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A new classification on parallel Ricci tensor for real hypersurfaces in the complex quadric

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First we introduce the notion of parallel Ricci tensor $\nabla\text{Ric} = 0$ for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ and show that the unit normal vector field N is singular. Next we give a new classification of real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ with parallel Ricci tensor.

Keywords: Parallel Ricci tensor; $\mathfrak{A}$-isotropic; $\mathfrak{A}$-principal; Kähler structure; complex conjugation; complex quadric

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1. Introduction

In the geometry of real hypersurfaces in complex space forms or quaternionic space forms it can be easily checked that there does not exist a real hypersurface with parallel shape operator S by virtue of the equation of Codazzi [16, 19]. In general, the shape operator S of a real hypersurface M in a Kähler manifold $\bar{M}$ with Kähler structure J is defined by $SX = \bar{\nabla}_X N$ for any vector field X on M where $\bar{\nabla}$ denotes the Levi-Civita connection of $\bar{M}$ and N a unit normal vector field of M in $\bar{M}$. When the Reeb vector field $\xi = -JN$ of M is principal, that is, $S\xi = \alpha\xi$, a real hypersurface M in $\bar{M}$ is said to be Hopf.

From this point of view many differential geometers have considered the notions of *parallel Ricci tensor* $\nabla\text{Ric} = 0$ or *harmonic curvature* $(\nabla_X \text{Ric})Y = (\nabla_Y \text{Ric})X$ on Riemannian manifolds, where ∇ denotes the induced connection on M from the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$. These notions are generalized conditions rather than parallel shape operator, $\nabla S = 0$ (see [7, 9, 13, 18, 31–33]). Recently, the Ricci tensors for real hypersurfaces in complex 2-plane Grassmannians, complex hyperbolic 2-plane Grassmannian, complex quadrics or complex hyperbolic quadric in the class of Hermitian symmetric spaces were investigated by many geometers (see [**1**–**3**, **5**, **6**, **11**, **20**, **21**, **24**, **25**, **28**–**30**]).

Among them, in the class of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ Suh [**24**] has given a non-existence property for Hopf real hypersurfaces with parallel Ricci tensor as follows:

Theorem A. There does not exist a Hopf real hypersurface with parallel Ricci tensor in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$.

Okumura [**17**] proved that the Reeb flow on a real hypersurface in the complex projective space $\mathbb{C}P^m = SU_{m+1}/S(U_m U_1)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^m$ for some $k \in \{0, \ldots, m-1\}$. For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_m U_2)$ some classifications were obtained by Berndt and Suh in [**1**] and [**2**]. Among them, the following assertion was given: the Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ (see [**2**]). Moreover, in [**25**] Suh has asserted that the Reeb flow on a real hypersurface in the non-compact Grassmannian $SU_{2,m}/S(U_2 U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2 U_{m-1})$ in $SU_{2,m}/S(U_2 U_m)$. Stimulated by these results we want to investigate such problems in the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$ which is a kind of Hermitian symmetric space with rank 2 different from the above ones. In view of the previous two results a natural expectation might be that the classification involves at least the totally geodesic Q^{m-1} in Q^m. But, remarkably, in [**3**] Berndt and Suh have proved the following result:

Theorem B. Let M be a real hypersurface of the complex quadric Q^m, $m \geqslant 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k$ in Q^{2k}.

The complex quadric Q^m is known to be a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$. Moreover, it can be regarded as a kind of real Grassmann manifold of compact type with rank 2 (see [**8**, **12**]). Accordingly, it admits both the Kähler structure J and the real structure A, which anti-commute with each other, that is, $AJ = -JA$. In addition, it has a special geometric structure $\mathfrak{A}$ named a parallel rank 2 vector bundle as the set of all real structures on the tangent spaces of Q^m. That is, the set is denoted by $\mathfrak{A}_{[z]} = \{A_\lambda \bar{z} \mid \lambda \in S^1 \subset \mathbb{C}\}$ at any point $[z]$ of Q^m. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m. Here the notion of parallel vector bundle $\mathfrak{A}$ means that $(\bar{\nabla}_U A)W = q(U)JAW$ for any vector fields U and W on Q^m, where $\bar{\nabla}$ and q denote a connection and a certain 1-form defined on $T_{[z]}Q^m$, $[z] \in Q^m$, respectively (see [**23**]).

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called *singular* if it is tangent to more than one maximal flat in Q^m. Since the complex quadric Q^m is a Hermitian symmetric space with rank 2, there are two types of singular tangent vectors for Q^m:

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1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A) := \{W \mid AW = W\}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-*principal*.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-*isotropic*.

When we consider a real hypersurface M in the complex quadric Q^m, the unit normal vector field N of M in Q^m can be either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [**2**, **3**, **26**]). In the first case where M has an $\mathfrak{A}$-isotropic unit normal vector field N, we have asserted in [**2**] that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k} as mentioned in theorem B. As the second case if the unit normal vector field N is $\mathfrak{A}$-principal, we have the following:

Theorem C [**4**]. *Let M be a connected orientable real hypersurface with constant mean curvature in the complex quadric Q^m, $m \geqslant 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of a tube around the m-dimensional sphere S^m which is embedded in Q^m as a real space form of Q^m.*

Motivated by theorem A and $\nabla \mathrm{R}ic = 0$ for a Hopf real hypersurface M in the complex quadric Q^m, we assert the following

Main theorem 1. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geqslant 3$, with parallel Ricci tensor. Then the unit normal vector field N is singular, that is, N is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

Now at each point $z \in M$ let us consider the maximal $\mathfrak{A}$-invariant subspace

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}$$

of T_zM, $z \in M$. Thus in the case that the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^\perp = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z \in M$, of the distribution $\mathcal{Q}$ in the complex subbundle $\mathcal{C} = \mathrm{Span}\{\xi\}^\perp$, becomes $\mathcal{Q}_z^\perp = \mathrm{Span}\{A\xi, AN\}$. Here it can be easily checked that the vector fields $A\xi$ and AN belong to the tangent space T_zM, $z \in M$ if the unit normal vector field N becomes $\mathfrak{A}$-isotropic. Then by virtue of theorems B and C, and our main theorem 1, in this paper we give a non-existence theorem for Hopf real hypersurfaces in the complex quadric Q^m with parallel Ricci tensor as follows:

Main theorem 2. *There does not exist a Hopf real hypersurface in the complex quadric Q^m, $m \geqslant 4$, with parallel Ricci tensor.*

This paper is composed as follows: In § 2 we give some basic material about the complex Q^m, including its Riemannian curvature tensor and a description of its singular vectors for $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic unit normal vector field. Apart from the complex structure J there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ which covers an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. A maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m is determined by one of these real structure A.

Accordingly, in § 3, we study the geometry of this subbundle $\mathcal{Q}$ for real hypersurfaces in Q^m and the equation of Codazzi from the curvature tensor of the complex quadric Q^m and some important formulas from the complex conjugation A of M in Q^m. Moreover, we give a remarkable proposition 3.4 which asserts that the unit normal vector field N of M in Q^m with parallel Ricci tensor must be singular. This gives a crucial point in the proof of our main theorem 1.

