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ON AN F-ALGEBRA OF HOLOMORPHIC FUNCTIONS

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0. Introduction. The complex maximal theorem of Hardy and Littlewood states:

(M_p). For $0 < p < \infty$, there exists a positive constant C_p such that if f is holomorphic in the unit disc U of the complex plane then

$$\int_0^{2\pi} Mf(\theta)^p \frac{d\theta}{2\pi} \leqq C_p \sup_{0 \leqq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi},$$

where

$$Mf(\theta) = \sup_{0 \leqq r < 1} |f(re^{i\theta})|.$$

The corresponding statement to the limiting case $p = 0$ can be stated as follows:

(M_0) There exists a positive constant C_0 such that if f is holomorphic in U

$$\int_0^{2\pi} \log^+ Mf(\theta) \frac{d\theta}{2\pi} \leqq C_0 \sup_{0 \leqq r < 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi},$$

where $\log^+ t = \max(\log t, 0)$.

The statement (M_0) is false as the following example shows.

Example. Consider

$$f(z) = \exp\left(\frac{1 + z}{1 - z}\right).$$

By a routine calculation we see that

$$\log^+ |f(re^{i\theta})| = \frac{1 - r^2}{1 - 2r \cos \theta + r^2},$$

and

$$\log^+ Mf(\theta) = \begin{cases} \dfrac{1}{|\sin \theta|}, & |\theta| \leqq \dfrac{\pi}{2}, \\ 1, & \dfrac{\pi}{2} \leqq |\theta| \leqq \pi. \end{cases}$$

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Therefore we have

$$\sup_{0 \leqq r < 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = 1,$$

but

$$\int_0^{2\pi} \log^+ Mf(\theta) \frac{d\theta}{2\pi} \geqq 2 \int_0^{\pi/2} \frac{1}{\sin\theta} \frac{d\theta}{2\pi} = \infty.$$

In this connection, we consider the following three successively stronger conditions on the functions f holomorphic in U:

(a) $\log^+|f_r|$, $(0 \leqq r < 1)$, are bounded in $L^1(T)$, where

$$f_r(e^{i\theta}) = f(re^{i\theta})$$

and T is the boundary of U.

(b) $\log^+|f_r|$, $(0 \leqq r < 1)$, are uniformly integrable, that is,

(i) $\log^+|f_r|$, $(0 \leqq r < 1)$, are bounded in $L^1(T)$, and

(ii) given $\epsilon > 0$, there exists a $\delta > 0$ so that

$$\int_E \log^+ |f_r(e^{i\theta})| \frac{d\theta}{2\pi} < \epsilon, \ (0 \leqq r < 1),$$

whenever $E \subset T$ with its Lebesgue measure $|E| < \delta$.

(c) $\log^+|f_r|$, $(0 \leqq r < 1)$, have an $L^1(T)$-majorant, or equivalently,

$$\int_0^{2\pi} \log^+ Mf(\theta) \frac{d\theta}{2\pi} < \infty.$$

The Nevanlinna class N is the class of all functions f holomorphic in U satisfying the condition (a), The Smirnov class N^+ is that of all functions f holomorphic in U satisfying (b). We denote by M the class of all functions f holomorphic in U satisfying the condition (c). The study on the classes N and N^+ has been well established (see **[3]**, **[4]**, **[8]**, **[9]** etc).

The study of the class M was suggested to the author by Professor P. R. Ahern, to whom the author wishes to express his sincere gratitude.

In Section 2, the containment relations with other classes and some relations with the class Re H^1 are given. In Sections 3, 4 and 5, we have some properties of M as an F-space. Our results are similar to those on N^+ by N. Yanagihara **[10, 11]** and by C. S. Davis **[1]**. In Section 6, we prove that M becomes an F-algebra and obtain some properties of M as an F-algebra which are also similar to those of N^+ obtained by J. W. Roberts and M. Stoll **[7]**. For example, the multiplicative linear functionals are determined and the invertible elements are characterized.

1. Preliminaries. We summarize some facts which will be needed in the sequel. The standard references are the books **[3, 4, 6]**.

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1.1. *Radial limits.* For $f \in N$, the radial limit

$$f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

exists for almost every $e^{i\theta}$ and $\log|f^*| \in L^1(T)$ unless $f \not\equiv 0$.

1.2. *Canonical factorization.* A function $f \in N$ can be factored as

$$f(z) = B(z)(S_1(z)/S_2(z))F(z),$$

where $B(z)$ is the Blaschke product with respect to zeros of $f(z)$, $S_k(z)$, $k = 1, 2$, are the singular inner functions with no common factor and $F(z)$ is an outer function for the class N, i.e.,

$$S_k(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu_k(t)\right)$$

with positive singular measures $d\mu_k$, $k = 1, 2$, and

$$F(z) = \omega \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log|f^*(e^{it})| \, dt\right)$$

with ω a constant of unit modulus. It is known that a function $f \in N$ belongs to N^+ if and only if $S_2 \equiv 0$.

1.3. H^p. The Hardy space H^p $(0 < p \leqq \infty)$ consists of all functions, f, holomorphic in U, for which $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \sup_{0 \leqq r < 1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}\right)^{1/p}, & 0 < p < \infty, \\ \sup_{z \in U} |f(z)|, & p = \infty. \end{cases}$$

It is known that

$$\bigcup_{p>0} H^p \subset N^+.$$

Identifying f with its radial limit f^* we can consider H^p as a closed subspace of $L^p(T)$ as

$$H^p = \left\{g \in L^p(T): \hat{g}(-n) \equiv \int_0^{2\pi} g(e^{it})e^{int} \frac{dt}{2\pi} = 0,\ n = 1, 2, \dots\right\}.$$

1.4. Re H^1. Re H^1 is defined to be the class of all real parts of the functions of the class H^1. For the class Re H^1, we have the well-known theorem of Burkholder, Gundy and Silverstein.

THEOREM A (Burkholder, Gundy and Silverstein). *A real-valued function $h \in L^1(T)$ belongs to* Re H^1 *if and only if*

$$\sup_{0 \leqq r < 1} |p_r * h(\theta)| \in L^1(T),$$

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where

$$p_r * h(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} h(e^{it})dt,$$

the Poisson integral of h. See [**6**, p. 249].

1.5. THEOREM B (Zygmund). *Let* $h \in L^1(T)$ *with* $h \geqq 0$. *Then* $h \in L \log L$ *if and only if*

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} h(e^{it})dt \in H^1,$$

or equivalently, $h \in L \log L$ *if and only if* $h \in \operatorname{Re} H^1$. *See* [**6**, pp. 135-136].

