0, I\}$ *where Q_t is the projection onto $[B[t, \infty)Q(K)]$ and Q is any non-trivial projection in* $\operatorname{lat}\pi(A(P))$.

(3) *If μ is supported on an orbit, then π is equivalent to* $\operatorname{Ind}\epsilon_u$ *for some $u \in G^{(0)}$.*

PROOF. Let $c \in Z^1(G, \mathbb{R})$ be a cocycle satisfying $P = \{(x, y) \in G : c(x, y) \geq 0\}$ and let $\alpha = \{\alpha_t\}$ be the corresponding automorphism group; that is, $\alpha_t(f)(\gamma) =$

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$e^{itc(\dot{\gamma})}f(\gamma)$, $f \in C^*(G; E)$. Let π be an irreducible representation of $C^*(G; E)$ and let P and Q be projections in $\operatorname{lat}\pi(A(P))$. For such a projection P we write

$$E_\lambda^P = \bigwedge_{t<\lambda} [\pi(B[t,\infty))P(K)],$$

where $B = C^*(G; E)$. When $\lambda_1 < \lambda_2$, $E_{\lambda_1}^P \geq E_{\lambda_2}^P$ and

$$E_s^P = \bigwedge_{\lambda<s}[\pi(B[\lambda,\infty))P(K)] = \bigwedge_{\lambda<s}\bigwedge_{t<\lambda}[\pi(B[t,\infty))P(K)] = \bigwedge_{t<s} E_t^P.$$

Also $\pi(B[t,\infty))E_\lambda^P(K) \subseteq E_{\lambda+t}^P(K)$, as $B[t,\infty)B[s,\infty) \subseteq B[t+s,\infty)$. Hence

$$\pi(B[t,\infty))\bigvee E_\lambda^P(K) \leq \bigvee E_\lambda^P(K)$$

(and $B[t,\infty)\bigwedge E_\lambda^P(K) \subseteq \bigwedge E_\lambda^P(K)$). Since this holds for all t, we conclude that $\bigvee E_t^P \in \pi(B)' = \mathbb{C}I$, and therefore that $\bigvee E_\lambda^P = I$, provided $P \neq 0$, while $\bigwedge E_\lambda^P = 0$ in any case. It follows that there is a spectral measure $E^P(\cdot)$ defined on the Borel sets of $\mathbb{R}$ such that $E^P[\lambda,\infty) = E_\lambda^P$. If we write $V_t^P = -\int e^{its}dE^P(s)$, we have

$$\pi(\alpha_t(a)) = V_t^P\pi(a)V_t^{P*} \quad \text{for } a \in B, t \in \mathbb{R},$$

by Forelli's Spectral-Commutation Principle (see Scholium 2.8 and Theorem 2.13 of [11]).

The same argument applies to the projection $Q \in \operatorname{lat}\pi(A(P))$ and so we may write

$$\pi(\alpha_t(a)) = V_t^Q\pi(a)V_t^{Q*} = V_t^P\pi(a)V_t^{P*}.$$

This, of course, implies that $V_t^P V_t^{Q*}$ lies in $\pi(B)' = \mathbb{C}I$. Hence we may write $V_t^P = \lambda(t)V_t^Q$ where λ is a character of R, that is, $\lambda(t) = e^{itr}$ for some $r \in \mathbb{R}$. We then find that

$$\int e^{its}dE^P(s) = \int e^{its}e^{itr}dE^Q(s) = \int e^{its}dE^Q(s-r).$$

Thus $E^P[s,\infty) = E^Q[s-r,\infty)$, for all $s \in \mathbb{R}$. This, however, implies that P and Q are comparable. Suppose, indeed, that $r > 0$ and take $0 < \epsilon < r$. Then $\epsilon - r < 0$ and $Q(K) \leq E_{\epsilon-r}^Q(K) = E_\epsilon^P(K) \subseteq P(K)$, so that $Q \leq P$. Similarly, if $r < 0$, $Q \geq P$. So we may assume that $r = 0$. Then $E^P(\cdot) = E^Q(\cdot)$. Write $E_t = E^P[t,\infty)$ and $\tilde{E}_t = \bigvee_{t<s} E^P[s,\infty)$, and let $F_t = E_t - \tilde{E}_t$. For every $s > 0$ we have $[\pi(B[s,\infty)E_t(K)] \subseteq [\pi(B[s,\infty))\pi(B[t-s/2,\infty))P(K)] \subseteq [\pi(B[t+s/2,\infty))P(K)] \subseteq \tilde{E}_t$. Hence $F_t\pi(B[s,\infty))F_t = 0$, $\forall s > 0$. If $f \in C_c(G; E)$ then we may write $f = f_1 + f_2 + f_3$, where $f_1 \in A(P) \cap C_c(G; E) \subseteq B[s,\infty)$ (for some $s > 0$), $f_2 \in A(P)^* \cap C_c(G; E) \subseteq B[s,\infty)^*$ and $f_3 \in C_0(G^{(0)})$. Hence

$F_t\pi(f)F_t \in F_t\pi(C_0(G^0))F_t$. Since $\pi(C_c(G; E))$ is σ-weakly dense in $B(K)$ (as π is irreducible), we have

$$F_tB(K)F_t \subseteq F_t\pi(C_0(G^{(0)}))''F_t.$$

Hence F_t is an abelian projection in $B(K)$ and therefore it is a projection of rank 1. Since $F_t \in \pi(C_0(G^{(0)}))'$, it corresponds to an atom of μ. As μ is ergodic we conclude that either μ is supported on an orbit or $E_t = \tilde{E}_t$. However, observe that we always have $\tilde{E}_0 \leq P \leq E_0$ and $\tilde{E}_0 \leq Q \leq E_0$. Hence, if μ is properly ergodic, then $P = Q$. On the other hand, if μ is supported on an orbit, $\tilde{E}_0$ and E_0 differ by at most one dimension. So again, in this case, either $P \leq Q$ or $Q \leq P$.

We have just seen that $\operatorname{lat}\pi(A(P))$ is always a nest and that if μ is properly ergodic, then the nest is continuous. Also we have seen that for every $P \in \operatorname{lat}\pi(A(P))$, $\operatorname{lat}(A(P)) = \{E^P_\lambda : \lambda \in \mathbb{R}\} \cup \{0, I\}$ if μ is properly ergodic, and $\operatorname{lat}\pi(A(P)) = \{E^P_\lambda, \tilde{E}^P_\lambda : \lambda \in R\} \cup \{0, I\}$ if μ is supported on an orbit. In fact, if μ is properly ergodic and $P \in \operatorname{lat}\pi(A(P))$, then $E^P_{t+\epsilon}(K) \subseteq [\pi(B[t, \infty))P(K)] \subseteq E^P_t(K)$ for every $\epsilon > 0$. From continuity we conclude that $E^P_t = [\pi(B[t, \infty))P(K)]$. Hence $\operatorname{lat}\pi(A(P)) = \{[\pi(B[t, \infty))P(K)] : t \in R\} \cup \{0, I\}$ in this case, as stated. On the other hand, if μ is supported on the orbit $[u]$, then $K = \sum^{\oplus}_{w\in[u]} K(w)$ and, by ergodicity $\dim K(w) \equiv 1$. We can then identify K with $\ell^2([u])$ and $U(w_1, w_2) = 1$. It follows that the integrated representation π is equivalent to $\operatorname{Ind}\epsilon_u$.

Theorem 3.2 tells us that $\operatorname{lat}\pi(A)$ consists of a nest of decomposable projections that is continuous if the measure μ is properly ergodic. To get further information about $\operatorname{lat}\pi(A)$ and, in particular, to make use of tools from ergodic theory, we have to 'get inside' each projection $Q \in \operatorname{lat}\pi(A)$ and analyze the components $Q(u)$. Scrutiny of the proof of Theorem 3.2 leads to the following somewhat technical result that will be useful to us in this analysis. Here again, we let π be an irreducible representation of $C^*(G; E)$ and we assume that $(\mu, G^{(0)} * K, U)$ is the triple associated with it. Fix some $Q \in \operatorname{lat}\pi(A(P))$ and assume $Q \notin \{0, I\}$. Then $Q = \int^{\oplus} Q(u)d\mu(u)$, $Q(u) \in B(K(u))$. The fact that $Q \in \operatorname{lat}\pi(A(P))$ is equivalent to the inequality:

$$U(u, w)Q(w)U(w, u) \leq Q(u) \quad \text{for } \nu\text{-a.e.}(u, w) \text{ in } P.$$

(Here ν is the measure obtained on G by integrating over μ.) We write Q_t for the projection onto $[B[t, \infty)Q(K)]$. Then $Q_t = \int^{\oplus} Q_t(u)d\mu(u)$ where

(1) $Q_t(u) = \bigvee_{t\leq c(u,w)} U(u, w)Q(w)U(w, u)$ μ-a.e. u. It follows that for ν-a.e. (u, w),

(2) $Q_{c(u,w)}(u) = U(u, w)Q(w)U(w, u)$, and

(3) $Q_{t+c(u,w)}(u) = \bigvee_{t\leq c(w,v)} U(u, v)Q(v)U(v, u) = U(u, w)Q_t(w)U(w, u)$.

We shall need the following proposition.

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PROPOSITION 3.3. *Suppose μ is properly ergodic, $Q \in \operatorname{lat}\pi(A(P))$, $Q \notin \{0, I\}$ and $A \subseteq X(= G^{(0)})$ is a Borel subset with $\mu(A) > 0$. Define for $u \in X$,*

$$d(u) = \sup\{t : Q_t(u) = Q(u)\}; \qquad b(A, u) = \inf\{c(u, w) : u \leq w \in A\};$$
$$t(A) = \operatorname{ess\,inf}\{d(u) : u \in A\}.$$

Then $Q_{t(A)}(u) = Q_{t(A)+b(A,u)}(u)$ μ-a.e. u.

