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NO COHOMOLOGICALLY TRIVIAL NONTRIVIAL AUTOMORPHISM OF GENERALIZED KUMMER MANIFOLDS

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Dedicated to Professor Tomohide Terasoma on the occasion of his sixtieth birthday

Abstract. For a hyper-Kähler manifold deformation equivalent to a generalized Kummer manifold, we prove that the action of the automorphism group on the total Betti cohomology group is faithful. This is a sort of generalization of a work of Beauville and a more recent work of Boissière, Nieper-Wisskirchen, and Sarti, concerning the action of the automorphism group of a generalized Kummer manifold on the second cohomology group.

§1. Introduction

Throughout this note, we work over $\mathbb{C}$. Our main result is Theorem 1.3.

The global Torelli theorem for K3 surfaces ([PS71, BR75], see also [BHPV04]) says that the contravariant action

$$\rho_2 : \mathrm{Aut}(S) \to \mathrm{GL}(H^2(S, \mathbb{Z})); \quad g \mapsto g^*|_{H^2(S,\mathbb{Z})}$$

is faithful for any K3 surface S. On the other hand, Dolgachev [Do84, 4.4] and Mukai and Namikawa [MN84, Mu10] show that there are Enriques surfaces E such that the action $\rho_2 : \mathrm{Aut}(E) \to \mathrm{GL}(H^2(E, \mathbb{Z}))$ is not faithful. Here and hereafter, we denote

$$\mathrm{GL}(L) := \mathrm{Aut}_{\mathrm{group}}(L)$$

for a finitely generated abelian group L, possibly with nontrivial torsion.

Throughout this note, by a hyper-Kähler manifold, we mean a simply connected compact Kähler manifold M admitting an everywhere nondegenerate global holomorphic 2-form ω_M such that $H^0(M, \Omega^2_M) = \mathbb{C}\omega_M$. Standard

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examples of hyper-Kähler manifolds are the Hilbert scheme $\mathrm{Hilb}^n(S)$ of 0-dimensional closed subschemes of length n on a K3 surface S, the generalized Kummer manifold $K_{n-1}(A)$, of dimension $2(n-1) \geqslant 4$, associated to a 2-dimensional complex torus A, and their deformations ([Be83, Sections 6, 7], see also Section 2).

Beauville [Be83-2, Propositions 9, 10] considered a similar question for hyper-Kähler manifolds and found the following.

THEOREM 1.1.

(1) *The action* $\rho_2 : \mathrm{Aut}(\mathrm{Hilb}^n(S)) \to \mathrm{GL}(H^2(\mathrm{Hilb}^n(S), \mathbb{Z}))$ *is faithful.*
(2) *The action* $\rho_2 : \mathrm{Aut}(K_{n-1}(A)) \to \mathrm{GL}(H^2(K_{n-1}(A), \mathbb{Z}))$ *is not faithful. More precisely,* $T(n) \subset \mathrm{Ker}\,\rho_2$.

Here $T(n) \simeq (\mathbb{Z}/n)^{\oplus 4}$ is the group of automorphisms induced by the group of n-torsion points $T[n] := \{a \in A | na = 0\}$ of $A = \mathrm{Aut}^0(A)$.

It is natural and interesting to determine $\mathrm{Ker}\,\rho_2$ in Theorem 1.1(2). In this direction, Boissière, Nieper-Wisskirchen, and Sarti [BNS11, Theorem 3, Corollary 5(2)] found the following complete answer.

THEOREM 1.2. $\mathrm{Ker}(\rho_2 : \mathrm{Aut}(K_{n-1}(A)) \to \mathrm{GL}(H^2(K_{n-1}(A), \mathbb{Z}))) = T(n) \rtimes \langle \iota \rangle$.

Here ι is the automorphism induced by the inversion -1 of A and $T(n) \rtimes \langle \iota \rangle$ is the semidirect product of $T(n)$ and $\langle \iota \rangle$, in which $T(n)$ is normal.