2. The class M. We have the following relations among the various classes. The proof of $M \subsetneqq N^+$ in the following theorem is due to P. R. Ahern.

2.1. THEOREM. $N \supsetneqq N^+ \supsetneqq M \supsetneqq \cup_{p>0} H^p$.

Proof. The relation $N \supsetneqq N^+$ is well known. The standard proof is by means of the canonical factorization. We give here another proof. Take as usual

$$f(z) = \exp\left(\frac{1 + z}{1 - z}\right).$$

Since $\log^+|f_r(e^{i\theta})| = P_r(\theta)$, we have

$$\int_0^{2\pi} \log^+|f_r(e^{i\theta})| \frac{d\theta}{2\pi} = 1, \quad 0 \leqq r < 1,$$

so $f \in N$. Now, fix $a > 0$. We have

$$\begin{aligned}\int_0^a \log^+|f_r(e^{i\theta})| \frac{d\theta}{2\pi} &\geqq \int_0^a \frac{1 - r^2}{(1 - r)^2 + \theta^2} \frac{d\theta}{2\pi} \\ &= \frac{1 + r}{2\pi}\int_0^{a/(1-r)} \frac{d\eta}{1 + \eta^2} \quad \left(\eta = \frac{\theta}{1 - r}\right) \\ &= \frac{1 + r}{2\pi}\tan^{-1}\left(\frac{a}{1 - r}\right) \to \infty \text{ as } r \to 1^-.\end{aligned}$$

Hence $\log^+|f_r|$, $0 \leqq r < 1$, are not uniformly integrable; so $f \notin N^+$.

The inclusion $N^+ \supset M$ is obvious. Now take a function

$$h \in L^1(T)\backslash\operatorname{Re} H^1$$

with $h \geqq 0$ and consider

$$f(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}h(e^{it})dt\right).$$

Then

$$\log^+ Mf(\theta) = \sup_{0 \leqq r < 1} P_r * h(\theta).$$

Since $h \notin \operatorname{Re} H^1$, we have, by Theorem A,

$$\int_0^{2\pi} \log^+ Mf(\theta)\frac{d\theta}{2\pi} = \infty;$$

so $f \notin M$. For $E \subset T$, we write

$$\int_E P_r * h(\theta)\frac{d\theta}{2\pi} \leqq \int_E |P_r * h(\theta) - h(\theta)|\frac{d\theta}{2\pi} + \int_E h(\theta)\frac{d\theta}{2\pi}$$
$$\leqq \int_0^{2\pi} |P_r * h(\theta) - h(\theta)|\frac{d\theta}{2\pi} + \int_E h(\theta)\frac{d\theta}{2\pi}.$$

Given $\epsilon > 0$, we can take $r_0 < 1$ so that

$$\int_0^{2\pi} |P_r * h(\theta) - h(\theta)|\frac{d\theta}{2\pi} < \epsilon/2,$$

whenever $r_0 < r < 1$, and $\delta > 0$ so that

$$\int_E h(\theta)\frac{d\theta}{2\pi} < \epsilon/2,$$

whenever $|E| < \delta$. Thus

$$\int_E P_r * h(\theta)\frac{d\theta}{2\pi} < \epsilon,$$

whenever $|E| < \delta$ and $r_0 < r < 1$. Therefore we can conclude that $\log^+|f_r|$, $0 \leqq r < 1$, form a uniformly integrable family; so $f \in N^+$.

The inclusion

$$M \supset \bigcup_{p>0} H^p$$

follows from the inequality

$$\log^+ a \leqq a^p \quad (a > 0, p > 0)$$

and the complex maximal theorem. For the proof of

$$M \neq \bigcup_{p>0} H^p,$$

we take the following example of N. Yanagihara **[12]**, who used it to show $N^+ \neq \cup_{p>0} H^p$,

$$f(z) = \exp\left(\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t) \frac{dt}{2\pi}\right),$$

where

$$\psi(\theta) = \begin{cases} \exp \frac{1}{\sqrt{\theta}}, & |\theta| \leqq 1, \\ e, & 1 \leqq |\theta| \leqq \pi. \end{cases}$$

Since $\psi \notin L^p(T)$ for any $p > 0$,

$$f \notin \bigcup_{p>0} H^p.$$

To show that $f \in M$, we note that

$$\log|f(e^{i\theta})| = \log \psi(\theta) = \begin{cases} \frac{1}{\sqrt{\theta}}, & |\theta| \leqq 1, \\ 1, & 1 \leqq |\theta| \leqq \pi \end{cases}$$

belongs to the Zygmund class $L \log L$; so that, by Theorem B,

$$\log^+ Mf(\theta) \in L^1(T).$$

This completes the proof.

2.2. THEOREM. *Let $h \in L^1(T)$, h real-valued and let*

$$f(z) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} h(e^{it}) dt\right).$$

If $h^+ \in \operatorname{Re} H^1$, then $f \in M$. The converse is false.

Proof. We see easily that

$$\log^+ Mf(\theta) \leqq \sup_{0 \leqq r < 1} P_r * h^+(\theta).$$

By Theorem A, $h^+ \in \operatorname{Re} H^1$ implies $f \in M$.

To show that the converse is false, we take a real-valued

$$h \in \operatorname{Re} H^1 \backslash L \log L.$$

We claim that $h^+ \notin \operatorname{Re} H^1$ and $h^- \notin \operatorname{Re} H^1$. Otherwise, $h^+ \in \operatorname{Re} H^1$, or $h^- \in \operatorname{Re} H^1$. Assume $h^+ \in \operatorname{Re} H^1$, say. Then $h^- = h^+ - h \in \operatorname{Re} H^1$, so $|h| = h^+ + h^- \in \operatorname{Re} H^1$. By Theorem B, $|h| \in L \log L$; so $h \in L \log L$, a contradiction. We note that

$$\log^+ Mf(\theta) = \sup_{0 \leqq r < 1} (P_r * h)^+ \leqq \sup_{0 \leqq r < 1} |P_r * h|.$$

Since $h \in \operatorname{Re} H^1$, we have, by Theorem A, $f \in M$. This completes the proof.

2.3. LEMMA [4, p. 97, Ex. 15]. *If $f \in N$, then*

$$m_{Mf}(\lambda) \leqq \frac{A}{\log \lambda} \sup_{0 \leqq r < 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi}, \quad \lambda > 1,$$

where A is a positive constant independent of f and

$$m_{Mf}(\lambda) = |\{\theta : Mf(\theta) > \lambda\}|.$$

The following theorem shows a difference of N and M.