In § 4, in order to prove our main theorem 2 for an $\mathfrak{A}$-principal normal vector field, the first step is to derive the Ricci tensor from the equation of Gauss for real hypersurfaces M in Q^m, and next by using the assumption of parallel Ricci tensor for $\mathfrak{A}$-principal normal vector field we will get some useful formulas and a remarkable proposition 4.2. As a final proof of main theorem 2, we will prove that a contact real hypersurface in Q^m, which are tubes over an m-dimensional unit sphere S^m in Q^m, does not admit a parallel Ricci tensor.

In § 5, we give a complete proof of our main theorem 2. The first part of this proof is to give some crucial equations from the parallel Ricci tensor for an $\mathfrak{A}$-isotropic unit normal vector field. Then in the middle part of the proof we will devote ourselves to the study of important formulas which can be derived from the parallelism of the Ricci tensor. Moreover, in the proof of our main theorem 2 we will show an important lemma 5.1 which assures that $SA\xi = 0$ and $SAN = 0$ on the distribution $\mathcal{Q}^{\perp} = \text{Span}\,\{A\xi, AN\}$ for the complex conjugation A of T_zQ^m, $z \in Q^m$.

Remark 1. Along the development of real hyersurfaces in the complex quadric Q^m over the years, we can make a more progress on the Ricci parallelism for real hypersurfaces in the complex quadric Q^m. So in this article we can give a new classification better than the contents given in the previous paper due to Suh [**27**].

2. The complex quadric

For more details in this section we refer to [**4**], [**10**], [**12**], [**15**], [**22**], [**27**] and [**28**]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $z_1, \ldots, z_{m+2}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric. For each $z \in Q^m$ we identify $T_z\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus \mathbb{C}z$ of $\mathbb{C}z$ in $\mathbb{C}^{m+2}$ (see [**12**]). The tangent space T_zQ^m can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus (\mathbb{C}z \oplus \mathbb{C}\rho)$ of $\mathbb{C}z \oplus \mathbb{C}\rho$ in $\mathbb{C}^{m+2}$, where $\rho \in \nu_zQ^m$ is a normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point z.

The complex projective space $\mathbb{C}P^{m+1}$ is defined by using the Hopf fibration

$$\pi : S^{2m+3} \to \mathbb{C}P^{m+1}, \quad z \to [z],$$

which is a Riemannian submersion. Then naturally we can consider the following diagram for the complex quadric Q^m:

$$\begin{array}{ccc} \tilde{Q} = \pi^{-1}(Q) & \xrightarrow{\tilde{i}} & S^{2m+3} \subset \mathbb{C}^{m+2} \\ \pi \downarrow & & \pi \downarrow \\ Q = Q^m & \xrightarrow{i} & \mathbb{C}P^{m+1} \end{array}$$

The submanifold $\tilde{Q}$ of codimension 2 in S^{2m+3} is called the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{m+2}$, and is given by

$$\tilde{Q} = \{x + iy \in \mathbb{C}^{m+2} | g(x, x) = g(y, y) = \frac{1}{2} \text{ and } g(x, y) = 0\},$$

where $g(x, y) = \sum_{i=1}^{m+2} x_i y_i$ for any $x = (x_1, \ldots, x_{m+2}), y = (y_1, \ldots, y_{m+2}) \in \mathbb{R}^{m+2}$. Then the tangent space is decomposed as $T_z S^{2m+3} = H_z \oplus F_z$ and $T_z\tilde{Q} = H_z(Q) \oplus F_z(Q)$ at $z = x + iy \in \tilde{Q}$ respectively, where the horizontal subspaces H_z and $H_z(Q)$ are given by $H_z = (\mathbb{C}z)^{\perp}$ and $H_z(Q) = (\mathbb{C}z \oplus \mathbb{C}\bar{z})^{\perp}$, and F_z and $F_z(Q)$ are fibers which are isomorphic to each other. Here $H_z(Q)$ is a subspace of H_z of real codimension 2 and orthogonal to the two unit normals $-\bar{z}$ and $-J\bar{z}$. Explicitly, at the point $z = x + iy \in \tilde{Q}$ it can be described as

$$H_z = \{u + iv \in \mathbb{C}^{m+2} \,|\, g(x, u) + g(y, v) = 0, \quad g(x, v) = g(y, u)\}$$

and

$$H_z(Q) = \{u + iv \in H_z \,|\, g(u, x) = g(u, y) = g(v, x) = g(v, y) = 0\},$$

where $\mathbb{C}^{m+2} = \mathbb{R}^{m+2} \oplus i\mathbb{R}^{m+2}$, and $g(u, x) = \sum_{i=1}^{m+2} u_i x_i$ for any $u = (u_1, \ldots, u_{m+2})$, $x = (x_1, \ldots, x_{m+2}) \in \mathbb{R}^{m+2}$.

These spaces can be naturally projected by the differential map π_* as $\pi_* H_z = T_{\pi(z)}\mathbb{C}P^{m+1}$ and $\pi_* H_z(Q) = T_{\pi(z)}Q$ respectively. Thus at the point $\pi(z) = [z]$ the tangent subspace $T_{[z]}Q^m$ becomes a complex subspace of $T_{[z]}\mathbb{C}P^{m+1}$ with complex codimension 1. The unit normal fields $-\pi_*\bar{z}$ and $-\pi_* J\bar{z}$ span the normal space of Q^m in $\mathbb{C}P^{m+1}$ at every point (see [22]).

Then let us denote by $A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to the unit normal $\pi_*\bar{z}$. It satisfies $A_{\bar{z}}\pi_* w = \tilde{\nabla}_{\pi_* w}\bar{z} = \pi_*\bar{w}$ for every $w \in H_z(Q)$, where $\tilde{\nabla}$ denotes the covariant derivative of $\mathbb{C}P^{m+1}$ induced by its Fubini-Study metric. That is, the shape operator $A_{\bar{z}}$ is just a complex conjugation restricted to $T_{[z]}Q^m$. Moreover, it satisfies the following for any $w \in T_{[z]}Q^m$ and any $\lambda \in S^1 \subset \mathbb{C}$

$$\begin{aligned} A_{\lambda\bar{z}}^2 w &= A_{\lambda\bar{z}} A_{\lambda\bar{z}} w = A_{\lambda\bar{z}} \lambda\bar{w} \\ &= \lambda A_{\bar{z}} \lambda\bar{w} = \lambda \bar{\nabla}_{\lambda\bar{w}} \bar{z} = \lambda\bar{\lambda}\bar{\bar{w}} \\ &= |\lambda|^2 w = w. \end{aligned}$$

Accordingly, $A_{\lambda\bar{z}}^2 = I$ for any $\lambda \in S^1$. So the shape operator $A_{\bar{z}}$ becomes an anti-commuting involution such that $A_{\bar{z}}^2 = I$ and $AJ = -JA$ on the complex vector

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space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \pi_*(\mathbb{R}^{m+2} \cap H_zQ)$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = \pi_*(i\mathbb{R}^{m+2} \cap H_z(Q))$ is the (-1)-eigenspace of $A_{\bar{z}}$. That is, $A_{\bar{z}}X = X$ and $A_{\bar{z}}JX = -JX$, respectively, for any $X \in V(A_{\bar{z}})$.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and any complex conjugation $A \in \mathfrak{A}$:

$$\begin{aligned}\bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY.\end{aligned}$$

Note that J and each complex conjugation A anti-commute, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

For every unit tangent vector $W \in T_zQ^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $0 < t < \pi/4$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$.