It is also natural and interesting to ask if the action of $\mathrm{Aut}(K_{n-1}(A))$ on the total cohomology group $H^*(K_{n-1}(A), \mathbb{Z}) := \bigoplus_{k=0}^{4(n-1)} H^k(K_{n-1}(A), \mathbb{Z})$ is faithful or not.

Our aim is to answer this question in a slightly more generalized form.

THEOREM 1.3. *The action* $\rho : \mathrm{Aut}(Y) \to \mathrm{GL}(H^*(Y, \mathbb{Z}))$ *is faithful for any hyper-Kähler manifold* Y *deformation equivalent to* $K_{n-1}(A)$.

First we prove Theorem 1.3 for $K_{n-1}(A)$. By Theorem 1.2, it suffices to show that $g^*|_{H^*(K_{n-1}(A), \mathbb{C})} \neq \mathrm{id}$ for each $g \in (T(n) \rtimes \langle \iota \rangle) \setminus \{\mathrm{id}\}$. This is checked in Section 3. We then prove Theorem 1.3 for any Y in Section 4, by using the density result due to Markman and Mehrotra [MM17]. In Section 5, among other things, we remark a similar result for deformation of the Hilbert scheme of a K3 surface (Theorem 5.1).

After posting this note on ArXiv (on 2012), Professor Y. Tschinkel kindly informed me that the action $T(n) \rtimes \langle \iota \rangle$ on $K_{n-1}(A)$ extends to a faithful action on any deformation Y of $K_{n-1}(A)$, in such a way that the extended action is trivial on $H^2(Y, \mathbb{Z})$ [HT13, Theorem 2.1, Proposition 3.1]. In particular, this shows that the action

$$\rho_2 : \mathrm{Aut}(Y) \to \mathrm{GL}(H^2(Y, \mathbb{Z}))$$

is not faithful even if Y is generic.

I should also mention that Theorem 1.3 is much motivated by the following question asked by Professor D. McDuff to me at the conference in Banff (July 2012), while the question itself is still completely open.

QUESTION 1.4. Is there an example of a compact Kähler manifold M such that the biholomorphic automorphism group is discrete, that is, $\mathrm{Aut}^0(M) = \{\mathrm{id}_M\}$, but with a biholomorphic automorphism $g \neq \mathrm{id}_M$ being homotopic to id_M in the group of diffeomorphisms?

§2. Preliminaries

In this section, we mainly fix notations we shall use. We follow [Be83] and [Be83-2]. So, $K_n(A)$ in [BNS11] is $K_{n-1}(A)$ in this note.

We refer to [Be83, Section 7] and [GHJ03, Part III] for more details on generalized Kummer manifolds and basic properties on hyper-Kähler manifolds.

Let A be a 2-dimensional complex torus and let n be an integer such that $n \geqslant 3$. Let $\mathrm{Hilb}^n(A)$ be the Hilbert scheme of 0-dimensional closed subschemes of A of length n. Then $\mathrm{Hilb}^n(A)$ is a smooth Kähler manifold of dimension $2n$. Let

$$\nu = \nu_A : \mathrm{Hilb}^n(A) \to \mathrm{Sym}^n(A) = A^n/S_n$$

be the Hilbert–Chow morphism. We denote the sum as 0-cycles by $\oplus$ and the sum in A by $+$. Then each element of $\mathrm{Sym}^n(A)$ is of the form

$$\bigoplus_{i=1}^{k} x_i^{\oplus m_i}.$$

Here, x_i are distinct points on A and m_i are positive integers such that $\sum_{i=1}^{k} m_i = n$. We have the following surjective morphism

$$s := s_A : \mathrm{Sym}^n(A) \to A; \qquad \bigoplus_{i=1}^{k} x_i^{\oplus m_i} \mapsto \sum_{i=1}^{k} m_i x_i.$$