2.4. THEOREM. *If $f \in N$, then*

$$\int_0^{2\pi} (\log^+ Mf(\theta))^\alpha \frac{d\theta}{2\pi} < \infty$$

for all α $(0 < \alpha < 1)$.

Proof. We set

$$E = \{\theta : Mf(\theta) \geqq e\}, \quad \text{and}$$

$$m^E_{Mf}(\lambda) = |\{\theta \in E : Mf(\theta) > \lambda\}|.$$

We have by Lemma 2.3

$$\begin{aligned}
\int_0^{2\pi} (\log^+ Mf(\theta))^\alpha \frac{d\theta}{2\pi} &= \int_{E^c} + \int_E \\
&\leqq 1 + \frac{1}{2\pi} \int_e^\infty (\log \lambda)^\alpha [-dm^E_{Mf}(\lambda)] \\
&\leqq 1 + \frac{\alpha}{2\pi} \int_e^\infty (\log \lambda)^{\alpha - 1} \frac{1}{\lambda} m^E_{Mf}(\lambda) d\lambda \\
&\leqq 1 + \frac{\alpha A}{2\pi} \left(\sup_{0 \leqq r < 1} \int_\theta^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} \right) \\
&\quad \times \int_e^\infty \frac{d\lambda}{\lambda (\log \lambda)^{2-\alpha}} < \infty,
\end{aligned}$$

since $0 < \alpha < 1$.

Unlike N or N^+ (see [9]), M is closed under integration as the following theorem shows.

2.5. THEOREM. *M is closed under integration.*

Proof. Let $f \in M$ and let

$$F(z) = \int_0^z f(z) dz = \int_0^r f(te^{i\theta}) e^{i\theta} dt.$$

Then $|F(re^{i\theta})| \leqq Mf(\theta)$; so $MF \leqq Mf$. Therefore $f \in M$.

3. ***M* as an *F*-space.** We define an invariant metric d on M by

$$d(f, g) = \int_0^{2\pi} \log(1 + M(f - g)(\theta)) \frac{d\theta}{2\pi}, \quad f, g \in M.$$

3.1. THEOREM. *M is an F-space, i.e., a complete topological vector space with an invariant metric.*

Proof. It suffices (see [**2**, p. 51]) to check the following properties:

(i) d is an invariant metric,

(ii) For a fixed $f \in M$, $c \mapsto cf$ is a continuous mapping from $\mathbf{C}$ into M,

(iii) For a fixed $c \in \mathbf{C}$, $f \mapsto cf$ is a continuous mapping from M into M,

(iv) M is complete.

(i) is obvious. (ii) follows from the dominated convergence theorem as

$$d(cf, 0) = \int_T \log(1 + |c|Mf(\theta)) \frac{d\theta}{2\pi} \to 0 \text{ as } c \to 0.$$

(iii) Take k an integer with $|c| \leqq k$. Then

$$d(cf, 0) \leqq d(kf, 0) \leqq kd(f, 0);$$

so $f \mapsto cf$ is continuous.

(iv) Let $\{f_n\}_{n=0}^{\infty}$ be a Cauchy sequence in M. Then $\{f_n\}$ is a Cauchy sequence in N^+. We know [**10**] that N^+ is complete. So $f_n \mapsto f \in N^+$ in the metric of N^+ for some $f \in N^+$. It suffices to show that $f \in M$ and $f_n \to f$ in M. Given $\epsilon > 0$, we can find an integer k so that

$$\int_0^{2\pi} \log(1 + M(f_n - f_m)(\theta)) \frac{d\theta}{2\pi} < \epsilon, \quad n, m \geqq k.$$

For $0 < r < 1$, we see that

$$\lim_{n\to\infty} M[(f_m)_r - (f_n)_r](\theta) = M[(f_m)_r - f_r](\theta).$$

By Fatou's lemma, we have

$$\begin{aligned} &\int_0^{2\pi} \log(1 + M[(f_m)_r - f_r](\theta)) \frac{d\theta}{2\pi} \\ &\leqq \underline{\lim}_{n\to\infty} \int_0^{2\pi} \log(1 + M[(f_m)_r - (f_n)_r](\theta)) \frac{d\theta}{2\pi} \\ &\leqq \underline{\lim}_{n\to\infty} \int_0^{2\pi} \log(1 + M(f_m - f_n)(\theta)) \frac{d\theta}{2\pi} \\ &\leqq \epsilon. \end{aligned}$$

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By the monotone convergence theorem,

$$\int_0^{2\pi} \log(1 + M(f_m - f)(\theta))\frac{d\theta}{2\pi}$$
$$= \lim_{r\to 1}\int_0^{2\pi} \log(1 + M[(f_m)_r - f_r](\theta))\frac{d\theta}{2\pi}$$
$$\leqq \epsilon.$$

We have $d(f_m - f, 0) \leqq \epsilon$ if $m \geqq k$. In particular, $f_m - f \in M$; so $f \in M$. This shows that $f_m \to f$ in M.

3.2. THEOREM. *The polynomials are dense in M. Therefore, M is separable.*

Proof. Let $f \in M$. We show that $f_r \to f$ in M as $r \to 1^-$. Since f has the radial limit at almost all θ, we see that for almost all θ, $t \mapsto f(te^{i\theta})$ is continuous on the closed interval $[0, 1]$; so uniformly continuous. For such a θ,

$$M(f - f_r)(\theta) \to 0 \quad \text{as } r \to 1^-.$$

Since

$$\log(1 + M(f - f_r)(\theta)) \leqq 2\log(1 + Mf(\theta)) \in L^1(T),$$

we have, by the dominated convergence theorem,

$$\int_0^{2\pi} \log(1 + M(f - f_r)(\theta))\frac{d\theta}{2\pi} \to 0 \quad \text{as } r \to 1^-,$$

that is, $f_r \to f$ in M as $r \to 1^-$. Since f_r can be uniformly approximated by polynomials on the closed unit disc, it can be approximated in M by polynomials. Hence the polynomials are dense in M.

4. Bounded subsets of M. The following characterization of boundedness in M and its proof are analogous to those in N^+. See [**11**].

4.1. THEOREM. *$L \subset M$ is bounded if and only if*

(i) $\displaystyle\int_0^{2\pi} \log^+ Mf(\theta)\frac{d\theta}{2\pi} < K(<\infty)$ *for all $f \in L$, and*

(ii) *given $\epsilon > 0$, there is a $\delta > 0$ such that*

$$\int_E \log^+ Mf(\theta)\frac{d\theta}{2\pi} < \epsilon \quad \textit{for all } f \in L$$

whenever $|E| < \delta$.