PROOF. Note that both d and $b(A, \cdot)$ are Borel functions. Consider $[\pi(A(P))\pi(\chi_A)Q(K)]$. (Here $\pi(\chi_A)Q(K)$ is the space $\int_A^{\oplus} Q(u)(K(u))d\mu(u)$). It lies in $\operatorname{lat}\pi(A(P))$, and thus, there is some t such that

$$[\pi(A(P))\pi(\chi_A)Q(K)] = [\pi(B[t, \infty))Q(K)].$$

Hence, for μ-a.e. u,

$$Q_t(u) = \bigvee_{u \leq w \in A} U(u, w)Q(w)U(w, u).$$

For $u \in A$ we get

$$Q_t(u) = Q(u).$$

Thus $t \leq d(u)$ for a.e. $u \in A$. Hence $t \leq t(A)$. Also, for $u \leq w \in A$,

$$U(u, w)Q(w)U(w, u) = U(u, w)Q_{t(A)}(w)U(w, u) \leq Q_{t(A)}(u)$$

(as $Q_{t(A)} \in \operatorname{lat}\pi(A(P))$ and $u \leq w$). Therefore $Q_t(u) \leq Q_{t(A)}(u) \leq Q_t(u)$ (the last inequality follows from $t \leq t(A)$). We conclude that

$$Q_{t(A)}(u) = \bigvee_{u \leq w \in A} U(u, w)Q(w)U(w, u).$$

Now, for $u \leq w \in A$, $Q(w) = Q_{t(A)}(w)$. Hence

$$\begin{aligned} Q_{t(A)}(u) &= \bigvee_{u \leq w \in A} U(u, w)Q(w)U(w, u) = \bigvee_{u \leq w \in A} U(u, w)Q_{t(A)}(w)U(u, w) \\ &= \bigvee_{u \leq w \in A} Q_{t(A)+c(u,w)}(u) = Q_{t(A)+b(A,u)}(u). \end{aligned}$$

The last equality follows from the definition of $b(A, u)$ and the continuity of $\{Q_t\}$.

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4. $\tilde{R}^{\mu}_{\infty}(c) = \{0, \infty\}$

We keep the assumptions and notation of Section 3. In particular, π is assumed to be irreducible, so the measure μ is ergodic. Our objective is to understand the structure of $\operatorname{lat}\pi(A(P))$ in terms of the 'asymptotic distribution' of the cocycle c.

THEOREM 4.1. *If* $\tilde{R}^{\mu}_{\infty}(c) = \{0, \infty\}$, *then* $\operatorname{lat}\pi(A(P)) = \{0, I\}$.

PROOF. Since $\tilde{R}^{\mu}_{\infty}(c) = \{0, \infty\}$, μ is not supported on an orbit, but is properly ergodic. Assume $\operatorname{lat}\pi(A(P)) \neq \{0, I\}$ and choose $Q \in \operatorname{lat}\pi(A(P))\backslash\{0, I\}$. For this Q, let $d(\cdot)$, $b(\cdot, \cdot)$ and $t(\cdot)$ be as defined in Proposition 3.3. Write $a = \operatorname{ess\,inf}_{u\in X} d(u)$. We claim that $a = 0$. Assume that this is not the case; that is, assume that $a \neq 0$. For a given u assume that there is some $w \in X$ such that $d(u) < c(u, w) < d(u) + a/2$. If $u \leq v \leq w$ and $d(u) \leq c(u, v)$, then

$$c(v, w) = c(u, w) - c(u, v) < d(u) + \frac{a}{2} - d(u) = \frac{a}{2} \leq \frac{d(v)}{2}.$$

Hence, $Q(v) = Q_{c(v,w)}(v)$ and $Q_{c(u,v)}(u) = Q_{c(u,w)}(u)$. Thus $Q(u) = Q_{d(u)}(u) = \bigvee_{u\leq v\leq w} Q_{c(u,v)}(u) = Q_{c(u,w)}(u)$. But then $c(u, w) \leq d(u)$. Since we assumed that $d(u) < c(u, w)$, we arrive at a contradiction showing that no such w exists. But then $Q(u) = Q_{d(u)}(u) = Q_{d(u)+a/2}(u)$, contradicting the definition of $d(u)$ and the assumption $a > 0$. This proves that $a = 0$, as claimed.

Our next goal is to produce a set B of positive measure such that the restriction of c to $G \cap (B \times B)$ is bounded. To this end, we write, for every $n \geq 1$,

$$B_n = \{u \in X \mid d(u) \leq 1/n\}.$$

Since $a = 0$, $\mu(B_n) > 0$ for every $n \geq 1$. Fix numbers δ and L, with $0 < \delta < L$. Using part (3) of Proposition 2.4 and the fact that $\tilde{R}^{\mu}_{\infty}(c) = \{0, \infty\}$, we can find, for every $n \geq 1$, a subset $A_n \subseteq B_n$, with

(1) $\mu(A_n) > 0$;
(2) $(A_n \times A_n) \cap c^{-1}[\delta, L] = \emptyset$;
(3) If $\mu(\cap B_n) > 0$ we require that $A_n = A_m$ for all n, m and write A_∞ for A_n; that is, $A_\infty \subseteq \bigcap_{n=1}^{\infty} B_n$. Using Proposition 3.3 we then have, for every $n \geq 1$,

$$Q_{t(A_n)}(u) = Q_{t(A_n)+b(A_n,u)}(u), \qquad u \in X \text{ and } t(A_n) \leq 1/n.$$

Write B_∞ for $\cap B_n$ and consider first the case when B_∞ has positive measure and then the case when it has measure zero.

Case I : If $\mu(B_\infty) > 0$, then $A_n \subseteq B_\infty$ for every n. Consequently, $d(u) = 0$ for every $u \in B_\infty$ and thus $t(A_\infty) = 0$. Hence, for $u \in B_\infty$, $Q(u) = Q_{b(A_\infty,u)}(u)$. Since

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$d(u) = 0$, $b(A_\infty, u) = 0$. If u_1, u_2 are both in B_∞, then there are $w_1, w_2 \in A_\infty$ such that $0 \leq c(u_i, w_i) < \delta$, $i = 1, 2$. But $(w_1, w_2) \in A_\infty \times A_\infty$, so either $|c(w_1, w_2)| < \delta$ or $|c(w_1, w_2)| > L$. It follows that either $|c(u_1, u_2)| < 3\delta$ or $|c(u_1, u_2)| > L - 2\delta$. With δ fixed and letting $L \to \infty$, we conclude that for every $(u_1, u_2) \in B_\infty \times B_\infty$, $|c(u_1, u_2)| < 3\delta$. Hence $c|_{B_\infty \times B_\infty}$ is bounded.

Case II : Suppose $\mu(B_\infty) = 0$. For $u \notin B_\infty$, $d(u) > 0$. Suppose $1/n < d(u)$. Then $u \notin B_n$. Also $b(B_n, u) > 0$ because if it equals 0, then for $0 < \epsilon < d(u) - 1/n$, we have $w \in B_n$ such that $0 \leq c(u, w) < \epsilon < d(u)$; hence $Q_{c(u,w)}(u) = Q(u)$. But also $Q_{d(u)} = Q(u)$; hence $Q_{c(u,w)}(u) = Q_{d(u)}(u)$. It follows that $Q(w) = Q_{d(u)-c(u,w)}(w)$. Hence $d(u) - c(u, w) \leq d(w) \leq 1/n$ and

$$d(u) \leq c(u, w) + \frac{1}{n} < \epsilon + \frac{1}{n} < \frac{1}{n} + \left(d(u) - \frac{1}{n}\right) = d(u).$$

This contradiction shows that $b(B_n, u) > 0$ for $n > 1/d(u)$. Now write $M(u) = [1/d(u)] + 1$, $c(u) = B_{M(u)}$, and $N(u) = \max(M(u), [1/b(c(u), u)] + 1)$. Then $b(c(u), u) > 0$ (as was shown above) and in fact $1/N(u) < b(c(u), u)$. Also $c(u) = B_{M(u)} \supseteq B_{N(u)}$; hence $b(c(u), u) \leq b(B_{N(u)}, u)$. Thus $1/N(u) \leq b(B_{N(u)}, u)$. Write N for $N(u)$. Then, for $n \geq N$,

$$t(A_N) \leq \frac{1}{N} \leq b(B_N, u) \leq b(B_n, u) \leq b(A_n, u) + t(A_n)$$

and thus $Q_{t(A_N)}(u) \geq Q_{b(A_n,u)+t(A_n)}(u)$. Using Proposition 3.3 we get $Q_{t(A_N)+b(A_N,u)}(u) \geq Q_{t(A_n)}(u)$, for a.e. u. Since $t(A_n) \leq 1/n \to 0$ (as $n \to \infty$), we have $Q(u) \leq Q_{t(A_N)+b(A_N,u)}(u)$.

But the reverse inequality is valid since $t(A_N) + b(A_n, u) \geq 0$. Hence $Q(u) = Q_{t(A_N)+b(A_N,u)}(u)$. We thus find that $t(A_N) + b(A_N, u) \leq d(u)$. Hence $b(A_N, u) \leq d(u)$, and so we can find some $w \in A_N$ with $0 \leq c(u, w) < d(u) + \delta$. Clearly this can also be done for every $n \geq N(u)$ in place of $N(u)$ (and for every $u \notin B_\infty$). Given u_1, u_2 in $X \backslash B_\infty$, we write $N = \max(N(u_1), N(u_2))$ and then we can find w_1, w_2 in A_N such that $0 \leq c(u_i, w_i) < d(u_i) + \delta$. We now argue as in Case I to get

$$|c(u_1, u_2)| \leq d(u_1) + d(u_2) + 3\delta.$$

If u_1, u_2 are in B_1, then $c|_{B_1 \times B_1}$ is bounded by $2 + 3\delta$.

We conclude that in both cases there is a set B of positive measure μ such that $c|_{B \times B}$ is bounded. It follows that $c|_{B \times B}$ is a coboundary; hence there is a Borel function $g : B \to \mathbb{R}$ such that

$$c(x, y) = g(y) - g(x), \quad (x, y) \in G \cap (B \times B).$$

But then we can find a Borel subset $A \subseteq B$ with:

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(1) $\mu(A) > 0$, $\mu(B\backslash A) > 0$;
(2) $u \in A$, $u \leq v \in B$ implies $v \in A$.

So writing $Y = \{u \in X : \exists w \in A \text{ such that } w \leq u\}$, we have:

(i) Y is increasing in X (that is, if $u \in Y$ and $v \geq u$ then $v \in Y$);
(ii) Y is Borel; and
(iii) $\mu(Y) > 0$ and $\mu(X\backslash Y) > 0$ (as $B\backslash A \subseteq X\backslash Y$).