The generalized Kummer manifold $K_{n-1}(A)$ is defined by

$$K_{n-1}(A) := (s \circ \nu)^{-1}(0).$$

Note that the morphism

$$s \circ \nu = s_A \circ \nu_A : \mathrm{Hilb}^n(A) \to A$$

is a smooth surjective morphism such that all fibers are isomorphic [Be83, Section 7]. So, $K_{n-1}(A)$ is isomorphic to any fiber of $s_A \circ \nu_A$. One can also describe $K_{n-1}(A)$ in a slightly different way, as follows. Let

$$A(n-1) := \left\{ (P_1, P_2, \ldots, P_n) \in A^n \;\middle|\; \sum_{i=1}^{n} P_i = 0 \right\}.$$

Then $A(n-1)$ is a closed submanifold of A^n and $A(n-1) \simeq A^{n-1}$. Moreover, $A(n-1)$ is stable under the action of S_n on A^n and

$$\mathrm{Sym}^n(A) \supset A^{(n-1)} := s^{-1}(0) = A(n-1)/S_n.$$

From this, we deduce that

$$K_{n-1}(A) = \nu^{-1}(A^{(n-1)}) = \mathrm{Hilb}^n(A) \times_{\mathrm{Sym}^n(A)} A^{(n-1)}.$$

Recall that $\dim \mathrm{Def}(A) = 4$, while $\dim \mathrm{Def}(K_{n-1}(A)) = 5$ for $n \geqslant 3$ and any local deformation of a hyper-Kähler manifold is a hyper-Kähler manifold [Be83, Section 7]. So, there are hyper-Kähler manifolds which are deformation equivalent to $K_{n-1}(A)$ but are not isomorphic to any generalized Kummer manifold.

From now until the end of this note, we denote by $X := K_{n-1}(A)$ $(n \geqslant 3)$ the generalized Kummer manifold, of dimension $2(n-1)$, associated to a 2-dimensional complex torus A and by $K := T(n) \lhd \langle \iota \rangle$ the subgroup of $\mathrm{Aut}(K_{n-1}(A))$ defined in the Introduction. We also use the notations introduced in this section freely in the remaining sections.

§3. Proof of Theorem 1.3 for $K_{n-1}(A)$

In this section, we prove Theorem 1.3 for $K_{n-1}(A)$.

First we prove the following.

PROPOSITION 3.1. *Let $g \in K \setminus T(n)$. Then $g^*|_{H^3(X,\mathbb{C})} \neq \mathrm{id}$.*

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Proof. Let (z_i^1, z_i^2) $(1 \leqslant i \leqslant n)$ be the standard global coordinates of the universal cover $\mathbb{C}_i^2$ of the ith factor $A_i = A$ of A^n. Then the universal cover $\mathbb{C}^{2(n-1)}$ of $A(n-1) \simeq A^{n-1}$ is a closed submanifold of $\mathbb{C}^{2n}$ defined by

$$z_1^1 + z_2^1 + \cdots + z_{n-1}^1 + z_n^1 = 0, \qquad z_1^2 + z_2^2 + \cdots + z_{n-1}^2 + z_n^2 = 0. \tag{3.1}$$

In particular, (z_i^1, z_i^2) $(1 \leqslant i \leqslant n-1)$ give the global coordinates of the universal cover $\mathbb{C}^{2(n-1)}$ of $A(n-1)$. Note that 1-forms dz_i^1 and dz_i^2 $(1 \leqslant i \leqslant n)$ can be regarded as global 1-forms on $A(n-1)$. They satisfy

$$dz_1^1 + dz_2^1 + \cdots + dz_{n-1}^1 + dz_n^1 = 0, \qquad dz_1^2 + dz_2^2 + \cdots + dz_{n-1}^2 + dz_n^2 = 0, \tag{3.2}$$

and $\{dz_i^1, dz_i^2 (1 \leqslant i \leqslant n-1)\}$ forms a basis of the space of global holomorphic 1-forms on $A(n-1) \simeq A^{n-1}$. Consider the following global $(2,1)$-form $\tilde{\tau}$ on $A(n-1)$:

$$\tilde{\tau} = dz_1^1 \wedge dz_1^2 \wedge d\overline{z}_1^2 + \cdots + dz_{n-1}^1 \wedge dz_{n-1}^2 \wedge d\overline{z}_{n-1}^2 + dz_n^1 \wedge dz_n^2 \wedge d\overline{z}_n^2.$$

LEMMA 3.2. *$\tilde{\tau}$ descends to a nonzero element τ of $H^{2,1}(X)$.*

Proof. Recall that, for compact Kähler orbifolds, the Hodge decomposition is pure and the Hodge theory works in the same way as smooth compact manifolds [St77].