Proof. ($\Leftarrow$) Given a neighborhood

$$V = \{g \in M : d(g, 0) < \eta\}$$

of 0 in M, we take $\epsilon > 0$ so that

$$\log(1 + \epsilon) + \frac{\epsilon}{2\pi} \log 2 + \epsilon < \eta.$$

Find a δ ($0 < \delta < \epsilon$) so that (ii) is satisfied. For $f \in L$, we can find an $E_f \subset T$ so that

$$|T \setminus E_f| < \delta \quad \text{and} \quad \log^+ Mf(\theta) \leqq K/\delta \quad \text{on } E_f.$$

We have

$$Mf(\theta) \leqq \exp\left(\frac{K}{\delta}\right) \equiv L(\delta) \quad \text{on } E_f.$$

Now, if $0 < \alpha < \min(1, \epsilon/L(\delta))$, then, for $f \in L$, we have

$$\begin{aligned} d(\alpha f, 0) &= \int_0^{2\pi} \log(1 + |\alpha| Mf(\theta))\frac{d\theta}{2\pi} \\ &\leqq \int_{E_f} \log(1 + \epsilon)\frac{d\theta}{2\pi} + \int_{T\setminus E_f} [\log 2 + \log^+ Mf(\theta)]\frac{d\theta}{2\pi} \\ &\leqq \log(1 + \epsilon) + \frac{\delta}{2\pi}\log 2 + \int_{T\setminus E_f} \log^+ Mf(\theta)\frac{d\theta}{2\pi} \\ &\leqq \log(1 + \epsilon) + \frac{\delta}{2\pi}\log 2 + \epsilon \\ &< \eta. \end{aligned}$$

Thus $\alpha L \subset V$. Therefore L is bounded in M.

($\Rightarrow$) Suppose that $L (\subset M)$ is bounded in M. Given $\eta > 0$, we can find an $\alpha_0 = \alpha_0(\eta)$ ($0 < \alpha_0 < 1$) so that

$$\int_0^{2\pi} \log(1 + |\alpha| Mf(\theta))\frac{d\theta}{2\pi} < \eta \quad \text{for all } f \in L$$

whenever $|\alpha| \leqq \alpha_0$. We have

$$\int_0^{2\pi} \log^+ |\alpha|\, Mf(\theta)\frac{d\theta}{2\pi} < \eta \quad \text{for all } f \in L,\ |\alpha| \leqq \alpha_0.$$

Since

$$\log^+ Mf \leqq \log^+ \alpha_0 Mf + \log\frac{1}{\alpha_0},$$

we have, for all $f \in L$,

$$\int_0^{2\pi} \log^+ Mf(\theta)\frac{d\theta}{2\pi} \leqq \int_0^{2\pi} \log^+ \alpha_0 Mf(\theta)\frac{d\theta}{2\pi} + \log\frac{1}{\alpha_0}$$

$$\leqq \eta + \log\frac{1}{\alpha_0}$$

$$\equiv K < \infty.$$

Thus (i) is satisfied. For (ii) given $\epsilon > 0$, take $\eta < \epsilon/2$ and $\alpha_0 = \alpha_0(\eta)$ as above. We choose $\delta > 0$ so that

$$\frac{\delta}{2\pi}\log\frac{1}{\alpha_0} < \epsilon/2.$$

If $|E| < \delta$, then

$$\int_E \log^+ Mf(\theta)\frac{d\theta}{2\pi} \leqq \int_E \log^+ \alpha_0 Mf(\theta)\frac{d\theta}{2\pi} + \int_E \log\frac{1}{\alpha_0}\frac{d\theta}{2\pi}$$

$$\leqq \eta + |E|\log\frac{1}{\alpha_0}\frac{1}{2\pi}$$

$$< \epsilon.$$

Thus (ii) is satisfied. This completes the proof.

Let $c_k \downarrow 0$ and let $r_k \uparrow 1$, $k = 1, 2, \ldots$. We consider

$$f_k(z) = \exp\left(c_k^2\frac{1 + r_k z}{1 - r_k z}\right), \quad k = 1, 2, \ldots.$$

4.2. Lemma.

$$\log^+ Mf_k(\theta) = c_k^2 \sup_{0 \leqq r < 1} \frac{1 - r_k^2 r^2}{1 - 2r_k r\cos\theta + r_k^2 r^2}$$

$$= \begin{cases} c_k^2\dfrac{1 - r_k^2}{1 - 2r_k\cos\theta + r_k^2}, & |\theta| \leqq \sin^{-1}\left(\dfrac{1 - r_k^2}{1 + r_k^2}\right), \\ c_k^2\dfrac{1}{\sin\theta}, & \sin^{-1}\left(\dfrac{1 - r_k^2}{1 + r_k^2}\right) \leqq |\theta| \leqq \pi/2, \\ c_k^2, & \pi/2 \leqq \theta \leqq \pi. \end{cases}$$

The proof is routine and is omitted.

4.3. LEMMA. *$\log^+ Mf_k$, $k = 1, 2, \dots$ are bounded in $L^1(T)$ if and only if*

$$c_k^2 \log \frac{1 + r_k^2}{1 - r_k^2}, \ k = 1, 2, \dots$$

are bounded.

Proof. Let

$$A_k = \left[0, \sin^{-1}\left(\frac{1 - r_k^2}{1 + r_k^2}\right)\right], \quad B_k = \left[\sin^{-1}\left(\frac{1 - r_k^2}{1 + r_k^2}\right), \pi/2\right]$$

and let $C_k = [\pi/2, \pi]$. The proof will follow from the following estimates:

(i) $$\int_{A_k} \log^+ Mf_k(\theta)\frac{d\theta}{2\pi} \leqq c_k^2 \int_0^{2\pi} \frac{1 - r_k^2}{1 - 2r_k \cos\theta + r_k^2}\frac{d\theta}{2\pi} = c_k^2.$$

(ii) $$\int_{A_k} \log^+ Mf_k(\theta)\frac{d\theta}{2\pi}$$

$$\geqq \frac{1}{2\pi}c_k^2 \frac{1 - r_k^2}{1 - 2r_k \cos \sin^{-1}\left(\dfrac{1 - r_k^2}{1 + r_k^2}\right) + r_k^2} \sin^{-1}\left(\frac{1 - r_k^2}{1 + r_k^2}\right)$$

$$\geqq \frac{1}{2\pi}c_k^2 \frac{1 - r_k^2}{1 - 2r_k\dfrac{2r_k}{1 + r_k^2} + r_k^2}\frac{1 - r_k^2}{1 + r_k^2}$$

$$\geqq \frac{1}{2\pi}c_k^2.$$

(iii) $$\int_{C_k} \log^+ Mf_k(\theta)\frac{d\theta}{2\pi} = \frac{1}{2\pi}c_k^2\frac{\pi}{2} = \frac{1}{4}c_k^2.$$