But then it follows ([13, Lemma 3.8]) that c is a Borel coboundary on all of X contradicting the assumption that $\tilde{R}^{\mu}_{\infty}(c) = \{0, \infty\}$.

5. $\tilde{R}^{\mu}_{\infty}(c) = \{0\}$

In this case, c is a coboundary, so c may be written as $c(u, v) = g(v) - g(u)$ ν a.e. where $g : X \to \mathbb{R}$ is a Borel function. It follows that there is a subset $Y \subseteq X$ that is decreasing; that is, if $z \leq y \in Y$, then $z \in Y$ and $\mu(Y)\mu(X\backslash Y) \neq 0$. Let Q be $\pi(\chi_Y)$. Then $Q \in \operatorname{lat}\pi(A(P))$ and $Q \notin \{0, 1\}$. Given an $a \in C^*(G, E)$ whose support is an open G-set $\tau \subseteq P$, and $f \in C_0(G^{(0)})$, we have,

$$(af)(\gamma) = a(\gamma)f(s(\gamma)) = f(\tau(r(\gamma))a(\gamma) = ((f \circ \tau)a)(\gamma).$$

Hence also

$$\pi(a)\pi(\chi_Y) = \pi(\chi_y \circ \tau)\pi(a) = \pi(\chi_{\tau^{-1}(Y)})\pi(a)$$

and

$$\pi(a)Q(K) \subseteq \pi(\chi_{\tau^{-1}(Y)})(K).$$

In fact $[\pi(a)\pi(a^*)(K)] = [\pi(\chi_{r(\tau)})(K)]$ and

$$\begin{aligned}\pi(\chi_{\tau^{-1}(Y)})(K) &= \pi(\chi_{\tau^{-1}(Y)})\pi(\chi_{r(\tau)})(K) = [\pi(\chi_{\tau^{-1}(Y)})\pi(a)\pi(a^*)K] \\ &= [\pi(a)\pi(\chi_Y)\pi(a^*)K] \subseteq [\pi(a)Q(K)].\end{aligned}$$

Hence $[\pi(a)Q(K)] = \pi(\chi_{\tau^{-1}(Y)})(K)$. Since $[\pi(B[t, \infty))Q(K)]$ can be written as a supremum of a countable family $\{[\pi(a_n)Q(K)]\}$ where each a_n is supported on an open G-set (and these supports have $c^{-1}[t, \infty)$ as their union), we conclude that there is a Borel set $Z_t \subseteq X$ with $\pi(\chi_{Z_t})(K) = [\pi(B[t, \infty))Q(K)]$; that is, $Q_t = \pi(\chi_{Z_t})$. From Theorem 3.2 it now follows that $\operatorname{lat}\pi(A(P)) = \{\pi(\chi_{Z_t})\}_{t\in\mathbb{R}}$ and, in particular, $\operatorname{lat}\pi(A(P)) \subseteq \pi(C_0(G^{(0)}))''$. We conclude:

THEOREM 5.1. *If* $\tilde{R}^{\mu}_{\infty}(c) = \{0\}$ *then* $\operatorname{lat}\pi(A(P)) \subseteq \pi(C_0(G^{(0)}))''$.

REMARK. Note that if μ is concentrated on an orbit $[u]$, then c is a coboundary and so $\tilde{R}^{\mu}_{\infty}(c) = \{0\}$. Hence, in this case, $\operatorname{lat}\pi(A(P)) \subseteq \pi(C_0(G^{(0)}))''$.

6. $\tilde{R}^{\mu}_{\infty}(c) = \lambda\mathbb{Z}$

In order to discuss the case where $R^{\mu}_{\infty}(c) = \lambda\mathbb{Z}$ we shall have to 'manipulate' c in a measure theoretic way, sometimes replacing it with a Borel cocycle c_1 that is cohomologous to c. The set $P_1 := c_1^{-1}([0, \infty))$, then, is a Borel partial order in G that no longer need be open. This fact is a nuisance, but causes us no material difficulty. The way to handle this additional complexity is suggested in [12] and we digress momentarily to deal with it. Our representation π of $C^*(G, E)$ will be fixed, as will the associated representation $(\mu, G^{(0)} * K, U)$ of (G, E). We also let $\nu = \int \lambda^u d\mu(u)$ and $\tilde{\nu} = \int \sigma^u d\mu(u)$, as is customary. Then (G, ν) is an ergodic measured equivalence relation in the sense of Feldman and Moore [6, 7]. We write M for the algebra generated by the functions in $L^{\infty}(\tilde{\nu})$ that are supported on the inverse images under j of non-singular Borel G-sets in the sense of Renault [26] and that transform under the action of $\mathbf{T}$ in the same way that functions in $C_c(G, E)$ transform. (The algebra operations are the same ones used to define $C_c(G, E)$.) Thus M is a bit larger than $B_c(G, E)$, the bounded Borel functions with compact support on E that transform appropriately, if X fails to be compact, but it is not much larger. In any event, it is evident from the fact that π can be expressed in integrated form that π may be extended to M using Lebesgue's dominated convergence theorem and the fact that the image of M under π lies in the strong closure of $\pi(C^*(G, E))$. It should be emphasized that M is not a von Neumann algebra, in general; in particular it is not the von Neumann algebra that Feldman and Moore associate to (G, ν) (and a 2-cocycle on G). While it is related to such a von Neumann algebra, we believe it will cause no confusion to use this notation; in fact, we believe that it will be helpfully suggestive.

It may also be helpful to note that because our C^*-algebra $C^*(G, E)$ is assumed to be nuclear, the measured equivalence relation (G, ν) is hyperfinite [12]. This allows us to assert (after taking an inessential contraction, if necessary) that there is a multiplicative cross section $a : G \longmapsto E$ to the map j. Using a we obtain, by composition, an isomorphism between M and the space of functions in $L^{\infty}(\nu)$ that are supported on non-singular Borel G-sets: For $f \in M$, $f \circ a$ is in the latter space, and the map $f \longmapsto f \circ a$ is an algebra isomorphism. As a result, we shall pass from one algebra to the other with little more than a brief acknowledgment.

Given a Borel set $Q \subseteq G$ of positive ν measure, we let $\mathscr{T}(Q)$ be the set of functions in M that are supported on $j^{-1}(Q)$. It is then evident that the map $Q \longmapsto \mathscr{T}(Q)$ is one-to-one (modulo ν-null sets). (It is perhaps worthwhile to insert here that this assertion is not the spectral theorem for bimodules in its measure theoretic form [13]. It is a much weaker statement. No topologies on or closures of the space $\mathscr{T}(Q)$ are mentioned, and the range of the map is not claimed to be all the bi-modules over $\mathscr{A} := \mathscr{T}(\Delta)$.) Note in particular that if P is a Borel subset of G containing Δ that is transitive, meaning that $P \circ P \subseteq P$, then $\mathscr{T}(P)$ is an algebra containing $\mathscr{A}$, which in turn may be viewed as $L^\infty(X,\mu)$. In fact, one only needs to assume that $\nu(P\backslash P\circ P)=0$ and $\nu(P\backslash\Delta)=0$, by arguments similar to those in Theorem 3.2 of [13]. If, moreover, $P\cup P^{-1}=G$, then $\mathscr{T}(P)+\mathscr{T}(P)^*=M$ because every non-singular Borel G-set may be expressed as the union of such sets, one contained in P and the other contained in P^{-1}. Note, in particular, that if c is a Borel cocycle and if P is $c^{-1}([0,\infty))$, then $\mathscr{T}(P)+\mathscr{T}(P)^*=M$. The cocycle is faithful in the same sense used with continuous cocycles if and only if $\mathscr{T}(P)\cap\mathscr{T}(P)^*=\mathscr{A}$. The cocycle induces a one-parameter automorphism group of M in the same way as a continuous cocycle induces one on $C_c(G,E)$ and $C^*(G,E)$: Namely, $\alpha_t(f)(\gamma)=e^{itc(\dot\gamma)}f(\gamma)$, $f\in M$. Observe that when this automorphism group is spatially implemented in the representation space of π by a unitary group, say $\{U_t\}_{t\in\mathbb{R}}$, then, writing β_t for $\mathrm{Ad}(U_t)$, $\mathrm{sp}_\beta(\pi(f))$ is contained in a set F if f is supported on $c^{-1}(F)$, for any f in M. Given a Borel cocycle c, and $t\in\mathbb{R}$, we write $M[t,\infty)$ or $M^c[t,\infty)$ for the space of those $f\in M$ that are supported a.e. ν on $c^{-1}([t,\infty))$.

We require the following measure theoretic version of Theorem 3.2.

THEOREM 6.1. *Let π be an irreducible representation of $C^*(G,E)$ and let $(\mu, G^{(0)}*K,U)$ be the associated representation of (G,E). Further, let M be the algebra associated with π and μ above and extend π to M using the fact that π is the integrated form of $(\mu,G^{(0)}*K,U)$. Let c be a Borel cocycle on G, let $P=c^{-1}([0,\infty))$, let $\mathscr{T}(P)$ be the associated subalgebra of M. Then*

(1) $\mathrm{lat}\,\pi(\mathscr{T}(P))$ *is a nest.*

(2) *If μ is properly ergodic and if* $\mathrm{lat}\,\pi(\mathscr{T}(P))$ *is not trivial, then it is a continuous nest and, given a non-trivial $Q\in\mathrm{lat}\,\pi(T(P))$, we have*

$$\mathrm{lat}\,\pi(\mathscr{T}(P))=\{Q_t : t\in\mathbb{R}\}\cup\{0,I\}$$

where Q_t is the projection onto $[M^c[t,\infty)Q(K)]$.