Since $\tilde{\tau}$ is S_n-invariant, it descends to a global $(2,1)$-form, say $\overline{\tau}$, on the compact Kähler orbifold $A^{(n-1)}$. Then $\tau = (\nu|_X)^*\overline{\tau} \in H^{2,1}(X)$ under $\nu|_X : X \to A^{(n-1)}$. It remains to show that $\tau \neq 0$. Since $(\nu|_X)^*$ is injective, it suffices to show that $\overline{\tau} \neq 0$ in $H^{2,1}(A^{(n-1)})$. For this, it suffices to show that $\tilde{\tau} \neq 0$ in $H^{2,1}(A(n-1))$, as q^* is also injective for the quotient map $q : A(n-1) \to A^{(n-1)}$. By Equation (3.2), we have

$$\begin{aligned}\tilde{\tau} &= dz_1^1 \wedge dz_1^2 \wedge d\overline{z}_1^2 + \cdots + dz_{n-1}^1 \wedge dz_{n-1}^2 \wedge d\overline{z}_{n-1}^2 \\ &\quad - \left(\sum_{k=1}^{n-1} dz_k^1\right) \wedge \left(\sum_{k=1}^{n-1} dz_k^2\right) \wedge \left(\sum_{k=1}^{n-1} d\overline{z}_k^2\right).\end{aligned}$$

This is the expression of $\tilde{\tau}$ in terms of the standard basis of $H^{2,1}(A(n-1))$. As $n-1 \geqslant 2$, the term

$$dz_1^1 \wedge dz_2^2 \wedge d\overline{z}_2^2$$

appears with coefficient -1 in this expression. Hence $\tilde{\tau} \neq 0$ in $H^{2,1}(A(n-1))$. This proves Lemma 3.2. ∎

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LEMMA 3.3. *Let $g \in K \setminus T(n)$ and $\tau \in H^{2,1}(X)$ be as in Lemma 3.2. Then $g^*\tau = -\tau$. In particular, $g^*|_{H^3(X,\mathbb{C})} \neq \mathrm{id}$.*

Proof. The automorphism g acts equivariantly on $A(n-1) \to A^{(n-1)} \leftarrow X$. For ι, hence for $g \in K \setminus T(n)$, we have

$$\iota^* dz_i^q = -dz_i^q, \qquad g^* dz_i^q = -dz_i^q \ (1 \leqslant i \leqslant n, q = 1, 2).$$

Hence $g^*\tilde{\tau} = -\tilde{\tau}$ by the shape of $\tilde{\tau}$. Thus $g^*\tau = -\tau$. By Lemma 3.2, $\tau \neq 0$ in $H^{2,1}(X)$. Hence $g^*|_{H^3(X,\mathbb{C})} \neq \mathrm{id}$ as claimed. □

Lemma 3.3 completes the proof of Proposition 3.1. □

Next we prove the following.

PROPOSITION 3.4. *Let $a \in T(n) \setminus \{\mathrm{id}\}$. Then $a^*|_{H^*(X,\mathbb{C})} \neq \mathrm{id}$.*

Proof. Let $a \in T(n) \simeq (\mathbb{Z}/n\mathbb{Z})^{\oplus 4}$ be an element of order $p \neq 1$ (p is *not* necessarily a prime number). Set $d = n/p$. Then d is a positive integer such that $d < n$. We freely regard a also as a torsion element of order p in A and automorphisms of various spaces which are naturally and equivariantly induced by the translation automorphism $x \mapsto x + a$ of A.

We will show first Lemma 3.5, Theorem 3.6 and Lemma 3.7, and then we will conclude the proof of Proposition 3.4.