(iv) $$\int_{B_k} \log^+ Mf_k(\theta)\frac{d\theta}{2\pi} \leqq c_k^2 \int_{\sin^{-1}((1-r_k^2)/(1+r_k^2))}^{\pi/2} \frac{\pi}{2}\frac{1}{\theta}\frac{d\theta}{2\pi}$$

$$\leqq \frac{c_k^2}{4}\int_{(1-r_k^2)/(1+r_k^2)}^{\pi/2} \frac{1}{\theta}d\theta$$

$$= \frac{c_k^2}{4}\left[\log\frac{1 + r_k^2}{1 - r_k^2} + \log\frac{\pi}{2}\right].$$

(v) $$\int_{B_k} \log^+ Mf_k(\theta)\frac{d\theta}{2\pi}$$

$$\geqq c_k^2 \int_{(\pi/2)(1-r_k^2)/(1+r_k^2)}^{\pi/2} \frac{1}{\theta}\frac{d\theta}{2\pi} = \frac{c_k^2}{2\pi}\log\left(\frac{1 + r_k^2}{1 - r_k^2}\right).$$

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This completes the proof.

4.4. THEOREM. *f_k, $k = 1, 2, \ldots$, form a bounded subset of M if and only if*

$$c_k^2 \log \frac{1 + r_k^2}{1 - r_k^2} \to 0 \quad \textit{as } k \to \infty.$$

Proof. ($\Rightarrow$) Suppose that f_k, $k = 1, 2, \ldots$, form a bounded subset of M. Given $\epsilon > 0$, we can take $\delta > 0$ so that

$$\int_E \log^+ Mf_k(\theta)\frac{d\theta}{2\pi} < \epsilon, \quad k = 1, 2, \ldots,$$

whenever $|E| < \delta$. Choose k_0 so that

$$\frac{\pi}{2}\frac{1 - r_k^2}{1 + r_k^2} < \delta \quad \text{whenever } k \geqq k_0.$$

For $k \geqq k_0$, we have

$$\begin{aligned}
\epsilon &> \int_{\sin^{-1}((1-r_k^2)/(1+r_k^2))}^{\delta} c_k^2 \frac{1}{\sin\theta}\frac{d\theta}{2\pi} \\
&\geqq \frac{c_k^2}{2\pi}\int_{(\pi/2)(1-r_k^2)/(1+r_k^2)}^{\delta} \frac{d\theta}{\theta} = \frac{c_k^2}{2\pi}\left[\log\delta - \log\frac{\pi}{2}\frac{1 - r_k^2}{1 + r_k^2}\right] \\
&= \left[c_k^2 \log \frac{1 + r_k^2}{1 - r_k^2} + c_k^2 \log\delta - c_k^2 \log\frac{\pi}{2}\right]/2\pi.
\end{aligned}$$

We can choose k_1 so that

$$c_k^2 \log \frac{1 + r_k^2}{1 - r_k^2} < \epsilon \quad \text{whenever } k \geqq k_1.$$

This shows that

$$c_k^2 \log \frac{1 + r_k^2}{1 - r_k^2} \to 0 \quad \text{as } k \to \infty.$$

($\Leftarrow$) Suppose that

$$c_k^2 \log \frac{1 + r_k^2}{1 - r_k^2} \to 0.$$

By Lemma 4.3, the condition (i) of boundedness is satisfied. For the condition (ii), we write, for $E \subset [0, \pi]$,

$$\int_E \log^+ Mf_k(\theta)\frac{d\theta}{2\pi} = \int_{E\cap A_k} + \int_{E\cap B_k} + \int_{E\cap C_k}$$

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where A_k, B_k and C_k are as in the proof of Lemma 4.3. We have the following estimates.

$$\int_{E\cap A_k} \leqq c_k^2 \int_0^{2\pi} \frac{1 - r_k^2}{1 - 2r_k \cos\theta + r_k^2}\frac{d\theta}{2\pi} = c_k^2,$$

$$\int_{E\cap C_k} \leqq c_k^2, \quad \text{and}$$

$$\int_{E\cap B_k} \leqq \int_{\sin^{-1}(1-r_k^2)/(1+r_k^2)}^{\pi/2} c_k^2 \frac{1}{\sin\theta}\frac{d\theta}{2\pi}$$

$$\leqq \frac{1}{2\pi} c_k^2 \int_{(1-r_k^2)/(1+r_k^2)}^{\pi/2} \frac{\pi}{2}\frac{1}{\theta} d\theta$$

$$= \frac{1}{4} c_k^2 \log \frac{1 + r_k^2}{1 - r_k^2} + \frac{1}{4} c_k^2 \log \frac{\pi}{2}.$$

Take k_0 so that $c_k^2 < \epsilon/3$ and

$$\frac{1}{4} c_k^2 \log \frac{1 + r_k^2}{1 - r_k^2} + \frac{1}{4} c_k^2 \log \frac{\pi}{2} < \epsilon/3$$

whenever $k \geqq k_0$. For $k \leqq k_0$, we have

$$\int_{E\cap A_k} \leqq c_1^2 \frac{1 + r_{k_0}}{1 - r_{k_0}} |E| \frac{1}{2\pi},$$

$$\int_{E\cap C_k} \leqq c_1^2 |E| \frac{1}{2\pi}, \quad \text{and}$$

$$\int_{E\cap B_k} \leqq c_1^2 \int_{\sin^{-1}((1-r_k^2)/(1+r_k^2))}^{\sin^{-1}((1-r_k^2)/(1+r_k^2))+|E|} \frac{1}{\sin\theta}\frac{d\theta}{2\pi}$$

$$\leqq c_k^2 \int_{\sin^{-1}((1-r_k^2)/(1+r_k^2))}^{\sin^{-1}((1-r_k^2)/(1+r_k^2))+|E|} \frac{\pi}{2}\frac{1}{\theta}\frac{d\theta}{2\pi}$$

$$= \frac{c_k^2}{4}\left[\log\left(\sin^{-1}\left(\frac{1 - r_k^2}{1 + r_k^2}\right) + |E|\right) - \log \sin^{-1}\left(\frac{1 - r_k^2}{1 + r_k^2}\right)\right]$$

$$= \frac{c_k^2}{4} \log\left[1 + |E|/\sin^{-1}\left(\frac{1 - r_k^2}{1 + r_k^2}\right)\right]$$

$$\leqq \frac{c_1^2}{4} \log\left(1 + |E|/\sin^{-1}\left(\frac{1 - r_{k_0}^2}{1 + r_{k_0}}\right)\right).$$

If we take $\delta > 0$ small enough, we can have

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$$\int_E \log^+ Mf_k(\theta)\frac{d\theta}{2\pi} < \epsilon, \quad \text{all } k,$$

whenever, $|E| < \delta$. Therefore f_k, $k = 1, 2, \ldots$, form a bounded subset of M.