PROOF. The proof is essentially the same as the proof of Theorem 3.2. The only change is in the proof of the fact that for $f\in M$, $F_t\pi(f)F_t\in F_t\pi(\mathscr{A})''F_t$. For this, note that, as was shown there, $F_t\pi(M[s,\infty))F_t=0$, $\forall s>0$. Let a be in M with support contained in $j^{-1}(\tau)$, where $\tau\subseteq P$ is a non-singular Borel G-set. So a is an element of $\mathscr{T}(P)$. Let $b_s\in L^\infty(X,\mu)$ be the characteristic function of the set $\{x\in X : c(x,\tau(x))\geq s\}$. Then $b_s\cdot a$ is in $M[s,\infty)$, where $b_s\cdot a(\gamma):=b_s(r(\gamma))a(\gamma)$. Therefore $F_t\pi(b_s\cdot a)F_t=0$, $s>0$. But then $\pi(b_s)F_t\pi(a)F_t=0$ for all $s>0$. It follows that $\pi(b)F_t\pi(a)F_t=0$, where b is the characteristic function of $r(\tau)$. But $b\cdot a=a$; hence $F_t\pi(a)F_t=0$. By taking adjoints this holds also for every element supported on $j^{-1}(\tau)$, for any non-singular Borel G-set $\tau\subseteq P^{-1}$. It follows that for every $f\in M$, $F_t\pi(f)F_t\in F_t\pi(\mathscr{A})F_t$. From irreducibility it now follows that $F_tB(K)F_t\subseteq F_t\pi(\mathscr{A})''F_t$.

REMARK. We observe in passing that Theorem 6.1 may have all reference to the topological structures on G and E removed. What is at issue really is the Borel structure. In fact, it applies to the context where (G, ν) is a measured equivalence relation with countable equivalence classes in the sense of Feldman and Moore [6, 7] determined by a quasi-invariant measure μ (so ν is obtained by integrating counting measures against μ) and where M is the algebra generated by the functions in $L^\infty(\nu)$ that are supported on non-singular Borel G-sets. If P is given by a Borel cocycle and if π is an irreducible representation of M that may be expressed in terms of a Borel representation $(\mu, G^{(0)} * K, U)$ of G, then the assertions (1) and (2) in Theorem 6.1 about $\operatorname{lat}\pi(\mathscr{T}(P))$ remain valid. In our application of this generalization of Theorem 6.1 there is no need to mention E because, as we noted above, our (G, ν) is hyperfinite, so E is measure theoretically trivial.

We now turn our attention to the case where $R^\mu_\infty(c) = \lambda\mathbb{Z}$. To simplify the writing, we may assume without loss of generality that $\lambda = 1$. Using Proposition 2.4(5), there is a Borel function $g : X \to \mathbb{R}$ such that $c(u, v) + g(v) - g(u) \in \mathbb{Z}$ for all $(u, v) \in G$. In fact we can assume $0 \le g(x) < 1$ (simply replace $g(x)$ by $g(x) - [g(x)]$). Write

$$d(u, v) = c(u, v) + g(v) - g(u).$$

Then d is a cocycle with values in $\mathbb{Z}$. Write $G_1 = d^{-1}(0)$. Then G_1 is a Borel subgroupoid of G containing Δ. The restriction $c|_{G_1}$ is a coboundary (for $(u, v) \in G_1$, $c(u, v) = g(u) - g(v)$).

LEMMA 6.2. *If* $c(u, v) \ge 0$, *then* $d(u, v) \ge 0$.

PROOF. Suppose $c(u, v) \ge 0$ and $d(u, v) < 0$; then $d(u, v) \le -1$ and $g(v) - g(u) = d(u, v) - c(u, v) \le -1$. But $0 \le g(u), g(v) < 1$, and thus $g(v) - g(u) \le -1$ is impossible.

Let π and M be as in our discussion preceding Theorem 6.1. Write $M^d[n, \infty)$ for $\mathscr{T}(d^{-1}[n, \infty))$ and $P_1 = d^{-1}[0, \infty)$. Fix a $Q \in \operatorname{lat}\pi(A(P))$ and assume that Q is different from 0 and I. Let F_n be the projection with range $[\pi(M^d[n, \infty))Q(K)]$. From Lemma 6.2 it follows that $P \subseteq P_1$, and clearly, $\mathscr{T}(P_1)M^d[n, \infty) \subseteq M^d[n, \infty)$. Hence each $F_n \in \operatorname{lat}\pi(\mathscr{T}(P))$. In fact, we have,

$$\cdots \subseteq F_1 \subseteq Q \subseteq F_0 \subseteq F_{-1} \subseteq \cdots .$$

Using Theorem 6.1(2) we can find $\{t_k\} \subseteq \mathbb{R}$, $t_k \le t_{k+1}$ such that $F_n = Q_{t_n}$. Write $L_n = F_n - F_{n+1}$. The algebra $M_1 := \mathscr{T}(G_1)$ is a $*-$subalgebra of M containing A that may be viewed to be of the kind discussed in the remark following the proof of Theorem 6.1. We define, for every n, a representation π_n of M_1 on $L_n(K)$ by

$\pi_n(f)L_n\xi = \pi(f)L_n\xi$, $f \in M_1$. Since $G_1 \circ d^{-1}[n,\infty) = d^{-1}(0) \circ d^{-1}[n,\infty) \subseteq d^{-1}[n,\infty)$, $\pi(M_1)F_n(K) \subseteq F_n(K)$. Hence $L_n, F_n \in \pi(M_1)'$. In fact we claim L_n is a minimal projection in $\pi(M_1)'$ because if $L_n = E_1 + E_2$, where each E_i is a projection in $\pi(M_1)'$, then first of all, $\pi(M^d[1,\infty))E_1(K) \subseteq \pi(M^d[1,\infty))F_n(K) \subseteq F_{n+1}(K) \subseteq K \ominus E_2(K)$. Similarly, $\langle \pi(a)\xi_2, \xi_1\rangle = 0$ for $\xi_i \in E_i(K)$ and $a \in M^d[1,\infty)$. Therefore, $\langle \xi_2, \pi(a^*)\xi_1\rangle = 0$ and we conclude that $\pi(M^d(-\infty,-1])E_1(K) \subseteq H \ominus E_2(K)$. Finally, $\pi(M^d(0))E_1(K) = \pi(M_1)E_1(K) \subseteq E_1(K) \subseteq K \ominus E_2(K)$. Since the cocycle d takes its values in the integers, this shows that $\pi(M)E_1(K) \subseteq K \ominus E_2(K)$. Since π is an irreducible representation of M, this shows that one of E_1 and E_2 is zero. We thus arrive at the following

COROLLARY 6.3. *Each π_n is an irreducible representation of M_1.*

In fact, we may assert that the representation π_n satisfies the conditions in the remark amending Theorem 6.1. The representation V_n of G_1 whose integral gives π_n is simply given by the formula $V_n(\gamma) = [F_n(r(\gamma)) - F_{n+1}(r(\gamma))]U(\gamma)[F_n(s(\gamma)) - F_{n+1}(s(\gamma))]$, $\gamma \in G_1$. Moreover, since $c|_{G_1}$ is a Borel coboundary, we can modify Theorem 5.1 to get $\operatorname{lat}\pi_n(\mathscr{T}(P\cap G_1)) \subseteq \pi_n(\mathscr{A})''$. The following lemma, then, allows us to interpolate the F_n with elements of $\operatorname{lat}\pi_n(\mathscr{T}(P\cap G_1))$.

LEMMA 6.4. $\{F \in \operatorname{lat}\pi(\mathscr{T}(P)) : F_{n+1} \le F \le F_n\} = \{F_{n+1} + L : L \in \operatorname{lat}\pi_n(\mathscr{T}(P\cap G_1))\}$.

PROOF. If $F_{n+1} \le F \le F_n$, then $F - F_{n+1} \in B(L_n(K))$. For every $\xi \in L_n(K)$ and $f \in \mathscr{T}(P\cap G_1)$,

$$\begin{aligned}\pi_n(f)(F - F_{n+1})\xi &= \pi(f)(F - F_{n+1})\xi \\ &= (F_n - F_{n+1})\pi(f)F\xi \in L_nF(K) = (F - F_{n+1})(K).\end{aligned}$$

Hence, $F - F_{n+1} \in \operatorname{lat}\pi_n(\mathscr{T}(P\cap G_1))$. Conversely, if $F - F_{n+1} \in \operatorname{lat}\pi_n(\mathscr{T}(P\cap G_1))$ and if $f \in \mathscr{T}(P)$, then $\pi(f)F\xi = \pi(f)(F - F_{n+1})\xi + \pi(f)F_{n+1}\xi$. Since $F_{n+1} \in \operatorname{lat}\pi(\mathscr{T}(P))$, $\pi(f)F_{n+1}\xi \in F_{n+1}(K) \subseteq F(K)$. It remains to show that $\pi(f)(F - F_{n+1})\xi \in F(K)$. We can write f_0 for the restriction of f to G_1 and then $f - f_0 \in M^d[1,\infty)$. Hence $\pi(f - f_0)(F - F_{n+1})\xi = \pi(f - f_0)F_n(F - F_{n+1})\xi \in F_{n+1}(K) \subseteq F(K)$ as $\pi(M^d[1,\infty))F_n(K) \subseteq F_{n+1}(K) \subseteq F(K)$. Also $\pi(f_0)(F - F_{n+1})\xi = \pi_n(f_0)(F - F_{n+1})\xi \subseteq (F - F_{n+1})(K)$ (as $F - F_{n+1} \in \operatorname{lat}\pi_n(\mathscr{T}(P\cap G_1))$. This completes the proof.

COROLLARY 6.5. *If $R^{\mu}_{\infty}(c) = \mathbb{Z}$, then either*

(a) $\operatorname{lat}\pi(A(P)) \subseteq \{F_{n+1} + \pi(\chi_Y)(F_n - F_{n+1}) : n \in \mathbb{Z},\ Y \subseteq X$ *Borel correction*$\}$, *or*
(b) $\operatorname{lat}\pi(A(P)) = \{0, I\}$.

REMARK. Note that if Q is a projection in $\operatorname{lat}\pi(A(P))$, different from 0 and I and if the F_n are as above, then we cannot have $F_0 = 0$ and $F_1 = I$ (implying that each F_n is 0 or I) because then, by the Corollary, $\operatorname{lat}\pi(A(P))$ would be contained in $\pi(C_0(G^0))''$ and this would imply the existence of a Borel decreasing subset of $G^{(0)}$. This, in turn, would imply that c is a Borel coboundary ([14, Lemma 3.8]).

If $R^\mu_\infty(c) = \mathbb{Z}$ then μ is properly ergodic and $\operatorname{lat}\pi(A(P))$ is either trivial or a continuous nest by Theorem 3.2. The following corollary shows that in the latter case, the nest is 'uniformly distributed' in a certain sense and has uniform infinite multiplicity.