LEMMA 3.5. *The fixed locus X^a consists of p^3 connected components F_i $(1 \leqslant i \leqslant p^3)$. Moreover, each F_i is isomorphic to the generalized Kummer manifold $K_{d-1}(A/\langle a\rangle)$ associated to the 2-dimensional complex torus $A/\langle a\rangle$.*

Proof. Let $\mathcal{S} \subset A$ be a 0-dimensional closed subscheme of length n. As $\langle a\rangle$ acts freely on A, the quotient map $\pi : A \to A/\langle a\rangle$ is étale of degree p. It follows that $a_*\mathcal{S} = \mathcal{S}$ if and only if there is a 0-dimensional closed subscheme $\mathcal{T} \subset A/\langle a\rangle$ of length $d = n/p$ such that $\mathcal{S} = \pi^*\mathcal{T}$. This $\mathcal{T}$ is clearly unique and we obtain an isomorphism

$$\mathrm{Hilb}^d(A/\langle a\rangle) \simeq (\mathrm{Hilb}^n(A))^a; \qquad \mathcal{T} \mapsto \pi^*\mathcal{T}. \tag{3.3}$$

Let $\mathcal{S} \in (\mathrm{Hilb}^n(A))^a$. Then $\nu(\mathcal{S}) \in (\mathrm{Sym}^n(A))^a$ as well, and $\nu(\mathcal{S})$ is then of the form

$$\nu(\mathcal{S}) = \bigoplus_{i=1}^{k} \bigoplus_{j=0}^{p-1} (x_i + ja)^{\oplus m_i}$$

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(and *vice versa*). Here $\sum_{i=1}^{k} m_i = d$ and all points $x_i + ja$ are distinct. Observe also that

$$K_{n-1}(A)^a = (\mathrm{Hilb}^n(A))^a \cap K_{n-1}(A).$$

As $\mathcal{S} \in (\mathrm{Hilb}^n(A))^a$ by our choice of $\mathcal{S}$, it follows from the equality above that $\mathcal{S} \in K_{n-1}(A)^a$ if and only if $\mathcal{S} \in K_{n-1}(A)$, that is, $\mathcal{S} \in (\mathrm{Hilb}^n(A))^a$ satisfies (by the definition of $K_{n-1}(A)$ and by the shape of $\nu(\mathcal{S})$) that

$$p(m_1x_1 + m_2x_2 + \cdots + m_kx_k + \alpha) = 0 \tag{3.4}$$

in A. Here $n(p-1)/2 \in \mathbb{Z}$ and $\alpha \in A$ is an element such that

$$p\alpha = (n(p-1)/2)a$$

in A. We choose and fix such α.

Let $A[p]$ be the group of p-torsion points of A. Then, Equation (3.4) is equivalent to

$$m_1x_1 + m_2x_2 + \cdots + m_kx_k + \alpha \in A[p]. \tag{3.5}$$

Since a is also a p-torsion point, Equation (3.5) is also equivalent to

$$m_1\pi(x_1) + m_2\pi(x_2) + \cdots + m_k\pi(x_k) + \pi(\alpha) \in \pi(A[p]) = A[p]/\langle a\rangle. \tag{3.6}$$

Write $\mathcal{S} = \pi^*\mathcal{T}$. Then, Equation (3.6) holds if and only if $\mathcal{T}$ is in the fibers of

$$s_{A/\langle a\rangle} \circ \nu_{A/\langle a\rangle} : \mathrm{Hilb}^d(A/\langle a\rangle) \to A/\langle a\rangle$$

over $\pi(A[p])$. We have $|\pi(A[p])| = p^3$, as a is also p-torsion. Hence, by the isomorphism (3.3), the fixed locus $K_{n-1}(A)^a$ is isomorphic to the union of p^3 fibers of $s_{A/\langle a\rangle} \circ \nu_{A/\langle a\rangle}$ and each fiber is isomorphic to $K_{d-1}(A/\langle a\rangle)$ as remarked in Section 2. This completes the proof of Lemma 3.5. □

Set

$$\sigma(n) = \sum_{1 \leqslant b | n} b,$$

the sum of all positive divisors of a positive integer n. The following fundamental result due to Göttche and Soergel ([GS93, Corollary 1], see also [Go94, De10]) is crucial in our proof.