The proof of the following theorem was communicated to the author by Professor K. Izuchi, to whom the author expresses his sincere thanks.

4.5. THEOREM. *M is not locally bounded.*

Proof. We take $c_k \downarrow 0$ and $r_k \uparrow 1$ ($k \to \infty$) so that

(i) $\quad c_k^2 \log\dfrac{1 + r_k^2}{1 - r_k^2}, \quad k = 1, 2, \ldots,$

are bounded but

(ii) $\quad c_k^2 \log\dfrac{1 + r_k^2}{1 - r_k^2} \not\to 0 \quad \text{as } k \to \infty.$

For $\alpha > 0$, we put

$$f_k^\alpha(z) = \exp\left(\alpha c_k^2 \frac{1 + r_k z}{1 - r_k z}\right).$$

By Theorem 4.4 and (ii), f_k^α, $k = 1, 2, \ldots$, are not bounded in M for each $\alpha > 0$. By Lemma 4.3 and (i), $\log^+ Mf_k^\alpha$, $k = 1, 2, \ldots$, are bounded in $L^1(T)$. If we set

$$F_{n,\alpha,k}(z) = f_k^\alpha(z)/n \quad \text{for } n = 1, 2, \ldots,$$

we see that $F_{n,\alpha,k}$, $k = 1, 2, \ldots$, are not bounded in M for each n and $\alpha > 0$. On the other hand, given $\epsilon > 0$ we can show that

$$\{F_{n,\alpha,k}\}_{k=1}^\infty \subset \{g \in M : d(g, 0) < \epsilon\} \equiv V_\epsilon$$

for sufficiently large n and sufficiently small $\alpha > 0$. In fact, if we set

$$E = \left\{\theta \in T : Mf_k^\alpha(\theta) \leqq \frac{1 + \sqrt{5}}{2}\right\},$$

we have

$$\begin{aligned} d(F_{n,\alpha,k}, 0) &= \int_T \log(1 + MF_{n,\alpha,k}) \frac{d\theta}{2\pi} \\ &\leqq \int_{E_k} \log\left(1 + \frac{1}{n}\frac{1 + \sqrt{5}}{2}\right) \frac{d\theta}{2\pi} \\ &\quad + \int_{T \setminus E_k} \log(1 + Mf_k^\alpha(\theta)) \frac{d\theta}{2\pi} \end{aligned}$$

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$$\leqq \log\left(1 + \frac{1}{n}\frac{1+\sqrt{5}}{2}\right) + 2\alpha \int_T \log^+ Mf_k(\theta)\frac{d\theta}{2\pi}.$$

Since $\log^+ Mf_k$, $k = 1, 2, \ldots$, are bounded in $L^1(T)$,

$$d(F_{n,\alpha,k}, 0) < \epsilon$$

if n is large enough and $\alpha > 0$ is small enough. Therefore V_ϵ is not bounded in M. This completes the proof.

4.6. THEOREM. *If $f \in M$, then $f_\zeta(z) = f(\zeta z)$, $|\zeta| < 1$, form a bounded set in M.*

Proof. Let $V = \{g \in M : d(g, 0) < \eta\}$ be a neighborhood of 0. We write $z = re^{i\theta}$ and $\zeta = \rho e^{i\psi}$. Since

$$Mf_\zeta(\theta) = Mf_\rho(\theta + \psi) \leqq Mf(\theta + \psi),$$

we have, for $|\alpha| \leqq 1$,

$$\begin{aligned}\log(1 + |\alpha| Mf_\zeta(\theta)) &\leqq \log(1 + |\alpha| Mf(\theta + \psi)) \\ &\leqq \log(1 + Mf(\theta + \psi)), \quad |\zeta| < 1.\end{aligned}$$

By the dominated convergence theorem,

$$\int_0^{2\pi} \log(1 + |\alpha| Mf_\zeta(\theta)) \frac{d\theta}{2\pi}$$

$$\leqq \int_0^{2\pi} \log(1 + |\alpha| Mf(\theta + \psi)) \frac{d\theta}{2\pi} \to 0 \quad \text{as } |\alpha| \to 0,$$

uniformly on $|\zeta| < 1$. There exists an α_0 such that if $|\alpha| \leqq \alpha_0$,

$$\int_0^{2\pi} \log(1 + |\alpha| Mf_\zeta(\theta)) \frac{d\theta}{2\pi} < \eta, \quad |\zeta| < 1,$$

i.e., $\alpha f_\zeta \in V$ for all $|\zeta| < 1$, if $|\alpha| \leqq \alpha_0$. Therefore $\{f_\zeta\}_{|\zeta|<1}$ form a bounded set in M.

5. Linear functionals on *M*. We follow the methods of N. Yanagihara [**10**], in the proofs of Theorems 5.2 and 5.3. For the proof of the following lemma, see [**10**, Lemma 1].

5.1. LEMMA. *Let $m_k \uparrow \infty$ as $k \to \infty$. If*

$$\lambda_k = O \exp(-\epsilon_k m_k)$$

for any sequence $\epsilon_k \downarrow 0$, then

$$\lambda_k = O \exp(-\epsilon m_k) \quad \textit{for some } \epsilon > 0.$$

5.2. THEOREM. *If $\{\lambda_k\}$ multiplies M into H^∞, that is*

$$f(z) = \sum a_k z^k \in M \quad \text{implies } \sum \lambda_k a_k z^k \in H^\infty,$$

then

$$\lambda_k = O \exp\left(-\epsilon \sqrt{\frac{k}{\log k}}\right) \quad \text{as } k \to \infty, \text{ for some } \epsilon > 0.$$

Proof. Given $\epsilon_k \downarrow 0$, let

$$\epsilon_k' = \max\left(\epsilon_k, \sqrt{\frac{\log k}{\sqrt{k}}}\right).$$

It suffices by Lemma 5.1 to show that

$$\lambda_k = O \exp\left(-\epsilon_k \sqrt{\frac{k}{\log k}}\right) \quad \text{as } k \to \infty.$$