COROLLARY 6.6. *If $R^\mu_\infty(c) = \mathbb{Z}$, then, for μ-a.e. $u \in X$, $\{F(u) : F \in \operatorname{lat}\pi(A(P))\}$ is either trivial or order isomorphic to $\mathbb{Z} \cup \{\pm\infty\}$.*

PROOF. We can assume $\operatorname{lat}\pi(A(P)) \neq \{0, I\}$ and then $\operatorname{lat}\pi((P)) \subseteq \{F_{n+1} + \pi(\chi_Y)(F_n - F_{n+1}) | n \in \mathbb{Z},\ Y \subseteq X \text{ Borel}\}$. If $F \in \operatorname{lat}\pi((P))$, then

$$\begin{aligned} F(u) &= F_{n+1}(u) + \chi_Y(u)(F_n(u) - F_{n+1}(u)) \\ &= \begin{cases} F_{n+1}(u) & u \notin Y \\ F_n(u) & u \in Y. \end{cases} \end{aligned}$$

Hence $\{F(u) | F \in \operatorname{lat}\pi(A(P))\} = \{F_n(u)\}$. It remains to show that, for $m \neq n$, $F_n(u) \neq F_m(u)$ for μ-a.e. $u \in X$. Write $Z_n = \{u \in X | F_n(u) = F_{n+1}(u)\}$ and assume that for some $m \in \mathbb{Z}$, $\mu(Z_m) > 0$. For $u \in X$, $F_n(u) = \bigvee\{U(u, w)Q(w)U(w, u) \mid d(u, w) \geq n\}$ as $F_n(K) = [\pi(M^d[n, \infty))Q(K)]$. Hence, for ν-a.e. (u, v) in $d^{-1}(\{1\})$, $F_{n+1}(u) = F_n(v)$ and, for $(u, v) \in (Z_m \times Z_m) \cap d^{-1}(\{1\})$, we get

$$F_m(u) = F_{m+1}(u) = F_m(v) = F_{m+1}(v) = F_{m+2}(u).$$

Hence, for a.e. $u \in r((Z_m \times Z_m) \cap d^{-1}(\{1\}))$ we have $F_m(u) = F_{m+2}(u)$. But $1 \in R^\mu_\infty(d)$ and that implies that $r((Z_m \times Z_m) \cap d^{-1}(\{1\})) = Z_m$ (up to a set of μ measure zero). We conclude that for a.e. $u \in Z_m$, $F_m(u) = F_{m+2}(u)$. We can apply this argument repeatedly to show that for every $k \in \mathbb{Z}$, $F_m(u) = F_{m+k}(u)$ for a.e. $u \in Z_m$. But this implies that for a.e. $w \in X$ and $u \in Z_m$, $U(u, w)Q(w)U(w, u) \leq F_0(u) \leq Q(u)$. By irreducibility, $Q(u) = I$ for a.e. $u \in Z_m$ and thus, for a.e. $u \in X$; contradicting the fact that Q was chosen to be non-trivial.

7. $R^\mu_\infty(c) = \mathbb{R}$

Our objective in this section is to prove a 'continuous' analogue of Corollary 6.6. That is, we shall show in Theorem 7.1 that if $R^\mu_\infty(c) = \mathbb{R}$ and if $\operatorname{lat}\pi(A(P))$ is different from $\{0, I\}$, then the nest, which then is $\operatorname{lat}\pi(A(P))$ and is decomposable, has the property that (almost) each component is continuous. This again means that the nest is 'uniformly distributed' in a very strong sense and that it has infinite uniform multiplicity.

We begin by noting that when $R^{\mu}_{\infty}(c) = \mathbb{R}$, given $a < b$ and subsets Y and Z of $G^{(0)}$ of positive μ-measure, then

$$\nu((Y \times Z) \cap c^{-1}(a, b)) > 0.$$

To see this take $\epsilon < (b - a)/2$ and fix some $t \in \mathbb{R}$ such that $\nu((Y \times Z) \cap c^{-1}(t - \epsilon, t + \epsilon)) > 0$ (by ergodicity this can be done). Since

$$(Y \times Z) \cap c^{-1}(a, b) \supseteq ((Y \times Y) \cap c^{-1}(a - t + \epsilon, b - t - \epsilon)) \circ ((Y \times Z) \cap c^{-1}(t - \epsilon, t + \epsilon))$$

we are done. This will be used in Lemma 7.3 and Lemma 7.5 below, which lie at the heart of the proof of

THEOREM 7.1. *Suppose* $R^{\mu}_{\infty}(c) = \mathbb{R}$ *and* $\operatorname{lat}\pi(A(P)) \neq \{0, I\}$, *so that* $\operatorname{lat}\pi(A(P)) = \{Q_t : t \in R\}$ *for some fixed* Q *in* $\operatorname{lat}\pi(A(P))$. *Then for* μ-*almost all* $u \in G^{(0)}$ *and every* $t \neq s$ *in* $\mathbb{R}$, $Q_t(u) \neq Q_s(u)$.

PROOF. The proof will use four lemmas. Throughout, a $Q \in \operatorname{lat}\pi(A(P))$ different from 0 and I will be fixed.

LEMMA 7.2. *Write* $J(Q) = \{(u, v) \in P : U(u, v)Q(v)U(v, u) \neq Q(u)\}$. *Then* $P \circ J(Q) \circ P \subseteq J(Q)$ *(up to a set of* ν-*measure zero).*

Thus $J(Q)$ is a *Borel ideal* subset of P, meaning that in the notation of the previous section, $\mathscr{T}(J(Q))$ is a 2-sided ideal in $\mathscr{T}(P)$.

PROOF. For almost every $(w, z) \in P$ we have $U(w, z)Q(z)U(z, w) \leq Q(w)$. Hence the lemma is obvious.

LEMMA 7.3. *For every ideal set* J *of positive measure there is a* t *with* $0 \leq t$ *such that* $c^{-1}(t, \infty) \subseteq J \subseteq c^{-1}[t, \infty)$ *(up to a set of* ν *measure zero).*

PROOF. Write $t = \operatorname{ess\,inf} c|_J$. Clearly $J \subseteq c^{-1}[t, \infty)$. Suppose $J \not\supseteq c^{-1}(t, \infty)$. Then there is a Borel G-set $\tau \subseteq P$ of positive measure such that $\tau \subseteq c^{-1}(t, \infty)$, and $\nu(J \cap \tau) = 0$. In fact, by making τ smaller if necessary, we can find some $t < t_1$ such that $\tau \subseteq c^{-1}(t_1, \infty)$ and $\nu(J \cap \tau) = 0$. Take $t < t_2 < t_1$. Since $t_2 > t = \operatorname{ess\,inf} c|_J$, we can find a Borel G-set $\tau_1 \subseteq J$ of positive measure such that for almost every

$(u, v) \in \tau_1$, $t \leq c(u, v) < t_2$. Consider $[r(\tau) \times r(\tau_1)] \cap G$. Since $R^{\mu}_{\infty}(c) = \mathbb{R}$, we have

$$\nu\left(c^{-1}[0, t_1 - t_2) \cap (r(\tau) \times r(\tau_1))\right) > 0.$$

If $(w_1, w_2) \in c^{-1}[0, t_1 - t_2) \cap (r(\tau) \times r(\tau_1))$, then

$$c(w_1, \tau_1(w_2)) = c(w_1, w_2) + c(w_2, \tau_1(w_2)) < t_1 - t_2 + t_2 = t_1,$$

while $c(w_1, \tau(w_1)) \geq t_1$. Hence $(\tau_1(w_2), \tau(w_1)) \in P$ and

$$(w_1, \tau(w_1)) = (w_1, w_2) \circ (w_2, \tau_1, (w_2)) \circ (\tau_1(w_2), \tau(w_1)) \in P \circ J \circ P \subseteq J.$$

We then get $\nu(\tau \cap J) > 0$ - a contradiction.

LEMMA 7.4. *Let $t > 0$ and $F \subseteq P$ satisfy $F \circ F \subseteq F$ and $c^{-1}[0, t) \subseteq F$. Then $\nu(P \backslash F) = 0$.*

PROOF. We may assume that F is the minimal set satisfying these conditions; that is, we may assume that

$$F = \{(w_1, w_2) \in P : \text{ there are } \quad x_1, \dots, x_n \in G^{(0)} \quad \text{such that}$$
$$(w_1, x_1), (x_n, w_2), (x_i, x_{i+1}) \in c^{-1}(0, t) \quad \text{for all } i = 1, \dots, n-1\}.$$

Suppose that $(w_1, w_2) \in F$ and that $x_1, \dots, x_n$ are as above; suppose, too, that $(w_1, u) \in P$ and $(v, w_2) \in P$. Then if $(u, v) \in P$, we can find a $k \leq j$ such that $x_k \leq u \leq x_{k+1}$ and $x_j \leq v \leq x_{j+1}$, implying that $(u, v) \in F$. It follows that $P \circ F^c \circ P \subseteq F^c$; that is, F^c is an ideal set (here $F^c = P \backslash F$). Hence, if $\nu(P \backslash F) > 0$, then there is a $b \geq 0$ such that $c^{-1}(b, \infty) \subseteq P \backslash F \subseteq c^{-1}[b, \infty)$. In fact, $P \backslash F \subseteq c^{-1}[t, \infty)$; so $\nu(c^{-1}(b, t)) = 0$. But since $R^{\mu}_{\infty}(c) = \mathbb{R}$, this can happen only if $b \geq t$; that is, only if $b \geq t > 0$. Now choose a Borel G-set $\tau_1 \subseteq P$ with positive measure such that $\tau_1 \subseteq c^{-1}[2b/3, b)$. Also note that $\nu(c^{-1}[2b/3, b) \cap (s(\tau_1) \times X)) > 0$ and we can find a Borel G- set $\tau_2 \subseteq c^{-1}[2b/3, b) \cap (s(\tau_1) \times X)$ having a positive measure. But then the G-set $\tau_1 \circ \tau_2$ has a positive measure and is contained in $c^{-1}[4b/3, 2b) \subseteq c^{-1}(b, \infty) \subseteq P \backslash F$. But $\tau_i \subseteq c^{-1}[2b/3, b) \subseteq c^{-1}[0, b) \subseteq F$, so $\tau_1 \circ \tau_2 \subseteq F \circ F \subseteq F$. This contradicts the assumption that $\nu(P \backslash F) > 0$.