3. Some general equations

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define the maximal $\mathfrak{A}$-invariant subspace of T_zM, $z \in M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Lemma 3.1 see [**26**]. *For each $z \in M$ we have*

(*i*) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*

(*ii*) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

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We now assume that M is a Hopf hypersurface. Then we have

$$S\xi = \alpha\xi$$

with the smooth function $\alpha = g(S\xi, \xi)$ on M. When we consider JX by the Kaehler structure J on Q^m for any vector field X on M in Q^m, we may write

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M. Then we now consider the Codazzi equation

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) \\ = \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) + g(X, AN)g(AY, Z) \\ - g(Y, AN)g(AX, Z) + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned} \tag{3.1}$$

Putting $Z = \xi$ we get

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ = -2g(\phi X, Y) + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ = g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

By comparing the previous two equations and putting $X = \xi$, we have the following

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi). \tag{3.2}$$

Reinserting (3.2) into the previous equation yields

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ = -2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ + 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Altogether this implies

$$\begin{aligned} 0 = 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ + 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned}$$

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At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leqslant t \leqslant \pi/4$ (see proposition 3 in [**22**]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$\begin{aligned} \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\ AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1. \end{aligned} \tag{3.3}$$

On the other hand, we have $JA\xi = -AJ\xi = -AN$, and inserting this formula into the previous equation implies

LEMMA 3.2. *Let M be a Hopf hypersurface in Q^m with (local) unit normal vector field N. For each point $z \in M$ we choose $A \in \mathfrak{A}_z$ such that $N_z = \cos(t)Z_1 + \sin(t)JZ_2$ holds for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leqslant t \leqslant \pi/4$. Then*

$$\begin{aligned} 0 = 2g(S\phi SX, Y) &- \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) + 2g(X, AN)g(Y, A\xi) \\ &- 2g(Y, AN)g(X, A\xi) + 2g(\xi, A\xi)\{g(Y, AN)\eta(X) - g(X, AN)\eta(Y)\} \end{aligned}$$

holds for all vector fields X and Y on M.

By the equation of Gauss, the curvature tensor $R(X, Y)Z$ for a real hypersurface M in Q^m induced from the curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$ as follows: for any tangent vector fields X, Y and Z of M

$$\begin{aligned} R(X, Y)Z = g(Y, Z)X &- g(X, Z)Y + g(JY, Z)(JX)^T - g(JX, Z)(JY)^T \\ &- 2g(JX, Y)(JZ)^T + g(AY, Z)(AX)^T - g(AX, Z)(AY)^T \\ &+ g(JAY, Z)(JAX)^T - g(JAX, Z)(JAY)^T \\ &+ g(SY, Z)SX - g(SX, Z)SY, \end{aligned} \tag{3.4}$$

where $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^m.

Let $\{e_1, e_2, \ldots, e_{2m-1}, e_{2m} := N\}$ be a basis of the tangent vector space T_zQ^m of Q^m at $z \in Q^m$. From (3.4), contracting Y and Z on M in Q^m, we have

$$\begin{aligned} \mathrm{Ric}(X) = (2m-1)X &- 3\eta(X)\xi - g(AN, N)(AX)^T + g(AX, N)(AN)^T \\ &- g(JAN, N)(JAX)^T + g(JAX, N)(JAN)^T \\ &+ (\mathrm{Tr}\, S)SX - S^2X, \end{aligned} \tag{3.5}$$

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where we have used the following

$$\begin{aligned}
\sum_{i=1}^{2m-1} g(Je_i, e_i) &= \sum_{i=1}^{2m} g(Je_i, e_i) - g(JN, N) = \mathrm{Tr}J - g(JN, N) = 0, \\
\sum_{i=1}^{2m-1} g(JX, e_i)(Je_i)^T &= \sum_{i=1}^{2m} g(JX, e_i)(Je_i)^T - g(JX, N)(JN)^T \\
&= (J^2X)^T - g(JX, N)(JN)^T = -X + g(X, \xi)\xi, \\
\sum_{i=1}^{2m-1} g(Ae_i, e_i) &= \sum_{i=1}^{2m} g(Ae_i, e_i) - g(AN, N) \\
&= \mathrm{Tr}A - g(AN, N) = -g(AN, N), \\
\sum_{i=1}^{2m-1} g(AX, e_i)(Ae_i)^T &= \sum_{i=1}^{2m} g(AX, e_i)(Ae_i)^T - g(AX, N)AN \\
&= X - g(AX, N)(AN)^T, \\
\sum_{i=1}^{2m-1} g(JAe_i, e_i)(JAX)^T &= \sum_{i=1}^{2m} g(JAe_i, e_i)(JAX)^T - g(JAN, N)(JAX)^T \\
&= (\mathrm{Tr}\, JA)(JAX)^T - g(JAN, N)(JAX)^T \\
&= -g(JAN, N)(JAX)^T,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{2m-1} g(JAX, e_i)(JAe_i)^T &= \sum_{i=1}^{2m} g(JAX, e_i)(JAe_i)^T - g(JAX, N)(JAN)^T \\
&= (JAJAX)^T - g(JAX, N)(JAN)^T \\
&= X - g(JAX, N)(JAN)^T.
\end{aligned}$$

On the other hand, for a real structure A of Q^m we decompose AX into its tangential and normal components given by $AX = BX + g(AX, N)N$. From this and the anti-commuting property between the complex structure J and real structure A, we get

$$AN = AJ\xi = -JA\xi = -\phi A\xi - g(A\xi, \xi)N. \tag{3.6}$$

In addition, from (3.3) we obtain that $g(A\xi, N) = 0$, which means that the unit vector field $A\xi$ is tangent to M. Thus, by using the Gauss formula, $\bar{\nabla}_X Y = \nabla_X Y +$

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$g(SX, Y)N$, we get

$$\begin{aligned}\nabla_X(A\xi) &= \bar{\nabla}_X(A\xi) - g(SX, A\xi)N \\ &= (\bar{\nabla}_X A)\xi + A(\bar{\nabla}_X\xi) - g(SX, A\xi)N \\ &= q(X)JA\xi + A(\nabla_X\xi + g(SX,\xi)N) - g(SX, A\xi)N \\ &= q(X)JA\xi + A\phi SX + g(SX,\xi)AN - g(SX, A\xi)N \\ &= q(X)\phi A\xi + q(X)g(A\xi,\xi)N + B\phi SX + g(\phi SX, AN)N \\ &\quad - g(SX,\xi)\phi A\xi - g(SX,\xi)g(A\xi,\xi)N - g(SX, A\xi)N \\ &= q(X)\phi A\xi + q(X)g(A\xi,\xi)N + B\phi SX - g(A\xi, SX)N \\ &\quad + g(A\xi,\xi)g(SX,\xi)N - g(SX,\xi)\phi A\xi \\ &\quad - g(SX,\xi)g(A\xi,\xi)N - g(SX, A\xi)N,\end{aligned}$$

where we have used the formulas $(\bar{\nabla}_X A)Y = q(X)JAY$ and (3.6). From this, *by comparing the tangential and normal parts of both sides*, we can assert the following:

LEMMA 3.3. *Let M be a real hypersurface in the complex quadric Q^m, $m \geqslant 3$. Then we obtain*

$$\nabla_X(A\xi) = q(X)\phi A\xi + B\phi SX - g(SX,\xi)\phi A\xi \tag{3.7}$$

and

$$q(X)g(A\xi,\xi) = 2g(SX, A\xi) \tag{3.8}$$

for any tangent vector field X of M.