THEOREM 3.6. *The topological Euler number $\chi_{\mathrm{top}}(K_{n-1}(A))$ of $K_{n-1}(A)$ is $n^3\sigma(n)$. (This is also valid for $n = 1, 2$.)*

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Now we consider the Lefschetz number of $h \in \mathrm{Aut}(X)$:

$$L(h) := \sum_{k=0}^{4(n-1)} (-1)^k \operatorname{tr} h^*|_{H^k(X,\mathbb{C})}.$$

LEMMA 3.7.

(1) *If $h \in \mathrm{Aut}(X)$ is cohomologically trivial, then $L(h) = n^3\sigma(n)$.*
(2) *$L(a) = n^3\sigma(d)$ for any element a of order p in $T(n) \setminus \{\mathrm{id}\}$ with $d = n/p$.*

Proof. If h is cohomologically trivial, then $\operatorname{tr} h^*|_{H^k(X,\mathbb{C})} = b_k(X)$. This implies (1). By the topological Lefschetz fixed point formula, Lemma 3.5 and Theorem 3.6, we obtain

$$L(a) = \chi_{\mathrm{top}}(X^a) = p^3\chi_{\mathrm{top}}(K_{d-1}(A/\langle a\rangle)) = p^3 \cdot d^3\sigma(d) = n^3\sigma(d).$$

This is nothing but the assertion (2). This proves Lemma 3.7. □

Since $d|n$ and $d \neq n$, it follows that

$$\sigma(d) \leqslant \sigma(n) - n < \sigma(n).$$

Hence $a \in T(n) \setminus \{\mathrm{id}\}$ is not cohomologically trivial by Lemma 3.7. This proves Proposition 3.4. □

Theorem 1.3 for $K_{n-1}(A)$ now follows from Theorem 1.2, Propositions 3.1 and 3.4. This completes the proof of Theorem 1.3 for $K_{n-1}(A)$.

§4. Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3 for any Y.

Let $\Lambda = (\Lambda, (*, **))$ be a fixed abstract lattice isometric to $(H^2(K_{n-1}(A), \mathbb{Z}), b)$. Here b is the Beauville–Bogomolov form of $K_{n-1}(A)$ (see e.g. [GHJ03, Example 23.20]).

Let Y be a hyper-Kähler manifold deformation equivalent to a generalized Kummer manifold $X = K_{n-1}(A)$. Let $g \in \mathrm{Aut}(Y)$ such that $g^*|_{H^*(Y,\mathbb{Z})} = \mathrm{id}$. We are going to show that $g = \mathrm{id}_Y$.

Let $\mathcal{M}^0$ be the connected component of the marked moduli space of $\mathcal{M}_\Lambda$, containing (Y, η). Here $\eta : H^2(Y, \mathbb{Z}) \to \Lambda$ is a marking. Huybrechts constructed the marked moduli space $\mathcal{M}_\Lambda$ [Hu99, 1.18] by patching Kuranishi spaces via local Torelli theorem for hyper-Kähler manifolds ([Be83, Theorem 5], [GHJ03, 25.2]). By construction, $\mathcal{M}_\Lambda$ is smooth, but highly non-Hausdorff. He also showed that the period map

$$p : \mathcal{M}^0 \to \mathcal{D} = \{[\omega] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | (\omega, \omega) = 0, (\omega, \overline{\omega}) > 0\}$$

is a surjective holomorphic map of degree 1 ([Hu99, Theorem 8.1], see also [Ve13, Hu12] for degree and further development). Let $[\omega] \in \mathcal{D}$. If $p^{-1}([\omega])$ $(\subset \mathcal{M}^0)$ is not a single point, then $p^{-1}([\omega])$ consists of points, being mutually inseparable, corresponding to birational hyper-Kähler manifolds [Hu99, Theorem 8.1].