LEMMA 7.5. *Let $J(Q)$ be as in Lemma 7.2. Then $\nu(P \backslash (J(Q) \cup \Delta)) = 0$.*

PROOF. $J(Q)$ is an ideal set and, applying Lemma 7.3, we may find a $t \geq 0$ such that $c^{-1}(t, \infty) \subseteq J(Q) \subseteq c^{-1}[t, \infty)$. (Note that $\nu(J(Q)) > 0$ because Q is not reducing.) Hence $F = P \backslash J(Q)$ contains $c^{-1}[0, t)$ and clearly $F \circ F \subseteq F$. Therefore $t = 0$ by Lemma 7.4 and thus $J(Q) = P \backslash \Delta$, up to a set of measure zero.

COMPLETION OF THE PROOF OF THEOREM 7.1 We conclude from the definition of $J(Q)$ and Lemma 7.5 that for ν-a.e. $(u, v) \in P\backslash\Delta$,

$$U(u,v)Q(v)U(v,u) \lneq Q(u). \tag{1}$$

Hence, we can find a μ-null set $N \subseteq X$ such that for every $u \notin N$ and every $v \geq u$, (1) holds. Now, for every pair of rational numbers $q < p$ we write $Y = r(c^{-1}(q,p) \cap (X \times (X\backslash N)))$. If $\mu(X\backslash Y) > 0$, then $\nu([(X\backslash Y) \times X] \cap c^{-1}(q,p)) > 0$ (as $R_\infty^\mu(c) = \mathbb{R}$). But then we can find a Borel G-set τ of positive measure contained in $[(X\backslash Y) \times X] \cap c^{-1}(q,p)$. For every $u \in r(\tau)$ we have $u \notin Y$ but $(u, \tau(u)) \in c^{-1}(q,p)$. From the definition of Y, it follows that $(u, \tau(u)) \notin X \times (X\backslash N)$ (otherwise $(u, \tau(u)) \in c^{-1}(q,p) \cap (X \times (X\backslash N))$ and $u \in Y$). Hence $\tau(u) \in N$. Since this holds for every $u \in r(\tau)$, we have $s(\tau) \subseteq N$. But this is impossible since N is null and $\nu(\tau) > 0$. This shows that $\mu(X\backslash Y) = 0$. We now write $N_0 = N \cup \left(\bigcup_{q<p}(X\backslash Y_{q,p})\right)$ where $Y_{q,p}$ is the Y associated with q, p as above. Then $\mu(N_0) = 0$. For every pair of real numbers, t, s, with $t < s$, we can find rational numbers $q < p$ such that $t < q < p < s$. Hence, for every $u \notin N_0$ and every pair t, s, with $t < s$, $u \in r(c^{-1}(t,s) \cap (X \times (X\backslash N)))$.

Now, fix $u \notin N_0$ and $0 < t < s$. Let $q \in \mathbb{R}$ satisfy $t < q < s$. Then $u \in r(c^{-1}(t,q) \cap (X \times (X\backslash N)))$. Hence we can find w_1 such that $t < c(u, w_1) < q$ and $w_1 \notin N$. Similarly we can find w_2 such that $q < c(u, w_2) < s$ and $w_2 \notin N$. Using (1) and the fact that u and w_2 are not in N, we get

$$\begin{aligned} U(u,w_2)Q(w_2)U(w_2,u) &\lneq U(u,w_2)U(w_2,w_1)Q(w_1)U(w_1,w_2)U(w_2,u) \\ &= U(u,w_1)Q(w_1)U(w_1,u) \lneq Q(u). \end{aligned}$$

Now, for every w with $c(u,w) \geq s$ we have $w_2 \leq w$, and thus,

$$\begin{aligned} Q_t(u) &= \bigvee_{c(u,v)\geq t} U(u,v)Q(v)U(v,u) \geq U(u,w_1)Q(w_1)U(w_1,u) \\ &\gneq U(u,w_2)Q(w_2)U(w_2,u) \geq U(u,w_2)U(w_2,w)Q(w)U(w,w_2)U(w_2,u) \\ &= U(u,w)Q(w)U(w,u) \end{aligned}$$

since $Q_s(u) = \bigvee_{c(u,w)\geq s} U(u,w)Q(w)U(w,u)$, $Q_s(u) < Q_t(u)$. This completes the proof of the theorem.

8. An example

We shall construct an example where π is an irreducible representation of the C^*-algebra and $\operatorname{lat} \pi(A(P)) \neq \{0, I\}$. This requires $\tilde{R}_\infty^\mu(c) \neq \{0, \infty\}$. In the case where

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$\tilde{R}^{\mu}_{\infty}(c) = \{0\}$ this can easily be done, since then c is a coboundary Proposition 2.4(4). For an explicit example, take $\pi = \operatorname{Ind}\epsilon_u$. In the following example we shall find that $R^{\mu}_{\infty}(c) = \mathbb{Z}$. Using Corollary 6.6, then, we see that the multiplicity of the nest lat $\pi(A(P))$ is infinite.

The C^*-algebra is the CAR algebra. So let X be $\{0, 1\}^{\infty}$ and $G \subseteq X \times X$ be defined as follows: $(x, y) \in G$, where $x = (x_i)_{i=0}^{\infty}$, $(y_i)_{i=0}^{\infty}$, if there is some N such that $x_i = y_i$ for every $i \geq N$. Then $B = C^*(G)$ can be identified with the closure of $\bigcup_{k=1}^{\infty} M_{2^k}$, where M_{2^k} is a subalgebra of B isomorphic to the $2^k \times 2^k$ matrices. We write $\{e_{ij}^{(k)} : 1 \leq i,\ j \leq 2^k\}$ for the matrix units of M_{2^k}. In fact $e_{ij}^{(k)}$ is the characteristic function of $\tau_{ij}^{(k)}$ which, in turn, is composed of all pairs $(x, y) \in G$, such that $x_m = y_m$ for $m > k$ and $i = 1 + \sum_{m=0}^{k} x_m 2^m$, while $j = 1 + \sum_{m=0}^{k} y_m 2^m$. We have $G = \bigcup_{i,j,k} \tau_{ij}^{(k)}$ and $G^{(0)} = \bigcup_i \tau_{ii}^{(k)}$ for every k.

We shall now define representations π_k of M_{2^k}. The representation space for each is $H = L^2(X, \mu, \ell^2(\mathbb{Z})) = \int^{\oplus} H(u)d\mu(u)$, where $H(u) \equiv \ell^2(\mathbb{Z})$ and $\mu = \prod_{k=0}^{\infty} \mu_k$, $\mu_k(\{0\}) = \mu_k(\{1\}) = 1\backslash 2$. (Note that μ is quasi-invariant. Indeed, μ is invariant.)

We let

$$(\pi_1(e_{ij}^{(1)})f)(x) = \begin{cases} U^{j-i} f(\tau_{ij}^{(1)}(x)) & x \in r(\tau_{ij}^{(1)}) \\ 0 & x \notin r(\tau_{ij}^{(1)}), \end{cases}$$

where $U \in B(\ell^2(\mathbb{Z}))$ is the shift $(Ua)_i = a_{i+1}$. Now let

$$(\pi_2(e_{1,2}^{(2)})f)(x) = \begin{cases} VUf(\tau_{12}^{(2)}(x)) & x \in r(\tau_{12}^{(2)}) \\ 0 & x \notin r(\tau_{12}^{(2)}), \end{cases}$$

and $(\pi_2(e_{jj}^{(2)})f)(x) = f(x)\chi_{\tau_{jj}^{(2)}}(x)$, where $V \in B(\ell^2(\mathbb{Z}))$ is defined by the diagonal matrix, whose jj element is $e^{2\pi i j\alpha}$, where α is a fixed irrational number. Noting that $e_{1,2}^{(1)} = e_{1,3}^{(1)} + e_{2,4}^{(2)}$ and, in fact, $e_{1,3}^{(2)} = e_{1,1}^{(1)}e_{1,2}^{(1)}$ and $e_{2,4}^{(2)} = e_{2,2}^{(2)}e_{1,2}^{(1)}$, we define $\pi_2(e_{1,3}^{(2)}) = \pi_2(e_{1,1}^{(1)})\pi_1(e_{1,2}^{(1)})$ and $\pi_2(e_{2,4}^{(2)}) = \pi_2(e_{2,2}^{(2)})\pi_1(e_{1,2}^{(1)})$. Finally, let $\pi_2(e_{2,3}^{(2)}) = \pi_2(e_{1,2}^{(2)})\pi_2(e_{1,3}^{(2)})$, $\pi_2(e_{3,4}^{(2)}) = \pi_2(e_{2,3}^{(2)})^*\pi_2(e_{2,4}^{(2)})$, $\pi_2(e_{1,4}^{(2)}) = \pi_2(e_{1,3}^{(2)})\pi_2(e_{3,4}^{(2)})$ and $\pi(e_{i,j}^{(2)}) = \pi(e_{j,i}^{(2)})^*$ if $i > j$. This defines a representation π_2 of M_4 that extends π_1. (Note that $M_2 \subseteq M_4$.)

For π_3 we define $(\pi_3(e_{jj}^{(3)})f)(x) = \chi_{\tau_{jj}^{(3)}}(x)f(x)$, and

$$(\pi_3(e_{1,2}^{(3)})f)(x) = \begin{cases} Uf(\tau_{1,3}^{(3)}(x)) & x \in r(\tau_{1,3}^{(3)}(x)) \\ 0 & x \notin r(\tau_{1,3}^{(3)}(x)). \end{cases}$$

Using similar arguments as above, we see that this defines π_3 on M_8, extending π_2.