By virtue of lemma 3.3 and the equations related to the Ricci tensor for real hypersurfaces in the complex quadric Q^m, we can prove our main theorem 1 in the introduction as follows:

PROPOSITION 3.4. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geqslant 3$, with parallel Ricci tensor. Then the unit normal vector field N becomes either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Proof. From the properties of complex structure J and real structure A, we get

$$JAX = \phi BX + g(\phi A\xi, X)\xi + g(A\xi, X)N$$

for any tangent vector field X of M. By using this formula and (3.6), the equation (3.5) can be rearranged:

$$\begin{aligned}\operatorname{Ric} X = (2m-1)X - 3\eta(X)\xi + g(A\xi,\xi)BX + g(\phi A\xi, X)\phi A\xi \\ + g(A\xi, X)A\xi + (\mathrm{Tr}S)SX - S^2X,\end{aligned}$$

together with $g(AN, N) = -g(A\xi,\xi)$ and $g(JAN, N) = g(A\xi, N) = 0$. Substituting the Reeb vector field ξ in this equation, we have

$$\operatorname{Ric}\xi = 2(m-2)\xi + 2g(A\xi,\xi)A\xi + (\alpha h - \alpha^2)\xi, \tag{3.9}$$

where h denotes the trace of the shape operator S, that is, $h = \mathrm{Tr}S$.

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From this, by using the assumption of Ricci parallelism, $\nabla \mathrm{Ric} = 0$, and (3.7), it follows that

$$\begin{aligned}
0 &= (\nabla_X \mathrm{Ric})\xi \\
&= 2(m-2)\nabla_X \xi + 2g\big(\nabla_X(A\xi), \xi\big)A\xi + 2g(A\xi, \nabla_X \xi)A\xi \\
&\quad + 2g(A\xi, \xi)\nabla_X(A\xi) + X(\alpha h - \alpha^2)\xi + (\alpha h - \alpha^2)\nabla_X \xi - \mathrm{Ric}(\nabla_X \xi) \\
&= 2(m-2)\phi SX + g(B\phi SX, \xi)A\xi + 2g(A\xi, \phi SX)A\xi \\
&\quad + 2g(A\xi, \xi)\big\{q(X)\phi A\xi + B\phi SX - \alpha\eta(X)\phi A\xi\big\} \\
&\quad + \big(X(\alpha h - \alpha^2)\big)\xi + (\alpha h - \alpha^2)\phi SX - \mathrm{Ric}(\phi SX).
\end{aligned}$$

By putting $X = \xi$ in the above equation, it yields that

$$2g(A\xi, \xi)\big(q(\xi) - \alpha\big)\phi A\xi + \big(\xi(\alpha h - \alpha^2)\big)\xi = 0,$$

because M is Hopf. Taking the inner product with ξ, it gives us $\xi(\alpha h - \alpha^2) = 0$. Therefore, we consequently have

$$g(A\xi, \xi)\big(q(\xi) - \alpha\big)\phi A\xi = 0. \tag{3.10}$$

If $g(A\xi, \xi) = 0$, the unit normal vector field N is $\mathfrak{A}$-isotropic.

From now on, we assume that the normal vector field N is not $\mathfrak{A}$-isotropic, that is, $g(A\xi, \xi) \neq 0$. Then, by virtue of (3.8), we obtain $q(\xi) = 2\alpha$. Then the equation (3.10) gives

$$\alpha\phi A\xi = 0.$$

From this, firstly, assume that the Reeb function $\alpha = g(S\xi, \xi)$ vanishes on M. Then (3.2) gives that $g(Y, AN)g(A\xi, \xi) = 0$ for any $Y \in T_x M$, $x \in M$. So, $g(A\xi, \xi) \neq 0$ gives $g(Y, AN) = 0$. This implies that $AN = g(AN, N)N$. Accordingly, $A^2 N = g(AN, N)AN = g(AN, N)^2 N$ gives $AN = \pm N$. That is, the unit normal vector field N is $\mathfrak{A}$-principal.

Now, let us consider the case of $\phi A\xi = 0$. We assert that the normal vector field N becomes $\mathfrak{A}$-principal. In fact, applying the structure tensor ϕ, it leads to $A\xi = g(A\xi, \xi)\xi$. Taking the inner product with $A\xi$ and using the self-adjoint property for real structure, that is, $A^2 = I$, it implies the following

$$g(A\xi, \xi)^2 = g(A\xi, A\xi) = g(\xi, \xi) = 1.$$

On the other hand, from (3.3) and as $g(A\xi, \xi) \neq 0$, we see that $g(A\xi, \xi) = -\cos 2t$, $t \in [0, \pi/4)$. According to these facts, $g(A\xi, \xi) = -1$, that is, $t = 0$. It implies that the normal vector field N is $\mathfrak{A}$-principal. □

Summing up lemma 3.3 and proposition 3.4, we give a complete proof of our main theorem 1 in the introduction.

Now let us give more information on Hopf real hypersurfaces in the complex quadric Q^m with $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic normal vector field. By using the formulas given in this section, we want to introduce well-known lemmas which are key roles in the proof of main theorem 2 as follows:

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LEMMA 3.5 [**26**]. *Let M be a Hopf hypersurface in Q^m such that the normal vector field N is $\mathfrak{A}$-principal everywhere. Then α is constant. Moreover, if $X \in \mathcal{C}$ is a principal vector field of M with principal curvature λ, then $2\lambda \neq \alpha$ and ϕX is a principal vector field of M with principal curvature $(\alpha\lambda + 2)/(2\lambda - \alpha)$.*

LEMMA 3.6 [**26**]. *Let M be a Hopf hypersurface in Q^m, $m \geqslant 3$, such that the normal vector field N is $\mathfrak{A}$-isotropic everywhere. Then α is constant.*

In § 4 and 5, the above two lemmas 3.5 and 3.6 will give a contribution in the proof of our main theorem 2 in the introduction.

4. Proof of main theorem 2 with $\mathfrak{A}$-principal unit normal vector field

Now in this section we consider only an $\mathfrak{A}$-principal unit normal vector field N for a real hypersurface M in Q^m with parallel Ricci tensor. Then from the curvature tensor in § 3 the Ricci tensor is given by

$$\begin{aligned}\mathrm{Ric}X &= (2m-1)X - 3\eta(X)\xi - g(AN,N)(AX)^T + g(AX,N)(AN)^T \\ &\quad - g(JAN,N)(JAX)^T + g(JAX,N)(JAN)^T + (\mathrm{Tr}S)SX - S^2X, \qquad (4.1)\end{aligned}$$

where $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^m.

