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Sharp upper and lower bounds for the gamma function

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We prove that for all $x > 0$, we have

$$\sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\alpha}{x^5}\right) < \Gamma(x+1) < \sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\beta}{x^5}\right)$$

with the best possible constants $\alpha = 0$ and $\beta = \frac{1}{1620}$.

1. Introduction

In 1916, Ramanujan [6–8], [9, p. 339] published the following elegant double inequality for Euler's gamma function in the *Journal of the Indian Mathematical Society* without proof:

$$\sqrt{\pi}\left(\frac{x}{\mathrm{e}}\right)^x\left(8x^3+4x^2+x+\frac{1}{100}\right)^{1/6} < \Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{\mathrm{e}}\right)^x\left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6}, \quad x \geqslant 0. \tag{1.1}$$

Karatsuba [5] proved in 2001 that (1.1) holds for all $x \geqslant 1$. Two years later, Alzer [4] showed that (1.1) is also valid for all $x \in [0, 1]$. Moreover, it was proved in [4] that on the left-hand side the constant $1/100$ can be replaced by the slightly larger value

$$\min_{0.6\leqslant x\leqslant 0.7} \Delta(x) = 0.010\,045\,0\ldots,$$

where

$$\Delta(x) = \left(\frac{1}{\pi}\right)^3\left[\Gamma(x+1)\left(\frac{\mathrm{e}}{x}\right)^x\right]^6 - 8x^3 - 4x^2 - x.$$

In 2002, Windschitl [10] discovered a remarkable approximation formula, which connects the gamma function with the hyperbolic sine function:

$$\Gamma(x+1) \cong \sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^x\left(x\sinh\frac{1}{x}+\frac{1}{810x^6}\right)^{x/2}. \tag{1.2}$$

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He pointed out that for $x > 8$, this formula is good to more than 8 decimal places, and therefore suggested it for computing the values of the gamma function on calculators with limited program or register memory.

Formula (1.2) inspired us to look for a counterpart of Ramanujan's inequalities (1.1). We ask: what are the best possible constants α and β such that

$$\sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\alpha}{x^5}\right) < \Gamma(x+1) < \sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\beta}{x^5}\right) \quad (1.3)$$

is valid for all $x > 0$? It is the aim of this paper to answer this question. To prove our main result we need several lemmas. They are collected in the next section. In § 3 we determine the optimal values α, β in (1.3), and we compare the lower and upper bounds for $\Gamma(x+1)$ given in (1.1) and (1.3).

The numerical and algebraic computations have been carried out with the computer program Maple V, Release 5.1.

2. Lemmas

Our first lemma presents some asymptotic formulae and a limit relation. Let ψ denote the logarithmic derivative of the gamma function.

Lemma 2.1. *For $x \to \infty$, we have*

$$\log\Gamma(x) \sim (x-\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log(2\pi) + \frac{1}{12x} - \frac{1}{360x^3} + \cdots, \quad (2.1)$$

$$\psi(x) \sim \log x - \frac{1}{2x} - \frac{1}{12x^2} + \cdots, \quad (2.2)$$

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^x\left[1+\frac{c_1}{x}+\frac{c_2}{x^2}+\frac{c_3}{x^3}+\frac{c_4}{x^4}+\frac{c_5}{x^5}+\cdots\right], \quad (2.3)$$

where

$$c_1 = \frac{1}{12},\quad c_2 = \frac{1}{288},\quad c_3 = -\frac{139}{51\,840},\quad c_4 = -\frac{571}{2\,488\,320},\quad c_5 = \frac{163\,879}{209\,018\,880},$$

$$\lim_{x\to\infty} x^5\left[\left(x\sinh\frac{1}{x}\right)^{x/2} - \left(1+\frac{c_1}{x}+\frac{c_2}{x^2}+\frac{c_3}{x^3}+\frac{c_4}{x^4}\right)\right] = \frac{6971}{41\,803\,776}. \quad (2.4)$$

Proof. Formulae (2.1)–(2.3) are given in [1, pp. 257, 259] and [11]. Let

$$h(t) = \left(\frac{\sinh t}{t}\right)^{1/(2t)}.$$

Then we have

$$h(0) = 1 \quad \text{and} \quad h^{(k)}(0) = c_k k! \text{ for } k = 1, 2, 3, 4.$$