By using the Hodge theoretic Torelli type theorem [Ma11], Markman and Mehrotra [MM17, Theorem 4.1] proved that the marked generalized Kummer manifolds are dense in $\mathcal{M}^0$. Actually they proved the following stronger density result:

THEOREM 4.1. *There is a dense subset $\mathcal{D}' \subset \mathcal{D}$ such that if $[\omega] \in \mathcal{D}'$, then any point of $p^{-1}([\omega])$ corresponds to a marked generalized Kummer manifold.*

Consider the Kuranishi family $u : \mathcal{U} \to \mathcal{K}$ of Y. Here and hereafter we freely shrink $\mathcal{K}$ around $0 = [Y]$. Since the Kuranishi family is universal, $g \in \mathrm{Aut}(Y)$ induces automorphisms $\tilde{g} \in \mathrm{Aut}(\mathcal{U})$ and $\overline{g} \in \mathrm{Aut}(\mathcal{K})$ such that $u \circ \tilde{g} = \overline{g} \circ u$ and $\tilde{g}|Y = g$. Since $\mathcal{K}$ is locally isomorphic to $\mathcal{D}$ by the local Torelli theorem, the locus

$$\mathcal{K}' \subset \mathcal{K},$$

consisting of the point t such that $u^{-1}(t)$ is a generalized Kummer manifold, is dense in $\mathcal{K}$. This is a direct consequence of Theorem 4.1 and the construction of $\mathcal{M}^0$ explained above. *Here we also emphasize that the density in $\mathcal{M}^0$ is not sufficient to conclude this.*

From now, we follow Beauville's argument [Be83-2, proof of Proposition 10].

Let T_Y be the tangent bundle of Y. Then, one can take $\mathcal{K}$ as a small polydisk in $H^1(Y, T_Y)$ with center 0. As ω_Y is everywhere nondegenerate, we have an isomorphism

$$H^1(Y, T_Y) \simeq H^1(X, \Omega_Y^1)$$

induced by the isomorphism

$$T_Y \simeq \Omega_Y^1 = T_Y^* : v \mapsto \omega_Y(v, *).$$

As g is cohomologically trivial and $H^{2,0}(Y) = H^0(Y, \Omega_Y^2) = \mathbb{C}\omega_Y$, we have $g^*\omega_Y = \omega_Y$ and $g^*|_{H^1(Y,\Omega_Y^1)} = \mathrm{id}$. Hence by the isomorphism above, we obtain $g^*|_{H^1(Y,T_Y)} = \mathrm{id}$, and therefore, $\overline{g} = \mathrm{id}_{\mathcal{K}}$.

Let $t \in \mathcal{K}$ be any point of $\mathcal{K}$. Then, by $\overline{g} = \mathrm{id}_{\mathcal{K}}$, the morphism $\tilde{g}$ preserves the fiber $Y_t = u^{-1}(t)$, that is,

$$\tilde{g}|_{Y_t} \in \mathrm{Aut}(Y_t).$$

Put $g_t := \tilde{g}|_{Y_t}$. Then g_t is also cohomologically trivial, because $g_t^*|_{H^*(Y_t,\mathbb{Z})}$ is derived from the action of $\tilde{g}$ on the constant system $\bigoplus_{k=0}^{4(n-1)} R^k u_* \mathbb{Z}$. Then $g_t = \mathrm{id}_{Y_t}$ for all $t \in \mathcal{K}'$, as we already proved Theorem 1.3 for $K_{n-1}(A)$ in Section 3. Since $\mathcal{U}$ is Hausdorff and $\tilde{g}$ is continuous, it follows that $\tilde{g} = \mathrm{id}_{\mathcal{U}}$. Hence $g = g_0 = \mathrm{id}_Y$ as well. This completes the proof of Theorem 1.3.

§5. A few concluding remarks

In this section, we remark a few relevant facts, which should be known to some experts.

Our first remark is about an analogue of Theorem 1.3 for a hyper-Kähler manifold deformation equivalent to the Hilbert scheme $\mathrm{Hilb}^n(S)$ of a K3 surface S.