We continue inductively and get $\{\pi_k\}$ by defining

$$(\pi_{2m}(e_{1,2}^{(2m)})f)(x) = \begin{cases} VUf(\tau_{1,2}^{(2m)}(x)) & x \in r(\tau_{1,2}^{(2m)}) \\ 0 & \text{otherwise; and} \end{cases}$$

$$(\pi_{2m-1}(e_{1,2}^{(2m-1)})f)(x) = \begin{cases} Uf(\tau_{1,2}^{(2m-1)}(x)) & x \in r(\tau_{1,2}^{(2m-1)}(x)) \\ 0 & \text{otherwise,} \end{cases}$$

where $m = 1, 2, 3, \ldots$. We also write

$$(\pi(e_{ij}^{(k)})f)(x) = \begin{cases} f(\tau_{ij}^{(k)}(x)) & x \in r(\tau_{ij}^{(k)}(x)) \\ 0 & \text{otherwise,} \end{cases}$$

and, for an operator $W \in B(\ell^2(\mathbb{Z}))$ we write $(\tilde{W}f)(x) = Wf(x),\ f \in L^2(X, \ell^2(\mathbb{Z}))$. Then, it is clear that for every i, j, k there is a unitary $u_{ij}^{(k)} \in B(\ell^2(\mathbb{Z}))$ such that

$$\pi_k(e_{ij}^{(k)}) = \tilde{u}_{ij}^{(k)}\rho(e_{ij}^{(k)}).$$

For example, $u_{1,2}^{(1)} = U$, $u_{1,2}^{(2)} = VU$ and, in fact,

$$u_{1,2}^{(k)} = \begin{cases} I & k = 0 \\ U & k = 2m-1,\ m \geq 1 \\ VU & k = 2m,\ m \geq 1. \end{cases}$$

In general, we have $u_{ij}^{(k)}u_{jl}^{(k)} = u_{il}^{(k)}$ and $u_{j,j+2m}^{(k)} = u_{l,l+m}^{(k-1)}$ where $l = [(j+1)/2]$ (because π_k extends π_{k-1}).

LEMMA 8.1. *For k odd and $0 \leq m \leq 2^{k-2} - 1$, each $u_{2+4m,3+4m}^{(k)}$ is a scalar multiple of V, that is,*

$$u_{2+4m,3+4m}^{(k)} \in \mathbb{C}V.$$

PROOF. We fix odd k and use induction on m. For $m = 0$ we have

$$u_{2,3}^{(k)} = u_{2,1}^{(k)}u_{1,3}^{(k)} = (u_{1,2}^{(k)})^* u_{1,2}^{(k-1)} = U^*VU.$$

But $U^*VU = \lambda^{-1}V$ where $\lambda = e^{2\pi i\alpha}$; hence the claim holds for $m = 0$. Now we have

$$\begin{aligned} u_{2+4(m+1),3+4(m+1)}^{(k)} &= u_{6+4m,2+4m}^{(k)}u_{2+4m,3+4m}^{(k)}u_{3+4m,7+4m}^{(k)} \\ &= u_{2+m,1+m}^{(k-2)}u_{2+4m,3+4m}^{(k)}u_{1+m,2+m}^{(k-2)}. \end{aligned}$$

But, by induction, $u^{(k)}_{2+4m,3+4m} \in \mathbb{C}V$ and, by construction, $u^{(k-2)}_{2+m,1+m} = (u^{(k-2)}_{1-m,2+m})^*$ and $u^{(k-2)}_{1+m,2+m}$ is a product of powers of U, V, U^* and V^*. Since U^*VU, UVU^*, V^*VV and VVV^* are all in $\mathbb{C}V$, we are done.

LEMMA 8.2. *For k odd and* $0 \leq m \leq 2^{k-1} - 1$, $u^{(k)}_{1+2m,2+2m} \in \mathbb{C}U$.

PROOF. We fix k and use induction on m. For $m = 0$ we have, $u^{(k)}_{1,2} = U$ by definition. For the induction step, we write

$$\begin{aligned} u^{(k)}_{1+2(m+1),2+2(m+1)} &= u^{(k)}_{3+2m,1+2m} u^{(k)}_{1+2m,2+2m} u^{(k)}_{2+2m,4+2m} \\ &= (u^{(k-1)}_{1+m,2+m})^* u^{(k)}_{1+2m,2+2m} u^{(k-1)}_{1+m,2+m} \in \mathbb{C}U \end{aligned}$$

because $u^{(k)}_{1+2m,2+2m} \in \mathbb{C}U$ (induction) and $u^{(k-1)}_{1+m,2+m}$ is a product of powers of U, U^*, V, V^*. However, UUU^*, U^*UU, $V^*UV = V^*UVU^*U = \lambda V^*VU = \lambda U$ and VUV^* are all in $\mathbb{C}U$.

LEMMA 8.3. *In the weak operator topology,*

$$\sum_{m=0}^{2^{k-1}-1} \rho(e^{(k)}_{1+2m,2+2m}) \xrightarrow[k\to\infty]{} \frac{1}{2} I.$$

PROOF. Write $W_k = S_k + T_k$, where

$$S_k = \sum_{m=0}^{2^{k-1}-1} \rho(e^{(k)}_{1+2m,2+2m}), \qquad T_k = \sum_{m=0}^{2^{k-1}-2} \rho(e^{(k)}_{2+2m,3+2m}) + \rho(e^{(k)}_{2^k,1}).$$

Then W_k is a unitary operator corresponding to a translation by $1/2^k$ (mod 1) when we think of $x \in X$ as a number in $[0, 1]$ (given by $\sum_{m=0}^{\infty} x_m/2^{m+1}$). It is clear, then, that $W_k \to I$ in the strong operator topology. It is easy to check that for every k, $W_k^* S_k W_k = T_k$.

Since $\| S_k \| \leq 1$ for all k we can find a subsequence $S_{k_n} \to S$ in the weak operator topology. But then $T_{k_n} = W^*_{k_n} S_{k_n} W_{k_n} \to S$ in the weak operator topology (as $W_{k_n} \to I$ strongly). We have $S + S = \lim_n S_{k_n} + T_{k_n} = \lim W_{k_n} = I$; hence $S = I/2$. Since this is the case for every weakly converging subsequence, $S_k \to I/2$ in the weak operator topology.

LEMMA 8.4. *In the weak operator topology,*

$$\sum_{m=0}^{2^{k-2}-1} \rho(e^{(k)}_{2+4m,3+4m}) \to \frac{1}{4} I.$$

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The proof of this lemma is very similar to the proof of Lemma 8.3 and is omitted.

LEMMA 8.5. $$\tilde{U}, \tilde{V} \in \left(\bigcup_{k=1}^{\infty} \pi_k(M_{2^k})\right)''.$$

PROOF. Write $\mathscr{U} = (\bigcup_{k=1}^{\infty} \pi_k(M_{2^k}))''$. For each k odd and $0 \leq m \leq 2^{k-2} - 1$, $u^{(k)}_{2+4m,3+4m}$ is in $\mathbb{C}V$ (Lemma 8.1); hence $\tilde{V}\rho(e^{(k)}_{2+4m,3+4m})$ is a multiple of $\pi_k(e^{(k)}_{2+4m,3+4m})$. Thus $\tilde{V}\rho(e^{(k)}_{2+4m,3+4m}) \in \mathscr{U}$ for every such k, m. Since

$$\sum_{m=0}^{2^{k-2}-1} \tilde{V}\rho(e^{(k)}_{2+4m,3+4m}) \to \frac{1}{4}\tilde{V}$$

in the weak operator topology (Lemma 8.4), $\tilde{V} \in \mathscr{U}$. The proof that $\tilde{U} \in \mathscr{U}$ is similar (using Lemma 8.2 and Lemma 8.3) and is omitted.

Having defined the sequence $\{\pi_k\}$ of representations of M_{2^k} with the property that π_{k+1} extends π_k, we get a representation π of $B = C^*(G)$ and from Lemma 8.5 we conclude that $\tilde{U}, \tilde{V} \in \pi(B)''$. Since U and V generate $B(\ell^2(\mathbb{Z}))$ as a von Neumann algebra, it is clear that for every $T \in B(\ell^2(\mathbb{Z}))$, $\tilde{T} \in \pi(B)''$. But $B(H) = B(\int_X^{\oplus} H(u)d\mu(u))$ is generated, as a von Neumann algebra, by $L^\infty(X, \mu)$ and $\{\tilde{T} : T \in B(H(u)) = B(\ell^2(\mathbb{Z}))\}$. Thus $\pi(B)'' = B(H)$. We conclude

COROLLARY 8.6. *π is an irreducible representation.*

We now turn to the partial order P. The partial order we define here is the same as in [29, Example 7.5]. We define a cocycle $c : G \to \mathbb{R}$ as follows:

$$c(x, y) = \sum c_i(x_i, y_i)$$

where c_i is a cocycle on $\{0, 1\}^2$ defined by $c_i(0, 1) = 1+2^{-i}$ (and $c_i(0, 0) = c_i(1, 1) = 0$, $c_i(1, 0) = -c_i(0, 1)$). We let $P = \{(x, y) \in G : c(x, y) \geq 0\}$. It is not hard to check that the partial order defined by P is the following: Let (x, y) be in G; that is, $x_i = y_i$ for all $i \geq$ some N. Then $x \leq y$ if either $\sum_{i=0}^{N-1}(y_i - x_i) > 0$ (that is, y has more 1's in the positions from 0 to $N-1$ than x has) or $\sum_{i=0}^{N-1}(y_i - x_i) = 0$ and x is smaller than y with respect to the lexicographic order.

Now write F for the projection in $B(\ell^2(\mathbb{Z}))$ with range $\ell^2(\mathbb{Z}_-)$ where $\mathbb{Z}_- = \{k \in \mathbb{Z} : k \leq 0\}$. Then $UFU^* \leq F$ and $VFV^* = F$.

LEMMA 8.7. $\tilde{F} \in \operatorname{lat} \pi(A(P))$.

PROOF. Suppose first that $(x, y) \in G$ and $x_i = y_i$ for all i, except that $x_k = 0$, $y_k = 1$. Write $L = \{i : x_i = 1, i < k\}$ and order L as $m_1 < m_2 < \cdots < m_p (< k)$. We define $x^{(1)}, x^{(2)}, \ldots, x^{(p)}$ in X by: $x^{(l)}$ is x except that $x_i^{(l)} = 0$ for $i = m_1, \ldots, m_l$. Similarly we define $y^{(1)}, \ldots, y^{(p)}$. We get $(x, x^{(1)}) \in \tau_{1,2}^{(m_1)}$, $(x^{(1)}, x^{(2)}) \in \tau_{1,2}^{(m_2)}, \ldots, (x^{(p-1)}, x^{(p)}) \in \tau_{1,2}^{(m_p)}$ and $(x^{(p)}, y^{(p)}) \in \tau_{1,2}^{(k)}$, $(y^{(p)}, y^{(p-1)}) \in \tau_{2,1}^{(m_p)}, \ldots, (y^{(1)}, y)$
$\in \tau_{2,1}^{(m_1)}$. But $(x, y) \in \tau_{i,i+1}^{(k)}$ for some i and what we have seen here is that for this i,

$$\tau_{i,i+1}^{(k)} \subseteq \tau_{1,2}^{(m_1)} \circ \tau_{1,2}^{(m_2)} \circ \cdots \circ \tau_{1,2}^{(m_p)} \circ \tau_{1,2}^{(k)} \circ \tau_{2,1}^{(m_p)} \circ \cdots \circ \tau_{2,1}^{(m_1)}.$$

Since $u_{1,2}^{(m)}$ is either U or VU, we see that for such i, $u_{i,i+1}^{(k)} = W_{i,k}U$ for some diagonal matrix $W_{i,k}$.