From this, by using $AN = N$, $A\xi = -\xi$ and $AX = BX$ for an $\mathfrak{A}$-principal unit normal vector field, we have

$$\mathrm{Ric}\,X = (2m-1)X - 2\eta(X)\xi - AX + hSX - S^2X, \qquad (4.2)$$

where the mean curvature $h = \mathrm{Tr}\,S$ is defined by the trace of the shape operator S of M in Q^m. From this, let us use the assumption of parallel Ricci tensor, that is, $\nabla_X\mathrm{Ric} = 0$ for any $X \in T_xM$, $x \in M$. Then it follows that

$$\begin{aligned}0 &= -2g(\nabla_X\xi, Y)\xi - 2\eta(Y)\nabla_X\xi - (\nabla_XA)Y + (Xh)SY \\ &\quad + h(\nabla_XS)Y - (\nabla_XS^2)Y \\ &= -2g(\phi SX, Y)\xi - 2\eta(Y)\phi SX - (\nabla_XA)Y + (Xh)SY \\ &\quad + h(\nabla_XS)Y - (\nabla_XS^2)Y,\end{aligned} \qquad (4.3)$$

where $(\nabla_XA)Y = \nabla_X(AY) - A\nabla_XY$, Here, AY belongs to T_xM, $x \in M$, from the fact that $g(AY, N) = g(Y, AN) = g(Y, N) = 0$ for any tangent vector Y on M. Then by putting $Y = \xi$ in (4.3), we know that

$$\begin{aligned}2\phi SX &= -(\nabla_XA)\xi + (Xh)S\xi + h(\nabla_XS)\xi - (\nabla_XS^2)\xi \\ &= -q(X)JA\xi - \alpha\eta(X)AN + \alpha(Xh)\xi \\ &\quad + h(\nabla_XS)\xi - (\nabla_XS^2)\xi\end{aligned} \qquad (4.4)$$

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In order to get the equation (4.4) we have used the following

$$\begin{aligned}(\nabla_X A)\xi &= \nabla_X(A\xi) - A\nabla_X\xi \\ &= (\bar{\nabla}_X(A\xi))^T - A\nabla_X\xi \\ &= \left\{(\bar{\nabla}_X A)\xi + A\bar{\nabla}_X\xi\right\}^T - A\phi SX \\ &= q(X)JA\xi + A\phi SX + g(SX,\xi)AN - A\phi SX \\ &= q(X)JA\xi + \alpha\eta(X)AN,\end{aligned}$$

where $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^m. Moreover, we get

$$(\nabla_X S)\xi = \nabla_X(S\xi) - S\nabla_X\xi = (X\alpha)\xi + \alpha\phi SX - S\phi SX,$$

and

$$(\nabla_X S^2)\xi = \nabla_X(S^2\xi) - S^2\nabla_X\xi = (X\alpha^2)\xi + \alpha^2\phi SX - S^2\phi SX.$$

Then (4.4) can be written as follows:

$$\begin{aligned}2\phi SX = &-q(X)JA\xi - \alpha\eta(X)AN + \alpha(Xh)\xi + h(X\alpha)\xi + h\alpha\phi SX - hS\phi SX \\ &- (X\alpha^2)\xi - \alpha^2\phi SX + S^2\phi SX.\end{aligned}$$

As its scalar product with ξ yields $X(\alpha h - \alpha^2) = 0$, it can be reduced as follows:

$$\begin{aligned}2\phi SX = &-q(X)JA\xi - \alpha\eta(X)AN + h\alpha\phi SX - hS\phi SX \\ &- \alpha^2\phi SX + S^2\phi SX.\end{aligned}$$

From this, if we take the tangential part, we have the following:

$$(2 + \alpha^2 - h\alpha)\phi SX = -hS\phi SX + S^2\phi SX \tag{4.5}$$

for any tangent vector $X \in T_xM$, $x \in M$, because we have assumed that the unit vector field N is $\mathfrak{A}$-principal, that is, $AN = N$, and $JA\xi = -AJ\xi = -AN$.

On the other hand, by lemma 4.2 in Berndt and Suh [**3**] for a contact hypersurface in complex quadric Q^m with $\mathfrak{A}$-principal normal vector field N we have

$$2S\phi SX = \alpha(\phi S + S\phi)X + 2\phi X.$$

From this, it follows that

$$\begin{aligned}2S^2\phi SX &= \alpha(S\phi S + S^2\phi)X + 2S\phi X \\ &= \alpha\left(\left\{\frac{\alpha}{2}(S\phi + \phi S)X + \phi X\right\}\right) + \alpha S^2\phi X + 2S\phi X \\ &= \frac{\alpha^2}{2}(S\phi + \phi S)X + \alpha\phi X + \alpha S^2\phi X + 2S\phi X.\end{aligned} \tag{4.6}$$

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Then summing up (4.5) and (4.6), we have

$$(2+\alpha^2-h\alpha)\phi SX = -h\Big\{\frac{\alpha}{2}(S\phi+\phi S)X+\phi X\Big\}+\frac{\alpha^2}{4}(S\phi+\phi S)X \\ +\frac{\alpha}{2}\phi X+\frac{\alpha}{2}S^2\phi X+S\phi X. \tag{4.7}$$

On the other hand, we give the following important lemma which will be useful in the proof of our main theorem 1. Then for $\mathfrak{A}$-principal unit normal vector field we assert the following

Lemma 4.1. *Let M be a real hypersurface in the complex quadric Q^m, $m \geqslant 3$, with $\mathfrak{A}$-principal singular normal vector field N. Then we obtain:*

(i) $AX = BX$

(ii) $A\phi X = -\phi AX$

(iii) $A\phi SX = -\phi SX$ *and* $q(X) = 2g(SX, \xi)$

(iv) $ASX = SX - 2g(SX, \xi)\xi = SAX$

for any $X \in T_xM$, $x \in M$.

Proof. In this case we must have $g(AX, N) = 0$ for an $\mathfrak{A}$-principal normal N. This means that $AX \in T_xM$, $x \in M$. So it gives (i) in lemma 4.1.

On the other hand, the complex structure J and the real structure A satisfy the anti-commuting property, $JA = -AJ$. From this and $JX = \phi X + \eta(X)N$, we get

$$\phi AX - \eta(X)N = \phi AX + \eta(AX)N = JAX \\ = -AJX = -A\big(\phi X + \eta(X)N\big) = -A\phi X - \eta(X)N.$$

Hence it implies $A\phi X = -\phi AX$. That is, we get (ii) in lemma 4.1. Moreover, differentiating the equations $A\xi = -\xi$ and $AN = N$ with respect to the Levi-Civita connection $\bar\nabla$ of Q^m, respectively, it follows that

$$-q(X)N + A\phi SX + g(SX,\xi)N = q(X)JA\xi + A\phi SX + g(SX,\xi)AN \\ = (\bar\nabla_X A)\xi + A(\bar\nabla_X\xi) = -\bar\nabla_X\xi = -\phi SX - g(SX,\xi)N$$

and

$$-q(X)\xi - ASX = q(X)JAN - ASX \\ = (\bar\nabla_X A)N + A(\bar\nabla_X N) = \bar\nabla_X N = -SX,$$

together with $\bar\nabla_X Y = \nabla_X Y + g(SX, Y)N$, $\bar\nabla_X N = -SX$ (so-called the Gauss and Weingarten formulae) and $\nabla_X\xi = \phi SX$. The tangential and normal part of the above two equations give (iii) and (iv) in lemma 4.1, respectively. Since the shape operator S of M and the real structure A are symmetric, we also obtain $ASX = SAX$.