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Taylor's theorem gives

$$\frac{1}{t^5}\left[h(t)-\sum_{k=0}^{5}\frac{h^{(k)}(0)}{k!}t^k\right]=\frac{h^{(6)}(\epsilon t)}{6!}t,\quad 0<\epsilon<1.$$

It follows that

$$\lim_{t\to 0}\frac{1}{t^5}\left[h(t)-\left(1+\sum_{k=1}^{4}c_k t^k\right)\right]=\frac{h^{(5)}(0)}{5!}=\frac{6971}{41\,803\,776}.$$

This yields (2.4). □

The following double inequality for ψ' is proved in [2].

LEMMA 2.2. *For all real numbers $x>0$ and integers $n\geqslant 0$, we have*

$$T_{2n}(x)<\psi'(x)<T_{2n+1}(x), \tag{2.5}$$

where

$$T_p(x)=\frac{1}{x}+\frac{1}{2x^2}+\sum_{k=1}^{p}\frac{B_{2k}}{x^{2k+1}}.$$

Here, B_n denotes the nth Bernoulli number.

The next lemma provides rational bounds for $1/\sinh^2$.

LEMMA 2.3. *Let*

$$\theta(t)=\frac{1}{t^2}-\frac{1}{3}+\frac{1}{15}t^2-\frac{2}{189}t^4+\frac{1}{675}t^6\quad\text{and}\quad\chi(t)=\theta(t)-\frac{2}{10\,395}t^8.$$

For all real numbers $t>0$, we have

$$\chi(t)<\frac{1}{(\sinh t)^2}<\theta(t). \tag{2.6}$$

Proof. First, we prove the right-hand side of (2.6). We define

$$\phi(t)=\theta(t)(\sinh t)^2-1.$$

Differentiation gives

$$\frac{2\mathrm{e}^t}{\sinh t}\phi'(t)=\theta'(t)(\mathrm{e}^{2t}-1)+2\theta(t)(\mathrm{e}^{2t}+1). \tag{2.7}$$

A short calculation reveals that

$$\theta(t)>0\quad\text{for } t>0 \tag{2.8}$$

and that there exists a number $t_0\in(2.052,2.053)$ such that

$$(t-t_0)\theta'(t)>0\quad\text{for } 0<t\neq t_0. \tag{2.9}$$

Using (2.7)–(2.9), we conclude that ϕ' is positive on (t_0,∞).

Next, we prove that $\phi'(t)$ is also positive for $t \in (0, t_0]$. Let

$$\delta(t) = \sum_{k=0}^{4} \gamma_{2k} t^{2k} \quad \text{with } \gamma_k = \frac{2^k B_k}{k!}.$$

We have

$$t \coth t - \delta(t) = \sum_{k=5}^{\infty} \gamma_{2k} t^{2k} = \sum_{k=3}^{\infty} (\gamma_{4k-2} + \gamma_{4k} t^2) t^{4k-2}. \tag{2.10}$$

Applying the inequalities

$$\frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{-2n}} < (-1)^{n+1} B_{2n} \leqslant \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{\rho-2n}}, \quad n \geqslant 1, \tag{2.11}$$

where

$$\rho = 2 + \frac{\log(1 - 6/\pi^2)}{\log 2} = 0.649\ldots$$

(see [3]), for $k \geqslant 3$ we obtain

$$\frac{-\gamma_{4k}}{\gamma_{4k-2}} t^2 = \frac{-B_{4k}}{k(4k-1) B_{4k-2}} t^2 < \frac{1 - 2^{2-4k}}{1 - 2^{\rho-4k}} \left(\frac{t}{\pi}\right)^2 < 1, \quad 0 < t \leqslant \pi. \tag{2.12}$$

From (2.10) and (2.12) we conclude that

$$\coth t > \frac{\delta(t)}{t}, \quad 0 < t \leqslant \pi. \tag{2.13}$$

Let $0 < t \leqslant t_0$. Using (2.7), (2.8) and (2.13) gives

$$\frac{2\mathrm{e}^t}{(\mathrm{e}^{2t} - 1)\sinh t} \phi'(t) = \theta'(t) + 2\theta(t) \coth t > \theta'(t) + 2\theta(t) \frac{\delta(t)}{t} = \frac{2t^7}{22\,325\,625} \nu(t),$$

where

$$\nu(t) = -7t^6 + 120t^4 - 1550t^2 + 21\,000$$

Since ν is positive on $(0, t_0]$, we conclude that $\phi'(t) > 0$ for $0 < t \leqslant t_0$.