Now assume that $(x, y) \in G$ satisfies $\sum^{N-1}(y_i - x_i) \geq 0$ and $x_i = y_i$ for $i \geq N$. Then $(x, y) \in \tau_{i,j}^{(N)}$ and we can find $z^{(0)} = x$, $z^{(1)}$, $z^{(2)}, \ldots, z^{(q)} = y$ such that for each i, $z^{(i)}$ and $z^{(i+1)}$ differ only in one coordinate (so that, as we have seen above, $(z^{(i)}, z^{(i+1)})$ belongs to a G-set τ whose corresponding u is a diagonal multiple of U or of U^*) and in most i's, $z^{(i+1)} \geq z^{(i)}$. This shows that $u_{ij}^{(N)}$ is a diagonal multiple of a non-negative power of U. Hence, if we write $P_1 = \{(x, y) : \Sigma^{N-1}(y_i - x_i) \geq 0$ and $x_i = y_i$ for $i \geq N\}$, then $\tilde{F} \in \operatorname{lat} \pi(a(P_1))$. Since $P_1 \supseteq P$, we are done.

In [29, Example 7.5] it was shown that $R_\infty(c) = \mathbb{Z}$. We shall now show that, in fact, $R_\infty^\mu(c) = \mathbb{Z}$ (with μ as above).

PROPOSITION 8.8. *In the example, $R_\infty^\mu(c) = \mathbb{Z}$.*

PROOF. Firstly, as just noted, $R_\infty(c) = \mathbb{Z}$ by [29, Example 7.5], where, we recall, $R_\infty(c)$ is defined to be the set of all $a \in R$ such that for every $\epsilon > 0$ and every non-empty open set $U \subseteq G^{(0)}$, $(U \times U) \cap c^{-1}(a - \epsilon, a + \epsilon) \neq \emptyset$. It follows that $R_\infty(c) \supseteq R_\infty^\mu(c)$ (as $\mu(U) > 0$ for every non-empty open set $U \subseteq G^{(0)}$). Hence $R_\infty^\mu(c) \subseteq \mathbb{Z}$. To show that $R_\infty^\mu(c) = \mathbb{Z}$, then, it suffices to show that $1 \in R_\infty^\mu(c)$.

Recall [29, Example 7.5] that $c(x, y) = \Sigma(1 + 2^{-n})(y_n - x_n)$ where $x = (x_n)$, $y = (y_n)$. Write $d(x, y) = \Sigma(y_n - x_n)$. Then d is a cocycle (but $d^{-1}(0) \neq G^{(0)}$) and $c - d$ is a coboundary. Hence [28, Lemma 3.2] $R_\infty^\mu(c) = R_\infty^\mu(c) = R_\infty^\mu(d)$ and it is left to show that $1 \in R_\infty^\mu(d)$. For this we have to show that, for every Borel set $Y \subseteq G^{(0)}$ with $\mu(Y) > 0$, $\nu((Y \times Y) \cap d^{-1}(1)) > 0$. It is enough in fact to assume that Y is a G_δ set; that is, $Y = \bigcap_n U_n$ where $U_n \subseteq U_{n-1} \subseteq G^{(0)}$ is open. Let V be an open set that is the support of some diagonal matrix unit; that is, there is some $m \geq 1$ and $v \in \prod_{k=1}^m \{0, 1\}$, $v = (v_k)_{k=1}^m$, such that $V = \{u \in G^0 : u_k = v_k \quad \forall k \leq m\}$. For $u \in V$ there is some $w \in V$ such that $d(u, w) = 1$, except when $u_k = 1$ for every $k > m$ (which is just one point). Since μ is not concentrated on an orbit, for μ-a.e. $u \in V$ there is $w \in V$ such that $d(u, w) = 1$. Thus $\nu(d^{-1}(1) \cap (V \times V)) \geq \mu(V)$.

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Since every open set $U \subseteq G^0$ can be written as a disjoint unit of such $V's$, we have $\nu(d^{-1}(1) \cap (U \times U)) \geq \mu(U)$ for every open subset $U \subseteq G^{(0)}$. For a G_δ set Y as above,

$$\begin{aligned}\nu(d^{-1}(1) \cap (Y \times Y)) &= \nu(d^{-1}(1) \cap (\cap_n U_n \times U_n)) = \nu(\cap_n (d^{-1}(1) \cap (U_n \times U_n))) \\ &= \lim_n \nu(d^{-1}(1) \cap (U_n \times U_n)) \geq \limsup_n \mu(U_n) = \mu(Y) > 0.\end{aligned}$$

This completes the proof.

9. $\mathbb{Z}$-analytic algebras

Let $A = A(P)$ be an analytic subalgebra of $C^*(G, E)$ associated with a faithful $\mathbb{Z}$-valued cocycle c. Then A is said to be *$\mathbb{Z}$-analytic*. For such algebras we have the following.

THEOREM 9.1. *Let $A = A(P)$ be a $\mathbb{Z}$-analytic subalgebra of $B = C^*(G, E)$. Then, for every irreducible representation π of B either* $\operatorname{lat}\pi(A(P)) = \{0, I\}$ *(in which case μ is properly ergodic) or* $\operatorname{lat}\pi(A(P))$ *is a totally atomic nest whose atoms are ordered as one of the orbits and are of rank 1 (in which case μ is concentrated on an orbit).*

PROOF. First note that for a $\mathbb{Z}$-valued faithful cocycle c, $\tilde{R}_\infty(c) \subseteq \{0, \infty\}$. Indeed, assume $m \in \tilde{R}_\infty(c)$ is a positive integer and find $(u, v) \in G$ such that $c(u, v) = m$. Since $c^{-1}(\{m\})$ is a (closed and) open set, we find an open G set, τ, containing (u, v) such that $c|\tau \equiv m$. Since $u \neq v$ we can take τ such that $r(\tau) \cap s(\tau) = \emptyset$. Since $m \in \tilde{R}_\infty(c)$, there is a pair (x, y) in $r(\tau) \times r(\tau)$ such that $c(x, y) = m$. But then $c(x, y) = c(x, \tau(x))$ and $y \neq \tau(x)$ (as $y \in r(\tau)$, $\tau(x) \in s(\tau)$). This is impossible, since c is faithful.

For every cocycle and every Borel measure μ we have $\tilde{R}^\mu_\infty(c) \subseteq \tilde{R}_\infty(c)$; hence here $\tilde{R}^\mu_\infty(c) \subseteq \{0, \infty\}$ and there are two possibilities: either $\tilde{R}^\mu_\infty(c) = \{0\}$ or $\tilde{R}^\mu_\infty(c) = \{0, \infty\}$. In the latter case we know (Theorem 4.1) that $\operatorname{lat}\pi(A(P)) = \{0, I\}$. So we now assume $\tilde{R}^\mu_\infty(c) = \{0\}$. Then c is a coboundary. In fact, it is a coboundary as a $\mathbb{Z}$-valued cocycle and thus, there is a Borel function $g : X \to \mathbb{Z}$ such that for ν-a.e. $(u, v) \in G$, $c(u, v) = g(v) - g(u)$ [28, Theorem 3.9(4)]. Since $\{g^{-1}(\{m\})\}_m$ is a countable partition of $G^{(0)}$, there are some $m \in \mathbb{Z}$ with $\mu(g^{-1}(\{m\})) > 0$. Write F for $g^{-1}(\{m\})$. Then on $G \cap (F \times F)$, $c = 0$, ν- a.e., but c is faithful and μ is ergodic so this can happen only if F is a singleton. Hence μ has an atom and, by ergodicity, μ is concentrated on an equivalence class, say $[u]$, and $\mu(\{v\}) > 0$ for every $v \in [u]$ since μ is quasi-invariant. Hence π is a representation on $\int_{G^{(0)}} K(w)d\mu(w) = \sum^{\oplus}_{w\in[u]} K(w)$

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and the irreducibility of π implies that dim $K(w) = 1$ for every $w = [u]$. The projections in lat $\pi(A(P))$ are precisely the projections onto subspaces of the form $\sum_{w \in M}^{\oplus} K(w)$ where $M \subseteq [u]$ is a decreasing set. This completes the proof.

REMARK. Theorem 9.1 generalizes both Theorem III.2.1 and Proposition III.3.2 of [19]. Note that, in addition to the fact that we do not assume here that the groupoid is AF, we also do not assume that π is masa preserving (as in [19, Proposition III.3.2]) and we do not restrict ourselves to the standard embedding algebra (as in [19, Theorem III.2.1]).

From Theorem 9.1 we conclude:

COROLLARY 9.2. *Let A be a $\mathbb{Z}$-analytic subalgebra of $B = C^*(G, E)$ and let ρ be a representation of A with an irreducible C^*-dilation π. Then, either ρ is the compression of* $\operatorname{Ind} \epsilon_u$ *to a subinterval of the equivalence class $[u]$ of some $u \in G^{(0)}$ or ρ is the restriction of π to A.*

If G is the transformation group groupoid determined by an action of $\mathbb{Z}$ on a locally compact space X and if c is the position cocycle of this action, so that $c(x, n) = n$, then the $\mathbb{Z}$-analytic subalgebra A of $B = C^*(G) = C^*(X, \mathbb{Z})$ determined by c is nothing but the analytic crossed product determined by the action of $\mathbb{Z}$. Our analysis, then, recaptures, extends, and explains the results in [1] and [4]. In these papers, ergodicity is used to show that certain special representations of A lead to transitive algebras. Our results show that for every irreducible representation π of B where the associated measure is not concentrated on an orbit, the algebra $\pi(A)$ is transitive.

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