Markman and Mehrotra [MM17, Theorem 1.1] also proved the strong density result for $\mathrm{Hilb}^n(S)$ of K3 surfaces S. So, the same argument as in Section 4 together with Beauville's result (Theorem 1.1(1)) implies the following result due to Mongardi [Mo13, Lemma 1.2]:

THEOREM 5.1. *Let W be a hyper-Kähler manifold deformation equivalent to* $\mathrm{Hilb}^n(S)$*. Then, the action $\rho_2 : \mathrm{Aut}(W) \to \mathrm{GL}(H^2(W, \mathbb{Z}))$ is faithful.*

Our second remark is about the fixed locus of symplectic automorphism of finite order.

In Lemma 3.5, we described the fixed locus X^a. Our description shows that X^a is a disjoint union of smooth hyper-Kähler manifolds. However, this is not accidental:

PROPOSITION 5.2. *Let (M, ω_M) be a holomorphic symplectic manifold of dimension $2d$, that is, M is a compact Kähler manifold and ω_M is an everywhere nondegenerate holomorphic 2-form on M (not necessarily unique up to $\mathbb{C}^\times$). Let $h \in \mathrm{Aut}(M)$ such that $h^*\omega_M = \omega_M$ and h is of finite order m. Let F be a connected component of the fixed locus $M^h = \{P \in M | h(P) = P\}$. Then $(F, \omega_M|_F)$ is a holomorphic symplectic manifold (possibly a point).*

Proof. F is isomorphic to the intersection of the graph of h and the diagonal Δ in $M \times M$. So it is compact and Kähler, possibly singular. Let $P \in F$. Since h is of finite order, h is locally linearizable at P (see the proof of [Ka84, Lemma 1.3]). That is, there are local coordinates $(y_1, y_2, \ldots, y_{2d})$ at P such that

$$h^*y_i = y_i \ (1 \leqslant \forall i \leqslant r), \qquad h^*y_j = c_j y_j \ (r+1 \leqslant \forall j \leqslant 2d). \tag{5.1}$$

Here $c_j \neq 1$ and satisfies $c_j^m = 1$. Then F is locally defined by $y_j = 0$ $(r+1 \leqslant j \leqslant 2d)$ in M. Hence F is smooth. Consider the linear differential map

$$dh_P : T_{M,P} \to T_{M,P}$$

of the tangent space $T_{M,P}$ of M at P. By Equation (5.1), we have the decomposition

$$T_{M,P} = T_{F,P} \oplus N. \tag{5.2}$$

Here, $N = \bigoplus_{j=r+1}^{2d} \mathbb{C}v_j$ for some v_j $(r+1 \leqslant j \leqslant 2d)$ such that $dh_P(v_j) = c_j^{-1}v_j$ and the tangent space $T_{F,P}$ of F at P is exactly the invariant subspace $(T_{M,P})^{dh_P}$. Using $h^*\omega_M = \omega_M$, we deduce that

$$\omega_{M,P}(v, v_j) = \omega_{M,P}(dh_P(v), dh_P(v_j)) = \omega_{M,P}(v, c_j^{-1}v_j) = c_j^{-1}\omega_{M,P}(v, v_j),$$

for any $v \in T_{F,P}$ and v_j $(r+1 \leqslant j \leqslant 2d)$. As $c_j \neq 1$, it follows that

$$\omega_{M,P}(v, v_j) = 0$$

for all $v \in T_{F,P}$ and v_j with $r+1 \leqslant j \leqslant 2d$. Hence, the decomposition (5.2) is orthogonal with respect to $\omega_{M,P}$. As $\omega_{M,P}$ is nondegenerate, it follows from the orthogonality of the decomposition that $\omega_{M,P}|_{T_{F,P}}$ is also nondegenerate on $T_{F,P}$ (possibly $\{0\}$). Hence $(F, \omega|_F)$ is a smooth symplectic manifold (possibly a point) as well. This completes the proof of Proposition 5.2. □

REMARK 5.3. Proposition 5.2 is a formal generalization of a result of Camere [Ca12, Proposition 3] for a symplectic involution.

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