Thus, ϕ is strictly increasing on $(0, \infty)$, so that, for $t > 0$, we obtain

$$\phi(t) > \lim_{z \to 0} \phi(z) = 0.$$

This completes the proof of the right-hand side of (2.6).

Now we show that

$$\eta(t) = 1 - (\sinh t)^2 \chi(t)$$

is positive on $(0, \infty)$. By differentiation we get

$$-\frac{2\mathrm{e}^t}{\sinh t} \eta'(t) = \chi'(t)(\mathrm{e}^{2t} - 1) + 2\chi(t)(\mathrm{e}^{2t} + 1). \tag{2.14}$$

We have

$$\chi'(t) < 0 \quad \text{for } t > 0. \tag{2.15}$$

Furthermore, there exists a number $t_1 \in (2.235, 2.239)$ such that

$$(t - t_1)\chi(t) < 0 \quad \text{for } 0 < t \neq t_1. \tag{2.16}$$

It follows from (2.14)–(2.16) that $\eta'(t) > 0$ for $t > t_1$. Let

$$\kappa(t) = \sum_{k=0}^{5} \gamma_{2k} t^{2k}.$$

Then we obtain

$$t \coth t - \kappa(t) = \sum_{k=3}^{\infty} (\gamma_{4k} + \gamma_{4k+2} t^2) t^{4k}.$$

Using (2.11) for $k \geqslant 3$ gives

$$\frac{\gamma_{4k+2}}{-\gamma_{4k}} t^2 < \frac{1 - 2^{-4k}}{1 - 2^{\rho-2-4k}} \left(\frac{t}{\pi}\right)^2 < 1, \quad 0 < t \leqslant \pi.$$

This leads to

$$\coth t < \frac{\kappa(t)}{t}, \quad 0 < t \leqslant \pi. \tag{2.17}$$

Let $0 < t \leqslant t_1$. From (2.14), (2.15) and (2.17) we get

$$-\frac{2\mathrm{e}^t}{(\mathrm{e}^{2t} + 1)\sinh t} \eta'(t) = \frac{\chi'(t)}{\coth t} + 2\chi(t) < \frac{t\chi'(t)}{\kappa(t)} + 2\chi(t) = -\frac{2t^{10}}{51\,975} \frac{p_1(t)}{p_2(t)},$$

where

$$p_1(t) = 100t^8 - 1760t^6 + 23\,023t^4 - 269\,280t^2 + 3\,420\,450$$

and

$$p_2(t) = 10t^{10} - 99t^8 + 990t^6 - 10\,395t^4 + 155\,925t^2 + 467\,775.$$

Since p_1 and p_2 are positive on $[0, 2.239]$, we conclude that $\eta'(t) > 0$ for $t \in (0, t_1]$. Hence, for $t > 0$, we have

$$\eta(t) > \lim_{z \to 0} \eta(z) = 0.$$

This implies the left-hand side of (2.6). □

Moreover, we need inequalities for certain polynomials of degree 13 and 21, respectively.

Lemma 2.4. *Let*

$$\begin{aligned}\lambda(t) = {} & 10t^{13} + 90t^{12} + 283t^{11} + 147t^{10} - 962t^9 - 258t^8 + 7473t^7 + 5673t^6 \\ & - 44\,493t^5 - 120\,572t^4 - 131\,208t^3 - 65\,912t^2 - 17\,325t - 1925. \end{aligned} \tag{2.18}$$

For all $t \in [0, 2.1]$ *we have* $\lambda(t) < 0$.