Let $c_k \downarrow 0$, $r_k \uparrow 1$ and let

$$c_k^2 \log \frac{1}{1 - r_k} \downarrow 0 \quad \text{as } k \to \infty.$$

Then we know by Theorem 4.4 that

$$f_k(z) = \exp\left(c_k^2 \frac{1 + r_k z}{1 - r_k z}\right)$$
$$= \sum_n a_n(k) r_k^n z^n,$$

$k = 1, 2, \ldots$ form a bounded sequence in M. Therefore, we have

$$\left|\sum_n \lambda_n a_n(k) r_k^n z^n\right| \leqq L < \infty, \quad |z| \leqq 1, k = 1, 2, \ldots.$$

In particular,

$$|\lambda_n a_n(k)| r_k^n \leqq L, \quad k = 1, 2, \ldots, n = 1, 2, \ldots.$$

Take c_k so that

$$\frac{1}{\sqrt{k}} \leqq 2c_k^2 \leqq 1.$$

Then

$$\log|a_k(k)| \geqq \sqrt{2c_k^2 k}(1 + o(1)).$$

See [**10**, Remark 3]. That is,

$$|a_k(k)| \geqq \exp(\sqrt{2c_k^2 k}(1 + o(1))).$$

Therefore, we have

$$|\lambda_k| \exp(\sqrt{2c_k^2 k}(1 + o(1)))\, r_k^k \leqq L,$$

or

$$|\lambda_k| \leqq L r_k^{-k} \exp(-\sqrt{2c_k^2 k}(1 + o(1))).$$

If we take $r_k = 1 - 1/k$ and c_k so that

$$c_k^2 \log \frac{1}{1 - r_k} = c_k^2 \log k = (\epsilon_k')^2/2$$

i.e.,

$$c_k^2 = (\epsilon_k')^2/2 \log k \geqq \frac{1}{2\sqrt{k}},$$

then r_k and c_k satisfy those conditions imposed on them. Thus we have,

$$\begin{aligned} |\lambda_k| &\leqq L\left(1 - \frac{1}{k}\right)^{-k} \exp\left(-\epsilon_k' \sqrt{\frac{k}{\log k}}(1 + o(1))\right) \\ &\leqq C \exp\left(-\epsilon_k \sqrt{\frac{k}{\log k}}\right). \end{aligned}$$

This completes the proof.

Now we have the following characterizations of continuous linear functionals on M. $A^\infty(U)$ denotes the class of all holomorphic functions continuous up to the boundary which is of class C^∞ on T.

5.3. THEOREM. *If Λ is a continuous linear functional on M, then there exists a $g \in A^\infty(U)$ such that*

$$\Lambda f = \lim_{r \to 1} \int_0^{2\pi} f(re^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi}, \quad f \in M.$$

Conversely, if $g \in A^\infty(U)$ and

$$\Lambda f = \lim_{r \to \infty} \int_0^{2\pi} f(re^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi}$$

exists for all $f \in M$, then Λ defines a continuous linear functional on M.

Proof. Suppose Λ is a continuous linear functional on M and let

$$b_k = \overline{\Lambda(z^k)}, \quad k = 1, 2, \dots .$$

Since $\{z^k\}_{k=1}^\infty$ is bounded in M and Λ is continuous, b_k, $k = 1, 2, \dots$ form a bounded sequence, so that

$$g(z) = \sum b_k z^k$$

defines a holomorphic function in U. Let

$$f(z) = \sum a_k z^k \in M.$$

Since $\{f_\zeta\}_{\zeta \in U}$ form a bounded set in M by Theorem 4.6, $\{\Lambda f_\zeta\}_{\zeta \in U}$ form a bounded set in $\mathbf{C}$. Hence

$$\Lambda f_\zeta = \lim_{k \to \infty} T\left(\sum_0^k a_k \zeta^k z^k\right)$$

$$= \sum a_k \bar{b}_k \zeta^k$$

is a bounded holomorphic function of ζ, $|\zeta| < 1$. Thus $\{\bar{b}_k\}$ multiplies M into H^∞. By Theorem 5.2, we have

$$|b_k| = O \exp\left(-\epsilon \sqrt{\frac{k}{\log k}}\right).$$

This shows that $g \in A^\infty(U)$.

Next, let $f(z) = \sum a_k z^k \in M$. Then

$$\sum_0^n a_k r^k z^k \to f_r \quad \text{in } M \text{ as } n \to \infty$$

and $f_r \to f$ in M as $r \to 1^-$. Therefore

$$\Lambda f_r = \lim_{n \to \infty} \Lambda\left(\sum_0^n a_k r^k z^k\right)$$

$$= \lim_{n \to \infty} \sum_0^n a_k r^k \bar{b}_k$$

$$= \sum_0^\infty a_k \bar{b}_k r^k;$$

so

$$\Lambda f = \lim_{r \to 1} \Lambda f_r = \lim_{r \to 1} \sum_0^\infty a_k \bar{b}_k r^k$$

$$= \lim_{r \to 1} \int_0^{2\pi} f(re^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi}.$$

Conversely, we assume that $g \in A^\infty(U)$ and

$$\Lambda f = \lim_{r \to 1} \int_0^{2\pi} f(re^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi}$$

exists for all $f \in M$. We define

$$\Lambda_r f = \int_0^{2\pi} f_r(e^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi}, \quad f \in M.$$

Then Λ_r is a continuous linear functional on M. Since

$$\lim_{r\to 1} \Lambda_r f$$

exists for all $f \in M$, by the uniform boundedness principle, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sup_r |\Lambda_r f| < \epsilon$$

whenever $d(f, 0) < \delta$. Therefore,

$$|\Lambda f| \leqq \overline{\lim_{r\to 1}} |\Lambda_r f| < \epsilon$$

whenever $d(f, 0) < \delta$. This shows that Λ is continuous.

As an application we have the following theorem. We follow Davis's proof [**1**] of the corresponding theorem for N^+.

5.4. THEOREM. *M is not locally convex.*

Proof. We know [**3**, Theorem 7.6] that there exists a singular inner function S with the following property: If

$$g(z) = \sum b_n z^n \in H^2$$

is orthogonal to SH^2, then

$$\sum_n n^\gamma |b_n|^2 = \infty \quad \text{for any } \gamma > 0.$$

In fact, if μ is a singular measure with modulus of continuity

$$\omega(t, \mu) = O\left(t \log \frac{1}{t}\right)$$

then

$$S_\mu(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\right)$$

has the property. We claim SM is not dense in M. Otherwise, $Sf_n \to 1$ in M for some sequence $\{f_n\}$ in M. Then $Sf_n \to 1$ in N^+. But we know [**7**] that SN^+ is closed in N^+; so $1 \in SN^+$, which is a contradiction.