Summing up all the facts above, we give a complete proof of our lemma 4.1. □

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By virtue of lemma 4.1, some characterizations of Hopf hypersurfaces in terms of singularity of the normal vector field are being investigated. Among them, as a new characterization of $\mathfrak{A}$-principal singular normal vector field, Lee and Suh [14] have proved a remarkable result as follows:

PROPOSITION 4.2. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geqslant 3$. Then M has an $\mathfrak{A}$-principal singular normal vector field N if and only if M is a contact real hypersurface with constant mean curvature and non-vanishing Reeb function in Q^m.*

Moreover, Berndt and Suh [3] have proved that a real hypersurface M is locally congruent to a tube over S^m in Q^m if and only if the shape operator S of M satisfies $S\phi + \phi S = k\phi$ for a non-zero constant k. Here we note that $k\alpha = -2$ for the constant Reeb function α on M. Then let us check whether a tube over S^m could satisfy (4.7) or not. Then (4.7) gives

$$\begin{aligned}(2+\alpha^2-h\alpha)\phi SX &= -h\left\{\frac{\alpha k}{2}+1\right\}\phi X + \frac{\alpha^2}{4}k\phi X\\ &\quad + \frac{\alpha}{2}\phi X + \frac{\alpha}{2}S^2\phi X + S\phi X.\end{aligned}$$

If we consider an eigenvector such that $SX = \lambda X$, then $(S\phi + \phi S)X = k\phi X$ gives that $S\phi X = (k-\lambda)\phi X$. From this, together with (4.7) using $\alpha k = -2$, it follows that

$$\alpha\lambda^2 - 2(\alpha^2 - h\alpha + 1)\lambda = 0.$$

Then either $\lambda = 0$ or $\lambda = \sqrt{2}\tan\sqrt{2}r$. Moreover, the trace h of the shape operator becomes $h = \alpha + (m-1)k$ (see also [3]). But for a tube over a sphere S^m we know that

$$\begin{aligned}\sqrt{2}\tan\sqrt{2}r &= \frac{2}{\alpha}(\alpha^2 - h\alpha + 1)\\ &= 2(\alpha - h) + \frac{2}{\alpha}\\ &= \frac{4(m-1)}{\alpha} + \frac{2}{\alpha}\\ &= \frac{2(2m-1)}{\alpha}\\ &= -(2m-1)\sqrt{2}\tan\sqrt{2}r,\end{aligned}$$

where in the third equality we have used $\alpha - h = -(m-1)k = 2(m-1)/\alpha$. This gives that $2m\sqrt{2}\tan\sqrt{2}r = 0$, which gives us a contradiction. So we conclude that a real hypersurface in Q^m which is a tube over a m-dimensional sphere S^m does not admit parallel Ricci tensor. Of course, in this case the unit normal N is $\mathfrak{A}$-principal.

Summing up the above documents, we conclude that there does not exist a Hopf real hypersurface in the complex quadric Q^m with parallel Ricci tensor when the unit normal vector field N is $\mathfrak{A}$-principal.

5. Proof of main theorem with $\mathfrak{A}$-isotropic unit normal vector field

In § 4, we proved that there does not exist a Hopf real hypersurface with *parallel Ricci tensor* in the complex quadric Q^m with $\mathfrak{A}$-principal unit normal vector field. Motivated by the result of § 4, in this section we give a complete proof of our main theorem for real hypersurfaces with parallel Ricci tensor when M has an $\mathfrak{A}$-isotropic unit normal vector field.

Since we assumed that the unit normal N is $\mathfrak{A}$-isotropic, by the definition in § 3 we know that $t = \pi/4$. Then by the expression of the $\mathfrak{A}$-isotropic unit normal vector field, the equation (3.3) gives $N = (1/\sqrt{2})Z_1 + (1/\sqrt{2})JZ_2$. This implies that

$$g(\xi, A\xi) = 0,\ g(\xi, AN) = 0,\ g(AN, N) = 0,\ g(A\xi, N) = 0,$$

and

$$g(JAN, \xi) = -g(AN, N) = 0.$$

Then the vector fields AN and $A\xi$ become tangent vector fields to M in Q^m. Moreover, by using these equations, we take the derivative of the Ricci tensor as follows:

$$\begin{aligned}(\nabla_Y \mathrm{Ric})X &= \nabla_Y(\mathrm{Ric}(X)) - \mathrm{Ric}(\nabla_Y X)\\ &= -3(\nabla_Y \eta)(X)\xi - 3\eta(X)\nabla_Y\xi + g(X, \nabla_Y(AN))AN\\ &\quad + g(AX, N)\nabla_Y(AN) + g((\nabla_Y(A\xi), X)A\xi\\ &\quad + g(A\xi, X)\nabla_Y(A\xi) + (Yh)SX + h(\nabla_Y S)X - (\nabla_Y S^2)X.\end{aligned} \tag{5.1}$$

Since AN is a tangent vector field for an $\mathfrak{A}$-isotropic normal vector field, we know that

$$\nabla_Y(AN) = \{(\bar{\nabla}_Y A)N + A\bar{\nabla}_Y N\}^T = \{q(Y)JAN - ASY\}^T,$$

and

$$\nabla_Y(A\xi) = -q(Y)AN + B\phi SY + g(SY, \xi)AN,$$

where we have used (3.6) and (3.7), and $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^m. By our assumption of Ricci parallelism, the above formula becomes

$$\begin{aligned}0 = &-3g(\phi SY, X)\xi - 3\eta(X)\phi SY + \{q(Y)g(JAN, X) - g(ASY, X)\}AN\\ &+ g(AX, N)\{q(Y)JAN - ASY\}^T\\ &+ \{-q(Y)g(AN, X) + g(B\phi SY, X) + \alpha\eta(Y)g(AN, X)\}A\xi\\ &+ g(A\xi, X)\{-q(Y)AN + B\phi SY + \alpha\eta(Y)AN\}\\ &+ (Yh)SX + h(\nabla_Y S)X - (\nabla_Y S^2)X.\end{aligned} \tag{5.2}$$

Now we assert an important lemma which gives a key role in the proof of our main theorem 2 as follows:

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LEMMA 5.1. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geqslant 3$, with $\mathfrak{A}$-isotropic unit normal vector field N. Then*

$$SA\xi = 0 \quad \textit{and} \quad SAN = -S\phi A\xi = 0.$$

Proof. Let us denote by $\mathcal{Q}^{\perp} = \text{Span}\{A\xi, AN\}$, where $\mathcal{Q}$ is the maximal $\mathfrak{A}$-invariant subspace in the complex subbundle of $\mathcal{C}$. By differentiating $g(AN, N) = 0$ and using $(\bar{\nabla}_X A)Y = q(X)JAY$ and the equation of Weingarten, we know that

$$\begin{aligned} 0 &= g(\bar{\nabla}_X(AN), N) + g(AN, \bar{\nabla}_X N) \\ &= g(q(X)JAN - ASX, N) - g(AN, SX) \\ &= -2g(ASX, N). \end{aligned}$$

Then $SAN = 0$. From (3.6), we obtain $AN = -\phi A\xi$. So, it implies that $S\phi A\xi = 0$. Moreover, by differentiating $g(A\xi, N) = 0$ and using $g(AN, N) = 0$, we have:

$$\begin{aligned} 0 &= g(\bar{\nabla}_X(A\xi), N) + g(A\xi, \bar{\nabla}_X N) \\ &= g(q(X)JA\xi + A(\phi SX + g(SX, \xi)N), N) - g(SA\xi, X) \\ &= -2g(SA\xi, X) \end{aligned}$$

for any $X \in T_z M$, $z \in M$, where in the third equality we have used $\phi AN = JAN = -AJN = A\xi$. Then it follows that

$$SA\xi = 0.$$

It completes the proof of our assertion. □

By taking the inner product of (5.2) with the Reeb vector field ξ, we have

$$\begin{aligned} 0 = &-3g(\phi SY, X) - g(AX, N)g(ASY, \xi) + g(AX, \xi)g(B\phi SY, \xi) \\ &+ (Yh)\alpha\eta(X) + hg((\nabla_Y S)X, \xi) - g((\nabla_Y S^2)X, \xi). \end{aligned} \tag{5.3}$$

On the other hand, let us use the following calculation

$$\begin{aligned} (\nabla_X S)\xi &= (X\alpha)\xi + \alpha\phi SX - S\phi SX, \\ (\nabla_X S^2)\xi &= (X\alpha^2)\xi + \alpha^2\phi SX - S^2\phi SX. \end{aligned}$$

By putting $X = \xi$ in (5.2) and using the above formulas, we get the following

$$\begin{aligned} 3\phi SY &= (Yh)S\xi + h(\nabla_Y S)\xi - (\nabla_Y S^2)\xi - g(ASY, \xi)AN + g(B\phi SY, \xi)A\xi \\ &= (Yh)\alpha\xi + h\{(Y\alpha)\xi + \alpha\phi SY - S\phi SY\} \\ &\quad - \{(Y\alpha^2)\xi + \alpha^2\phi SY - S^2\phi SY\} - g(ASY, \xi)AN + g(B\phi SY, \xi)A\xi \\ &= \alpha h\phi SY - hS\phi SY - \alpha^2\phi SY + S^2\phi SY - g(ASY, \xi)AN \\ &\quad + g(B\phi SY, \xi)A\xi + \big(\alpha(Yh) + h(Y\alpha) - (Y\alpha^2)\big)\xi, \end{aligned} \tag{5.4}$$

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where we have used $g(A\xi, N) = 0$ and $g(AN, N) = g(A\xi, \xi) = 0$. Besides, taking the inner product of (5.4) with ξ, it gives

$$(\alpha(Yh) + h(Y\alpha) - (Y\alpha^2)) = 0. \tag{5.5}$$

From this and by virtue of lemma 5.1, the equation (5.4) can be rearranged as follows:

$$(3 + \alpha^2 - \alpha h)\phi SY = -hS\phi SY + S^2\phi SY. \tag{5.6}$$

On the other hand, by virtue of lemma 4.2 in [**26**], we have the following

$$2S\phi SX = \alpha(S\phi + \phi S)X + 2\phi X - 2g(X, AN)A\xi + 2g(X, A\xi)AN. \tag{5.7}$$

Now let us consider the distribution $\mathcal{Q}^\perp$, which is an orthogonal complement of the maximal $\mathfrak{A}$-invariant subspace $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$ of T_xM, $x \in M$ in Q^m. Then by lemma 3.1 in § 3, the orthogonal complement $\mathcal{Q}^\perp = \mathcal{C}\ominus\mathcal{Q}$ becomes $\mathcal{C}\ominus\mathcal{Q} = \text{Span}\{AN, A\xi\}$. Then by lemma 5.1 the distribution $\mathcal{Q}^\perp$ is invariant by the shape operator S. Moreover, we have known that $SAN = 0$ and $SA\xi = 0$.

On the distribution $\mathcal{Q}$, we know that $AX \in T_xM$, $x \in M$, because $AN \in Q^\perp$. Moreover, by using the property of $g(X, A\xi) = 0$ for any $X \in \mathcal{Q}$, the equation (5.7) gives that

$$2S\phi SX = \alpha(S\phi + \phi S)X + 2\phi X \quad \forall X \in \mathcal{Q}. \tag{5.8}$$

Then the shape operator S can be expressed in such a way that

$$S = \text{diag}\,(\alpha, 0, 0, \lambda_1, \dots, \lambda_{m-2}, \mu_1, \dots, \mu_{m-2})$$

where $\text{diag}\,(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$ denotes a $(m \times m)$-diagonal matrix with the main diagonal entries ε_k, $k = 1, 2, \dots, m$.

Now, we can take an orthonormal basis $X_1, \dots, X_{2(m-2)} \in \mathcal{Q}$ such that $AX_i = \lambda_i X_i$ for $i = 1, \dots, m-2$. Then by (5.8) we know that $\alpha \neq 2\lambda_i$ for all i. Furthermore, we get

$$S\phi X_i = \frac{\alpha\lambda_i + 2}{2\lambda_i - \alpha}\phi X_i.$$

Hence, on the distribution $\mathcal{Q}$, for any X and $\phi X \in \mathcal{Q}$ such that $SX = \lambda X$ and $S\phi X = \mu\phi X$, $\mu = (\alpha\lambda + 2)/(2\lambda - \alpha)$, the equation (5.6) becomes the following

$$\begin{aligned} \lambda(3 + \alpha^2 - \alpha h)\phi X &= -h\lambda S\phi X + \lambda S^2\phi X \\ &= -h\lambda\mu\phi X + \lambda\mu^2\phi X \\ &= \lambda\mu(\mu - h)\phi X. \end{aligned} \tag{5.9}$$

From this, now we consider the following two cases.

- Case 1. For non-vanishing principal curvatures $\lambda \neq 0$.