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Proof. Let $0 \leqslant t \leqslant 2.1$. Then

$$\begin{aligned}\lambda(t) &< t^9(10t^4 - 962) + t^5(90t^7 - 44\,493) + t^3(283t^8 - 131\,208)\\ &\quad + t^2(147t^8 - 65\,912) + t^4(7473t^3 + 5673t^2 - 120\,572)\\ &< 0,\end{aligned}$$

since all terms in brackets are negative. □

LEMMA 2.5. *Let*

$$\mu(t) = a(t) - b(t), \tag{2.19}$$

where

$$a(t) = \sum_{k=0}^{17} a_k t^k \quad \textit{and} \quad b(t) = \sum_{k=9}^{21} b_k t^k,$$

with

$$\begin{gathered} a_0 = 8\,891\,467\,200, \qquad a_1 = 97\,806\,139\,200, \qquad a_2 = 468\,363\,546\,000,\\ a_3 = 1\,239\,747\,720\,750, \quad a_4 = 1\,797\,428\,036\,250, \quad a_5 = 1\,595\,137\,522\,770,\\ a_6 = 877\,527\,340\,470, \qquad a_7 = 280\,158\,434\,100, \qquad a_8 = 36\,995\,476\,375,\\ a_9 = a_{10} = a_{11} = 0, \qquad a_{12} = 32\,584\,465, \qquad a_{13} = a_{14} = a_{15} = a_{16} = 0,\\ a_{17} = 1375, \qquad b_9 = 5\,041\,210\,075, \qquad b_{10} = 2\,139\,981\,547,\\ b_{11} = 61\,322\,767, \qquad b_{12} = 0, \qquad b_{13} = 16\,261\,905,\\ b_{14} = 11\,624\,085, \qquad b_{15} = 2\,771\,604, \qquad b_{16} = 232\,419,\\ b_{17} = 0, \qquad b_{18} = 6655, \qquad b_{19} = 3685,\\ b_{20} = 847, \qquad b_{21} = 77.\end{gathered}$$

For all $t \in [0, 2]$, we have $\mu(t) > 0$.

Proof. If $0 \leqslant t \leqslant 1$, then

$$\mu(t) \geqslant a(0) - b(1) = 1\,618\,051\,534.$$

Let $1 \leqslant t \leqslant 2$. We have

$$\begin{aligned}\mu(t) &= t^{12}(a_{17}t^5 + a_{12} - b_{20}t^8 - b_{21}t^9) + t^8(a_8 - b_{19}t^{11})\\ &\quad + t^7(a_7 - b_{18}t^{11}) + t^6(a_6 - b_{16}t^{10}) + t^5(a_5 - b_{15}t^{10})\\ &\quad + t(a_4t^3 + a_1 - b_{14}t^{13} - b_{13}t^{12} - b_{10}t^9)\\ &\quad + t^2(a_2 - b_{11}t^9) + (a_3t^3 + a_0 - b_9t^9).\end{aligned}$$

A short calculation shows that the terms in brackets are positive. Hence, $\mu(t) > 0$. □

3. Main result

We are now in a position to determine the largest number α and the smallest number β such that (1.3) holds for all positive x.

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THEOREM 3.1. *For all real numbers $x > 0$, we have*

$$\sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^{x}\left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\alpha}{x^5}\right) < \Gamma(x+1) < \sqrt{2\pi x}\left(\frac{x}{\mathrm{e}}\right)^{x}\left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\beta}{x^5}\right) \tag{3.1}$$

with the best possible constants $\alpha = 0$ and $\beta = \frac{1}{1620}$.

Proof. First, we prove the left-hand side of (3.1) with $\alpha = 0$. For $x > 0$, we define

$$f(x) = \log\Gamma(x+1) + x - \tfrac{1}{2}x\log\sinh\left(\frac{1}{x}\right) - \tfrac{1}{2}(1+3x)\log x - \tfrac{1}{2}\log(2\pi).$$

Differentiation gives

$$2f'(x) = g\left(\frac{1}{x}\right), \tag{3.2}$$

where

$$g(t) = 2\psi\left(1+\frac{1}{t}\right) - 1 - t + 3\log t - \log\sinh t + t\coth t.$$