Now, suppose M is locally convex. Then there exists a nontrivial continuous linear functional Λ on M such that

$$\Lambda(Sf) = 0 \quad \text{for all } f \in M.$$

By Theorem 5.3, there exists a $g \in A^{\infty}(U)$ such that

$$\Lambda f = \lim_{r \to 1} \int_0^{2\pi} f_r(e^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi}, \quad f \in M.$$

Since $\Lambda(Sz^n) = 0$, $n = 1, 2, \ldots$, we have

$$\int_0^{2\pi} e^{in\theta}S(e^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi} = 0, \quad n = 1, 2, \ldots.$$

As $g \in H^2$, g is orthogonal to $SH^2(U)$. If $g(z) = \sum b_n z^n$ then

$$\sum n^{\gamma}|b_n|^2 = \infty \quad \text{for any } \gamma > 0,$$

which is a contradiction to the fact $g \in A^{\infty}(U)$. This shows that M is not locally convex.

6. M as an F-algebra.

6.1. THEOREM. *The multiplication in M is continuous. Therefore, M is an F-algebra.*

Proof. Fix $g \in M$. Suppose $f_k \to f$ in M and $f_k g \to h$ in M. Then $f_k \to f$ in N^+ and $f_k g \to h$ in N^+. Since the multiplication in N^+ is continuous [**1**], we have $h = fg$. By the closed graph theorem, $f \mapsto fg$ is continuous in M.

Now let $f_k \to f$ and $g_k \to g$ in M. Then

$$d(f_k g_m, fg) \leqq d(f_k(g_m - g), 0) + d(g(f_k - f), 0).$$

As we saw above,

$$d(g(f_k - f, 0) \to 0.$$

Define $\Lambda_k : M \to M$ by

$$\Lambda_k h = f_k h, \quad h \in M.$$

Then Λ_k is a continuous linear operator. Also, $\Lambda_k h$ is convergent for each $h \in M$. By the uniform boundedness principle, $\{\Lambda_k\}$ is bounded uniformly. Hence $\Lambda_k h \to 0$ uniformly on k as $h \to 0$. Therefore

$$\lim_{m \to \infty} d(f_k(g_m - g), 0) = 0$$

uniformly on k. Hence $f_k g_m \to 0$ as $m, k \to \infty$. This shows that the multiplication is continuous and hence M is an F-algebra.

6.2. LEMMA. *Let*

$$f(z) = \exp\left(\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log|f(e^{it})| \frac{dt}{2\pi}\right)$$

be an outer function. Then $\log|f| \in \operatorname{Re} H^1$ *if and only if*

$$\log^+ Mf, \log^+ M\left(\frac{1}{f}\right) \in L^1(T).$$

Proof. We easily see that

$$\log^+|f(re^{i\theta})| + \log^+ \frac{1}{|f(re^{i\theta})|} = |P_r * \log|f|(\theta)|.$$

It follows that

$$\left\{\begin{matrix} \log^+ Mf(\theta) \\ \log^+ M\left(\frac{1}{f}\right)(\theta) \end{matrix}\right\} \leqq \sup_{0 \leqq r < 1} |P_r * \log|f|(\theta)|$$

$$\leqq \log^+ Mf(\theta) + \log^+ M\left(\frac{1}{f}\right)(\theta).$$

Therefore $\log|f| \in \operatorname{Re} H^1$ if and only if

$$\log^+ Mf \in L^1(T) \quad \text{and} \quad \log^+ M\left(\frac{1}{f}\right) \in L^1(T).$$

This completes the proof.

The following theorem characterizes the invertible elements of M.

6.3. THEOREM. *The only invertible elements of M are those outer functions f with*

$$\log|f| \in \operatorname{Re} H^1.$$

Proof. If $f(z) = B(z)S(z)F(z)$ (the canonical factorization) is an invertible element of M, it is also an invertible element of N^+; so $B(z)S(z) \equiv 1$. That is, $f(z) = F(z)$, the outer function. Since $f \in M$ and $1/f \in M$,

$$\log^+ Mf \in L^1(T) \quad \text{and} \quad \log^+ M\left(\frac{1}{f}\right) \in L^1(T).$$

Therefore $\log|f| \in \operatorname{Re} H^1$ by Lemma 6.2. The converse is obvious.

For each $\lambda \in U$, we define γ_λ on M by $\gamma_\lambda(f) = f(\lambda)$. Since

$$\log(1 + |f(\lambda)|) \leqq 2d(f, 0)\frac{1}{1 - |\lambda|}, \quad \textbf{[8]}$$

we see that γ_λ defines a continuous multiplicative linear functional on M and

$$\mathfrak{m}_\lambda = \{f \in M : f(\lambda) = 0\} = \ker \gamma_\lambda$$

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is a closed maximal ideal.

The following result characterizes the multiplicative linear functionals on M. We follow [7] for the proof of theorems below.

6.4. THEOREM. *If γ is a nontrivial multiplicative linear functional on M, then $\gamma = \gamma_\lambda$ for some $\lambda \in U$ and hence λ is continuous.*

Proof. Let $\lambda = \gamma(z)$. Then $\gamma(z - \lambda) = 0$. If $|\lambda| \geqq 1$, then $z - \lambda$ is invertible in M; so $\gamma(z - \lambda) \neq 0$, a contradiction; so $|\lambda| < 1$. Consider

$$(z - \lambda)M = \{(z - \lambda)f{:}f \in M\}.$$

If $f \in M$ and $f(\lambda) = 0$, then

$$f(z) = (z - \lambda)g(z) \quad \text{for some } g \in M;$$

so $f \in (z - \lambda)M$. Therefore

$$\mathfrak{m}_\lambda = (z - \lambda)M \subset \ker \gamma.$$

But $\mathfrak{m}_\lambda$ is a closed maximal ideal in M. Hence $\mathfrak{m}_\lambda = \ker \gamma$. Since $\ker \gamma$ is closed, γ is continuous. Furthermore, $\gamma(f) = f(\lambda)$.

6.5. THEOREM. *There exists a maximal ideal $\mathfrak{m}$ which is not the kernel of a multiplicative linear functional.*

Proof. Let

$$S(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\right)$$

be a singular inner function. Since $1 \notin SM$, SM is a proper ideal of M. By Zorn's lemma, there exists a maximal ideal $\mathfrak{m}$ containing SM. If $\mathfrak{m} = \mathfrak{m}_\lambda$ then $Sf(\lambda) = 0$ for any $f \in M$. But $S(\lambda) \neq 0$ for any λ; so $\mathfrak{m} \neq \mathfrak{m}_\lambda$ for any $\lambda \in U$.

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