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Then in this case, by (5.9), any principal curvatures of the shape operator S on the distribution $\mathcal{Q}$ satisfy the following quadratic equation

$$x^2 - hx + (\alpha h - \alpha^2 - 3) = 0. \tag{5.10}$$

Since the discriminant of (5.10) is $D = (h - 2\alpha)^2 + 12 > 0$, we conclude that there exist two distinct principal curvatures λ and μ satisfying

$$\lambda = \frac{h + \sqrt{D}}{2} \quad \text{and} \quad \mu = \frac{h - \sqrt{D}}{2}.$$

Moreover, by Vieta's formula for quadratics, we get $\lambda + \mu = h$. So, it follows

$$\begin{aligned} h &= \mathrm{Tr} S \\ &= \alpha + (m-2)(\lambda + \mu) \\ &= \alpha + (m-2)h. \end{aligned} \tag{5.11}$$

From this, the mean curvature h is given by

$$h = -\frac{\alpha}{m-3} \tag{5.12}$$

On the other hand, by (5.5) we know that the function $(\alpha h - \alpha^2)$ is constant. Then the constant $(\alpha h - \alpha^2)$ implies

$$\begin{aligned} \alpha h - \alpha^2 &= \alpha\{\alpha + (m-2)(\lambda + \mu)\} - \alpha^2 \\ &= (m-2)(\lambda + \mu)\alpha \\ &= (m-2)h\alpha. \end{aligned}$$

From this, it follows that the function αh is constant. Then the constant αh, together with the constancy of $\alpha h - \alpha^2$, implies that the Reeb function α must be constant. Thus the trace h of the shape operator S should be constant. So the above quadratic equation (5.10) has constant coefficients. This gives that all of principal curvatures λ and μ are constant principal curvatures. Accordingly, the expression of the shape operator S of M in Q^m with parallel Ricci tensor is given by

$$S = \mathrm{diag}\left(\alpha, 0, 0, \underbrace{\frac{h+\sqrt{D}}{2}, \ldots, \frac{h+\sqrt{D}}{2}}_{(m-1)}, \underbrace{\frac{h-\sqrt{D}}{2}, \ldots, \frac{h-\sqrt{D}}{2}}_{(m-1)}\right)$$

This means that on the distribution $\mathcal{Q}$ the following holds

$$S\phi + \phi S = h\phi.$$

From this, together with (5.7), we get the following for any $X \in \mathcal{Q}$

$$2S\phi SX = \alpha h\phi X + 2\phi X \tag{5.13}$$

Since $SX = \lambda X$, $S\phi X = \mu\phi X$, $\mu = (\alpha\lambda + 2)/(2\lambda - \alpha)$, (5.13) becomes

$$2\lambda\mu = \alpha h + 2. \tag{5.14}$$

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From this we can assert that the Reeb function α is non-vanishing. In fact, if the constant Reeb function α vanishes, then (5.14) gives $\lambda\mu = 1$. But λ and μ are roots of the quadratic equation (5.10). So they satisfy $\lambda + \mu = h$. Moreover, (5.12) gives $h = 0$. So it follows that $\lambda = -\mu$. Then this gives a contradiction.

Now let us put $k = \alpha h + 2$. Then from (5.12) it follows that $k = -(\alpha^2/(m-3+2))$. Then (5.14) becomes

$$2\alpha\lambda^2 + (4-2k)\lambda + \alpha k = 0, \tag{5.15}$$

where $4-2k = 2\alpha^2/(m-3)$ and $\alpha k = -(\alpha^3 - 2\alpha(m-3))/(m-3)$. Then (5.15) and the non-vanishing Reeb function $\alpha \neq 0$ imply that the principal curvatures λ and μ are roots of the following quadratic equation

$$x^2 + \frac{\alpha}{m-3}x - \left\{\frac{\alpha^2}{2(m-3)} - 1\right\} = 0. \tag{5.16}$$

Comparing (5.16) with the quadratic equation (5.10), we have the following

$$\frac{\alpha^2}{2(m-3)} - 1 = \alpha^2 + 3 - \alpha h. \tag{5.17}$$

From this, together with $h = -\alpha/(m-3)$ in (5.12), it follows that

$$\begin{aligned}\frac{\alpha^2}{2(m-3)} - 1 &= \alpha^2 + 3 - \alpha h \\ &= \alpha^2 + 3 + \frac{\alpha^2}{m-3} \\ &= \frac{2(m-2)\alpha^2}{2(m-3)} + 3.\end{aligned} \tag{5.18}$$

Then it implies that

$$\frac{(2m-5)\alpha^2}{2(m-3)} = -4,$$

which gives a contradiction for $m \geqslant 4$. So this case does not happen for real hypersurfaces with parallel Ricci tensor in the complex quadric Q^m.

- Case 2. For vanishing principal curvatures $\lambda = 0$.

Then in this case, by using lemma 5.1, we conclude that on the distribution $\xi \oplus [\mathcal{C} \ominus \mathcal{Q}] \oplus \mathcal{Q}$, where $\mathcal{C}\ominus\mathcal{Q} = \mathrm{Span}\{AN, A\xi\}$ the shape operator becomes

$$S = \mathrm{diag}\,(\alpha, 0, 0, \underbrace{-\frac{2}{\alpha}, \dots, -\frac{2}{\alpha}}_{(m-1)}, \underbrace{0, \dots, 0}_{(m-1)}).$$

Then the vanishing principal curvature $\lambda = 0$ gives $\mu = -2/\alpha \neq 0$. From this, together with (5.9), the other principal curvatures also satisfy

$$x^2 - hx + (\alpha h - \alpha^2 - 3) = 0. \tag{5.19}$$

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So, the quadratic equation (5.19) gives that $h = \lambda + \mu = \mu = -2/\alpha$, because one root is assumed $\lambda = 0$. Moreover, the constant term of (5.9) identically vanishes $\alpha h - \alpha^2 - 3 = 0$, so it follows that $h = (\alpha^2 + 3)/\alpha$. Then these two formulas for the mean curvature h gives that $\alpha^2 + 5 = 0$. This implies a contradiction. So this case also can not be considered for a real hypersurface M in the complex quadric Q^m with parallel Ricci tensor.

Summing up the above two cases and all of discussions, we give a complete proof of our main theorem 2 in the introduction.

Remark 2. In this remark let us check whether the Ricci tensor of the tube M over a totally geodesic complex projective space $\mathbb{C}P^k$ in the complex quadric Q^m, $m = 2k$, mentioned in Berndt and Suh [**2**] is parallel or not. Then by a theorem in [**2**], the shape operator S commutes with the structure tensor ϕ, that is, $S\phi = \phi S$. In this case we know that the normal vector field N of M in Q^{2k} is $\mathfrak{A}$-isotropic. So let us suppose that the Ricci tensor of M is parallel. Then for any $X \in \mathcal{Q}$ such that $SY = \lambda Y$ the equation (5.6) gives that

$$(3 + \alpha^2 - \alpha h)\lambda\phi Y = -h\lambda^2\phi Y + \lambda^3\phi Y. \tag{5.20}$$

On the other hand, by (5.7), together with the commuting property $S\phi = \phi S$, we know that

$$(2\lambda - \alpha)\lambda\phi Y = (\alpha\lambda + 2)\phi Y \tag{5.21}$$

From this, we see that the function $\lambda \neq 0$. Then by (5.9) with $\lambda \neq 0$, we get

$$\lambda^2 - h\lambda + (\alpha h - \alpha^2 - 3) = 0.$$

Moreover, (5.21) gives that

$$\lambda^2 - \alpha\lambda - 1 = 0.$$

From these two equations we know that $h = \alpha$ and $\alpha h - \alpha^2 - 3 = -1$. Then these two equations give us a contradiction. So a tube over a complex k-dimensional projective space $\mathbb{C}P^k$ never has parallel Ricci tensor.

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