We obtain

$$g'(t) = -\frac{2}{t^2}\psi'\left(1+\frac{1}{t}\right) + \frac{3}{t} - 1 + t - t(\coth t)^2. \tag{3.3}$$

Lemma 2.2 yields

$$T_4(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} < \psi'(x). \tag{3.4}$$

Using (3.4) and (2.6) leads to

$$\begin{aligned} g'(t) &< -\frac{2}{t^2}T_4\left(1+\frac{1}{t}\right) + \frac{3}{t} - 1 - \frac{t}{(\sinh t)^2} \\ &< -\frac{2}{t^2}T_4\left(1+\frac{1}{t}\right) + \frac{3}{t} - 1 - t\chi(t) \\ &= \frac{t^5}{51\,975(t+1)^9}\lambda(t), \end{aligned}$$

where λ is defined in (2.18). From lemma 2.4 we conclude that g' is negative on $(0, 2.1]$.

Applying (3.4) and $\coth t > 1$ $(t > 0)$, from (3.3) we obtain

$$g'(t) < -\frac{2}{t^2}T_4\left(1+\frac{1}{t}\right) + \frac{3}{t} - 1 = \nu(t), \quad \text{say.} \tag{3.5}$$

Let

$$\omega(t) = 105t(t+1)^9\nu(t). \tag{3.6}$$

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Then,

$$\omega(t) = -105t^{10} - 630t^9 - 1286t^8 - 87t^7 + 4657t^6 + 10\,353t^5 + 11\,977t^4 + 8505t^3 + 3745t^2 + 945t + 105.$$

Since

$$\omega^{(k)}(2.1) < 0 \text{ for } k = 0, 1, \dots, 6 \quad \text{and} \quad \omega^{(7)}(t) < 0 \text{ for } t > 0,$$

we conclude that $\omega(t) < 0$ for $t \geqslant 2.1$. It follows from (3.5) and (3.6) that g' is also negative on $[2.1, \infty)$.

Hence, g is strictly decreasing on $(0, \infty)$. Using (2.2) yields

$$g(t) < \lim_{z \to 0} g(z) = 0, \quad t > 0.$$

From (3.2) we conclude that f' is negative on $(0, \infty)$. Applying (2.1) gives

$$f(x) > \lim_{z \to \infty} f(z) = 0.$$

This leads to the first inequality in (3.1) with $\alpha = 0$.

Next, we prove the right-hand side of (3.1) with $\beta = \frac{1}{1620}$. Let $x > 0$ and

$$u(x) = \log \Gamma(x+1) + x - \tfrac{1}{2} x \log \sinh\left(\frac{1}{x}\right) - \tfrac{1}{2}(1+3x)\log x - \log\left(1 + \frac{\beta}{x^5}\right) - \tfrac{1}{2}\log(2\pi).$$

We have

$$2u'(x) = v\left(\frac{1}{x}\right), \tag{3.7}$$

where

$$v(t) = 2\psi\left(1 + \frac{1}{t}\right) - 1 - t + \frac{10\beta t^6}{\beta t^5 + 1} + 3\log t - \log\sinh t + t\coth t.$$

From (2.5) we obtain

$$\psi'(x) < T_5(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{66x^{11}}. \tag{3.8}$$

Applying (3.8) and (2.6) gives

$$\begin{aligned} v'(t) &= -\frac{2}{t^2}\psi'\left(1 + \frac{1}{t}\right) + \frac{3}{t} - 1 + t + \frac{10\beta t^5(\beta t^5 + 6)}{(\beta t^5 + 1)^2} - t(\coth t)^2 \\ &> -\frac{2}{t^2}T_5\left(1 + \frac{1}{t}\right) + \frac{3}{t} - 1 + t + \frac{10\beta t^5(\beta t^5 + 6)}{(\beta t^5 + 1)^2} - t\left(1 + \frac{1}{(\sinh t)^2}\right) \\ &> -\frac{2}{t^2}T_5\left(1 + \frac{1}{t}\right) + \frac{3}{t} - 1 + \frac{10\beta t^5(\beta t^5 + 6)}{(\beta t^5 + 1)^2} - t\theta(t) \\ &= \frac{t^7}{51\,975(t+1)^{11}(t^5 + 1620)^2}\mu(t), \end{aligned} \tag{3.9}$$

where μ is defined in (2.19). Lemma 2.5 yields that $v'(t) > 0$ for $0 < t \leqslant 2$.

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Using

$$\psi'(x) < T_1(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} \quad \text{and} \quad t < \sinh t, \quad t > 0,$$

we obtain from (3.9) that

$$\begin{aligned} v'(t) &> -\frac{2}{t^2} T_1\left(1 + \frac{1}{t}\right) + \frac{3}{t} - 1 + t + \frac{10\beta t^5(\beta t^5 + 6)}{(\beta t^5 + 1)^2} - t\left(1 + \frac{1}{(\sinh t)^2}\right) \\ &> -\frac{2}{t^2} T_1\left(1 + \frac{1}{t}\right) + \frac{2}{t} - 1 + \frac{10\beta t^5(\beta t^5 + 6)}{(\beta t^5 + 1)^2} = \sigma(t), \quad \text{say.} \end{aligned}$$

We define

$$\tau(t) = \frac{3}{t}(t+1)^3(t^5 + 1620)^2 \sigma(t).$$

Then we have

$$\begin{aligned} \tau(t) = {} & 27t^{12} + 87t^{11} + 89t^{10} + 30t^9 + 281\,880t^7 + 865\,080t^6 \\ & + 871\,560t^5 + 291\,600t^4 - 7\,873\,200t^2 - 7\,873\,200t - 2\,624\,400. \end{aligned}$$

Since

$$\tau^{(k)}(2) > 0 \quad \text{for } k = 0, 1, 2 \quad \text{and} \quad \tau'''(t) > 0 \quad \text{for } t > 0,$$

we conclude that τ, σ and v' are positive on $[2, \infty)$.

Thus, v is strictly increasing on $(0, \infty)$ with

$$\lim_{z \to 0} v(z) = 0,$$

so that (3.7) implies that $u'(x) > 0$. It follows that

$$u(x) < \lim_{z \to \infty} u(z) = 0.$$

This yields the second inequality in (3.1) with $\beta = \frac{1}{1620}$.

The double inequality (3.1) is equivalent to

$$\alpha < F(x) < \beta,$$

where

$$F(x) = x^5 \left(\frac{\Gamma(x+1)(\mathrm{e}/x)^x}{\sqrt{2\pi x}(x \sinh(1/x))^{x/2}} - 1 \right).$$

We have

$$\lim_{x \to 0} F(x) = 0$$

and using (2.3) and (2.4) gives

$$\lim_{x \to \infty} F(x) = \frac{163\,879}{209\,018\,880} - \frac{6971}{41\,803\,776} = \frac{1}{1620}.$$

This implies that in (3.1) the best possible constants are $\alpha = 0$ and $\beta = \frac{1}{1620}$. $\square$

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REMARK 3.2. We denote the lower and upper bounds for $\Gamma(x+1)$ given in (1.1) by $L_1(x)$ and $U_1(x)$, respectively, and the lower and upper bounds given in (3.1) by $L_2(x)$ and $U_2(x)$, respectively. Then we have the following limit relations:

$$\lim_{x\to 0} \frac{L_1(x) - L_2(x)}{(x/\mathrm{e})^x} = \frac{\sqrt{\pi}}{\sqrt[3]{10}}, \qquad \lim_{x\to\infty} x^{5/2} \frac{L_1(x) - L_2(x)}{(x/\mathrm{e})^x} = -\frac{7}{14\,400}\sqrt{2\pi},$$

$$\lim_{x\to 0} x^{9/2} \frac{U_1(x) - U_2(x)}{(x/\mathrm{e})^x} = -\frac{\sqrt{2\pi\mathrm{e}}}{1620}, \qquad \lim_{x\to\infty} x^{7/2} \frac{U_1(x) - U_2(x)}{(x/\mathrm{e})^x} = \frac{11}{11\,520}\sqrt{2\pi}.$$

This implies that for all sufficiently small x the lower and upper bounds in (1.1) are better than those in (3.1), whereas for all sufficiently large x, the lower and upper bounds in (3.1) improve those given in (1.1).

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